An Economic Nonlinear Model Predictive Control Approach for Mitigating Epidemic Spreading on Networks

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Lorenzo Calogero, Michele Pagone, Lorenzo Zino, and Alessandro Rizzo

***Abstract*— We consider a discrete-time susceptible–infected–susceptible epidemic model on a network, in which we incorporate two control actions: vaccination of part of the population and implementation of non-pharmaceutical interventions. Then, we formulate the problem of devising an optimal control strategy for the epidemic disease using the two actions, with a tradeoff between public healthcare impact of the disease and social and economic costs associated with interventions. The control problem is solved by leveraging an economic nonlinear model-predictive control scheme, for which we prove the closed-loop stability using a dissipativity argument.**

I. Introduction

In the last decade, the systems and control community has witnessed a growing interest in epidemic modeling [1]–[5], which was further boosted by the collective effort during the COVID-19 health crisis [6]–[9]. Within this context, particular interest has been devoted to network epidemic models, which leverage graph theory to capture the complex spatial and social heterogeneity of populations and, consequently, are capable of reproducing and analyzing the nontrivial patterns that characterize real-world epidemic spreading.

Mathematical modeling has enabled the scientific community to leverage control-theoretic tools to design and assess different control policies for epidemic outbreaks [1], [4]. Different aspects of epidemic control have been investigated, including optimal drug distribution [10], vaccination campaigns planning [11]–[13], optimizing non-pharmaceutical interventions (NPIs) [6]–[9], [14], also accounting for collective behavioral responses [15]–[19].

In this context, Model Predictive Control (MPC) has emerged as a powerful tool [12], [14], [20]. The reason for such a success relies on its capability to deal with nonlinear dynamics and to provide optimal control commands for multi-variable systems in the presence of inputs, outputs, and state constraints [21]–[23]. In the past few years, especially starting from the COVID-19 health crisis, MPC has started being adopted in the contest of epidemic control [8], [9], [12], [14], [24], [25]. However, standard MPC for regulation/tracking may not be sufficiently effective in scenarios where economic performance is the primary goal rather than a fast convergence to the equilibrium. Indeed, standard MPC is unaware of the transient behavior of the system, being designed for ensuring asymptotic tracking of the steady-state equilibrium. Hence, the controller tries to push the system to the target, disregarding the economic properties of the plant, away from non steady economic optimum [26]. Tailoring this limitations to the application at hand, it is worth to stress that, in epidemic models, besides being important to control the number of infections, it is often also key to consider the social and economic impact associated with interventions, which often leads to a nontrivial cost function.

To address these limitations, we propose a novel method to control network epidemic models by leveraging Economic Nonlinear MPC. Economic MPC (E-MPC) [27], [28] is arisen as an advanced control tool where the economic cost of the system is employed as the stage cost for the dynamic regulation purpose. The main goal of E-MPC is to provide an optimal control law which allows the plant to operate as close as possible to its economically optimal operation point, while ensuring stability of the closed loop (see, e.g., [29]).

Our main contribution is threefold. First, we cast the optimal control problem for an endemic epidemic disease within the framework of Economic Nonlinear MPC (E-NMPC), by defining a suitable cost-function that trades-off the impact to the healthcare system of the disease, the social and economical impact of NPIs, and the costs associated with pharmaceutical interventions such as vaccination. Second, we use a dissipativity argument [27], [30] to prove closed-loop stability of the E-NMPC scheme. Third, we demonstrate our framework in a realistic scenario of epidemic spreading on a real-world network reconstructed using mobility data between Italian regions [31]. In this case study, we compare the performance of the proposed E-NMPC scheme with a classical NMPC for regulation and quadratic cost function, illustrating the main advantages of the proposed approach.

Notation: We denote by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$ the real, real nonnegative, strictly positive real, positive integer, and strictly positive integer numbers, respectively. The all-1 and all-0 vectors and the identity matrix are denoted by $\mathbf{1}$, $\mathbf{0}$, and I, respectively, with dimensions omitted when clear from the context. Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ its spectral radius (maximum eigenvalue in modulus).

