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# A Continuous Opinions Discrete Actions Model on Line Graphs

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**Abstract** The starting point of this work is a well-known class of linear systems on a graph that asymptotically converge to a consensus state. We consider a variation of this dynamics, by modifying some of the states through the nearest integer function: this change sets the dynamics apart from the consensus dynamics. We focus our study on the case in which the underlying graph is a line, which is particularly significant as it exhibits asymptotic behaviors that are far from consensus. In this case, we compute the equilibria of the system and prove convergence of solutions.

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## 1 Introduction

The systems of ordinary differential equations studied in this paper have the main feature that discrete and continuous states coexist. In fact, the $i$-th equation is linear in $x_i$, whereas other components enter the equation as arguments

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of the nearest integer function, thus taking values in a discrete set. Similar systems have been previously considered in the contest of control with limited information. The basic idea was that when implementing a control law, the information on the state may not be precise, or control laws may be constrained to take values in a discrete set [5,12,19]. Such systems have been addressed as quantized systems and control laws as quantized controls: in the following we talk about quantized system to mean a system with discretized states. The focus in [5,12] and related papers was to preserve stability, despite quantization, whereas in the systems we consider in this paper there are no controls to be chosen, but quantization is inherent to the model, so we focus on analysis.

Our reason for considering systems with quantized states comes from the opinion dynamics literature ([22,23]). One of the most basic principle in opinion dynamics is that opinions of individuals who communicate approach each other [13]. Here we assume that communication of one's opinion is not precise. The reason for such imprecision may be poor language, or the fact that opinion is not expressed verbally, but through a behavior or a choice among a finite number of options or actions, namely the quantized states. In the opinion dynamics literature we talk about CODA-models [17,18,9,10], where the acronym CODA stands for Continuous Opinion Discrete Actions. A discrete set of choices is also typical in economic literature where the discrete choice can be associated to a utility function.

The aim of the model we present here is not to obtain a realistic description of the evolution of opinions in a group, but to highlight the effect of communicating by means of a finite number of options or actions. To introduce the system that we study through this paper, we take a step back to Abelson's linear model [1,22], which is the continuous time counterpart of seminal French-DeGroot model [11]. We mainly use the term interaction to include communication as a special case. Let the indices $1,\dots,n$ represent $n$ individuals. The opinion of the $i$-th individual on a certain subject at time $t$ is expressed by means of the real number $x_i(t)$. The evolution of $x_i$ is assumed to depend on the opinions of the individuals interacting with $i$ and is regulated by the following equation:

$$\dot{x}_i = \sum_{j\neq i} a_{ij}(x_j - x_i),\ \ i,j = 1,\dots n, \tag{1}$$

where the coefficients $a_{ij} \in \mathbb{R}$ represent the interactions between individuals $i$ and $j$. The number $a_{ij}$ can be null if no communication occurs, positive if $i$ is influenced by $j$, or even negative, in the case of antagonistic interactions [2]. Here we consider the case $a_{ij} \in \{0,1\}$, thus only distinguishing whether interaction does or does not occur: if $a_{ij} = 0$, then agent $i$ is not influenced by $j$, whereas if $a_{ij} = 1$, then agent $j$ influences agent $i$ . Matrix $A = (a_{ij})$ can be seen as the adjacency matrix of the graph describing interactions among individuals. Under appropriate assumptions, namely the graph having a directed spanning tree, the states of the system converge to a common value, that is, a consensus state.

Being the tendency to reach consensus disproved by experience, a vast literature has developed to represent opinion dynamics in a more accurate way and to explain agreement and disagreement at the same time [22,23]. Our contribution, begun in [7], is situated in this framework and, as already mentioned, is aimed to highlight the effect of communicating opinions by means of a finite number of options or actions. In terms of equations, this is represented by the fact that in (1) the term $x_j$ is replaced by $q(x_j)$, where the function $q : \mathbb{R} \to \mathbb{Z}$ is defined by $q(s) := \lfloor s + \frac{1}{2} \rfloor$ and is called *quantizer.* The system of interest is then

$$\dot{x}_i = \sum_{j \neq i} a_{ij} [q(x_j) - x_i], \; i, j = 1, \dots n. \tag{2}$$

We sometimes call (2) *quantized actions system* or *quantized model.* It can be also written as

$$\dot{x} = A\mathbf{q}(x) - Dx, \tag{3}$$

where $A = (a_{ij})$, $\mathbf{q}(x) := (q(x_1), \dots, q(x_n))$ and $D$ is the diagonal matrix whose diagonal entries are $d_i := \sum_{j \neq i} a_{ij}$. The number $d_i$ is also called the *out-degree* of node $i$.

System (2)-(3) is a patchwork of affine systems, as $A\mathbf{q}(x)$ is constant on the hypercubes where $\mathbf{q}$ is constant. Patchworks of regular vector fields have been widely studied in the literature, mainly in connection with control problems ([3,16,4]). As already mentioned, here we focus on analysis. In fact, despite an apparently minor difference between (1) and (2), the underlying graphs being the same, qualitative behavior of the modified system may change a lot. The ultimate goal of our investigation is to understand asymptotic behavior for any topology. The general result in [7] (Theorem 4.2) is not fully satisfactory, in the sense that it proves convergence to an invariant set that is "large", and its statement cannot substantially be improved (as discussed in the comments following the proof in the same paper). Therefore, making assumptions on the graph topology is necessary for any substantial advance. In this paper, after introducing some notation and reviewing basic results, we focus on line graphs. The reason for this focus is that, among graphs having a directed spanning tree, we can see the line graph as the farthest graph from the complete graph in terms of connectivity. In [7], it was proved that in the case of the complete graph one has consensus despite quantization, whereas in the case of the line graph it was only shown an example of an equilibrium point far from consensus. In this paper, we advance the understanding of the latter case, by characterizing equilibria, counting them and proving that all solutions converge. After this analysis, we briefly review some examples emphasizing the distance between Abelson's linear model (1) and the quantized actions model (2) and show a new example that demonstrates how quantization can cause polarization of opinions.

To describe the novelty of this paper with respect to the literature, we first compare system (2) with [9]. The model in the latter paper is in discrete time, it originates by introducing quantization in a nonlinear (non-quantized) model, it emphasizes extreme opinions and allows for discrete states only in a binary set, whereas our model is continuous in time, originates from a linear (non-quantized) model and admits a non-binary discrete states.

We now briefly compare this work with previous work of ours [6, 7, 21], which consider CODA counterparts of Abelson's linear opinion dynamics model. In [6], a discrete-time version of (3) was introduced but not investigated, as it was evident that in general consensus could not be achieved and convergence to cycles showed up in simulations. The continuous-time CODA model (3) was instead introduced in [7]. In comparison to the latter paper, we provide a full characterization of the dynamics on the line graph and we enlarge the variety of non-consensus behaviors, including polarization.

In the conference paper [21], we focused on providing several examples of peculiar behaviors under the hypothesis of binary action, where $x_i \in [0, 1]$ for all $i$. These examples included cycles and Zeno solutions, which are not reported in this paper. In this paper, instead, we lift the restriction to binary actions and (i) focus on the analysis of the dynamics on line graphs; (ii) provide several non-consensus examples beyond line graphs, not present in the conference paper.

