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CONFORMAL GEOMETRY OF QUASI-UMBILICAL TIMELIKE SURFACES

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Abstract. The paper focuses on the conformal Lorentz geometry of quasi-umbilical timelike surfaces in the $(1+2)$-Einstein universe, the conformal compactification of Minkowski 3-space realized as the space of oriented null lines through the origin of $\mathbb{R}^{2,3}$. A timelike immersion of a surface X in the Einstein universe is quasi-umbilical if its shape operator at any point of X is non-diagonalizable over $\mathbb{C}$. We prove that quasi-umbilical surfaces are isothermic, that their conformal deformations depend on one arbitrary function in one variable, and show that their conformal Gauss map is harmonic. We investigate their geometric structure and show how to construct all quasi-umbilical surfaces from null curves in the 4-dimensional neutral space form $S^{2,2} = \{x \in \mathbb{R}^{2,3} \mid \langle x, x\rangle = 1\}$.

1. Introduction

This paper is a contribution to the subject of conformal Lorentz geometry of timelike surfaces in the Einstein universe $\mathcal{E}^{1,2}$, the conformal compactification of Minkowski 3-space realized as the space of null rays through the origin of $\mathbb{R}^{2,3}$. $\mathcal{E}^{1,2}$ is endowed with the Lorentz metric $g_{\mathcal{E}}$ induced from $\mathbb{R}^{2,3}$. It is diffeomorphic to $S^1 \times S^2$ and with respect to the conformal class $[g_{\mathcal{E}}]$ is an example of an oriented, time-oriented, conformally flat Lorentz manifold. As such, it admits a transitive action by the identity component $\mathrm{O}^{\uparrow}_{+}(2,3)$ of the pseudo-orthogonal group of $\mathbb{R}^{2,3}$, which is isomorphic to the group of conformal transformations of $(\mathcal{E}^{1,2}, [g_{\mathcal{E}}])$ preserving orientation and time-orientation. In the paper we will also use the realization of $\mathcal{E}^{1,2}$ as the symmetric R-space [8, 26, 48] defined by the conjugacy class of parabolic subalgebras of $\mathfrak{o}(2,3)$, the Lie algebra of $\mathrm{O}^{\uparrow}_{+}(2,3)$. In dimension three, another model for the Einstein universe is the Grassmannian of oriented Lagrangian planes of $\mathbb{R}^4$ [2, 41, 46]. If we write $\mathbb{R}^{2,3} = \mathbb{V}^2_- \oplus \mathbb{V}^3_+$ as the

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direct sum of a negative-definite 2-dimensional subspace $\mathbb{V}^2_-$ and a positive-definite 3-dimensional subspace $\mathbb{V}^3_+$ and denote by S^1_- and S^2_-, respectively, the unit spheres of $\mathbb{V}^2_-$ and $\mathbb{V}^3_+$, then the orientable, time-orientable submanifold $S^1_- \times S^2_+$, with the Lorentzian structure inherited from $\mathbb{R}^{2,3}$, provides another model (the *pseudo-metric model*) of the Einstein universe.

As in the case of definite signature, the 3-dimensional Lorentzian space forms, namely Minkowski, de Sitter and anti-de Sitter 3-spaces, can be conformally embedded as open domains of $\mathcal{E}^{1,2}$, the so-called Minkowski, de Sitter, and anti-de Sitter chambers [2, 20, 23, 24].

Background and motivations. Let $f : X \to \mathbb{R}^{1,2}$ be a timelike immersion of a surface X in Minkowski 3-space. At umbilical points, the second fundamental form II of f is proportional to the first fundamental form I and the mean and Gauss curvatures H and K satisfy $H^2 - K = 0$. For timelike surfaces in Minkowski space, $H^2 - K$ can vanish also at nonumbilical points [35], in which case I and II must share a null direction and the shape operator is non-diagonalizable over $\mathbb{C}$. Following Clelland [17], a timelike immersion f is called (totally) *quasi-umbilical* if at every point of X its shape operator A is non-diagonalizable over $\mathbb{C}$, or equivalently, if at every point of X the trace-free part A^0 of the shape operator satisfies $\det(A^0) = 0$, with $A^0 \neq 0$.

Conformal invariance. Since the property of being quasi-umbilical only depends on the conformal class of the induced Lorentz metric, we shall profit from studying it in the more appropriate context of conformal Lorentz geometry by considering quasi-umbilical timelike immersions in the Einstein universe $\mathcal{E}^{1,2}$. Beside quasi-umbilical surfaces, there are three other classes of timelike immersions of constant type,[1] namely the class of elliptic surfaces, for which $\det(A^0) > 0$ on X, the class of hyperbolic surfaces, for which $\det(A^0) < 0$ on X, and the class of surfaces for which A^0 vanishes identically on X. The latter consists of totally umbilical embedded Lorentzian tori.

Related work. Clelland [17] classified quasi-umbilical surfaces in Minkowski 3-space as ruled surfaces whose rulings are all null lines, with the additional property that any null curve γ on the surface transversal to the rulings is non-degenerate (i.e., the vectors γ', γ'' are linearly independent at each point). At about the same time, The [46] investigated the conformal Lorentz geometry of timelike surfaces, including quasi-umbilical ones, which he called 2-*parabolic*, in connection with his study of the geometry of second-order Monge-Ampere equations. More recently, Burstall, Musso, and Pember [11] have shown that quasi-umbilical surfaces arise as one of the four classes of orthogonal surfaces of codimension 2 harmonic sphere congruences, which also include S-Willmore surfaces in the sense of Ejiri [22], constant mean curvature surfaces in 3-dimensional space forms, and surfaces of constant lightcone mean curvature in 3-dimensional lightcones. There has recently been also a considerable interest in elliptic and hyperbolic timelike surfaces,

[1] Cf. [31, p. 9] for the notion of immersion of constant type in a homogeneous space.

especially in connection with the study of the Lorentz counterparts of Willmore and isothermic immersions [3, 12, 18, 19, 36, 43, 44, 46, 49]. Finally, we observe that conformal Lorentz geometry is related to Lie sphere geometry, a classical topic in differential geometry which has recently received much attention especially in relation with the study of Dupin submanifolds [14, 15, 16], Lie applicable surfaces [38, 45], and the theory of integrable systems [9, 25]. Actually, the Einstein universe $\mathcal{E}^{1,2}$ can be viewed as a double covering of the Lie quadric $\mathcal{Q}^3$ in Lie sphere geometry, whose elements correspond to oriented circles and points in $\mathbb{R}^2$ [5, 34].

Purposes. A first purpose of this paper is to underline and clarify the links of quasi-umbilical immersions with the general theory of isothermic submanifolds in symmetric R-spaces (in the sense of [8]) and with the classical deformation problem of submanifolds in homogeneous spaces [13, 10, 31, 32, 37, 38, 45]. A second purpose is to give a detailed geometric description of quasi-umbilical surfaces by showing that all quasi-umbilical surfaces can be constructed from suitable null curves in the neutral four-space form

$$S^{2,2} = \{V \in \mathbb{R}^{2,3} \mid \langle V, V\rangle = 1\} \cong \mathrm{O}^{\uparrow}_{+}(2,3)/\mathrm{O}^{\uparrow}_{+}(2,2).$$

Description of results. Adapting to our case a basic construction in Möbius geometry that goes back to the work of Blaschke and Thomsen [5, 6, 28, 47], we can define the conformal Gauss map $\mathcal{N}_f : X \to S^{2,2}$ of a timelike immersion. (For the study of spacelike immersions in conformal Lorentz geometry, and more specifically of spacelike Willmore surfaces, we refer to [1] and [30].) Geometrically, $\mathcal{N}_f$ associates to a point $x \in X$ the unique totally umbilical timelike torus of the Einstein universe having second order analytic contact with f at $f(x)$, the so-called *central torus* of the surface at $f(x)$. If f is quasi-umbilical, then $\mathcal{N}_f$ has rank 1, so that f is the envelope of a 1-parameter family of totally umbilical Lorentz tori. The second envelope of the family, denoted by $f^\sharp : X \to \mathcal{E}^{1,2}$, is referred to as the *dual* of f. A quasi-umbilical immersion f is called *regular* if Λ, the leaf space of the vertical distribution $\operatorname{Ker} d\mathcal{N}_f$, is a connected 1-dimensional manifold, so that $\mathcal{N}_f$ factors on Λ through a submersion $\pi_\Lambda : X \to \Lambda$ with connected fibers and a null curve $\Gamma : \Lambda \to S^{2,2}$, that is, $\mathcal{N}_f = \Gamma \circ \pi_\Lambda$. The curve Γ is called the *directrix curve* of f. Locally, every quasi-umbilical immersion is regular. The first main result of the paper is the following.

Theorem A. *Let $f : X \to \mathcal{E}^{1,2}$ be a quasi-umbilical timelike immersion. Then the conformal Gauss map $\mathcal{N}_f : X \to S^{2,2}$ of f is harmonic. In addition, if f is regular, then f is isothermic and its infinitesimal deformations depend on one arbitrary function in one variable.*

According to whether the quadratic form $\langle d\mathcal{N}_f, df^\sharp\rangle$ is identically zero or not, a quasi-umbilical immersion f is called *exceptional*, respectively, *general*. We will separately describe the geometric structure of the two classes of immersions under the assumption of regularity.

Exceptional quasi-umbilical surfaces. First, we prove that a quasi-umbilical f is exceptional if and only if its directrix curve Γ is a *biisotropic* curve of $S^{2,2}$, meaning that Γ' and Γ'' are linearly independent null vector fields (cf. Proposition 4.2). We show that biisotropic curves can be constructed from suitable plane curves of a unimodular affine plane. Then, associated to any biisotropic curve $\Gamma : \Lambda \to S^{2,2}$, we consider the *normal tube* $\mathbb{T}_\Gamma \cong \Lambda \times S^1$ of Γ and construct a canonical map $f_\Gamma : \mathbb{T}_\Gamma \to \mathcal{E}^{1,2}$, the *tautological map* of Γ (cf. Section 4.1 and Definition 4.7). The second main result accounts for the geometric structure of exceptional quasi-umbilical surfaces.

Theorem B. *Let $\Gamma : \Lambda \to S^{2,2}$ be a biisotropic curve. The tautological map f_Γ of Γ is a timelike immersion with conformal Gauss map $\Gamma \circ \pi_\Lambda$. The umbilic locus of f_Γ is the disjoint union of two null curves $\mathrm{C}_\pm \subset \mathbb{T}_\Gamma$ that disconnect $\mathbb{T}_\Gamma$ into two open sets, $\mathbb{T}^\pm_\Gamma$. Moreover, the restrictions of f_Γ to $\mathbb{T}^\pm_\Gamma$ are exceptional quasi-umbilical immersions. Conversely, if $f : X \to \mathcal{E}^{1,2}$ is a regular quasi-umbilical immersion of exceptional type and $\Gamma : \Lambda \to S^{2,2}$ is its directrix curve, then $f(X)$ is contained in either $f_\Gamma(\mathbb{T}^+_\Gamma)$ or $f_\Gamma(\mathbb{T}^-_\Gamma)$.*

This implies that, locally, exceptional quasi-umbilical surfaces arise as immersions of a component of the normal tube of a biisotropic curve of $S^{2,2}$. Another consequence is that quasi-umbilical immersions of exceptional type depend on one function in one variable, namely the affine curvature of the affine curve generating the directrix curve. Moreover, they are second order deformations of each other (cf. Proposition 4.9). The general construction will be illustrated for the case in which the generating affine curve is a conic.

General quasi-umbilical surfaces. We start by showing that the directrix curve Γ of a general quasi-umbilical immersion is a *generic* null curve, meaning that Γ'' can be either timelike or spacelike. Then, proceeding in analogy with [39, 40, 42], for a generic Γ we introduce a preferred parametrization (proper time) and construct a canonical moving frame along the curve, from which one determines the two fundamental differential invariants of Γ, namely the *left and right curvatures* κ_λ and κ_ϱ. The canonical moving frame is used to build two normal tubes $\mathbb{T}_\lambda$ and $\mathbb{T}_\varrho$ along Γ, the *left and right normal tubes* of Γ, naturally identified by a diffeomorphism J^ϱ_λ. The left and right tubes intersect along two null curves $\mathrm{C}_\pm$, whose complementary set has two connected components, $\mathbb{T}^\pm_\lambda$ and $\mathbb{T}^\pm_\varrho$, the *parabolic components* of the normal tubes. The canonical projections $\pi_\lambda : \mathbb{T}_\lambda \to \Lambda$ and $\pi_\varrho : \mathbb{T}_\varrho \to \Lambda$ make $\mathbb{T}_\lambda$ and $\mathbb{T}_\varrho$ into circle bundles over Λ. We then construct two canonical maps, $f_\lambda : \mathbb{T}_\lambda \to \mathcal{E}^{1,2}$ and $f_\varrho : \mathbb{T}_\varrho \to \mathcal{E}^{1,2}$, the *left and right tautological maps* of Γ. The third main result accounts for the structure of general quasi-umbilical surfaces.