II. Controlled Epidemic Model

We consider a population stratified into n subpopulations, each representing geographic displacement of the individuals, risk classes, or age cohorts. The interactions between and within subpopulations are captured by of a (weighted) adjacency matrix $A \in \mathbb{R}^{n\times n}_{\geq 0}$, where entry A_{ij} represents the amount of contacts that individuals in the i-th subpopulation

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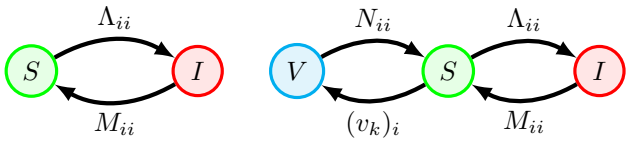

(a) Uncontrolled model (b) Controlled model

Fig. 1. Schematic of the (a) uncontrolled and (b) controlled epidemic models, representing the transitions between the states of susceptible (S, in green), infected (I, in red), and vaccinated (V, in cyan), with the corresponding transition rates.

establish with those in the j-th subpopulation. Without any loss in generality, we assume that the amounts of contact are normalized to 1, so that A is stochastic ($A\mathbf{1} = \mathbf{1}$).

We assume that subpopulations are large, so that we can approximate them as a continuum of individuals. Consequently, we can define the state vector $y_k \in [0,1]^n$, which quantifies the fraction of infected population at time $k \in \mathbb{Z}_{\geq 0}$ for each subpopulation. We assume that the fraction of infected individuals evolves according to a standard discrete-time SIS model [3], whereby recovered individuals are immediately susceptible again to the disease. Hence, the fraction of susceptible individuals at time k is equal to $\mathbf{1}-y_k$. This is a good proxy for many real-world diseases with limited (or even without) natural immunity, such as several sexually transmitted diseases or tuberculosis.

According to the SIS epidemic model, the uncontrolled dynamics is governed by the following nonlinear dynamics:

$$y_{k+1} = (I - hM)y_k + h\Lambda \operatorname{diag}(\mathbf{1} - y_k)Ay_k, \qquad (1)$$

where $M \in \mathbb{R}^{n\times n}_{>0}$ and $\Lambda \in \mathbb{R}^{n\times n}_{>0}$ are a diagonal matrices of recovery and infection rates, respectively, and $h \in \mathbb{R}_{>0}$ is the time-step of the discrete-time dynamics. In other words, Eq. (1) posits that the fraction of infected individuals in a generic i-th subpopulation at time $k+1$ is given by the sum of two contrasting mechanisms. First, individuals in the i-th subpopulations recovers with rate M_{ii}. Hence, the fraction of infected individuals decreases by hM_{ii} at each time step. Second, susceptible individuals in the i-th subpopulations become infected with rate Λ_{ii}, upon interactions with infected individuals in their subpopulation or in other subpopulations. This leads to an increase in the fraction of infected individuals captured by the second, positive term. A schematic of the SIS epidemic model is reported in Fig. 1a.

A. Control Actions

We incorporate two control actions within Eq. (1):

1) Vaccination: At time k, a rate $v_k \in [0,1]^n$ of the population is vaccinated, with each component representing the vaccinations in each subpopulation (a fraction hv_k of vaccinated population at the k-th time step). We assume that only susceptible individuals can be vaccinated and that vaccination provides full immunity to the disease, but it wanes at rate $N_{ii} \in \mathbb{R}_{>0}$ for the i-th subpopulation.

2) NPIs: At time k, social activity of individuals is reduced by a factor $n_k \in [0,1]^n$, with each component representing the reduction in each subpopulation.

To guarantee that the model is realistic in terms of the recovery and immunity-waning processes, we need to make the following assumption on the discretization time-step.

Assumption 1. *Let us assume that the time-step satisfies* $h \leq \min_{i\in\{1,\dots,n\}}\{1, M_{ii}^{-1}, N_{ii}^{-1}, (1+\Lambda_{ii})^{-1}\}$.

Hence, the controlled SIS dynamics follows:

$$\begin{aligned} y_{k+1} &= (I - hM)y_k \\ &\quad + h\Lambda \operatorname{diag}(\mathbf{1} - y_k - z_k)A \operatorname{diag}(\mathbf{1} - n_k)y_k, \quad (2a) \\ z_{k+1} &= (I - hN)z_k + hv_k, \quad (2b) \end{aligned}$$

where vector $z_k \in [0,1]^n$ represents the fraction of vaccinated population in each subpopulation at time k. In summary, the state vector of the controlled system is $x_k = [y_k, z_k]^\top \in [0,1]^{2n}$, the control input is $u_k = [v_k, n_k]^\top \in [0,1]^{2n}$. A schematic for the controlled epidemic model is reported in Fig. 1b. It is important to observe that, for sufficiently small time steps h, we can guarantee that Eq. (2) is well-defined, as proved in the following.