**Contents.** In Section 2 we collect definitions and basic results needed to understand the rest of the paper. In Sections 3 and 4 we focus on the undirected and directed line graphs: equilibria are characterized in Section 3, and convergence of solutions is proved in Section 4. In Section 5, behaviors occurring on less extreme graphs are briefly recalled and some examples are presented, including one showing polarization, together with some possible opinion dynamics interpretation. Finally, in Section 6 we conclude the paper and present future investigation directions. [1]

**Notation.** We denote by $\mathbf{0}$ the null vector in $\mathbb{R}^n$ and $\mathbf{1}$ the vector with all entries equal to 1. In general we do not distinguish between row vectors and column vectors, since whether a vector is a row or a column will be evident from the context.

## 2 Preliminaries

The function $\mathbf{q}$ in (3) causes the right-hand side of the equation to be discontinuous. For this reason, solutions must be intended in a generalized sense. For a presentation of this theory that is tailored to consensus dynamics, we refer the reader to [8]. Here we focus on Carathéodory solutions which, among generalized solutions, are the most similar to classical ones.

**Definition 2.1 (Carathéodory solution)** Given a differential equation

$$\dot{x}(t) = f(x(t)), \tag{4}$$

[1] Our manuscript has no associated data.

with initial condition $x(t_0) = x_0$, a *Carathéodory solution* on $I = [t_0, T) \subset \mathbb{R}$, $T \leq +\infty$, is an absolutely continuous function $\varphi : I \to \mathbb{R}^n$ such that

$$\dot{\varphi}(t) = f(\varphi(t)) \text{ for almost every } t \in I,$$

and $\varphi(t_0) = x_0$. Equivalently, $\varphi$ satisfies

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(\varphi(\tau))d\tau.$$

Existence of Carathéodory solutions of (2) is guaranteed by the following theorem.

**Theorem 2.1 (Existence and boundedness of solutions [7])** *For any initial condition, a Carathéodory solution of* (2) *exists on* $[t_0, +\infty)$ *and is bounded.*

In the rest of the paper, if not differently stated, we simply write solutions to intend Carathéodory solutions.

The standard definition of equilibrium point is given in the following.

**Definition 2.2 (Equilibrium)** Point $x^* \in \mathbb{R}^n$ is an *equilibrium* if the constant function $\varphi(t) = x^*$ is a solution of (4).

The states that have all coincident components play a special role in the systems considered in this paper and deserve definition.

**Definition 2.3 (Consensus state, quantized consensus state)** Point $\bar{x}$ is a *consensus state* if $\bar{x} = (\alpha, \ldots, \alpha) = \alpha\mathbf{1}$. Point $\bar{q}$ is a *quantized consensus state*, if it is a consensus state with $\alpha \in \mathbb{Z}$.

The following theorem gives a characterization of the asymptotic behavior of the Abelson's linear model under mild assumption on the underlying graph.

**Theorem 2.2 (Convergence to a consensus state for Abelson's linear model [22])** *Any solution of* (1) *asymptotically converges to a consensus state if and only if the graph with adjacency matrix $A$ has a directed spanning tree.*

In the case of the quantized model (2), convergence to consensus is so far been proved only in the cases of complete and bipartite complete graphs.

**Theorem 2.3 (Convergence to a consensus state for the quantized actions model [7])** *If the graph with adjacency matrix $A$ is complete or complete bipartite* [2]*, then, for any initial condition, solutions of* (2) *asymptotically converge to a consensus state. In particular, solutions converge to a quantized consensus state in the case of a complete graph.*

[2] A complete bipartite graph is a graph such that the nodes can be divided into two disjoint subsets $V_1, V_2$ such that there are edges connecting every node in $V_1$ to every node in $V_2$ and there are no edges connecting nodes within each of the two sets.

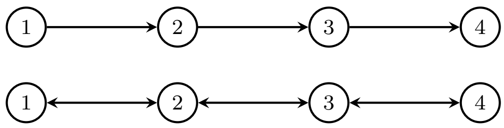


**Fig. 1** Directed and undirected line graphs with four nodes

If we do not restrict to complete and complete bipartite graphs, we need to introduce a new type of equilibrium that accounts for attractive points which are not zeros of the vector field. For this purpose, it is convenient to associate to (3) the sets where $\mathbf{q}$ is constant.

**Definition 2.4** Given $h \in \mathbb{Z}^n$, we define $S_h \subset \mathbb{R}^n$ as

$$S_h := \{x \in \mathbb{R}^n |\ h_i - \frac{1}{2} \leq x_i < h_i + \frac{1}{2}, i = 1, \dots, n\}.$$

System (3) is affine inside $S_h$, that we address as hypercubes with a slight abuse of language. We will often need to work with the closure of $S_h$, defined as

$$\overline{S_h} := \{x \in \mathbb{R}^n | h_i - \frac{1}{2} \leq x_i \leq h_i + \frac{1}{2}, i = 1, \dots, n\},$$

and with the vector fields $f_h$ which coincide with the restrictions of the right-hand side of (3) on $S_h$, namely

$$f_h(x) = Ah - Dx,\ (f_h(x))_i = \sum_{j \neq i} a_{ij}(h_j - x_i).$$

**Definition 2.5 (Extended equilibrium)** An *extended equilibrium* of (2) is a point $x^e \in \mathbb{R}^n$ such that there exists $h^e \in \mathbb{Z}^n$ such that $f_{h^e}(x^e) = 0$ and $x^e \in \overline{S_{h^e}}$.

In the following, we denote the set of extended equilibria with $\mathcal{E}_E$ and the set of Carathéodory equilibria with $\mathcal{E}_C$. We remark that $\mathcal{E}_C \subseteq \mathcal{E}_E$. Extended equilibria are important as their basin of attraction is a set with non-empty interior, as long as all nodes have out-degree at least 1. This fact is a consequence of the following result, whose proof is the same as the proof of [21, Prop. 1].

**Proposition 2.1 (Basin of attraction of extended equilibria [21])** *Let $x^e$ be an extended equilibrium and let $h^e \in \mathbb{Z}^n$ be such that $f_{h^e}(x^e) = 0$. If $d_i \neq 0$ for all $i = 1, \dots, n$, then all solutions of* (2) *with initial condition in $S_{h^e}$ converge to $x^e$.*

## 3 Equilibria on Line Graphs

In this section we fully characterize extended equilibria in the case of line graphs. Whereas for directed graph equilibria are easily obtained, for the undirected line this characterization requires some effort.

Equations (2) on the directed line become

$$\begin{cases} \dot{x}_i = q(x_{i+1}) - x_i, \ i = 1, \dots, n-1, \\ \dot{x}_n = 0. \end{cases} \tag{5}$$

The expressions of its equilibria are obtained by simply applying the definitions of Carathéodory and extended equilibria to (5).