Theorem C. *Let $\Gamma : \Lambda \to S^{2,2}$ be a generic null curve.*

(1) *The tautological map f_λ is a timelike immersion with umbilic locus $\mathrm{C}_+ \cup \mathrm{C}_-$ and conformal Gauss map $\mathcal{N}_{f_\lambda} = \Gamma \circ \pi_\lambda$. The restrictions $f^\pm_\lambda$ of f_λ*

to the parabolic components $\mathbb{T}^{\pm}_{\lambda}$ of the left normal tube $\mathbb{T}_{\lambda}$ are quasi-umbilical immersions of general type with dual maps $(f^{\pm}_{\lambda})^{\sharp} = f^{\pm}_{\varrho} \circ J^{\varrho}_{\lambda}$.

(2) *The tautological map f_{ϱ} is a timelike immersion with umbilic locus $\mathrm{C}_+ \cup \mathrm{C}_-$ and conformal Gauss map $\mathcal{N}_{f_{\varrho}} = \Gamma \circ \pi_{\varrho}$. The restrictions $f^{\pm}_{\varrho}$ of f_{ϱ} to the parabolic components $\mathbb{T}^{\pm}_{\varrho}$ of the right normal tube $\mathbb{T}_{\varrho}$ are quasi-umbilical immersions of general type with dual maps $(f^{\pm}_{\varrho})^{\sharp} = f^{\pm}_{\lambda} \circ J^{\lambda}_{\varrho}$.*

Conversely, let $f : X \to \mathcal{E}^{1,2}$ be a regular quasi-umbilical immersion of general type. Then $f(X)$ is contained in either $f_{\lambda}(\mathbb{T}^{+}_{\lambda})$ or $f_{\lambda}(\mathbb{T}^{-}_{\lambda})$, where f_{λ} is the left tautological immersion originated by the directrix curve Γ of f.

Theorem C implies that quasi-umbilical immersions of general type depend on two functions in one variable, namely the left and right curvatures of their directrix curves. As for the deformation problem, we show that two quasi-umbilical immersions of general type f and $\tilde{f}$ are second order conformal deformations of each other if and only if they have the same helicity (to be specified) and if either $\kappa_{\lambda} = \tilde{\kappa}_{\lambda}$ or $\kappa_{\varrho} = \tilde{\kappa}_{\varrho}$ (cf. Remark 5.5). The construction will be illustrated by an example at the end of Section 5.

It is interesting to note that the construction of the canonical timelike immersions arising from biisotropic or generic null curves of $S^{2,2}$ is algorithmic and only involves algebraic manipulations and derivatives. The parametrizations of quasi-umbilical immersions given in [17] can be obtained restricting the maps f_{Γ} to the subdomains of $\Lambda \times S^1$ that are sent into a Minkowski chamber of $\mathcal{E}^{1,2}$ [20, 23]. Moreover, our construction allows us to exhibit quasi-umbilical immersed and embedded tori or cylinders, a phenomenon that can never occur in 3-dimensional Minkowski geometry.

The material is organized as follows. Section 2 briefly reviews some basic facts about the geometry of the Einstein universe (cf. [2, 20, 23, 26, 27, 41, 46]) and describes the AdS chambers and the "toroidal model" of the Einstein universe which, in turn, is used to visualize the geometric features of quasi-umbilical surfaces. Section 3 recalls the construction of the conformal Gauss map and of the dual map of a quasi-umbilical immersion. We then recall the notion of an isothermic surface, adapting to the context of timelike surfaces in $\mathcal{E}^{1,2}$ the terminology of the general theory of isothermic submanifolds in symmetric R-spaces, and discuss the related concept of second order deformation. Section 3 concludes with the proof of Theorem A. Section 4 describes the geometric structure of exceptional quasi-umbilical immersions and proves Theorem B. Section 5 accounts for the geometric structure of general quasi-umbilical immersions and proves Theorem C.

2. Preliminaries

2.1. **The automorphism group.** Consider $\mathbb{R}^5$ with the scalar product of signature $(2, 3)$

$$\langle X, Y\rangle = -(X^0Y^4 + X^4Y^0) - X^1Y^1 + X^2Y^2 + X^3Y^3 = {}^tX\, h\, Y, \tag{2.1}$$

where $h = (h_{ij})$ and $h_{ij} = h_{ji}$. Choose an oriented spacelike 3-dimensional vector subspace $\mathbb{V}^3_+ \subset \mathbb{R}^5$ and a positive-oriented orthogonal basis V_1, V_2, V_3 of $\mathbb{V}^3_+$.

Let $\mathtt{D}_2$ be the set of all nonzero decomposable bivectors $X \wedge Y \in \Lambda^2(\mathbb{R}^5)$ such that the restriction of (2.1) to $\mathrm{span}\{X, Y\} =: [X \wedge Y]$ is negative definite. An element $X \wedge Y \in \mathtt{D}_2$ is said to be *positive* if $\det(X, Y, V_1, V_2, V_3) > 0$. The set of all such positive bivectors is a connected component of $\mathtt{D}_2$, denoted by $\mathtt{D}_2^\uparrow$. The boundary $\partial\mathtt{D}_2^\uparrow$ of $\mathtt{D}_2^\uparrow$ consists of all isotropic bivectors $V \wedge W$ (i.e., $V \wedge W \neq 0$, $\langle V, V\rangle = \langle W, W\rangle = \langle V, W\rangle = 0$) satisfying the condition $\langle X \wedge Y, V \wedge W\rangle = \langle X, V\rangle\langle Y, W\rangle - \langle Y, V\rangle\langle X, W\rangle > 0$, for every $X \wedge Y \in \mathtt{D}_2^\uparrow$. The elements of $\partial\mathtt{D}_2^\uparrow$ are said to be *future-directed*.

Let $\mathbb{R}^{2,3}$ denote $\mathbb{R}^5$ with the scalar product (2.1), the orientation induced by the determinant, and the time-orientation determined by $\mathtt{D}_2^\uparrow$. The *automorphism group* of $\mathbb{R}^{2,3}$ is the 10-dimensional Lie group $\mathrm{Aut}(\mathbb{R}^{2,3})$ of linear isometries of $\mathbb{R}^{2,3}$ preserving the given orientation and time-orientation.

Let $M_{\mathfrak{B}}(\Phi)$ be the matrix representing $\Phi \in \mathrm{Aut}(\mathbb{R}^{2,3})$ with respect to a chosen basis $\mathfrak{B} = (B_0, \dots, B_4)$ of $\mathbb{R}^{2,3}$. Similarly, let $G_{\mathfrak{B}}$ be the symmetric matrix representing the scalar product $\langle\cdot\,,\cdot\rangle$ with respect to $\mathfrak{B}$. For every choice of $\mathfrak{B}$, the map $\mathrm{Aut}\left(\mathbb{R}^{2,3}\right) \ni \Phi \mapsto M_{\mathfrak{B}}(\Phi) \in \mathrm{GL}(5, \mathbb{R})$ is a faithful matrix representation of $\mathrm{Aut}\left(\mathbb{R}^{2,3}\right)$. We say that:

- $\mathfrak{B}$ is a *lightcone basis* if it is positive-oriented, if[2]
$$G_{\mathfrak{B}} = h = -(E^0_4 + E^4_0) - E^1_1 + E^2_2 + E^3_3,$$
and if the isotropic bivector $B_0 \wedge (B_1 + B_2)$ is future-directed;
- $\mathfrak{B}$ is a *pseudo-orthogonal basis* if it is positive-oriented, if
$$G_{\mathfrak{B}} = -E^0_0 - E^1_1 + E^2_2 + E^3_3 + E^4_4,$$
and if the isotropic bivector $(B_0 - B_4) \wedge (B_1 + B_2)$ is future-directed.

Let

$$T_{\mathrm{lp}} = \frac{1}{\sqrt{2}}(E^0_0 + E^4_0) + E^1_1 + E^2_2 + E^3_3 + \frac{1}{\sqrt{2}}(E^4_4 - E^0_4). \tag{2.2}$$

Then, $\mathfrak{B}$ is a lightcone basis if and only if $\mathfrak{B} \cdot T_{\mathrm{lp}}$ is a pseudo-orthogonal basis. In particular, if $\mathfrak{E} = (E_0, \dots, E_4)$ is the standard basis of $\mathbb{R}^{2,3}$, $\mathfrak{P} = (P_0, \dots, P_4) = \mathfrak{E} \cdot T_{\mathrm{lp}}$ is called the *standard pseudo-orthogonal basis* of $\mathbb{R}^{2,3}$.

Choosing the standard basis $\mathfrak{E}$, the automorphism group $\mathrm{Aut}(\mathbb{R}^{2,3})$ is identified with the identity component of the pseudo-orthogonal group of (2.1), namely

$$\mathrm{O}^\uparrow_+(2,3) = \left\{ F \in \mathrm{SL}(5, \mathbb{R}) \mid {}^tF\, h\, F = h,\ F_0 \wedge (F_1 + F_2) \in \partial\mathtt{D}_2^\uparrow \right\},$$

[2]Here E^a_b, $0 \leq a, b \leq 4$, denotes the elementary matrix with 1 in the (a, b) place and 0 elsewhere.

where $F_0, \dots, F_4 \in \mathbb{R}^{2,3}$ denote the column vectors of F. If $F \in \mathrm{O}^{\uparrow}_{+}(2,3)$, the ordered set $(F_0, \dots, F_4)$ defines a lightcone basis of $\mathbb{R}^{2,3}$. Conversely, any lightcone basis arises in this way. Thus the set of all lightcone basis of $\mathbb{R}^{2,3}$ can be identified with $\mathrm{O}^{\uparrow}_{+}(2,3)$.

The Lie algebra of $\mathrm{O}^{\uparrow}_{+}(2,3)$ is the vector space

$$\mathfrak{o}(2,3) = \{\mathbf{a} \in \mathfrak{sl}(5,\mathbb{R}) \mid {}^t\mathbf{a}\, h + h\, \mathbf{a} = 0\},$$

with the commutator of matrices as a bracket. Taking

$$\begin{array}{llll} M^0_0 = E^0_0 - E^4_4, & M^1_0 = E^1_0 - E^4_1, & M^2_0 = E^2_0 + E^4_2, & M^3_0 = E^3_0 + E^4_3, \\ M^2_1 = E^2_1 + E^1_2, & M^3_1 = E^3_1 + E^1_3, & M^3_2 = E^3_2 - E^2_3, & M^1_4 = E^1_4 - E^0_1, \\ M^2_4 = E^2_4 + E^0_2, & M^3_4 = E^3_4 + E^0_3 & & \end{array}$$

as a basis of $\mathfrak{o}(2,3)$, the *Maurer–Cartan form* $\varphi = (\varphi^i_j) = F^{-1}dF$ of $\mathrm{O}^{\uparrow}_{+}(2,3)$ can be written as

$$\begin{aligned} \varphi = {} & \varphi^0_0 M^0_0 + \varphi^1_0 M^1_0 + \varphi^2_0 M^2_0 + \varphi^3_0 M^3_0 + \varphi^2_1 M^2_1 \\ & + \varphi^3_1 M^3_1 + \varphi^3_2 M^3_2 + \varphi^1_4 M^1_4 + \varphi^2_4 M^2_4 + \varphi^3_4 M^3_4, \end{aligned}$$

where the left-invariant 1-forms $\varphi^0_0 = -\varphi^4_4$, $\varphi^1_0 = -\varphi^4_1$, $\varphi^2_0 = \varphi^4_2$, $\varphi^3_0 = \varphi^4_3$, $\varphi^2_1 = \varphi^1_2$, $\varphi^3_1 = \varphi^1_3$, $\varphi^3_2 = -\varphi^2_3$, $\varphi^1_4 = -\varphi^0_1$, $\varphi^2_4 = \varphi^0_2$, $\varphi^3_4 = \varphi^0_3$ are linearly independent and span the dual Lie algebra $\mathfrak{o}(2,3)^*$. The Maurer–Cartan form satisfies the *Maurer–Cartan equations*