Proposition 1. *Under Assumption 1,* $\mathcal{D} = \{x_k \in [0,1]^{2n} : \mathbf{0} \leq y_k + z_k \leq \mathbf{1}\}$ *is positively invariant under Eq. (2).*

Proof. We prove the statement by induction. Clearly if initial conditions are in $\mathcal{D}$, then $x_0 \in \mathcal{D}$. Let us assume that $x_k \in \mathcal{D}$ and consider a generic entry i. Then, clearly $(y_{k+1})_i \geq (1 - hM_{ii})(y_k)_i \geq 0$. On the other hand, $(z_{k+1})_i \geq (1-hN_{ii})(z_k)_i \geq 0$. On the other hand, using the fact that A is stochastic, we get $(y_{k+1})_i \leq (1-hM_{ii})(y_k)_i + h\Lambda_{ii}(1 - (y_k)_i) \leq 1$. Moreover, from the second equation we get $(z_{k+1})_i \leq (z_k)_i + h(v_k)_i \leq 1$, due to the constraint on v_k (and the fact that $h \leq 1$). Finally, we are left to prove that $(y_k + z_k)_i \leq 1$. By summing the two equations, we get $(y_{k+1} + z_{k+1})_i \leq (1 - hM_{ii})(y_k)_i + h\Lambda_{ii}(1 - (y_k + z_k)_i) + (1 - hN_{ii})z_k + hv_k \leq (1 - h\min\{M_{ii}, N_{ii}\})(y_k + z_k)_i + h(\Lambda_{ii} + 1)(1 - (y_k + z_k)_i) \leq 1$, which yields the claim. □

The theory of network epidemic processes [3] guarantees that, given a constant control input u_s, there is always a unique attractive equilibrium x_s of Eq. (2). This equilibrium is either the disease-free equilibrium (DFE) $x_s = \mathbf{0}$, or an endemic equilibrium (EE) with x_s, with at least an entry $(x_s)_i > 0$. In particular, we can make the following claim.

Proposition 2. *Consider Eq. (2) under Assumption 1 with constant control input* $u_s = (v_s, n_s)$. *Then:*

1) *If* $\sigma(I - h(\Lambda \operatorname{diag}(\mathbf{1} - N^{-1}v_s)A \operatorname{diag}(\mathbf{1} - n_s) - M)) \leq 1$, *then the DFE* $x_s = \mathbf{0}$ *is the only equilibrium of Eq. (2), and it is asymptotically stable;*
2) *If* $\sigma(I - h(\Lambda \operatorname{diag}(\mathbf{1} - N^{-1}v_s)A \operatorname{diag}(\mathbf{1} - n_s) - M)) > 1$, *then the DFE is unstable, and Eq. (2) has a unique EE* x_s, *which is asymptotically stable.*

Proof. From Eq. (2b), we observe that, at the equilibrium, it necessarily holds $z_s = N^{-1}v_s$. By inserting this expression into Eq. (2a), and introducing $\tilde{\Lambda} = \Lambda \operatorname{diag}(\mathbf{1} - N^{-1}v_s)$ and

$\tilde{A} = A\,\mathrm{diag}(\mathbf{1} - n_s)$, the dynamics obtained coincides with a network SIS model with heterogeneous population sizes (the total population in the i-th sub-population is equal to $1 - v_s/N_{ii}$). Hence, the claims follow from the theory of discrete-time SIS network models [3], [32]. □

Remark 1. *In the absence of control, the results in Proposition 2 reduces to the well-known condition on $\sigma(\Lambda A - M)$.*

The case in which the DFE is unstable is the most interesting, since it implies that control is not able to completely eradicate the disease. In that scenario, the EEs can be computed in closed form under the following homogeneity assumption, which is standard in epidemic modeling [4].