**Proposition 3.1 (Equilibria of directed line)** *Carathéodory and extended equilibria sets for* (5) *are given by*

$$\mathcal{E}_C = \{(q(\alpha), \dots, q(\alpha), \alpha), \alpha \in \mathbb{R}\},$$

$$\mathcal{E}_E = \mathcal{E}_C \cup \{(z, \dots, z, z + 1/2), z \in \mathbb{Z}\}.$$

We remark that the directed line may show extended equilibria with empty basin of attraction. This does not contradict Proposition 2.1, since in this case $d_1 = 0$.

*Example 3.1 (Non-attractive extended equilibrium)* Consider a directed line graph with 2 nodes, whose equations are

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_2 = 0. \end{cases} \tag{6}$$

Note that $d_2 = 0$. In $S_{(0,0)}$, equation (6) reads

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = 0. \end{cases}$$

The point $\bar{x} = (0, 1/2)$ is an extended equilibrium for (6) as $\bar{x} \in \overline{S_{(0,0)}}$ and $f_{(0,0)}(\overline{x}) = 0$. Despite being an extended equilibrium, $\bar{x}$ is not attractive for any solution. In fact, solutions of (6) evolves on lines parallel to the $x_1$-axis as $x_2(t)$ is constant, then also $q(x_2(t))$ and solutions starting in $S_{(0,0)}$ can not reach $\bar{x}$. On the other hand, $q(\bar{x}) = (0, 1) \in S_{(0,1)}$. The unique solution starting at $\bar{x}$ follows the equation

$$\begin{cases} \dot{x}_1 = 1 - x_1, \\ \dot{x}_2 = 0, \end{cases}$$

and then converges to $(1, 1/2)$.

We now proceed to consider the undirected line, whose equations (2) become

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_i = q(x_{i+1}) + q(x_{i-1}) - 2x_i,\ i = 2, \dots, n-1, \\ \dot{x}_n = q(x_{n-1}) - x_n. \end{cases} \tag{7}$$

In [7] it was shown that there exist extended equilibria of (7) which are far from consensus. Here we characterize them all.

**Proposition 3.2 (Equilibria of undirected line)** *Let $k \in \mathbb{Z}^n$ be such that $k_1 = k_n = 0, k_j \in \{-1, 0, 1\}$ for $j = 2, ..., n-1$, and $\sum_{j=2}^{n-1} k_j = 0$. Extended equilibria of* (7) *are the points of the form*

$$x^e = \alpha \mathbf{1} + x^*, \tag{8}$$

*where $\alpha \in \mathbb{Z}$ and $x^*$ is such that*

$$x_i^* = \frac{1}{2}k_i + \sum_{j=1}^{i}(i-j)k_j.$$

*Proof* Throughout this proof, in order to find the extended equilibria, we need to consider the dynamics inside each set $S_h$ and extend it to the set's closure. In particular, an extended equilibrium $x \in \overline{S_h}$, such that $x$ nullifies the dynamics in $S_h$, satisfies $h_i - \frac{1}{2} \leq x_i \leq h_i + \frac{1}{2}$.

We begin by setting to 0 the derivatives in (7), obtaining

$$\begin{cases} q(x_2) - x_1 = 0, \\ q(x_{i+1}) + q(x_{i-1}) - 2x_i = 0, \\ q(x_{n-1}) - x_n = 0. \end{cases}$$

As $x_1 = q(x_2)$, $x_1$ must be an integer. Let $x_1 = \alpha \in \mathbb{Z}$, we obviously have $q(x_1) = \alpha$, and, from the first line of the system, we have that $q(x_2) = \alpha$. Rearranging the term in the second line of the system, we see that

$$x_i = \frac{q(x_{i+1}) + q(x_{i-1})}{2}, \quad i = 2, \dots, n-1,$$

which means that $x_i$ is an integer or half of an integer.With this information, if we know the quantized value $q(x_i)$, we have three possible values of $x_i$: $q(x_i) - \frac{1}{2}, q(x_i)$ or $q(x_i) + \frac{1}{2}$ (remember that extended equilibria are in the closure of the hypercubes). We write this in compact form as $x_i = q(x_i) + \frac{1}{2}k_i$, $k_i \in \{-1, 0, 1\}$.
We apply this fact to $x_2$ and obtain $x_2 = \alpha + \frac{1}{2}k_2$.
Let us consider again the second equation of the system, this time centered in $x_{i-1}$. We have

$$q(x_i) + q(x_{i-2}) - 2x_{i-1} = 0 \Longrightarrow q(x_i) = 2x_{i-1} - q(x_{i-2}),$$

which allows us to find $q(x_i)$, for $i-1=2,\dots,n-1 \Rightarrow i=3,\dots,n$. By iterating, we can obtain

$$x_3 = \alpha + k_2 + \frac{1}{2}k_3, \quad x_4 = \alpha + 2k_2 + k_3 + \frac{1}{2}k_4.$$

We now prove by induction that $q(x_i) = \alpha + \sum_{j=2}^{i-1}(i-j)k_j$, and by consequence that $x_i = \alpha + \sum_{j=2}^{i-1}(i-j)k_j + \frac{1}{2}k_i$. From the previous computations, we see that the statement holds true for $i = 3, 4$.
Moving to the inductive step, we assume that the statement holds for $i-2$ and $i-1$, and we check that it still holds for $i$:

$$\begin{aligned}
q(x_i) &= 2x_{i-1} - q(x_{i-2}) \\
&= 2\alpha + 2(\sum_{j=2}^{i-2}(i-1-j)k_j + \frac{1}{2}k_{i-1}) - \alpha - \sum_{j=2}^{i-3}(i-2-j)k_j \\
&= \alpha + k_{i-1} + \sum_{j=2}^{i-2}2(i-1-j)k_j - \sum_{j=2}^{i-3}(i-2-j)k_j \\
&= \alpha + k_{i-1} + 2k_{i-2} + \sum_{j=2}^{i-3}(i-j)k_j = \alpha + \sum_{j=2}^{i-1}(i-j)k_j.
\end{aligned}$$

Finally, we consider the third line of the system, in which we substitute the expressions $q(x_{n-1}) = \alpha + \sum_{j=2}^{n-2}(n-1-j)k_j$ and $x_n = \alpha + \sum_{j=2}^{n-1}(n-j)k_j$. Notice that $x_n = q(x_{n-1})$, i.e. $x_n$ is an integer, and consequently $x_n = q(x_n) \Rightarrow k_n = 0$. The condition $q(x_{n-1}) - x_n = 0$ becomes

$$\alpha + \sum_{j=2}^{n-2}(n-1-j)k_j - \alpha - \sum_{j=2}^{n-1}(n-j)k_j = 0,$$

from which we obtain

$$\sum_{j=2}^{n-1}k_j = 0.$$

Lastly, both $x_1$ and $x_n$ must be integer, thus they are equal to their own quantized value, that is $k_1 = k_n = 0$. Notice that the sum $\sum_{j=2}^{i-1}(i-j)k_j$ can equally be written as $\sum_{j=1}^{i}(i-j)k_j$, as the first and last terms in the latter version are null. □

Note that the choice of $\alpha \in \mathbb{Z}$ is arbitrary, which allows us to assume $\alpha = 0$. This is not restrictive, as every extended equilibrium having $\alpha \neq 0$ can be obtained as a translation of the extended equilibria having $\alpha = 0$. To simplify the notation, throughout the rest of the paper we will refer to these points, denoting them and their corresponding hypercube with an asterisk: $h^* \coloneqq (h_1^*, h_2^*, \ldots, h_n^*)$ and $x^* \coloneqq (x_1^*, x_2^*, \ldots, x_n^*)$, where

$$h_i^* \coloneqq \sum_{j=1}^{i}(i-j)k_j \text{ and } x_i^* \coloneqq h_i^* + \frac{1}{2}k_i. \tag{9}$$

Mind that, in general, $h^* \neq q(x^*)$, as $h^*$ represents the hypercube on which the right-hand side is zero if $x = x^*$, that is, $f_{h^*}(x^*) = 0$. While by definition $x^*$ belongs to $\overline{S_{h^*}}$, it does not necessarily belong to $S_{h^*}$. In particular, through Proposition 2.1, $S_{h^*}$ can be interpreted as the hypercube attracted by $x^*$. Naturally, the same also holds for the generic $x^e$ and $h^e$.