$$d\varphi = -\varphi \wedge \varphi. \tag{2.3}$$

2.2. **The $(1+2)$-Einstein universe.** Consider the orthogonal direct sum decomposition $\mathbb{R}^{2,3} = \mathbb{V}^2_- \oplus \mathbb{V}^3_+$ into the oriented, negative definite subspace $\mathbb{V}^2_- = [P_0 \wedge P_1]$ and the spacelike subspace $\mathbb{V}^3_+ = [P_2 \wedge P_3 \wedge P_4]$, where $\mathfrak{P} = (P_0, \dots, P_4)$ is the standard pseudo-orthogonal basis of $\mathbb{R}^{2,3}$. Let $(x_0, \dots, x_4)$ be the Cartesian coordinates with respect to $\mathfrak{P}$. Denote by S^1_- the unit circle of $\mathbb{V}^2_-$, by S^2_+ the unit sphere of $\mathbb{V}^3_+$, by J the counterclockwise rotation of $\pi/2$ in the oriented plane $\mathbb{V}^2_-$, and by V_- and V_+ the orthogonal projections of $V \in \mathbb{R}^{2,3}$ onto $\mathbb{V}^2_-$ and $\mathbb{V}^3_+$, respectively. Let $\mathcal{E}^{1,2}$ be the Cartesian product $S^1_- \times S^2_+$. The scalar product on $\mathbb{R}^{2,3}$ induces a Lorentzian pseudo-metric $g_{\mathcal{E}}$ on $\mathcal{E}^{1,2}$. The normal bundle of $\mathcal{E}^{1,2}$ is spanned by the unit normal vector fields defined by $\mathbf{n}_1|_V = V_-$ and $\mathbf{n}_2|_V = V_+$, for each $V \in \mathcal{E}^{1,2}$. Contracting the volume form $dx^0 \wedge \cdots \wedge dx^4$ with $\mathbf{n}_1$ and $\mathbf{n}_2$ yields a volume form $dV_{\mathcal{E}}$ on $\mathcal{E}^{1,2}$. We time-orient $\mathcal{E}^{1,2}$ by requiring that the timelike vector field $\mathbf{t}|_V = JV_-$ is future-oriented.

Definition 2.1. The Lorentzian manifold $(\mathcal{E}^{1,2}, g_{\mathcal{E}})$, with the above specified orientation and time-orientation, is called the $(1+2)$-*Einstein universe*.

For each nonzero vector $X \in \mathbb{R}^{2,3}$, let $[X]$ denote the oriented line spanned by X (i.e., the ray of X). The map $\mathcal{E}^{1,2} \ni X \mapsto [X]$ allows us to identify $\mathcal{E}^{1,2}$ with the manifold of oriented null rays of $\mathbb{R}^{2,3}$. We will make no distinction between these two models.

Remark 2.2. As a model of the Einstein universe it is often used the non-orientable manifold $\widehat{\mathcal{E}}^{1,2}$ of null (unoriented) lines of $\mathbb{R}^{2,3}$. This is the quotient of $\mathcal{E}^{1,2}$ by the orientation-reversing involutory isometry $[X] \mapsto [-X]$.

Under the identification above, $\mathrm{O}^{\uparrow}_{+}(2,3)$ acts effectively and transitively on $\mathcal{E}^{1,2}$ on the left by $F \cdot [X] = [F \cdot X]$, for each $F \in \mathrm{O}^{\uparrow}_{+}(2,3)$ and $[X] \in \mathcal{E}^{1,2}$. This action preserves the oriented and time-oriented conformal Lorentzian structure of $\mathcal{E}^{1,2}$. It is a classical result that every restricted conformal transformation of $\mathcal{E}^{1,2}$ is induced by a unique element of $\mathrm{O}^{\uparrow}_{+}(2,3)$ (cf. [2, 23]). Accordingly, we call $\mathrm{O}^{\uparrow}_{+}(2,3)$ the (*restricted*) *conformal group* of $\mathcal{E}^{1,2}$.

2.3. **The AdS chambers and the neutral space form $S^{2,2}$.** Let $S \in \mathbb{R}^{2,3}$ be a unit spacelike vector. The *positive and negative AdS (anti-de Sitter) chambers* of $\mathcal{E}^{1,2}$ determined by S are the open sets

$$\mathcal{A}^{\varepsilon}_{S} = \{[V] \in \mathcal{E}^{1,2} \mid \operatorname{sgn}\langle V, S\rangle = \varepsilon\} \subset \mathcal{E}^{1,2}, \quad \varepsilon = \pm 1.$$

The common boundary of the AdS chambers is the 2-dimensional torus $\partial\mathcal{A}_S = \{[V] \in \mathcal{E}^{1,2} \mid \langle V, S\rangle = 0\} \cong S^1 \times S^1$, called the *AdS wall*. Accordingly, $\mathcal{E}^{1,2}$ splits as the disjoint union of the two AdS chambers and the AdS wall. The AdS walls are totally umbilical timelike surfaces (Lorentz tori) of $\mathcal{E}^{1,2}$. Contracting $dV_{\mathcal{E}}$ with S defines a volume form on $\partial\mathcal{A}_S$. Thus, the unit spacelike vectors of $\mathbb{R}^{2,3}$ are in 1:1 correspondence with the oriented AdS walls. In this way, the smooth hyperquadric

$$S^{2,2} = \{S \in \mathbb{R}^{2,3} \mid \langle S, S\rangle = 1\} \subset \mathbb{R}^{2,3}$$

can be viewed as the manifold of all oriented AdS walls of $\mathcal{E}^{1,2}$. The scalar product of $\mathbb{R}^{2,3}$ induces a neutral $(2,2)$ pseudo-metric g_S on $S^{2,2}$ [33]. The group $\mathrm{O}^{\uparrow}_{+}(2,3)$ acts transitively on $S^{2,2}$ on the left preserving g_S. With respect to such an action, $S^{2,2}$ is a symmetric space. The map

$$\mathbb{V}^2_{-} \oplus S^2_{+} \ni V_{-} + V_{+} \longmapsto V_{-} + \sqrt{1 - \langle V_{-}, V_{-}\rangle} V_{+} \in S^{2,2} \tag{2.4}$$

is a smooth diffeomorphism. So, as a manifold, $S^{2,2}$ is the Cartesian product of $\mathbb{R}^2$ with a 2-dimensional sphere.

Remark 2.3. As a model for the *anti-de Sitter 3-space*, we take the hyperquadric $\mathcal{A} \subset \mathbb{R}^4$ consisting of all vectors $\mathbf{p} = (x, y)$, $x = (x^1, x^2)$, $y = (y^1, y^2)$ of $\mathbb{R}^4$, such that $-(x^1)^2 - (x^2)^2 + (y^1)^2 + (y^2)^2 = -1$, equipped with the Lorentzian structure induced by the neutral scalar product $(\mathbf{p}, \tilde{\mathbf{p}})_{(2,2)} = -x^1\tilde{x}^1 - x^2\tilde{x}^2 + y^1\tilde{y}^1 + y^2\tilde{y}^2$. Let $\mathfrak{B}$ be a pseudo-orthogonal basis such that $B_4 = S$. Then, $\mathcal{A}^{\varepsilon}_{S}$ is the image of $\mathcal{A}$ via the conformal embedding

$$\mathcal{A} \ni (x, y) \mapsto \left[x^1 B_0 + x^2 B_1 + y^1 B_2 + y^2 B_3 + \varepsilon B_4\right] \in \mathcal{A}^{\varepsilon}_{S}.$$

Let $\mathrm{T} \subset \mathbb{R}^3$ be the open solid torus swept out by rotating around the z-axis the (open) unit disk in the xz-plane centered at $(2, 0, 0)$. Using local

coordinates $0 \leq r < 1$, ϕ, $\theta \in [0, 2\pi)$, the points of T can be parametrized by

$$P(r, \phi, \theta) = \Big((r\cos\phi + 2)\cos\theta, (r\cos\phi + 2)\sin\theta, r\sin\phi \Big).$$

The map $\mathcal{T}_\varepsilon : \mathrm{T} \to \mathcal{A}_S^\varepsilon$, defined by

$$P(r, \phi, \theta) \mapsto \Big[r\cos(\phi)B_0 + r\sin(\phi)B_1 + \cos(\theta)B_2 + \sin(\theta)B_3 + \varepsilon\sqrt{1 - r^2}B_4 \Big]$$

is a smooth diffeomorphism which extends by continuity to the boundary $\partial\mathrm{T}$ of the solid torus T, giving rise to a diffeomorphism

$$\partial\mathrm{T} \ni P(1, \phi, \theta) \mapsto [\cos(\phi)B_0 + \sin(\phi)B_1 + \cos(\theta)B_2 + \sin(\theta)B_3] \in \partial\mathcal{A}_S$$

between the boundary $\partial\mathrm{T}$ of the solid torus and the AdS wall $\partial\mathcal{A}_S$. Thus, topologically, $\mathcal{E}^{1,2}$ can be identified with the disjoint union $\overline{\mathrm{T}} \times \{-1, 1\}$ of two copies of the closed solid torus modulo the equivalence relation $[(P, \epsilon)]_\sim = \{(P, \epsilon)\}$, if $P \in \mathrm{Int}(\mathrm{T})$, and $[(P, \epsilon)]_\sim = \{(P, 1)(P, -1)\}$, if $P \in \partial\mathrm{T}$. In the following, we will use the "toroidal" projections $\mathcal{T}_\varepsilon^{-1} : \overline{\mathcal{A}}_S^\varepsilon \to \overline{\mathrm{T}}$ to visualize the geometric content of some of our results.

2.4. **Other models of the Einstein universe.**

2.4.1. *$\mathcal{E}^{1,2}$ as symmetric R-space.* The transitive action of $\mathrm{O}^\uparrow_+(2, 3)$ on $\mathcal{E}^{1,2}$ defines a principal H_0-bundle

$$\pi_{\mathcal{E}} : \mathrm{O}^\uparrow_+(2, 3) \ni F \longmapsto [F_0] \in \mathcal{E}^{1,2}, \tag{2.5}$$

where H_0 is the isotropy subgroup of $\mathrm{O}^\uparrow_+(2, 3)$ at $[E_0]$, namely the 7-dimensional connected Lie subgroup of $\mathrm{O}^\uparrow_+(2, 3)$

$$H_0 = \left\{ R(r, L, x) = \begin{pmatrix} e^r & e^r\, {}^t x\epsilon_{1,2}L & \frac{e^r}{2}\, {}^t x\epsilon_{1,2}x \\ 0 & L & x \\ 0 & 0 & e^{-r} \end{pmatrix} \;\Big|\; \begin{array}{l} r \in \mathbb{R},\, x \in \mathbb{R}^3 \\ {}^t L = \epsilon_{1,2}L^{-1}\epsilon_{1,2} \\ \epsilon_{1,2} = \mathrm{diag}(-1, 1, 1) \end{array} \right\}.$$

The Lie algebra of H_0 is the subalgebra of $\mathfrak{o}(2, 3)$

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} r & {}^t x\epsilon_{1,2} & 0 \\ 0 & A & x \\ 0 & 0 & -r \end{pmatrix} \;\Big|\; \begin{array}{l} r \in \mathbb{R},\, x \in \mathbb{R}^3 \\ {}^t A = -\epsilon_{1,2}A\epsilon_{1,2} \\ \epsilon_{1,2} = \mathrm{diag}(-1, 1, 1) \end{array} \right\}.$$

The polar space $\mathfrak{h}_0^\perp$ of $\mathfrak{h}_0$ with respect to the Killing form of $\mathfrak{o}(2, 3)$ is

$$\mathfrak{h}_0^\perp = \left\{ \begin{pmatrix} 0 & {}^t x\epsilon_{1,2} & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \;\Big|\; \begin{array}{l} x \in \mathbb{R}^3 \\ \epsilon_{1,2} = \mathrm{diag}(-1, 1, 1) \end{array} \right\},$$

which is an abelian subalgebra of $\mathfrak{o}(2, 3)$. Thus $\mathfrak{h}_0$ is a *height* 1 parabolic subalgebra of $\mathfrak{o}(2, 3)$ (cf. [8] for the definitions and more details). More generally, for each $[V] \in \mathcal{E}^{1,2}$, the polar of $\mathfrak{h}_{[V]}$, the Lie algebra of the isotropy subgroup $H_{[V]}$ at $[V]$, is an abelian subalgebra of $\mathfrak{o}(2, 3)$, and hence $\mathfrak{h}_{[V]}$ is a *height* 1 parabolic subalgebra of $\mathfrak{o}(2, 3)$.