Assumption 2. *Let us assume that, in Eq. (1) and Eq. (2), $M = \mu I$, $\Lambda = \lambda I$, and $N = \nu I$, with $\mu, \lambda, \nu \in \mathbb{R}_{>0}$.*

Proposition 3. *Let us consider the uncontrolled SIS dynamics in Eq. (1) under Assumptions 1 and 2. Then, the EE of Eq. (1) is given by $y_s = (1 - \frac{\mu}{\lambda})\mathbf{1}$.*

Proof. The EE $y_s \neq \mathbf{0}$ of Eq. (1) must satisfy $y_s = (I - hM)y_s + h\Lambda\,\mathrm{diag}(1 - y_s)Ay_s$, which simplifies as $My_s = \Lambda\,\mathrm{diag}(1 - y_s)Ay_s$; moreover, it can be rewritten in component-wise form as

$$\mu y_{s,i} = \lambda(1 - y_{s,i}) \sum_{j=0}^{n} A_{ij} y_{s,j}, \quad \forall\, i = 1, \dots, n. \tag{3}$$

Now, let $m = \min_j y_{s,j}$ and $\overline{M} = \max_j y_{s,j}$. Being A row stochastic, by convex combination it holds that $m \leq \sum_{j=0}^{n} A_{ij} y_{s,j} \leq \overline{M}$. Now, considering Eq. (3) for the minimum and maximum components of y_s, it holds that

$$\mu m \geq \lambda(1 - m)m \;\; \Rightarrow \;\; m \geq 1 - \frac{\mu}{\lambda}, \tag{4a}$$

$$\mu \overline{M} \leq \lambda(1 - \overline{M})\overline{M} \;\; \Rightarrow \;\; \overline{M} \leq 1 - \frac{\mu}{\lambda}, \tag{4b}$$

yielding $m = \overline{M} = 1 - \frac{\mu}{\lambda}$ and, thus, $y_s = (1 - \frac{\mu}{\lambda})\mathbf{1}$. □

Concerning the controlled SIS dynamics in Eq. (2), we restrict to a specific subset of EEs of interest, as follows:

Proposition 4. *Let us consider the controlled SIS dynamics in Eq. (2) under Assumptions 1 and 2. Let us pick a constant state, shaped as $(y_s, z_s) = (\mathbf{1}\overline{y}, \mathbf{1}\overline{z})$. Then, there exists a constant input $(v_s, n_s) = (\mathbf{1}\overline{v}, \mathbf{1}\overline{n})$ such that (y_s, z_s, v_s, n_s) is an EE of Eq. (2) and*

$$\overline{v} = \nu\overline{z}, \quad \overline{n} = 1 - \frac{\mu}{\lambda}\frac{1}{1 - \overline{y} - \overline{z}}. \tag{5}$$

Proof. By replacing z_s and v_s in Eq. (2b), it easily follows that $\overline{v} = \nu\overline{z}$. Now, replacing y_s, n_s, and z_s in Eq. (2a), and expressing it in component-wise form yields

$$\mu\overline{y} = \lambda(1 - \overline{y} - \overline{z}) \sum_{j=0}^{n} A_{ij}(1 - \overline{n})\overline{y}, \quad \forall\, i = 1, \dots, n. \tag{6}$$

Being A row stochastic, $\sum_{j=0}^{n} A_{ij} = 1$, $\forall\, i$. Then, solving Eq. (6) for $\overline{n}$ yields $\overline{n} = 1 - \frac{\mu}{\lambda}\frac{1}{1-\overline{y}-\overline{z}}$. □

III. Economic Model Predictive Control

Consider the controlled SIS dynamics in Eq. (2), compactly denoted by

$$x_{k+1} = f(x_k, u_k), \tag{7}$$

where $x_k = [y_k, z_k]^\top \in [0,1]^{2n} = \mathcal{X}$ and $u_k = [v_k, n_k] \in [0,1]^{2n} = \mathcal{U}$ are the state and input vectors at time k, respectively; also, let $\Xi = \mathcal{X} \times \mathcal{U} = [0,1]^{4n}$ and $\xi = [x^\top, u^\top]^\top = (x, u)$.