*Example 3.2* Let us compute all the extended equilibria (with $\alpha = 0$) on the line graph with 4 nodes. The only possible values for $k$ are $k_a = (0,0,0,0)$, $k_b = (0,1,-1,0)$, $k_c = (0,-1,1,0)$.

For $k_a$ we simply have $h_a^* = (0,0,0,0), x_a^* = (0,0,0,0)$, as all the term in the sums are null.

For $k_b$, the corresponding hypercube and extended equilibrium are $h_b^* = (0,0,1,1), x_b^* = (0,\frac{1}{2},\frac{1}{2},1)$. In fact, $h_{b1}^* = 0$ and $h_{b2}^* = k_{b1} = 0$, $h_{b3}^* = 2k_{b1} + k_{b2} = 0 + 1 = 1$, $h_{b4}^* = 3k_{b1} + 2k_{b2} + k_{b3} = 0 + 2 \cdot 1 - 1 = 1$, then, $x_b^* = h_b^* + \frac{1}{2}k_b = (0, 0 + \frac{1}{2}, 1 - \frac{1}{2}, 1) = (0, \frac{1}{2}, \frac{1}{2}, 1)$.

Analogously, $k_c = -k_b$ leads to $h_c^* = -h_b^* = (0,0,-1,-1)$,and $x_c^* = -x_b^* = (0,-\frac{1}{2},-\frac{1}{2},-1)$.

*Remark 3.1* To simplify the computation of $h^*$ in more complex cases, we point out that $h_i^*$ can be obtained from $h_{i-1}^*$ as

$$h_i^* = \sum_{j=1}^{i}(i-j)k_j = \sum_{j=1}^{i-1}(i-j)k_j = \sum_{j=1}^{i-1}(i-j-1)k_j + \sum_{j=1}^{i-1}k_j = h_{i-1}^* + \sum_{j=1}^{i-1}k_j.$$

*Remark 3.2* In [7] it was found that the furthest extended equilibrium from consensus, i.e. the extended equilibrium with the greatest difference between $x_1^*$ and $x_n^*$, is such that

$$|x_n^* - x_1^*| = \begin{cases} \frac{(n-2)^2}{4}, & \text{if } n \text{ even,} \\ \frac{(n-1)(n-3)}{4}, & \text{if } n \text{ odd.} \end{cases}$$

This is achieved for $k$ of the form

1. $k = (0,1,\ldots,1,-1,\ldots,-1,0)$, if $n$ is even,

2. $k = (0,1,\ldots,1,0,-1,\ldots,-1,0)$, if $n$ is odd,

and their opposites. In fact, if $x_1^* = 0$ in the first case we have

$$x_n^* = \sum_{j=2}^{n-1}(n-j)k_j = \sum_{j=2}^{n/2}(n-j) - \sum_{j=\frac{n}{2}+1}^{n-1}(n-j)$$
$$= \left(n-2+n-\frac{n}{2}\right)\left(\frac{n}{4}-\frac{1}{2}\right) - \left(n-1-\frac{n}{2}+1\right)\left(\frac{n}{4}-\frac{1}{2}\right) = \frac{(n-2)^2}{4},$$

while, similarly, in the second case

$$x_n^* = \sum_{j=2}^{n-1}(n-j)k_j = \frac{(n-1)(n-3)}{4}.$$

### 3.1 Cardinality of the Set of Extended Equilibria on an Undirected Line Graph

In this subsection we investigate the cardinality, denoted by #, of the set of extended equilibria, to see how it scales as $n$ grows larger. The extended equilibria are infinite due to the arbitrary choice of $\alpha$ in (8). Here, what we are interested in, is the cardinality of the set

$$\mathcal{E}_n^* \coloneqq \{x^*, \text{ with } x^* \text{defined in (9)}\}$$

for a line graph with $n$ nodes. To this end, as there is a one to one correspondence between extended equilibria and the $k$ defined in the hypothesis of Proposition 3.2, it suffices to count in how many different ways $k$ can be written while respecting such hypothesis.

**Proposition 3.3 (Number of equilibria)** *If $n$ is even, then*

$$\#\mathcal{E}_n^* = \sum_{i=0}^{\frac{n-2}{2}} \frac{(n-2)!}{((\frac{n-2-2i}{2})!)^2 \cdot (2i)!}.$$

*If $n$ is odd, then* $\#\mathcal{E}_n^* = \displaystyle\sum_{i=0}^{\frac{n-3}{2}} \frac{(n-2)!}{((\frac{n-3-2i}{2})!)^2 \cdot (2i+1)!}.$

*Proof* Given a line graph with $n$ nodes, we have that $k_1 = k_n = 0$, therefore the values that can vary are only $m = n-2$. We will find the cardinality by partitioning the set of all possible $k$s in subsets according to the number of components equal to 0 in $k$, i.e. $X_p^m$ is the set of all possible $k$s with $m$ components, of which $p$ components are null. Let's start by considering $m$ even, and by counting the cases for which $k_i \neq 0$ for all $i$. Since $k$ components must sum to 0, $k$ is formed by $\frac{m}{2}$ components with value 1 and $\frac{m}{2}$ with value $-1$.