The adjoint representation of $\mathrm{O}^{\uparrow}_{+}(2,3)$ induces an action on the Grassmannian of 7-dimensional linear subspaces of $\mathfrak{o}(2,3)$ which, in turn, gives rise to a transitive action of $\mathrm{O}^{\uparrow}_{+}(2,3)$ on the orbit $\mathcal{O}$ through $\mathfrak{h}_0 = \mathfrak{h}_{[E_0]}$. Accordingly, the orbit $\mathcal{O}$ acted upon transitively by $\mathrm{O}^{\uparrow}_{+}(2,3)$ is a self-dual symmetric R-space (cf. [8, 26, 48]).[3] Since $\mathfrak{h}_{[F\cdot V]} = Ad_F \cdot \mathfrak{h}_{[V]}$ and $\mathfrak{h}_{[V]} = \mathfrak{h}_{[W]}$ if and only if $[V] = [W]$, the map $\mathcal{E}^{1,2} \ni [V] \mapsto \mathfrak{h}_{[V]} \in \mathcal{O}$ defines an equivariant smooth diffeomorphism. This describes $\mathcal{E}^{1,2}$ as the symmetric R-space given by the conjugacy class $\mathcal{O}$ of height 1 parabolic subalgebras of $\mathfrak{o}(2,3)$.

2.4.2. $\mathcal{E}^{1,2}$ *as oriented Lagrangian Grassmannian.* The Einstein universe $\mathcal{E}^{1,2}$ can also be realized as the *Grassmannian of oriented Lagrangian planes* of $\mathbb{R}^4$. This description of $\mathcal{E}^{1,2}$ is specific of dimension three. Let $(\mathbb{R}^4, \omega)$ be $\mathbb{R}^4$ with the standard symplectic form $\omega(x,y) = {}^t x J y$, where $J = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$. Next, on the 6-dimensional vector space $\Lambda^2\left((\mathbb{R}^4)^*\right)$ consider the neutral scalar product (α, β) of signature $(3,3)$, defined by $(\alpha,\beta)\omega^2 = \alpha \wedge \beta$. By construction, ω is a spacelike vector, i.e., $(\omega,\omega) = 1$, which implies that the restriction of $(\cdot\,,\cdot)$ to the 5-dimensional polar space $[\omega]^{\perp}$ has signature $(2,3)$. Hence, $[\omega]^{\perp}$ together with $(\cdot\,,\cdot)$ can be identified with $\mathbb{R}^{2,3}$. Thus, the Einstein universe $\mathcal{E}^{1,2}$ (i.e., the manifold of null rays of $\mathbb{R}^{2,3} \cong [\omega]^{\perp}$) can be canonically identified with the Grassmannian of oriented Lagrangian planes of $(\mathbb{R}^4, \omega)$ (cf. [20, 46]).

3. Quasi-umbilical timelike immersions

3.1. **Timelike immersions.** Let X be an *oriented* connected surface. An immersion $f : X \to \mathcal{E}^{1,2}$ is said to be *timelike* if the induced metric $f^*(g_{\mathcal{E}})$ is Lorentzian. Observe that $(X, f^*(g_{\mathcal{E}}))$ is time-orientable. When no confusion can arise, we use f to denote both the map into $\mathcal{E}^{1,2}$ and a lift of f to $\mathbb{R}^{2,3}$.

Let $\pi_0 : \mathcal{F}_0(f) \to X$, where

$$\mathcal{F}_0(f) = \{(x, F) \in X \times \mathrm{O}^{\uparrow}_{+}(2,3) \mid f(x) = \pi_0(F) = [F_0]\},$$

be the pullback by f of the principal H_0-bundle $\pi_{\mathcal{E}} : \mathrm{O}^{\uparrow}_{+}(2,3) \to \mathcal{E}^{1,2}$. A *conformal frame field* along f is a local section of $\mathcal{F}_0(f)$, that is, a smooth map $F : U \to \mathrm{O}^{\uparrow}_{+}(2,3)$, where U is an open set of X, such that $f = \pi_0 \circ F$. For any such frame field we put $\phi = F^*\varphi = (\phi^i_j)$. Then[4]

$$dF_k = \sum_j \phi^j_k F_j \quad (k = 0, \dots, 4).$$

[3] A parabolic subalgebra $\mathfrak{q} \subset \mathfrak{o}(2,3)$ is said to be *complementary* to a given parabolic subalgebra $\mathfrak{p}$ if $\mathfrak{o}(2,3) = \mathfrak{p} + \mathfrak{q}$. If $\mathfrak{q}$ is complementary to $\mathfrak{p}$, the orbit $\mathcal{O}^*$ of $\mathfrak{q}$ coincides with the set of all parabolic subalgebras complementary to $\mathfrak{p}$. The orbit $\mathcal{O}^*$ is the *dual* of $\mathcal{O}$. The symmetric R-space $\mathcal{O}$ is *self-dual* if $\mathcal{O}^*$ and $\mathcal{O}$ coincide.

[4] We suppress the tensor symbol in the tensor products $\phi^j_k \otimes F_j$ occurring in the sum.

Given a conformal frame field on U, any other on U is given by

$$\tilde{F} = FR,$$

where $R : U \to H_0$ is a smooth map. If $\tilde{\phi} = (\tilde{F})^*\varphi$, then

$$\tilde{\phi} = R^{-1}\phi R + R^{-1}dR. \tag{3.1}$$

Definition 3.1. A conformal frame field $F : U \to \mathrm{O}^{\uparrow}_{+}(2,3)$ is of *first order* if at every point of U

$$\phi^3_0 = 0, \quad \phi^1_0 \wedge \phi^2_0 > 0.$$

It is easily seen that first order frame fields exist locally. Moreover, if F is a first order frame field, then any other on U is given by $\tilde{F} = FR$, for a smooth map $R : U \to H_1$, and

$$H_1 = \left\{ R(r,s,x,y) = \begin{pmatrix} e^r & e^r\, {}^t x \epsilon_{1,1} B & e^r y & e^r \frac{{}^t x \epsilon_{1,1} x + y^2}{2} \\ 0 & B & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & e^{-r} \end{pmatrix} \in H_0 \right\},$$

where

$$r \in \mathbb{R}, \quad B = \left(\begin{smallmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{smallmatrix}\right), \ s \in \mathbb{R}, \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R}, \quad \epsilon_{1,1} = \mathrm{diag}(-1,1).$$

The totality of first order frames along f gives rise to a principal H_1-bundle $\pi_1 : \mathcal{F}_1(f) \to X$, where $\mathcal{F}_1(f) = \{(x,F) \in X \times \mathrm{O}^{\uparrow}_{+}(2,3)\}$, being F any local first order frame field along f.

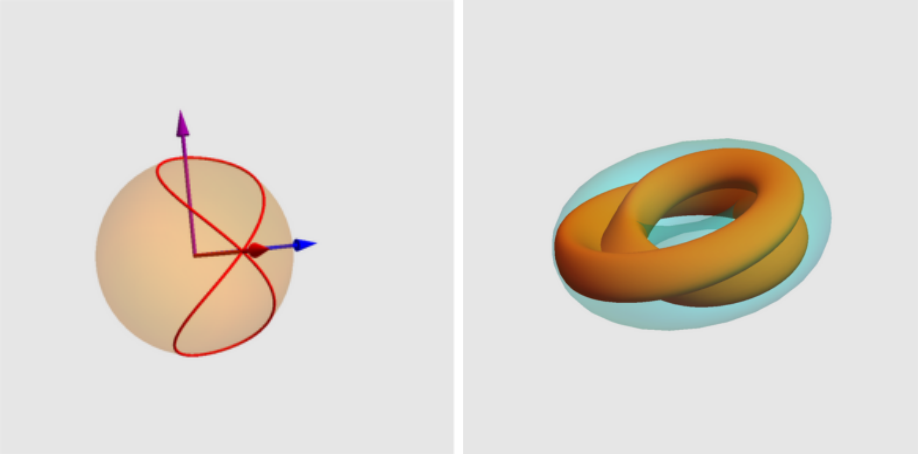

FIGURE 1. The Viviani curve (a spherical lemniscate) (cf. [29]) and the associated timelike immersion of the Klein bottle in the Einstein universe.

Definition 3.2. A first order frame field F along f is of *second order* if

$$\phi^3_1 \wedge \phi^2_0 + \phi^3_2 \wedge \phi^1_0 = 0. \tag{3.2}$$

A proof of the existence of second order frame fields, using different notation, can be found in [46, p. 99] (cf. also [6, 34, 37] for the Möbius case).

Proposition 3.3. *Second order frame fields exist near any point of X. Moreover, if $F, \hat{F} : U \to \mathrm{O}^{\uparrow}_{+}(2,3)$ are second order frame fields, then $\hat{F} = FR$, where $R : U \to H_2$ is a smooth map from U into the closed subgroup*

$$H_2 = \{R(r,s,x,y) \in H_1 \mid y = 0\}.$$

The second order frame fields along f are the local sections of a reduced subbundle $\pi_2 : \mathcal{F}_2(f) \to X$ of $\pi_0 : \mathcal{F}_0(f) \to X$, with structure group H_2.

Definition 3.4. Since the map $\mathcal{F}_2(f) \ni (x, F) \mapsto F_3 \in S^{2,2}$ is constant along the fibers of $\pi_2 : \mathcal{F}_2(f) \to X$, there exists a unique smooth map $\mathcal{N}_f : X \to S^{2,2}$, called the *conformal Gauss map* of f, such that $\mathcal{N}_f \circ \pi_2 = F_3$.

Remark 3.5. The AdS wall $\partial \mathcal{A}_{\mathcal{N}_f|_x}$ is characterized by the property of having second order analytic contact with f at $f(x)$ and is called the *central torus* of the surface at $f(x)$ (see Figure 2). The order of contact is strictly bigger than two if and only if x is an *umbilical point* of f (i.e., if $f(x) \wedge d\mathcal{N}_f|_x = 0$).

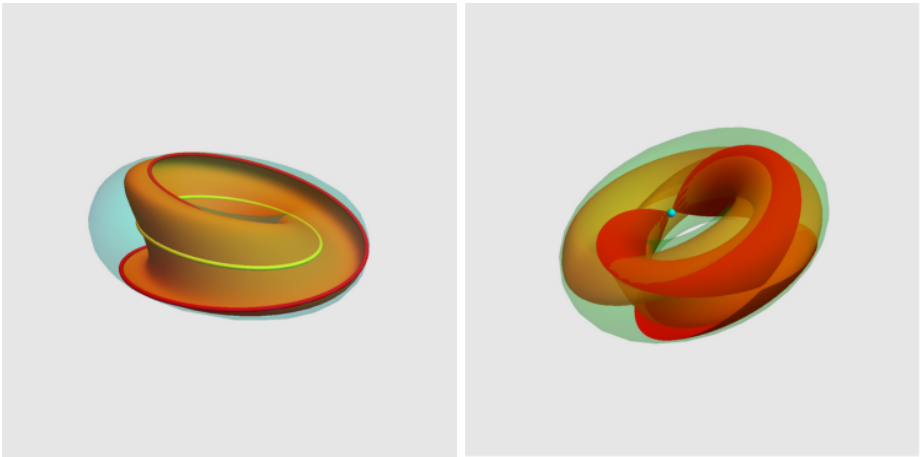

FIGURE 2. On the left: the Möbius strip bounded by the red curve, the intersection of the surface with the AdS wall. On the right: the portion of a central torus of the surface (at the point colored in cyan) contained in the positive AdS chamber.

Remark 3.6. On $\mathcal{F}_2(f)$, the 1-forms ϕ^3_0 and $\phi^3_1 \wedge \phi^2_0 + \phi^3_2 \wedge \phi^1_0$ are identically zero, and $\phi^1_0 \wedge \phi^2_0 > 0$. Differentiating $\phi^3_0 = 0$, by (2.3), we compute $\phi^3_1 \wedge \phi^1_0 + \phi^3_2 \wedge \phi^2_0 = 0$. It follows, by Cartan's Lemma, that there exist smooth functions h_{11}, h_{12}, h_{22} on $\mathcal{F}_2(f)$, such that

$$\phi^3_1 = h_{11}\phi^1_0 + h_{12}\phi^2_0, \quad \phi^3_2 = h_{12}\phi^1_0 + h_{22}\phi^2_0. \tag{3.3}$$

Because $\phi_1^3 \wedge \phi_0^2 + \phi_2^3 \wedge \phi_0^1 = 0$, by (3.3), it then follows that $h_{11} = h_{22}$. Next, on $\mathcal{F}_2(f)$, we have $\phi_3^4 = \phi_0^3 = 0$, so we compute

$$dF_3 = \phi_3^0 F_0 + \phi_1^3 F_1 - \phi_2^3 F_2. \tag{3.4}$$

Because $h_{11} = h_{22}$ on $\mathcal{F}_2(f)$, it follows from (3.4) that

$$\langle dF_3, dF_3 \rangle = (h_{11}^2 - h_{12}^2)\left(-(\phi_0^1)^2 + (\phi_0^2)^2\right). \tag{3.5}$$

Since $\mathcal{N}_f = F_3$ on $\mathcal{F}_2(f)$, we see that $\langle d\mathcal{N}_f, d\mathcal{N}_f \rangle$ is proportional to $-(\phi_0^1)^2 + (\phi_0^2)^2$, which defines the induced conformal structure on X. This shows that the conformal Gauss map $\mathcal{N}_f : X \to S^{2,2}$ is *weakly conformal*.