Let us now consider an equilibrium point $(x_s, u_s) = \xi_s$ of Eq. (2) and Eq. (7), satisfying $x_s = f(x_s, u_s)$, for $(x_s, u_s) = \xi_s \in \Xi$. Among the possible equilibria of Eq. (2) and Eq. (7), we restrict our interest to those given by Proposition 4. Thus, we define the following equilibrium manifold:

$$\Xi_s = \Big\{\xi_s \in \Xi : \;\; \xi_s = (\mathbf{1}\overline{y}, \mathbf{1}\overline{z}, \mathbf{1}\overline{v}, \mathbf{1}\overline{n}), \\ \overline{v} = \nu\overline{z}, \;\; \overline{n} = 1 - \frac{\mu}{\lambda}\frac{1}{1 - \overline{y} - \overline{z}}\Big\}. \tag{8}$$

Our control task is to stabilize the controlled SIS dynamics in Eq. (2) towards an equilibrium $\xi_s \in \Xi_s$, ensuring that the system operates with a profitable economic performance. To this aim, we introduce an economic criterion to be optimized, denoted by $\ell(x, u)$. The desired equilibrium ξ_s is chosen as that minimizing the economic criterion ℓ, i.e.,

$$(x_s, u_s) = \underset{(x,u)\in\Xi_s}{\arg\min}\; \ell(x, u) \tag{9}$$

Typically, $\ell(x, u)$ is non-quadratic and non-convex.

Assumption 3. *Let us assume that the economic criterion ℓ satisfies the following two properties:*

- *$\exists!\,(x_s, u_s) \in \Xi_s$ minimizing $\ell(x, u)$;*
- *ℓ is locally Lipschitz continuous on Ξ.*

Note that, the second property in Assumption 3 is needed to prevent the "blow-up" of the economic criterion, in finite time, for bounded variations of its variables.

We are now in position to formulate our E-NMPC problem. Let us define the following economic cost function:

$$J_N(\hat{x}, \hat{u}) = \sum_{j=0}^{N-1} \ell(\hat{x}_j, \hat{u}_j) + V_o(\hat{x}_N), \tag{10}$$

where $\hat{x} = \{\hat{x}_j\}_{j=0}^{N}$ and $\hat{u} = \{\hat{u}_j\}_{j=0}^{N-1}$ are state and input sequences over the discrete time interval $j \in \{0, \dots, N\}$. In its general formulation, the economic cost function J_N in Eq. (10) comprises the economic criterion ℓ as stage cost and a terminal offset cost V_o [33]. The offset cost $V_o(\hat{x}_N)$ has the purpose of facilitating the convergence of $\hat{x}_N$ to x_s.

Assumption 4. *Let us assume that V_o is non-negative and strictly convex on $\mathcal{X}$, and satisfies $x_s = \arg\min_{x\in\mathcal{X}} V_o(x)$.*

The E-NMPC optimal control law is derived by solving the following optimal control problem (OCP) at each time instant $k \geq 0$:

$$\hat{u}^* = \underset{\hat{x},\hat{u}}{\arg\min}\; J_N(\hat{x}, \hat{u}) \tag{11a}$$

$$\text{s.t.} \quad \hat{x}_0 = x_k, \quad \hat{x}_{j+1} = f(\hat{x}_j, \hat{u}_j), \tag{11b}$$
$$(\hat{x}_j, \hat{u}_j) \in \Xi, \tag{11c}$$
$$\hat{x}_N = x_s, \tag{11d}$$
$$j = 0, \dots, N-1.$$

The OCP in Eq. (11) takes as input the current system state x_k and provides the optimal predicted input sequence $\hat{u}^*$.

As in the usual MPC framework, the optimal control law is delivered to the plant in a receding horizon fashion. Hence, the optimal control input is given by $u_k = \hat{u}_0^*$.