The cardinality $\#X_0^m$ of this first subset is then given by a permutation with repetition:

$$\#X_0^m = P_m^{(\frac{m}{2},\frac{m}{2})} = \frac{m!}{\frac{m}{2}! \cdot \frac{m}{2}!}.$$

Consider now the subset of possible $k$s with 2 components equal to 0, and consequently $m-2$ components $\neq 0$. Just like in the previous case, these $m-2$ components can be ordered in

$$\#X_0^{m-2} = P_{m-2}^{(\frac{m-2}{2},\frac{m-2}{2})} = \frac{(m-2)!}{\frac{m-2}{2}! \cdot \frac{m-2}{2}!}.$$

From there we must add the 2 null components. From every result of the permutation of the $m-2$ non-null components, we can add the 2 null components in $P_m^{(2,m-2)}$ different ways, where in the apex the 2 represents the repetition of the null components and the $m-2$ the repetition of the non-null ones. This means that

$$\#X_2^m = P_{m-2}^{(\frac{m-2}{2},\frac{m-2}{2})} \cdot P_m^{(2,m-2)}.$$

We can repeat the same process to find $k$ with $p = 4, 6, 8, \ldots, m$ null components, to find

$$\#X_p^m = P_{m-p}^{(\frac{m-p}{2},\frac{m-p}{2})} \cdot P_m^{(p,m-p)}.$$

We then have, for $m$ even, that the cardinality we are looking for is the sum of the cardinalities of all these subsets, i.e. for $p = 2i$,

$$\begin{aligned}\#\mathcal{E}_n^* &= \sum_{i=0}^{\frac{m}{2}} \#X_{2i}^m = \sum_{i=0}^{\frac{m}{2}} P_{m-2i}^{(\frac{m-2i}{2},\frac{m-2i}{2})} \cdot P_m^{(2i,m-2i)} \\ &= \sum_{i=0}^{\frac{m}{2}} \frac{(m-2i)!}{((\frac{m-2i}{2})!)^2} \cdot \frac{m!}{(2i)! \cdot (m-2i)!} = \sum_{i=0}^{\frac{m}{2}} \frac{m!}{((\frac{m-2i}{2})!)^2 \cdot (2i)!}.\end{aligned}$$

The case for $m$ odd is similar. Notice that necessarily there is an even number of non-null components, as there must be an equal number of 1s and -1s. Consequently, if $m$ is odd there's at least one component with value 0. Elements in $X_1^m$ can be obtained by adding a null component to elements in $X_0^{m-1}$, which can be done in $m$ different ways (notice that $P_m^{(1,m-1)} = m$) per element in $X_0^{m-1}$, so that

$$\#X_1^m = \#X_0^{m-1} \cdot m = P_{m-1}^{(\frac{m-1}{2},\frac{m-1}{2})} \cdot P_m^{(1,m-1)}.$$

Analogously we can get the cardinality of the sets $X_p^m$ for $p = 3, 5, 7, \ldots, m$ and by summing them we obtain the statement. $\square$

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\#\mathcal{E}_n^*$ | 1 | 1 | 3 | 7 | 19 | 51 | 141 | 393 | 1107 |

**Table 1** Computed cardinality of the sets of extended equilibria on the line graphs with $n$ nodes

The results obtained using this formula are shown in Table 1. Notice that line graphs with 2 or 3 nodes are, respectively, complete and complete bipartite graph, for which convergence to consensus is guaranteed. For larger $n$, the number of equilibria grows fast: in fact, the growth is exponential in $n$. This can be be verified by taking the largest term of the sum and applying Stirling approximation to obtain $\#\mathcal{E}_n^* \geq g(n)$ with $g(n) \sim \frac{3^n}{18\pi n}$ as $n \to +\infty$. This is a relatively tight estimate considering that points in $\#\mathcal{E}_n^*$ depend on the choice of the parameters $k_j \in \{+1, -1, 0\}$, $j = 2, \ldots, n-1$, thus giving $\#\mathcal{E}_n^* \leq 3^n$.

## 4 Convergence on Line Graphs

In this section we study the asymptotic behaviors of (3) when the underlying graph is a line. We easily obtain convergence to Carathéodory equilibria in the case of the directed line, whereas convergence to extended equilibria in the case of the undirected line is more elaborate: a preliminary result was obtained in [21] in the special case when initial conditions belong to $[0, 1]^n$.

**Proposition 4.1 (Converge on directed line graph)** *For any initial condition, solutions of* (5) *converge to a Carathéodory equilibrium.*

*Proof* As $\dot{x}_n = 0$, it holds $x_n(t) = x_n(0)$ for all $t \geq 0$ and $q(x_n(t)) = q(x_n(0))$ for all $t \geq 0$. Consider now the equation for $x_{n-1}$:

$$\dot{x}_{n-1} = q(x_n) - x_{n-1} = q(x_n(0)) - x_{n-1}.$$

Its unique solution is $x_{n-1}(t) = e^{-t}x_{n-1}(0) - q(x_n(0))$ which tends to $q(x_n(0))$ as $t \to +\infty$. This implies that there exists $T_{n-1}$ such that $q(x_n(0)) - 1/2 \leq x_{n-1}(t) < q(x_n(0)) + 1/2$ for all $t \geq T_{n-1}$ and then $q(x_{n-1}(t)) = q(x_n(0))$ for all $t \geq T_{n-1}$. Repeating the same argument for $i = n-2, n-3, \ldots, 1$, we have that there exists $T > 0$ such that $q(x_i(t)) = q(x_n(0))$ for all $i = 1, \ldots, n-1$, and all $t \geq T$ and then $x_i(t) \to q(x_n(0))$ for all $i = 1, \ldots, n-1$, as $t \to +\infty$, thus the proof is complete. □

**Proposition 4.2 (Convergence for the undirected line graph)** *For any initial condition, solutions of* (7) *converge to an extended equilibrium.*

*Proof* It is known from [7], Proposition 2.2, that there exists a time $T$ after which $\max_i q(x_i(t))$ and $\min_i q(x_i(t))$ are constant for $t \geq T$. We denote the two constant values with $q_M, q_m$.

We start by showing that there exists a representative of $q_M$ with constant index, i.e. $x_{i^*}(t)$ such that $q(x_{i^*}(t)) = q_M$ for $t > T' > T$.

Consider the set $\mathcal{M}(t) := \{i | q(x_i(t)) = q_M\}$. $\mathcal{M}(t)$ is non-empty for all $t$ by definition of $q_M$. Being time dependent, indexes can enter and exit the set, we want to show that there is at least one $i^*$ that is definitely in the set.

Consider $i \in \mathcal{M}(T)$, either $i \in \mathcal{M}(t)$ for all $t > T$ (hence $i^* = i$), or there exists $T_1 > T$ such that $i \notin \mathcal{M}(T_1)$.

Without loss of generality consider $i \neq 1, n$, as $q(x_1(t)) = q_M$ for all $t > T$ if and only if the same holds for $q(x_2(t))$, and analogously for nodes $n$ and $n-1$.

If there exists $T_1 > T$ such that $i \notin \mathcal{M}(T_1)$, it means that $q(x_i)$ at some point drops from $q_M$ to $q_M - 1$ (it cannot increase to $q_M + 1$ by definition of $q_M$). This implies that $\frac{q(x_{i-1})+q(x_{i+1})}{2} < q_M - \frac{1}{2}$ (as $x_i$ converges to that value) for at least a time period, i.e. node $i$ has at least one neighbor outside of $\mathcal{M}(t)$. Without loss of generality, let $i+1 \notin \mathcal{M}(t)$, then there are two cases:

1. in general, $x_i$ switches quantization interval by reaching value $q_M - \frac{1}{2}$, and we end up with at least two neighboring nodes with quantized value lower than $q_M$,
2. $x_i$ reaches value $q_M - \frac{1}{2}$ at the same time as all the neighbors who previously had quantized value lower than $q_M$.