Remark 3.7. If $F : U \subset X \to \mathrm{O}_+^{\uparrow}(2,3)$ is a first order frame field along f, the shape operator of f with respect to F can be defined at every point of U by the self-adjoint endomorphism of the tangent space given by $A_F = -\phi_1^3 F_1 + \phi_2^3 F_2$. The shape operator depends on the choice of F. Indeed, if $\tilde{F}$ and F are first order frames on U and $\tilde{F} = FR$, for $R = R(r,s,x,y) : U \to H_1$, it follows from (3.1) that $A_{\tilde{F}} = e^{-r} A_F - y\,\mathrm{Id}$. It is not difficult to verify that a first order frame field is of second order if $\operatorname{tr} A_F = 0$. In particular, observe that the conformal factor in (3.5) amounts to the negative of $\det A_F$, the determinant of the trace-free part of the shape operator of f.

Definition 3.8. If $f : X \to \mathcal{E}^{1,2}$ is a timelike immersion, then

$$\langle d\mathcal{N}_f, d\mathcal{N}_f \rangle = \varrho_f f^*(g_{\mathcal{E}}),$$

where ϱ_f is a smooth real-valued function. A point $x \in X$ is called

- a 2-*elliptic point* if $\varrho_f(x) < 0$;
- a 2-*hyperbolic point* if $\varrho_f(x) > 0$;
- a *quasi-umbilical point* (or a 2-*parabolic point*) if $\varrho_f(x) = 0$, with $f(x) \wedge d\mathcal{N}_f|_x \neq 0$;
- an *umbilical point* if $f(x) \wedge d\mathcal{N}_f|_x = 0$. If x is umbilical, $\varrho_f(x) = 0$.

If all points of X are of a fixed type, i.e., if f is of *constant type*, f is said to be, respectively, *elliptic, hyperbolic, quasi-umbilical*, and *totally umbilical*.

Remark 3.9. It is easily seen that the (trace-free part of the) shape operator of f at x is non-diagonalizable over $\mathbb{C}$ if and only if x is quasi-umbilical. A timelike immersion f is totally umbilical if and only if $f(X)$ is an open set of an AdS wall. With respect to a second order frame field F, the 2nd fundamental form of f is locally given by $-\langle dF_0, d\mathcal{N}_f \rangle$.

Example 3.10. Let $(P_0, \ldots, P_4)$ be the standard pseudo-orthogonal basis of $\mathbb{R}^{2,3}$ and $\gamma : \mathbb{R} \to \mathcal{E}^{1,2}$ be the curve defined by

$$\gamma(s) = P_0 + \frac{1}{2}((1 + \cos(2s))P_2 + \sin(2s)P_3 + 2\sin(s)P_4).$$

Consider the 1-parameter group of isometries of $\mathcal{E}^{1,2}$ given by

$$\begin{aligned} Q(t) = \cos(2t)(E_0^0 + E_1^1) - \sin(2t)(E_0^1 - E_1^0) + E_2^2 + \\ + \cos(t)(E_3^3 + E_4^4) + \sin(t)(E_3^4 - E_4^3). \end{aligned}$$

Then the map

$$\hat{f} : \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2 \ni [(s,t)] \mapsto Q(t) \cdot \gamma(s) \in \mathcal{E}^{1,2}$$

is a timelike immersion trapped in the closure of the AdS chamber $\mathcal{A}_{P_3}$ (see Figure 2). In addition, $\hat{f}$ is invariant under the involutory orientation-reversing diffeomorphism $\psi : (s,t) \in \mathbb{T}^2 \to (-s, t+\pi) \in \mathbb{T}^2$. Hence, $\hat{f}$ descends to a generically 1:1 immersion of the Klein bottle $\mathbb{B}^2 = \mathbb{T}^2/\{\mathrm{Id}, \psi\}$ (see Figure 1). The immersion $\hat{f}$ has no umbilical points and there are two closed curves of quasi-umbilical points that disconnect $\mathbb{B}^2$ into two open sets consisting respectively of 2-elliptic and 2-hyperbolic points (see Figure 3). This construction works for every spherical lemniscate which is invariant under a reflection with respect to a great circle of the 2-sphere.

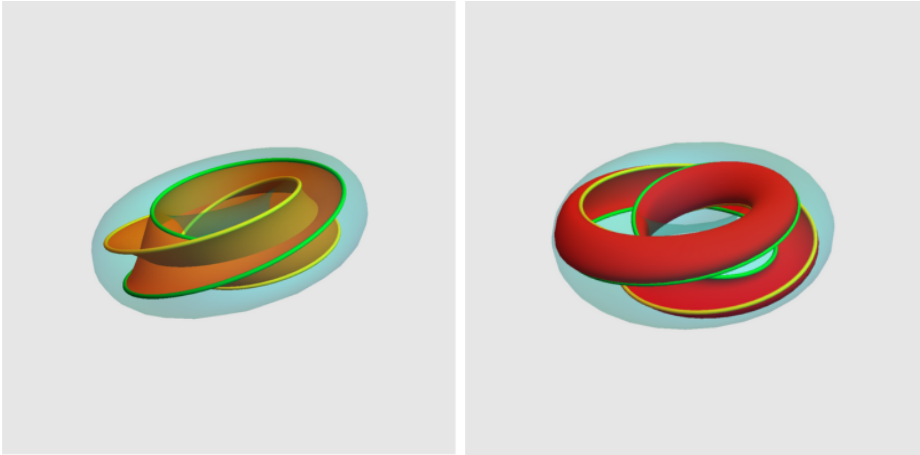

FIGURE 3. On the left: the hyperbolic region bounded by the two quasi-umbilical curves. On the right: the elliptic region bounded by the two quasi-umbilical curves.

3.2. **Isothermic maps.** Viewing the Einstein universe $\mathcal{E}^{1,2}$ as a symmetric R-space, i.e., as the orbit $\mathcal{O}$ of the height 1 parabolic subalgebra $\mathfrak{h}_0$ of $\mathfrak{o}(2,3)$ (cf. Section 2.4), we now discuss isothermic maps in $\mathcal{E}^{1,2}$. According to [8], the notion of isothermic map is strictly related to the structure of symmetric R-space of the target space. Let $f : X \to \mathcal{E}^{1,2} \cong \mathcal{O}$ be a timelike immersion. Consider the vector bundle

$$\mathfrak{H}_f^\perp = \left\{ (x, \mathbf{h}) \in X \times \mathfrak{o}(2,3) \mid \mathbf{h} \in \mathfrak{h}_x^\perp \right\},$$

where, for each $x \in X$, $\mathfrak{h}_x$ denotes the Lie algebra of the stabilizer of $f(x)$.

Definition 3.11. A timelike immersion $f : X \to \mathcal{E}^{1,2}$ is *isothermic* if there is a nonzero 1-form $\delta \in \Omega^1(X) \otimes \mathfrak{H}_f^\perp$ which is closed as an element of $\Omega^1(X) \otimes \mathfrak{o}(2,3)$. Thus δ takes values in $\mathfrak{H}_f^\perp$ and is closed when viewed as an $\mathfrak{o}(2,3)$-valued form. The 1-form δ is called an *infinitesimal deformation* of f.

Remark 3.12. The concept of isothermic map originates from the deformation problem of submanifolds in homogeneous spaces (cf. [10, 13, 31, 32, 37, 38, 41, 45]). Two maps $f : M \to G/H$ and $\hat{f} : \hat{M} \to G/H$ are said to be *kth order deformations of each other* (with unfixed parameters) if there exist a diffeomorphism $\Phi : M \to \hat{M}$, the *change of parameters*, and a nonconstant map $\Delta : M \to G$, the *deformation*, such that f and $\Delta(p) \cdot \hat{f} \circ \Phi$ have kth order contact at p, for every $p \in M$. In addition, if $M = \hat{M}$ and $\Phi = \mathrm{Id}_M$, then f and $\hat{f}$ are said to be *kth order deformations of each other* (with fixed parameters). If G/H is a symmetric R-space and if $f : M \to G/H$ is isothermic, then f admits second order deformations [8]. In the case under discussion, the situation is the following. Let $f : X \to \mathcal{E}^{1,2}$ be an isothermic timelike immersion, $p : \tilde{X} \to X$ be a simply connected covering of X, and δ be an infinitesimal deformation of f. Since $\mathfrak{h}_x^\perp$ is abelian, δ satisfies the Maurer–Cartan equation. So, there exists a smooth map $\Delta : \tilde{X} \to \mathrm{O}^\uparrow_+(2,3)$, such that $\Delta^{-1} d\Delta = p^*\delta$. Then,

$$\hat{f} : \tilde{X} \ni \tilde{x} \longmapsto \Delta(\tilde{x}) \cdot f|_{p(\tilde{x})} \in \mathcal{E}^{1,2}$$

is a second order deformation (with fixed parameters) of $f \circ p$. The deformability of f depends on the existence of an infinitesimal deformation originating a map $\Delta : \tilde{X} \to \mathrm{O}^\uparrow_+(2,3)$ which is invariant under the group of deck transformations of the covering.

3.3. **General properties of quasi-umbilical immersions.**

Lemma 3.13. *Let $f : X \to \mathcal{E}^{1,2}$ be a quasi-umbilical immersion and let g denote the induced Lorentzian metric $f^*(g_{\mathcal{E}})$. Then, the conformal Gauss map $\mathcal{N}_f$ has rank 1. In addition,* $\operatorname{Im} d\mathcal{N}_f$ *is a null line bundle such that* $\operatorname{Im} d\mathcal{N}_f = \operatorname{Ker} d\mathcal{N}_f$.[5]

Proof. Since the property is local in nature, we may assume the existence of a global cross section $F : X \to \mathcal{F}_2(f)$, such that $[F_0] = f$. Then, from (3.3), (3.4) and (3.5), we compute

$$dF_0 = \phi_0^0 F_0 + \phi_0^1 F_1 + \phi_0^2 F_2, \quad d\mathcal{N}_f = h(\phi_0^1 + \eta\phi_0^2)(F_1 - \eta F_2) + \phi_4^3 F_0,$$

where h is a nowhere vanishing smooth function and $\eta = \pm 1$. Let (∂_1, ∂_2) denote the trivialization of $T(X)$ dual to (ϕ_0^1, ϕ_0^2). Then, identifying $T(X)$ with the quotient bundle $\operatorname{span}\{F_0, F_1, F_2\}/\operatorname{span}\{F_0\}$, we obtain

$$d\mathcal{N}_f = h(\phi_0^1 + \eta\phi_0^2)(\partial_1 - \eta\partial_2).$$

This implies that $\operatorname{Im} d\mathcal{N}_f = \operatorname{Ker} d\mathcal{N}_f = \operatorname{span}\{\partial_1 - \eta\partial_2\}$. From the expression of $g = -(\phi_0^1)^2 + (\phi_0^2)^2$, it follows that $\operatorname{Im} d\mathcal{N}_f$ is a null line subbundle of $T(X)$. This concludes the proof. □

Definition 3.14. On X there is a unique *canonical orientation* such that (V, V_*) is a positive basis of $T_x(X)$, for every $x \in X$, for every nonzero

[5]Notice that $d\mathcal{N}_f|_x$ is a 2-step nilpotent self-adjoint endomorphism of $T_x(X)$, $\forall\, x \in X$.

element V of $\mathrm{Im}(d\mathcal{N}_f|_x)$, and for every null vector $V_* \in T_x(X)$, such that $g(V, V_*) = -1$. From now on, we consider on X the canonical orientation induced by f. By construction, the map

$$\Psi : \mathrm{Im}\, d\mathcal{N}_f \setminus \{0\} \ni \mathbf{V} \longmapsto -g(d\mathcal{N}_f(\mathbf{V}_*), \mathbf{V}_*) \in \mathbb{R}$$

is well defined and is either strictly positive or strictly negative. The sign of Ψ, denoted by ε_f, is called the *helicity* of f.