IV. The Epidemic Control Problem

By framing the E-NMPC approach within the context of epidemic dynamics, we specialize the economic cost function in Eq. (10) to the case under investigation. The idea is to design a suitable cost function balancing the public health interventions and the economic/social costs, while steering the epidemic evolution towards the economically-optimal equilibrium. Therefore, we define ℓ as follows:

$$\begin{aligned}\ell(x,u) &= q_1(\mathbf{1}-y)^\top n + q_2 \sum_{i=1}^{n} g(y_i) \\ &\quad + q_3 \mathbf{1}^\top v + q_4 \|x - x_s\|_2^2 \\ &= q_1 l_1(y,n) + q_2 l_2(y) + q_3 l_3(v) + q_4 l_4(x), \end{aligned} \tag{12}$$

where $q = [q_1, q_2, q_3, q_4]^\top \in \mathbb{R}^4_{>0}$ are weighting parameters. The stage cost ℓ encodes the following control objectives:

- The term $l_1(y,n) = (\mathbf{1}-y)^\top n$ represents a sort of Gross Domestic Product (GDP) loss if the non-infected fraction of population is subject to restrictions and/or lockdown.
- The term $l_2(y) = \sum_{i=1}^{n} g(y_i)$, with $g : [0,1] \to \mathbb{R}_{\geq 0}$, accounts for the hospital occupancy in each subpopulation. Its aim is to prevent that the quota of infected individuals needing hospitalization and/or intensive care treatments overcomes the capacity $\tilde{y} \in (0,1)$. The function g is designed to exhibit the following properties:
  - $g(0) = 0$, $g(1) = 1$, $g(\tilde{y}) = a$, $0 < a \ll 1$,
  - $\frac{dg}{dy}(y) \geq 0$, $\forall\, y \in [0,1]$ (g is non-decreasing),
  - $\frac{dg}{dy}(0) = 0$ (g has a stationary point in $y = 0$),
  - $\frac{d^2 g}{dy^2}(\tilde{y}) = 0$ (g has an inflection point at $\tilde{y}$),

  where a is a design parameter. The above properties can be univocally achieved by a fourth-degree polynomial, i.e., $g(y) = c_1 y^4 + c_2 y^3 + c_3 y^2 + c_4 y = c^\top [y^4, y^3, y^2, y]^\top$.
- The term $l_3(v) = \mathbf{1}^\top v = \sum_{i=1}^{n} v_i$ represents the cost of the vaccination campaign.
- The term $l_4(x) = q_4 \|x - x_s\|_2^2$ is a regularization term for enforcing convergence towards x_s (see Section IV-A), which is added after having computed x_s through Eq. (9) using the first three terms of ℓ.

Unlike the general formulation of E-NMPC in Eq. (10) and Eq. (11), we set $V_o(x) = 0$, since we have no need to further facilitate the convergence of the state towards x_s. Our formulation differ from [14], where the cost function only accounts for the economy, and hard constraints are imposed for the epidemic spreading. Here, instead, we have a tunable tradeoff of healthcare and economy in Eq. (12).

A. Closed-Loop Stability Guarantees

In the E-MPC framework, ensuring closed-loop stability is not as straightforward as in the classic MPC setting. This arises because the economic stage cost ℓ is typically not minimal at the equilibrium (x_s, u_s) in Eq. (9), i.e., $\ell(x,u) < \ell(x_s,u_s)$ for some $(x,u) \in \Xi \setminus \Xi_s$. This aspect does not allow to employ the optimal cost value J_N^* of the OCP in Eq. (11) as a Lyapunov function, since J_N^* may not be monotonically decreasing along the trajectories of the closed-loop system given by Eq. (7) and Eq. (11), even if the latter is stable. A widely-recognized approach to establish stability guarantees for E-NMPC is based on dissipativity arguments [27], [30].

Definition 1. *The system in Eq. (7) is strictly dissipative with respect to the supply rate $s : \Xi \to \mathbb{R}$ if there exist a storage function $\pi : \mathcal{X} \to \mathbb{R}$ and a positive definite function $\rho : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that $\pi(f(x,u)) - \pi(x) \leq -\rho(x) + s(x,u)$, $\forall\, (x,u) \in \Xi$.*

From Definition 1, the following result holds true:

Theorem 1 ([27], [30])**.** *Consider the closed-loop system in Eq. (7) and Eq. (11), where (x_s, u_s) is in Eq. (9). If Eq. (7) is strictly dissipative with respect to the supply rate*

$$s(x,u) = \ell(x,u) - \ell(x_s,u_s), \tag{13}$$

and the function ρ in Definition 1 is positive definite in x_s (i.e., $\rho(x_s) = 0$, $\rho(x) > 0$, $\forall\, x \in \mathcal{X} \setminus \{x_s\}$), then x_s is an asymptotically stable equilibrium of the closed-loop system.