In the first case, $i$ is trapped outside of $\mathcal{M}(t)$ forever. Notice that if $q(x_i) = q_M - 1$, to switch back to $q_M$ it must hold that $\frac{q(x_{i-1})+q(x_{i+1})}{2} > q_M - \frac{1}{2} \Rightarrow q(x_{i-1}) + q(x_{i+1}) = 2q_M$. Since $q_M$ is the maximum value attainable, this requires both $q(x_{i-1})$ and $q(x_{i+1})$ to be equal to $q_M$, but $q(x_{i-1}) < q_M$, and similarly $q(x_{i+1})$ cannot increase to $q_M$ unless $q(x_i)$ does so first, making both nodes unable to reach $q_M$.

In the second case, we have multiple further cases:

2a. if $q(x_{i-1}) = q(x_{i+1}) = q_M$, then $x_i$ will start moving back towards $q_M$ and does not leave the quantization interval,
2b. both neighbors have quantized value $q_M - 1$ so that $x_i$ drops to that quantization interval and cannot go back,
2c. one neighbor has quantized value $q_M$ while the other has quantized value $q_M - 1$; then, $x_i$ stays constant at value $q_M - \frac{1}{2}$ so it keeps quantized value $q_M$, at least until either neighbor changes quantized value, so that we fall under one of the previous cases.

This shows that nodes in $\mathcal{M}(t)$ either don't leave the set, or leave it for good, and since $\mathcal{M}(t)$ always has at least an element, there needs to be one element that never leaves.

An analogous results can be obtained for $q_m$, the only difference in the proof being that in case 2c opinion $q_m + \frac{1}{2}$ constant for an interval of time cannot be assigned quantized value $q_m$, so that the case considered falls in the category "out of the set for good" instead of "not leaving the set", which does not change the global end result.

Now that we know that two components have definitively constant quantized value, we show that so do all the other components. Let us consider $x_i$

such that $q(x_i) = q_M$ for all $t > T$. Now, either node $i+1$ has definitive constant quantized value, or it oscillates between different quantized values. By contradition, let us assume the latter. If node $i+1$ oscillates, it must necessary be between $q_M$ and $q_M - 1$, as if we had $q(x_{i+1}) = q_M - 2$ we would have $\frac{q(x_{i-1})+q(x_{i+1})}{2} \leq \frac{2q_M-2}{2} = q_M - 1$, which would allow $x_i$ to switch to quantized value $q_M - 1$ contradicting the hypothesis. The oscillation of $q(x_{i+1})$ causes an oscillation of $q(x_{i+2})$, in fact when $q(x_{i+1}) = q_M - 1$, for it to reach $q_M$ it must hold that both its neighbors have quantized value $q_M$, in particular that $q(x_{i+2}) = q_M$; analogously, when $q(x_{i+1}) = q_M$, to drop to $q_M - 1$ we need $\frac{q(x_i)+q(x_{i+2})}{2} < q_M - \frac{1}{2}$ which implies $q(x_{i+2}) \leq q_M - 2$. Thus, we have that node $i+2$ oscillates at least between quantized values $q_M$ and $q_M - 2$. By induction we can show that this implies that node $i+j$ oscillates at least between quantized values $q_M$ and $q_M - j$: this is true for $j = 0, 1, 2$ as shown, and if its true for $j-1$, then the jump from $q(x_{i+j-1}) = q_M - j + 2$ to $q(x_{i+j-1}) = q_M - j + 1$ requires that $\frac{q(x_{i+j-2})+q(x_{i+j})}{2} < q_M - j + \frac{3}{2}$, and since $q(x_{i+j-2})$ is at least equal to $q_M - j + 2$, $q(x_{i+j})$ is at most equal to $q_M - j$; furthermore, as shown before for $j = 1$ value $q_M$ must be attained, as such $q(x_{i+j})$ oscillates between $q_M$ and $q_M - j$, concluding by induction that node $i+j$ must oscillate at least between quantized values $q_M$ and $q_M - j$. However, such oscillation is actually not possible for all $j$. In fact, starting from $i$ and moving in either direction, at some point we either find $i^*$ such that $q(x_{i^*}) = q_m$, which does not oscillate, or reach a node 1 or $n$, which can only copy the quantized value of their only neighbor, allowing them to oscillate only as much as the neighbor, not more than it, as required by the induction formula.

This concludes by contradiction that no infinite oscillation can happen, and as such that there exists a time $T^*$ after which all quantized values are constants. This implies that after $T^*$ the solution enters a hypercube $S_h$ that it will never leave, and the dynamics in that hypercube has a globally asymptotically stable equilibrium point the solution will converge to. □

## 5 Between Consensus and Dissensus

The analysis conducted in the previous sections and in [7] examines two extreme situations from the point of view of graph topology: the line graph and the complete graph. These configurations yield contrasting asymptotic behaviors, specifically “far from” and “close to” consensus equilibria. This naturally raises the question: what occurs in intermediate scenarios?

Previous studies by the authors have already documented a range of peculiar behaviors. In particular in [21] it was shown that on directed graphs, solutions may exhibit periodic or Zeno behaviors. In this section, we shall present three new examples: the first example shows the phenomenon of “hidden consensus” on complete bipartite graphs; the second example shows the emergence of polarization, phenomenon often sought after by advanced opinion dynamics models; and the final example numerically investigates the dynamics

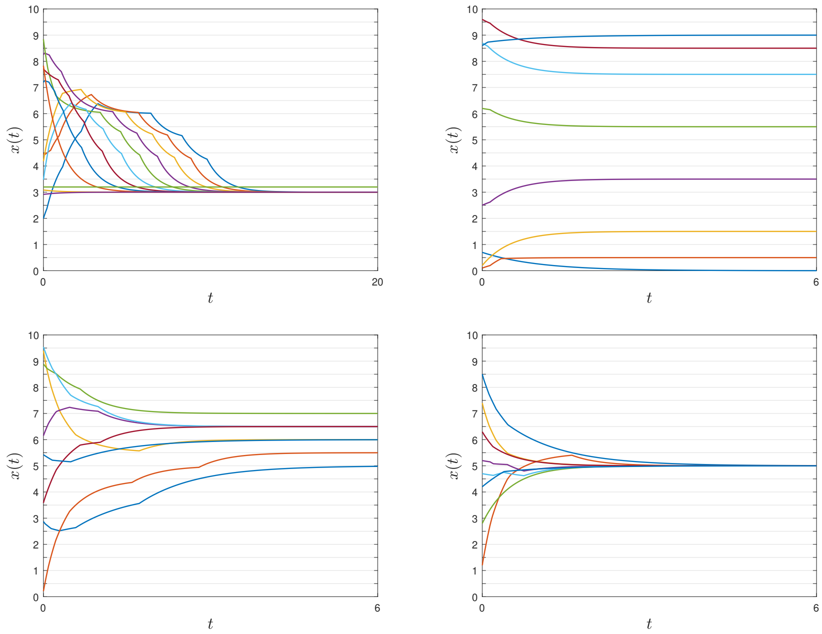


**Fig. 2** Top left: dynamics (5) on the directed line with uniformly random initial condition. Top right: dynamics (7) on the undirected line converging to the extended equilibrium furthest from consensus. Bottom left: undirected line, uniformly random initial condition. Bottom right: undirected line, convergence to consensus. The figures are generated with the software Matlab

on a sparse random graph, which is qualitatively similar to the behavior on the line graph, further motivating the study of the latter. The figures of this section are generated with the software Matlab.