Remark 3.15. If X is equipped with the canonical orientation induced by f and if F is a second order frame field along f, then $h_{11} = h_{12}$ and $h := h_{11}$ is nowhere vanishing. The sign of h is equal to the helicity of f. Replacing f by $-f$, the helicity changes sign.

Definition 3.16. Let $f : X \to \mathcal{E}^{1,2}$ be a quasi-umbilical immersion. A second order frame field along f is said to be *adapted* if

$$h = \varepsilon_f, \quad \phi_4^3 = \phi_0^0 - 2\phi_1^2 = 0, \quad (\phi_4^1 + \phi_4^2) \wedge (\phi_0^1 + \phi_0^2) = 0. \tag{3.6}$$

The following result can be found differently formulated in [46, p. 103].

Proposition 3.17. *Adapted frame fields exist near any point of X. In addition, if $F, \hat{F} : U \to \mathrm{O}^{\uparrow}_{+}(2,3)$ are adapted frame fields, then $\hat{F} = F\,R$, where $R : U \to H_*$ is a smooth map into the abelian 1-dimensional closed subgroup*

$$H_* = \{R(s) = R(r, s, x, 0) \in H_2 \mid r = 2s,\, x = 0\}.$$

The adapted frame fields along f are the local sections of a reduced subbundle $\mathcal{F}_*(f)$ of $\mathcal{F}_2(f)$, with structure group H_*.

Definition 3.18. The map $\mathcal{F}_*(f) \ni (x, F) \mapsto [F_4] \in \mathcal{E}^{1,2}$ is constant along the fibers. It descends to a map $f^\sharp : X \to \mathcal{E}^{1,2}$, called the *dual* of f.

Remark 3.19. The dual map is the second envelope of the 1-parameter family of central tori of f. That is,

$$f^\sharp(x) \in \partial\mathcal{A}|_x, \quad df^\sharp|_x(T_xX) \subset T_{f^\sharp(x)}(\partial\mathcal{A}|_x), \quad \forall\, x \in X,$$

where $\partial\mathcal{A}|_x$ is the AdS wall of $\mathcal{N}_f(x)$.

Definition 3.20. A quasi-umbilical immersion f is called *regular* if the leaf space Λ of the vertical distribution $\mathrm{Ker}\, d\mathcal{N}_f$ is Hausdorff with respect to the quotient topology. If f is regular, Λ is a connected 1-dimensional manifold and the natural projection $\pi : X \to \Lambda$ is a smooth submersion with connected fibers. Then, $\mathcal{N}_f = \Gamma \circ \pi$, where $\Gamma : \Lambda \to S^{2,2}$ is a null immersed curve, called the *directrix curve* of f. Note that locally every quasi-umbilical immersion is regular.

Remark 3.21. As we will show below, a regular quasi-umbilical immersion can be reconstructed from its directrix curve. This explain the key role played by null curves in the neutral space form $S^{2,2}$. By the identification of $S^{2,2}$ with $\mathbb{V}^2_- \oplus S^2_+$ described in (2.4), a null curve of $S^{2,2}$ has a parametrization of the form $\Gamma = \alpha + \sqrt{1 - \langle\alpha, \alpha\rangle}\beta$, where $\alpha : \mathbb{R} \to \mathbb{V}_-$ is an immersed

plane curve parametrized by arclength and $\beta : \mathbb{R} \to S^2_+$ is a spherical curve with speed

$$v_\beta = \frac{\sqrt{1 - \langle \alpha, \alpha \rangle \sin^2(\theta)}}{1 - \langle \alpha, \alpha \rangle},$$

where $\theta = \widehat{\alpha\alpha'}$. Thus, quasi-umbilical surfaces depend on two functions in one variable, the geodesic curvatures of the curves α and β.

Theorem A. *The conformal Gauss map $\mathcal{N}_f : X \to S^{2,2}$ of a quasi-umbilical immersion $f : X \to \mathcal{E}^{1,2}$ is harmonic. Moreover, if f is regular, then f is isothermic and its infinitesimal deformations depend on one arbitrary function in one variable.*

Proof. The first part of the theorem is local in nature, so we assume the existence of a global adapted frame field $F : X \to \mathrm{O}^{\uparrow}_+(2,3)$, such that $[F_0] = f$. We then have

$$\begin{cases} g = -(\phi^1_0)^2 + (\phi^2_0)^2 = f^*(g_{\mathcal{E}}), \\ \phi^3_0 = \phi^3_4 = \phi^0_0 - 2\phi^2_1 = 0, \\ \phi^3_1 = \phi^3_2 = \varepsilon_f(\phi^1_0 + \phi^2_0), \\ \phi^1_4 = r(\phi^1_0 + \phi^2_0) + \alpha, \\ \phi^2_4 = r(\phi^1_0 + \phi^2_0) - \alpha \end{cases} \tag{3.7}$$

where r is a smooth function and α is an exterior differential 1-form. Then

$$d\mathcal{N}_f = \phi^3_1 F_1 - \phi^3_2 F_2 = \varepsilon_f(\phi^1_0 + \phi^2_0)(F_1 - F_2). \tag{3.8}$$

Since X is 2-dimensional, the harmonicity of $\mathcal{N}_f$ only depends on the conformal structure induced by f on X. Thus, it suffices to compute the tension field of $\mathcal{N}_f$ with respect to the Lorentzian metric g. Let ∇ be the Levi-Civita covariant derivative of g and $\tilde{\nabla}$ be the covariant derivative on the pullback bundle $\mathcal{N}_f^*(T(S^{2,2}))$ induced by the Levi-Civita connetion of $S^{2,2}$. Next, let D be the covariant derivative on $T^*(X) \otimes \mathcal{N}_f^*(T(S^{2,2}))$ induced by ∇ and $\tilde{\nabla}$. Let $\eta^1 = \phi^1_0 + \phi^2_0$ and $\eta^2 = \phi^1_0 - \phi^2_0$. Then $\phi^2_1 = a\eta^1 + b\eta^2$, where a, b are smooth functions. We then have

$$\begin{cases} \tilde{\nabla}(F_1 - F_2) = \phi^2_1(F_1 - F_2) = (a\eta^1 + b\eta^2)(F_1 - F_2), \\ \nabla\eta^1 = (-3a\eta^1 + b\eta^2) \otimes \eta^1. \end{cases}$$

This implies

$$D(d\mathcal{N}_f) = -2a(\eta^1 \otimes \eta^1) \otimes (F_1 - F_2).$$

Taking into account that $2g = \eta^1 \otimes \eta^2$, we may conclude that the tension field $\mathrm{tr}_g(D(d\mathcal{N}_f))$ is identically zero. This proves the first claim.

Next, suppose that f is regular. Let $\Gamma : \Lambda \to S^{2,2}$ be the directrix curve of f and ζ be a nowhere vanishing 1-form on Λ. We put $\tilde{\zeta} = \pi^*(\zeta)$. Then there exists a unique adapted frame field $F : X \to \mathrm{O}^{\uparrow}_+(2,3)$ such

that $\tilde{\zeta} = \pm(\phi_0^1 + \phi_0^2)$. Possibly replacing ζ with $-\zeta$, we may assume that $\tilde{\zeta} = \phi_0^1 + \phi_0^2$. We call ζ a *line element* of Λ. Then,

$$\begin{aligned}\phi = 2\phi_1^2 M_0^0 + \phi_1^2 M_1^2 + \phi_0^1 M_0^1 + \phi_0^2 M_0^2 + \varepsilon\tilde{\zeta}(M_1^3 + M_2^3)\\ + r\tilde{\zeta}(M_4^1 + M_4^2) + \alpha(M_4^1 - M_4^2),\end{aligned} \tag{3.9}$$

where (ϕ_0^1, ϕ_0^2) is a positive-oriented pseudo-orthogonal coframe, $\varepsilon = \pm 1$ is the helicity of f, r is a smooth function, and α is a 1-form. From the structure equations, we obtain

$$0 = d\tilde{\zeta} = d(\phi_0^1 + \phi_0^2) = \phi_1^2 \wedge (\phi_0^1 + \phi_0^2) = \phi_1^2 \wedge \tilde{\zeta}, \tag{3.10}$$

which implies

$$\phi_1^2 = s\tilde{\zeta}, \tag{3.11}$$

where $s : X \to \mathbb{R}$ is a smooth function. Let $\ell : \Lambda \to \mathbb{R}$ be a nonzero smooth function. Put $\tilde{\ell} = \ell \circ \pi$ and denote by η the 1-form

$$\eta = \tilde{\ell}\tilde{\zeta}(M_4^1 - M_4^2). \tag{3.12}$$

This is a closed 1-form with values in the polar space $\mathfrak{h}_0^\perp$ of the parabolic subalgebra $\mathfrak{h}_0$. It follows from (3.9) and (3.10) that

$$\phi \wedge \eta + \eta \wedge \phi = 0.$$

Next, let $\delta := F \cdot \eta \cdot F^{-1}$. Then,

$$d\delta = dF \wedge \eta F^{-1} - F \cdot \eta \wedge dF^{-1} = F \cdot (\phi \wedge \eta + \eta \wedge \phi) \cdot F^{-1} = 0.$$

Since $\mathfrak{h}_x = Ad_{F(x)}(\mathfrak{h}_0)$ and $\mathfrak{h}_x^\perp = Ad_{F(x)}(\mathfrak{h}_0^\perp)$, then $\delta|_x \in T_x^*(X) \otimes \mathfrak{h}_x^\perp$, for every $x \in X$. This proves that δ is an infinitesimal deformation.

Every infinitesimal deformation arises in this way. In fact, if δ is an infinitesimal deformation, then $\eta = Ad_{F^{-1}}\delta$ is a $\mathfrak{h}_0^\perp$-valued 1-form. From $d\delta = 0$ it follows that $d\eta = \eta \wedge \phi + \phi \wedge \eta$. Using (3.9) it is easy to see that the latter equation holds true if and only if η is as in (3.12), for some nonzero smooth function $\ell : \Lambda \to \mathbb{R}$. This concludes the proof. □

Definition 3.22. Let $f : X \to \mathcal{E}^{1,2}$ be a quasi-umbilical immersion and $f^\sharp : X \to \mathcal{E}^{1,2}$ be its dual map. According to whether the quadratic form $\langle d\mathcal{N}_f, df^\sharp\rangle$ is identically zero or not, f is called *exceptional* or *general*.

4. Quasi-umbilical immersions: the exceptional case

4.1. **Biisotropic curves in the neutral space form $S^{2,2}$.** Let Λ be a connected 1-dimensional manifold and ζ be a nowhere zero 1-form on Λ. For a smooth map $G : \Lambda \to \mathbb{R}^k$, we denote by G', G'', etc., the derivatives of G with respect to ζ.

Definition 4.1. An immersed curve $\Gamma : \Lambda \to S^{2,2}$ is called *biisotropic* if $\langle \Gamma', \Gamma'\rangle = \langle \Gamma'', \Gamma''\rangle = 0$ and $(\Gamma' \wedge \Gamma'')|_\tau \neq 0$, for every $\tau \in \Lambda$.

Proposition 4.2. *Let $f : X \to \mathcal{E}^{1,2}$ be a regular quasi-umbilical immersion. Then, f is exceptional if and only if its directrix curve Γ is biisotropic.*

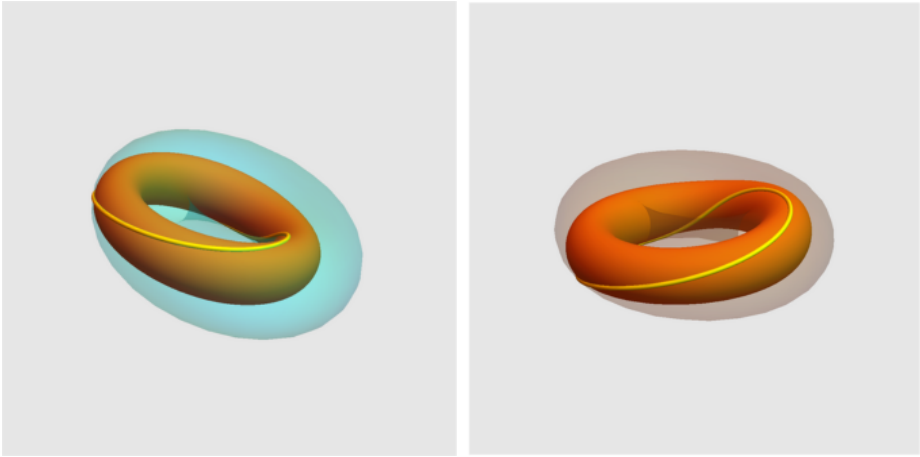

FIGURE 4. The timelike immersed tube corresponding to the "biisotropic circle" (4.15): on the left, the portion of the surface contained in the positive AdS chamber (a parabolic component); on the right, the portion contained in the negative AdS chamber (the other parabolic component). The yellow curve is the curve of umbilical points of the surface.