The result in Theorem 1 relies on introducing the so-called "rotated" stage cost [28], [33], [34], defined as $\tilde{\ell}(x,u) = \ell(x,u) + \pi(x) - \pi(f(x,u))$. The rotated stage cost allows to define a modified cost function $\tilde{J}_N$, whose optimal value $\tilde{J}_N^*$ can be employed as a suitable Lyapunov function.

Remark 2. *In order to make the E-NMPC problem "well-posed" and meaningful from a physical and practical point of view, we confine our discussion to the case in which the equilibrium $\xi = \mathbf{0}$ cannot be reached by any closed-loop trajectory. Indeed, such an equilibrium implies that the disease is not present among the population; furthermore, this equilibrium is unstable, as shown in Proposition 2.*

Remark 3. *In the following, for the sake of readability, we shall limit the theoretical analysis on dissipativity and closed-loop stability to the single-agent case (i.e., $n = 1$). Extension to the multi-agent case is quite straightforward with minor mathematical modifications.*

Theorem 2. *The controlled SIS dynamics in Eq. (2) is strictly dissipative with respect to the supply rate in Eq. (13).*

By Theorems 1 [27], [30] and 2, we can conclude that (x_s, u_s) given by Eq. (9) is an asymptotically stable equilibrium of the closed-loop system given by Eq. (7) and Eq. (11).

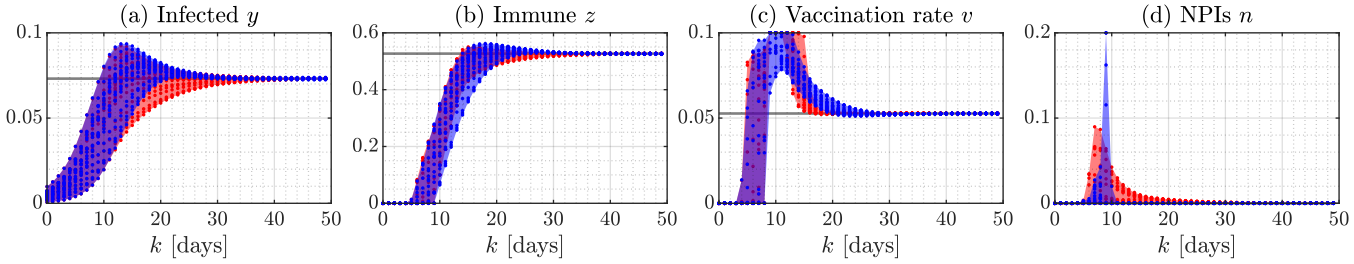

Fig. 2. Closed-loop trajectories of the SIS dynamics, under the control of standard NMPC (•) and our proposed E-NMPC (•).

V. Simulation Results

A. Settings

1) Implementation Details: The E-NMPC OCP in Eq. (11) is formulated with CasADi [35] and solved with Ipopt [36]. Simulations are performed in MATLAB® 2023b on a 13th Gen Intel® Core™ i7 CPU at 1.7 GHz.

2) Simulation Data: For the controlled SIS dynamics in Eq. (2), the adjacency matrix A in reconstructed from real-world data representing the amount of contacts between the $n = 21$ regions of Italy [31]. Further epidemic parameters are set as follows: $M = \mu I$, $\Lambda = \lambda I$, $N = \nu I$, $\mu = 0.2$, $\lambda = 0.5$, $\nu = 0.1$, $h = 1\,(\text{days})$. The initial state $x_0 = [y_0, z_0]^\top$ of Eq. (2) is set as follows: $z_0 = \mathbf{0}$, and each entry $(y_0)_i$ is sampled from a uniform probability distribution over $[10^{-4}, 10^{-2}]$, each entry independent of the others. The E-NMPC prediction horizon is set to $N = 10$. All simulations last $T_{\text{sim}} = N_{\text{sim}} h = 50\,\text{days}$ ($N_{\text{sim}} = 50$).

3) Parameters: The economic stage cost ℓ is given by Eq. (12). We select the weighting parameters as $q = [q_1, q_2, q_3, q_4]^\top = [1, 1, 0.5, 0.1]^\top$. For the term l_2 in Eq. (12), we set $a = 10^{-2}$, yielding $g(y) = c^\top [y^4, y^3, y^2, y]^\top$, $c = [2.23, -1.74, 0.51, 0.16]^\top$.