Before describing the first example, we highlight a difference between the dynamics on complete and complete bipartite graphs. Notice that Theorem 2.3 specifies that the consensus reached is integer only for complete graphs, not for complete bipartite graphs. In fact, for the latter, it is possible to converge to equilibrium points of the type $\bar{x} = (h + \frac{1}{2})\mathbf{1},\ \mathbf{h} \in \mathbb{Z}$, which is peculiar as this implies that whereas opinions asymptotically coincide, actions do not. As an observer, only actions would be seen, and as such it would be impossible to detect the consensus, hence the term "hidden". Another example of hidden consensus, on a non-complete bipartite graph, was provided in [21, Ex. 1].

*Example 5.1 (Hidden consensus)* Consider a complete bipartite graph with partition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, where the two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint and have even and non-null cardinalities, named $n_1$ and $n_2$ respectively. Take initial conditions in $S_h$, where $h = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0, 1, \dots, 1)$, such that each subset has half of its nodes quantized to 0 and half quantized to 1, that is, $h_i = 0$ for $i = 1, \dots, \frac{n_1}{2}$ and $i = n_1 + 1, \dots, n_1 + \frac{n_2}{2}$, and $h_i = 1$ for $i =$

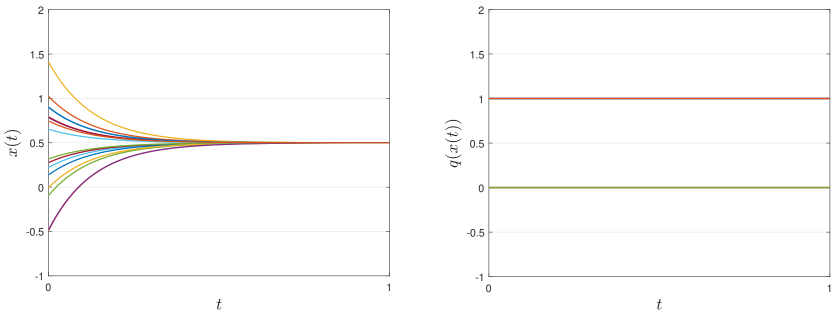


**Fig. 3** Dynamics on the complete bipartite graph with $n_1 = n_2 = 8$ and initial conditions as discussed in Example 5.1. Left: opinions, right: actions

$\frac{n_1}{2} + 1, \ldots, n_1$ and $i = n_1 + \frac{n_2}{2} + 1, \ldots, n$. For all $i \in \mathcal{V}_1$, the dynamics reads

$$\dot{x}_i = \sum_{j \in \mathcal{V}_2} (h_j - x_i) = \frac{n_2}{2} - n_2 x_i,$$

and analogously, for $i \in \mathcal{V}_2$, $\dot{x}_i = \frac{n_1}{2} - n_1 x_i$. As long as the dynamics does not switch, the solution is attracted by the equilibrium point with components $x_i = \frac{n_1}{2n_1} = \frac{n_2}{2n_2} = \frac{1}{2}$. Since this holds for all $i$, every node's opinion monotonically converges to $\frac{1}{2}$, never crossing the threshold and hence never inducing a dynamics switch, so the solution converges to the consensus point $\bar{x} = (\frac{1}{2}, \ldots, \frac{1}{2})$. Notice, however, that, for all $t \geq 0$, half of the opinions converge to $\frac{1}{2}$ from below, and as such are quantized to 0, while the other half converge from above, and are thus quantized to 1. This means that, despite the solution converging to consensus, actions (the only observable quantities) stay apart and could as such be interpreted as representing dissensus in the network. An example is shown in Figure 3.

The previous example shows that when a node converges to opinion $h+\frac{1}{2}$, it means that the node has no preference in choosing between actions $h$ and $h+1$; our model predicts that the node will keep taking the action that they were already taking. From a social dynamics point of view, this could be interpreted as a representation of the force of habit, making one reluctant to change unless strictly necessary.

In the next example, we highlight another peculiar behavior: polarization. Polarization of opinions is a common real-world occurrence, prompting many advanced opinion dynamics models to attempt to capture this phenomenon. The polarization example we present differs significantly from situations shown in [9]: while in [9] only two discrete states were allowed, here the number of discrete states is not limited. Other studies attribute polarization to antagonistic interactions among individuals [2] or to bounded confidence [14]. Here we suggest that polarization can be a consequence of the discretized actions and of the topology of the graph. Specifically, this behavior can be present when agents are divided into different communities with few interconnections.

In the following example, we analytically show polarization in the case of a Barbell graph, and then present a numerical simulation showing this type of behavior on a more complex graph.

*Example 5.2 (Polarization)* Consider a Barbell graph, the union of two complete graphs connected by an edge, having an even number $n$ of nodes and adjacency matrix with entries $a_{ij} = 1$ for $(i,j) = (\frac{n}{2}, \frac{n}{2}+1)$, $(i,j) = (\frac{n}{2}+1, \frac{n}{2})$, and $(i,j)$ such that $i \neq j$ and both $i$ and $j$ belong to $\mathcal{V}_1 := \{1, \ldots, \frac{n}{2}\}$ or both belong to $\mathcal{V}_2 := \{\frac{n}{2}+1, \ldots, n\}$. Take initial conditions in $S_h$, where $h = (0, \ldots, 0, c, \ldots, c)$, $c \in \mathbb{Z}^+$ i.e., the first complete graph is quantized to 0 and the second to a constant $c$. Notice that, except for the bridge nodes $\frac{n}{2}$ and $\frac{n}{2}+1$, every node is only connected to nodes with their same quantized value, so that, for $i \in \mathcal{V}_1,\ i \neq \frac{n}{2}$,

$$\dot{x}_i = \sum_{j=1}^{\frac{n}{2}-1} (h_j - x_i) \Longrightarrow x_i \to \frac{1}{d_i} \sum_{j=1}^{\frac{n}{2}-1} h_j = 0.$$

Similarly, for $i \in \mathcal{V}_2,\ i \neq \frac{n}{2}+1$,

$$\dot{x}_i = \sum_{\frac{n}{2}+1}^{n} (h_j - x_i) \Longrightarrow x_i \to \frac{1}{d_i} \sum_{\frac{n}{2}+1}^{n} h_j = \frac{1}{\frac{n}{2}-1}(\frac{n}{2}-1)c = c.$$

For the bridge nodes, we have

$$\dot{x}_{\frac{n}{2}} = \sum_j a_{\frac{n}{2}j} h_j - d_{\frac{n}{2}} x_{\frac{n}{2}} \Longrightarrow x_{\frac{n}{2}} \to \frac{2}{n} \sum_j a_{\frac{n}{2}j} h_j = \frac{2}{n} \sum_{j=1}^{\frac{n}{2}} h_j + \frac{2}{n} h_{\frac{n}{2}+1} = \frac{2c}{n}.$$

By symmetry, $x_{\frac{n}{2}+1} \to c - \frac{2c}{n}$.