Proof. Consider a line element ζ on Λ and the unique adapted frame field F along f such that $\phi_0^1 + \phi_0^2 = \tilde{\zeta} = \pi^*(\zeta)$ (cf. the proof of Theorem A). Now, the dual map $f^\sharp = \varrho F_4$, where ϱ is a smooth positive function. From (3.9) and (3.11), we compute

$$d\mathcal{N}_f = \varepsilon\tilde{\zeta}(F_1 - F_2), \quad dF_4 = r\tilde{\zeta}(F_1+F_2) + \alpha(F_1 - F_2) - 2s\tilde{\zeta}F_4, \tag{4.1}$$

which implies $\langle df^\sharp, d\mathcal{N}_f\rangle = -2\varepsilon\varrho r(\tilde{\zeta})^2$. Therefore, f is exceptional if and only if $r = 0$. On the other hand, $\Gamma' \circ \pi = \varepsilon(F_1 - F_2)$. Differentiating the latter identity yields

$$(\Gamma'' \circ \pi)\tilde{\zeta} = d(\Gamma' \circ \pi) = \varepsilon d(F_1 - F_2) = -\varepsilon\tilde{\zeta}\,(2rF_0 + s(F_1 - F_2) + F_4)\,.$$

Thus, $\langle \Gamma'' \circ \pi, \Gamma'' \circ \pi\rangle = -4r\varepsilon^2$, from which the claim follows. □

Lemma 4.3. *Let $\Gamma : \Lambda \to S^{2,2}$ be a biisotropic curve. Then there exist an isotropic (null) 2-dimensional subspace $\mathbb{V}_\Gamma \subset \mathbb{R}^{2,3}$ and a 3-dimensional subspace $\mathbb{W}_\Gamma \subset \mathbb{R}^{2,3}$, such that*

$$[(\Gamma' \wedge \Gamma'')|_\tau] = \mathbb{V}_\Gamma \subset [(\Gamma \wedge \Gamma' \wedge \Gamma'')|_\tau] = \mathbb{W}_\Gamma, \quad \forall\, \tau \in \Lambda. \tag{4.2}$$

Proof. Differentiating the scalar product relations

$$\langle \Gamma, \Gamma\rangle = 1, \quad \langle \Gamma', \Gamma'\rangle = \langle \Gamma'', \Gamma''\rangle = 0 \tag{4.3}$$

yields the following relations

$$\langle \Gamma, \Gamma'\rangle = \langle \Gamma, \Gamma''\rangle = \langle \Gamma, \Gamma'''\rangle = \langle \Gamma', \Gamma''\rangle = \langle \Gamma', \Gamma'''\rangle = \langle \Gamma'', \Gamma'''\rangle = 0. \tag{4.4}$$

It follows from (4.3) that $[(\Gamma' \wedge \Gamma'')|_\tau]$ is a null 2-dimensional subspace and that $[(\Gamma \wedge \Gamma' \wedge \Gamma'')|_\tau]$ is a 3-dimensional subspace of type $(+, 0, 0)$, for every

$\tau \in \Lambda$. By construction, $[(\Gamma' \wedge \Gamma'')|_\tau] = [(\Gamma \wedge \Gamma' \wedge \Gamma'')|_\tau]^\perp$. From (4.4) it then follows that $\Gamma'''|_\tau \in [(\Gamma' \wedge \Gamma'')|_\tau]$, for every τ. This implies that the 2-dimensional subspace $[(\Gamma' \wedge \Gamma'')|_\tau]$ and the 3-dimensional subspace $[(\Gamma \wedge \Gamma' \wedge \Gamma'')|_\tau]$ are constant, as claimed. □

The set of unit vectors lying in $\mathbb{W}_\Gamma$ is the union of two disjoint affine planes parallel to $\mathbb{V}_\Gamma$. The trajectory of Γ is contained in one of them, denoted by $\mathcal{V}_\Gamma$. The positive orientation of $\mathbb{V}_\Gamma$ induces on $\mathcal{V}_\Gamma$ the structure of an affine oriented plane and Γ can be considered as an affine plane curve. Let dA be a positive element of $\Lambda^2(\mathbb{V}_\Gamma)$. Since Γ has no flex points (i.e., points where $\Gamma' \wedge \Gamma'' = 0$), there exists a unique 1-form ζ, such that $dA(\Gamma', \Gamma'') = 1$, the *affine line element* of Γ relative to dA. If we fix an affine line element, then $\Gamma \wedge \Gamma' \wedge \Gamma''$ and $\Gamma' \wedge \Gamma''$ are constant. Differentiating $dA(\Gamma', \Gamma'') = 1$, we get $dA(\Gamma', \Gamma''') = 0$. Then, $\Gamma''' = h\Gamma'$, where $h : \Lambda \to \mathbb{R}$ is a smooth function, the *affine curvature* of Γ relative to dA.

Remark 4.4 (Curvature sign). Our definition of the affine curvature is not the only reasonable possibility; one can also change its sign, defining curvature to be $\tilde{h} = -h$, as in the classical treatise of Blaschke [4].

Definition 4.5. A lightcone basis $\mathfrak{C} = (C_0, \dots, C_4)$ is said to be *calibrated* to Γ and ζ if

$$\Gamma' \wedge \Gamma'' = C_4 \wedge (C_1 - C_2), \quad \Gamma \wedge \Gamma' \wedge \Gamma'' = C_3 \wedge C_4 \wedge (C_1 - C_2).$$

If $\mathfrak{C}$ is a calibrated basis, then

$$\Gamma = C_3 + x(C_1 - C_2) + yC_4,$$

where $\gamma = (x, y) : \Lambda \to \mathbb{R}^2$ is an affine plane curve ($x'y'' - x''y' = 1$), the *affine reduction* of Γ relative to $\mathfrak{C}$. Let

$$\upsilon = \sqrt{(x')^2 + (y')^2} \quad \text{and} \quad \mu = xy' - x'y$$

be the speed and the angular momentum of γ. The vector field

$$\mathrm{R} = -x'C_0 + \frac{1}{2}y'(C_1 + C_2) + \frac{\mu^2}{2\upsilon^2}y'(C_1 - C_2) + \mu C_3 - \frac{\mu^2}{2\upsilon^2}x'\mathrm{C}_4 \tag{4.5}$$

is called the *isotropic normal vector field* along Γ relative to $\mathfrak{C}$. By construction, we have

$$\langle \mathrm{R}, \mathrm{R} \rangle = \langle \mathrm{R}, \Gamma \rangle = \langle \mathrm{R}, \Gamma' \rangle = 0, \quad \langle \mathrm{R}, \Gamma'' \rangle = 1. \tag{4.6}$$

If R and $\hat{\mathrm{R}}$ are isotropic normal vector fields relative to different area elements and different calibrated bases, then $\tilde{\mathrm{R}} = c\mathrm{R} + r\Gamma'$, where c is a positive constant and r is a smooth function. Then, the null plane $[(\mathrm{R} \wedge \Gamma')|_\tau$ is independent of the choice of the null normal vector field.

Definition 4.6. The *normal tube* of Γ is the circle bundle over Λ given by

$$\pi_\Lambda : \mathbb{T}_\Gamma = \big\{(\tau, [V]) \in \Lambda \times \mathcal{E}^{1,2} \mid V \in [(\mathrm{R} \wedge \Gamma')|_\tau\big\} \to \Lambda. \tag{4.7}$$

If R is an isotropic normal vector, the elements of the fiber $\pi_\Lambda^{-1}(\tau)$ can be described by $[V(\tau,\theta)] = [\cos(\theta)\mathrm{R}|_\tau + \frac{1}{2}\sin(\theta)\Gamma'|_\tau]$, for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Hence, $\mathbb{T}_\Gamma$ can be identified with $\Lambda \times S^1$ by means of the mapping

$$\Lambda \times S^1 \ni (\tau,\theta) \mapsto (\tau,[V(\tau,\theta)]) \in \mathbb{T}_\Gamma. \tag{4.8}$$

Let

$$\mathrm{C}_\pm = \{(\tau,[V(\tau,\pm\pi/2)]) \mid \tau \in \Lambda\} \subset \mathbb{T}_\Gamma, \tag{4.9}$$

and call them the *umbilical curves* of $\mathbb{T}_\Gamma$. The complement of $\mathrm{C}_+ \cup \mathrm{C}_-$ consists of the two connected open sets

$$\mathbb{T}_\Gamma^\pm = \{(\tau,[V(\tau,\theta)]) \mid \tau \in \Lambda,\ \mathrm{sgn}(\cos\theta) = \pm 1\}, \tag{4.10}$$

referred to as the *positive and negative parabolic components* of $\mathbb{T}_\Gamma$.

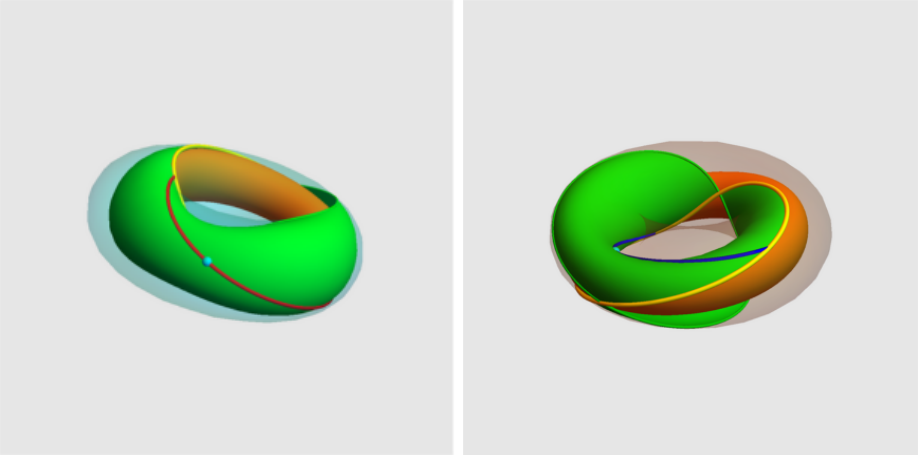

FIGURE 5. The positive (left) and negative (right) parabolic cylinders $f_\Gamma^+(\mathbb{T}_\Gamma^+)$ and $f_\Gamma^-(\mathbb{T}_\Gamma^-)$ and (in green) the central torus of $f_\Gamma(\mathbb{T}_\Gamma)$ for the tube $\mathbb{T}_\Gamma$ along the "biisotropic circle" (4.15). On the left, the positive part of the central torus; on the right, its negative part. The red and blue curves are the positive and negative parts of the line of tangency of the osculating torus with $f_\Gamma(\mathbb{T}_\Gamma)$. The yellow line is the umbilical line.

Definition 4.7. We call $f_\Gamma : \mathbb{T}_\Gamma \to \mathcal{E}^{1,2}$, $(\tau,[V]) \longmapsto [V]$, the *tautological map of* Γ. The restrictions of f_Γ to $\mathbb{T}_\Gamma^\pm$ are denoted by $f_\Gamma^\pm$, respectively.

4.2. **The structure of exceptional quasi-umbilical surfaces.** Using the notation and the constructions of the previous section, we are now in a position to describe the geometric structure of exceptional quasi-umbilical surfaces. We can prove the following.

Theorem B. *Let $\Gamma : \Lambda \to S^{2,2}$ be a biisotropic curve. Then, the tautological map f_Γ is a timelike immersion with conformal Gauss map $\Gamma\circ\pi_\Lambda$ and umbilic locus $\mathrm{C}_+ \cup \mathrm{C}_-$. In addition, $f_\Gamma^\pm$ are exceptional quasi-umbilical immersions*

with helicities ± 1 *and dual maps* $(f_\Gamma^\pm)^\sharp = [\mp\Gamma'' \circ \pi_\Lambda]$. *Conversely, let* $f : X \to \mathcal{E}^{1,2}$ *be a regular quasi-umbilical immersion with helicity* $\varepsilon = \pm 1$. *If* f *is exceptional, then* $f(X) \subset f_\Gamma^\pm\left(\mathbb{T}_\Gamma^\pm\right)$, *where* Γ *is the directrix curve of* f.