We then compute the optimal economic equilibrium $\xi_s = (y_s, z_s, v_s, n_s)$ through Eq. (9), obtaining $\xi_s = (0.0731, 0.5269, 0.0527, 0)$. We can observe how the term l_1 makes $n_s = 0$, in order not to have any steady-state GDP loss caused by continuously subjecting a constant fraction of non-infected people to restrictions/lockdown. The terms l_2 and l_3, instead, weight the other three quantities, favoring either a higher vaccination rate to reduce the fraction of infected population, or reduce vaccinations at the price of a higher endemic equilibrium. Concerning input and state constraints in Eq. (11c) of the E-NMPC OCP, we set an upper bound to the vaccination rate and NPIs, i.e., $\mathcal{U} = \{u = (v, n) \in [0, 1]^{2n} : u_{\text{lb}} \leq u \leq u_{\text{ub}}\}$, $u_{\text{lb}} = [0, 0]^\top$, $u_{\text{ub}} = [0.1, 0.2]^\top$.

B. Results

We assess the performance of E-NMPC, comparing it with standard NMPC, as reported in Figs. 2-3. Figure 2 reports the closed-loop trajectories of the SIS dynamics Eq. (2) under the control of E-NMPC in Eq. (11) and standard NMPC. Both schemes achieve regulation towards the prescribed equilibrium point (x_s, u_s), with different transient behaviors. The closed-loop performance are assessed in Fig. 3b in terms of point-wise and cumulative stage cost along the closed-loop trajectories, i.e., $\ell(x_k, u_k)$ and $\sum_{j=0}^{k} \ell(x_j, u_j)$, respectively, $\forall\, k = 1, \ldots, N_{\text{sim}}$. In general, the E-NMPC attains a lower cumulative cost by the end of the simulation for each stage costs term l_1, $i = 1, 2, 3$. The most noticeable difference is for term l_1, meaning that E-NMPC achieves a lower GDP loss due to the restrictive. Thus, it is clear how E-NMPC is capable of delivering a more profitable transient behavior in closed-loop, attenuating the number of NPI counter-measures while steering the system state towards the desired equilibrium at the same convergence rate of standard NMPC. Therefore, the simulation campaign confirms the effectiveness of our E-NMPC approach.

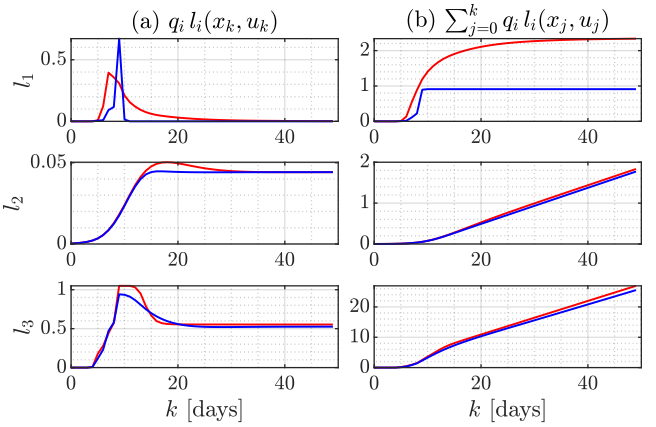

Fig. 3. Evolution of the (a) point-wise and (b) cumulative economic stage cost terms along the closed-loop trajectories: NMPC ——; E-NMPC ——.

VI. Conclusion

We designed an E-NMPC scheme for the control of an endemic disease by means of NPIs and vaccinations, and established closed-loop stability guarantees using a dissipativity argument. The use of an E-NMPC scheme allows us to trade-off the costs associated with the intervention policies, the healthcare burden, and the social and economical costs associated with NPIs. Our promising preliminary results pave the way for several avenues of future research. In fact, the proposed approach can be extended to more realistic scenarios of epidemic spreading, which may include other control actions. Furthermore, future research may focus on devising more sophisticated constraints set (e.g., dynamic constraints, terminal region different from the singleton, periodic references), for which further investigations on recursive feasibility of the optimal control problem will be required.

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