Now, if we take $c \leq \frac{n}{4}$, then $x_{\frac{n}{2}}$ does not reach the threshold $\frac{1}{2}$ and hence does not change quantization interval. Since the same holds for node $\frac{n}{2}+1$, the solution never leaves $S_h$ so the polarization is maintained. Notice that, given a sufficiently large $n$, we can have an arbitrarily large difference between the two opinion clusters. Fig. 4 shows a numerical example.

Simulations show the formation and persistence of polarization even for more widespread initial conditions, and for more general graphs with a 2-communities structure. An example is shown in Fig. 5, which shows the solution of the dynamics applied to a random network sampled from a “stochastic block model” [15]: two communities have strong interaction within themselves (90% chance to have an edge between any two nodes belonging to the same community) and weak interaction between each other (10% chance when the nodes are in different communities), with linearly distributed initial conditions.

Finally, we present a numerical example on a sparse random graph, called “random geometric graph” [20], to show that the behavior observed is similar to the behavior studied on the line graph. In fact, sparse graphs tend to show more dispersed opinions at equilibrium, covering a range of opinions

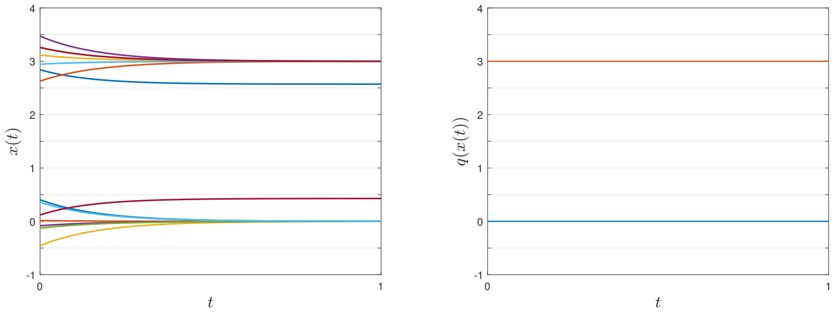


**Fig. 4** Simulations illustrating Ex. 5.2, on a Barbell graph with 2 communities of 7 agents each. Left: opinions, right: actions

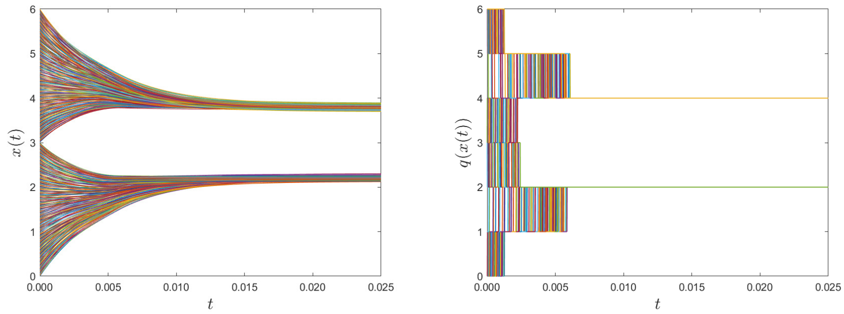


**Fig. 5** Plots of, respectively, opinions and actions of the solution of the dynamics on a random graph, sampled from a stochastic block model, with linearly distributed initial conditions

in a fairly uniform manner, as opposed to dense graphs with a community structures (e.g. one community for the complete graph, two communities for Example 5.2) which show a behavior close to clusterization, i.e., the formation of large groups, one per community, of individuals that share similar opinions.

*Example 5.3 (Random geometric graph)* A random geometric graph is an undirected random graph built in the following way: $n$ nodes are randomly generated, independently sampled from the uniform distribution, as points in a subset of the $\mathbb{R}^2$ space, e.g. the $[0,1] \times [0,1]$ square, then any two nodes are connected by an edge if their distance is smaller than a given radius $r$. For any given $n$, if $r$ is sufficiently small, then the graph is sparse. In fact, provided a node $i$ falls at a distance of least $r$ from the border of the $[0,1] \times [0,1]$ square, then its degree is equal to the number of nodes that fall in the circle of area $\pi r^2$ centered in node $i$, so that the expected degree is $(n-1)\pi r^2$.

Simulations show that the dynamics on these random graphs appears to behave qualitatively quite similarly to the dynamics on the undirected line. In fact, as seen in Figure 6, the asymptotic behavior generally is convergence to widespread extended equilibria. Just like for the line graph, the dispersion of

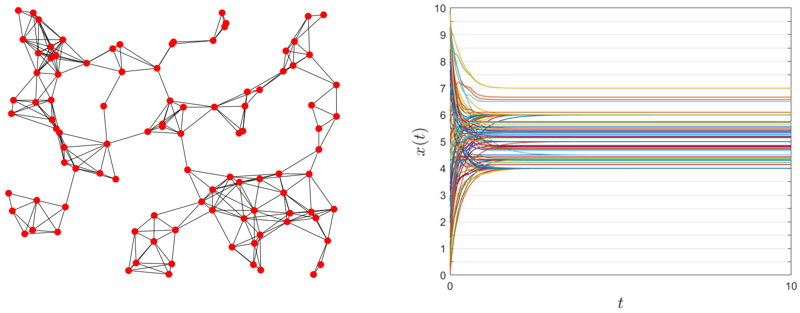


**Fig. 6** Random geometric graph with $n = 100$ nodes and $r = 0.15$, and uniformly random initial condition. Left: graph, right: solution of the dynamics

opinions tends to reduce at first, down to a certain interval, which represents the range of opinions attained at the equilibrium. In this type of graph, simulations show how components individually converge to many different points inside that range, covering well the entire interval, as opposed to the previous examples where opinions end up being concentrated around specific points. This further suggests that dense graphs facilitate the formation of few large clusters (e.g. consensus for complete graph, two clusters for Barbell), while sparse graphs tend to allow for a dispersion of agents into many small clusters.

## 6 Conclusions

The model that is studied in this paper shows how the quantization of the communicated states can drastically change the features of the linear consensus system. In fact, the quantized model presents a wide variety of behaviors, ranging from consensus to polarization to dispersion over multiple opinions. While consensus remains the rule in well-connected graphs like the complete graph, in this paper we focused on showing how different the evolution can be when connectivity is limited: for this case, the line graphs are the most suitable examples. Our main result was to show that on the undirected line solutions converge to one of many, of order at least $\frac{3^n}{n}$, extended equilibria, which can be very far from consensus and which all have non-empty basin of attraction.

In terms of the analysis of the model, many open questions remain, in particular proving convergence for general graphs. A plausible conjecture is that convergence to extended equilibria holds true for undirected graphs, while we know that it cannot hold for general directed graphs. It would also be interesting to characterize graphs giving rise to consensus. Finally, we mention that it would be desirable to treat the system with general methods (e.g. Lyapunov arguments within hybrid system theory) but until now our attempts have been unsuccessful: it is challenging to understand what is the appropriate general framework to approach these systems in an effective and synthetic way.

**Data Availability** Data sharing not applicable to this article.

## Declarations

**Conflict of interest** The authors declare that there is no conflict of interest.

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