Proof. Let $\mathbb{T}_\Gamma$ be identified with $\Lambda \times S^1$ as in (4.8). Take a positive area element dA on $\mathbb{V}_\Gamma$ and denote by ζ the corresponding affine arc element. Consider a calibrated lightcone basis $\mathfrak{C}$ and the associated null normal vector field R (cf. (4.5)). Let h be the affine curvature of Γ. Let $\mathrm{T} : \Lambda \times S^1 \to \mathbb{R}^{2,3}$ be the lift of f_Γ defined by $\mathrm{T}(\tau,\theta) = \cos(\theta)\mathrm{R}|_\tau + \frac{1}{2}\sin(\theta)\Gamma'|_\tau$. Then

$$d\mathrm{T} = \Big(-\sin(\theta)\mathrm{R} + \frac{1}{2}\cos(\theta)\Gamma'\Big)d\theta + \Big(\cos(\theta)\mathrm{R}' + \frac{1}{2}\sin(\theta)\Gamma''\Big)\zeta.$$

Differentiating (4.6), taking into account that $\Gamma''' = h\Gamma'$, yields $\langle \mathrm{R}, \mathrm{R}'\rangle = \langle \mathrm{R}', \Gamma''\rangle = 0$, and $\langle \mathrm{R}', \Gamma'\rangle = -1$, from which one computes

$$\langle d\mathrm{T}, d\mathrm{T}\rangle = -\zeta\left(d\theta - (\cos\theta)^2\langle \mathrm{R}', \mathrm{R}'\rangle\zeta\right).$$

This implies that the tautological map f_Γ is a timelike immersion.

Let $F_0^\pm : \mathbb{T}_\Gamma^\pm \to \mathrm{O}^\uparrow_+(2,3)$ be the lift of $f_\Gamma^\pm$ defined by

$$F_0^\pm = \frac{\pm 1}{\cos\theta}\mathrm{T} = \pm\Big(\mathrm{R} + \frac{1}{2}\tan(\theta)\Gamma'\Big). \tag{4.11}$$

Denote by (∂_1, ∂_2) the trivialization of $T(\mathbb{T}_\Gamma^\pm)$ dual to the coframe (ζ, du), where $u = \tan\theta$. Consider the maps $F_1^\pm, \dots, F_4^\pm : \mathbb{T}_\Gamma^\pm \to \mathbb{R}^{2,3}$, defined by

$$\begin{cases} F_1^\pm = -\partial_1 F_0^\pm - (\langle \partial_1 F_0^\pm, \partial_1 F_0^\pm\rangle + 1)\partial_2 F_0^\pm, \\ F_2^\pm = -\partial_1 F_0^\pm - (\langle \partial_1 F_0^\pm, \partial_1 F_0^\pm\rangle - 1)\partial_2 F_0^\pm, \\ F_3^\pm = \Gamma \circ \pi_{\mathbb{T}}, \\ F_4^\pm = \mp\Gamma'' \circ \pi_{\mathbb{T}}. \end{cases} \tag{4.12}$$

Then, $F_\pm = (F_0^\pm, \dots, F_4^\pm)$ is an adapted frame field along $f_\Gamma^\pm$. The 1-form $\phi_\pm = F_\pm^{-1} dF_\pm$ is as in (3.9), with $\phi_1^2 = 0$, $r = 0$, $\alpha = -h\zeta$, and

$$\begin{cases} \phi_0^1 = -\frac{1}{2}\left(du + (\frac{\mu^2 + (xx' - yy')^2}{\upsilon^4} + 1)\zeta\right), \\ \phi_0^2 = \frac{1}{2}\left(du + (\frac{\mu^2 + (xx' - yy')^2}{\upsilon^4} - 1)\zeta\right), \end{cases} \tag{4.13}$$

where $\gamma = (x, y)$ is the affine reduction of Γ with respect to $\mathfrak{C}$. This proves that $f_\Gamma^\pm$ are exceptional quasi-umbilical immersions with helicity $\varepsilon = \pm 1$, conformal Gauss maps $\Gamma \circ \pi_\Lambda$, and dual maps $[-\varepsilon\Gamma'' \circ \pi_{\mathbb{T}}]$. By continuity, $\Gamma \circ \pi_\Lambda$ is the conformal Gauss map of f_Γ. From the identity $(d\mathcal{N}_f \wedge \mathrm{T})|_{(\tau,\theta)} = \cos(\theta)(\mathrm{R} \wedge \Gamma')|_\tau \zeta$, it follows that (τ, θ) is an umbilical point if and only if $(\tau, \theta) \in \mathrm{C}_+ \cup \mathrm{C}_-$.

Conversely, let $f : X \to \mathcal{E}^{1,2}$ be a regular, quasi-umbilical immersion, with directrix curve $\Gamma : \Lambda \to S^{2,2}$, and let $\mathcal{F}_*(f)$ be the bundle of adapted frames along f. Let $\mathfrak{N}_2^+(\mathbb{R}^{2,3})$ denote the Grassmannian of positive null planes of $\mathbb{R}^{2,3}$. The map

$$\mathcal{F}_*(f) \ni (\tau, F) \longmapsto [F_4 \wedge (F_1 - F_2)] \in \mathfrak{N}_2^+(\mathbb{R}^{2,3})$$

is constant on the fibers of $\mathcal{F}_*(f)$. So, it descends to a smooth map $\mathbb{V}_f : X \to \mathfrak{N}_2^+(\mathbb{R}^{2,3})$. Let $F : U \subset X \to \mathrm{O}_+^\uparrow(2,3)$ be a local section of $\mathcal{F}_*(f)$. The 1-form $\phi = F^{-1}dF$ is as in (3.9), with $r = 0$ and $\alpha = -q\zeta$, for some smooth function q. Then $d(F_4 \wedge (F_1 - F_2)) = -3\phi_1^2(F_4 \wedge (F_1 - F_2))$. This implies that $[F_4 \wedge (F_1 - F_2)]$ is constant. By construction, $\mathbb{V}_f|_U = [F_4 \wedge (F_1 - F_2)]$. This proves that $\mathbb{V}_f$ is a constant positive null plane. Next, we choose a positive area element $dA \in \Lambda^2(\mathbb{V}_f)$ and consider the unique global cross section $F : X \to \mathrm{O}_+^\uparrow(2,3)$ of $\mathcal{F}_*(f)$, such that $dA(F_4, F_1 - F_2) = 1$. Then $F_4 \wedge (F_1 - F_2)$ is constant. Consequently, ϕ_1^2 vanishes identically. This implies that ϕ_0^1 and ϕ_0^2 are closed. Let $\tilde{\zeta}$ be a nowhere zero 1-form on Λ. Since $F_3 = \Gamma \circ \pi_\Lambda$, then $\phi_0^1 + \phi_0^2 = \rho\pi_\Lambda^*(\tilde{\zeta})$, where ρ is a nowhere vanishing smooth function. On the other hand, the 1-form $\phi_0^1 + \phi_0^2$ is closed. This implies that $d\rho \wedge \pi_\Lambda^*(\tilde{\zeta}) = 0$. Then, ρ is constant on the fibers of π_Λ. Therefore, there exists a nowhere zero smooth function $\varrho : \Lambda \to \mathbb{R}$, such that $\rho = \varrho \circ \pi_\Lambda$. Put $\zeta = \varrho\tilde{\zeta}$, then $\pi_\Lambda^*(\zeta) = \phi_0^1 + \phi_0^2$. Since $\phi_1^2 = 0$, the 1-form α is closed. Then, $dq \wedge \zeta = 0$. This in turn implies that $q = h \circ \pi_\Lambda$, where $h : \Lambda \to \mathbb{R}$ is a smooth function. Differentiating $F_3 = \Gamma \circ \pi_\Lambda$ yields

$$F_1 - F_2 = \Gamma' \circ \pi_\Lambda, \quad F_4 = -\varepsilon\Gamma'' \circ \pi_\Lambda, \quad d(F_3 \wedge (F_1 - F_2) \wedge F_4) = 0. \tag{4.14}$$

This shows that Γ is a biisotropic curve, such that $\mathbb{V}_\Gamma = \mathbb{V}_f$. In addition, ζ is the affine line element of Γ relative to the area element dA. Next, we choose a lightcone basis $\mathfrak{C}$ calibrated to Γ and ζ and consider the associated isotropic normal vector field R. By construction, R satisfies the identities (4.6). We write $\mathrm{R} \circ \pi_\Lambda = \sum_{j=0}^4 r_j F_j$, where $r_0, \ldots, r_4$ are smooth functions. From (4.14) and (4.6) it follows that $r_0 = \varepsilon$, $r_1 + r_2 = r_3 = r_4 = 0$. Then $F_0 = \varepsilon\mathrm{R} \circ \pi_\Lambda - r_1\Gamma' \circ \pi_\Lambda$. This implies that $f(x) = [F_0(x)] \in f_\Gamma^\pm(\mathbb{T}_\Gamma^\pm)$, for every $x \in X$. □

Remark 4.8. Theorem B shows that, locally, any exceptional quasi-umbilical immersion f is a parabolic component of the normal tube of a biisotropic curve. The rank of the dual map $f^\sharp$ is less than or equal to 1 and $df^\sharp|_x = 0$ if and only if the affine curvature of the directrix curve vanishes at $\pi_\Lambda(x)$.

Proposition 4.9. *Let $\Gamma : \Lambda \to S^{2,2}$ and $\tilde{\Gamma} : \tilde{\Lambda} \to S^{2,2}$ be two biisotropic curves defined on open intervals. Let ζ and $\tilde{\zeta}$ be affine line elements of Γ and $\tilde{\Gamma}$, respectively. If the affine lengths $\ell = \int_\Lambda \zeta$ and $\tilde{\ell} = \int_{\tilde{\Lambda}} \tilde{\zeta}$ are finite, then $f_\Gamma^\pm$ and $f_{\tilde{\Gamma}}^\pm$ are second order deformations of each other (with unfixed parameters).*

Proof. The affine line elements are defined up to a constant positive multiplicative factor. So, we may assume that $\ell = \tilde{\ell} = 2$. Let s and $\tilde{s}$ be smooth primitives of ζ and $\tilde{\zeta}$ such that $s(\Lambda) = \tilde{s}(\tilde{\Lambda}) = (-1, 1)$. By possibly replacing Γ with $\Gamma \circ s^{-1}$ and $\tilde{\Gamma}$ with $\tilde{\Gamma} \circ \tilde{s}^{-1}$, we may assume $\Lambda = \tilde{\Lambda} = (-1, 1)$ and $\zeta = \tilde{\zeta} = ds$. Let $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$ be lightcone bases calibrated to (Γ, ζ) and $(\tilde{\Gamma}, \tilde{\zeta})$, respectively. Denote by R and $\tilde{\mathrm{R}}$ the corresponding isotropic normal vectors

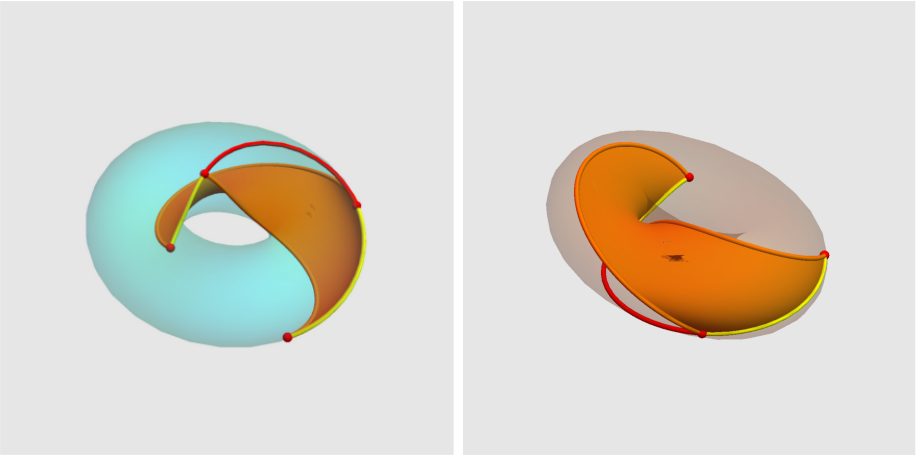

FIGURE 6. The positive (on the left) and the negative (on the right) parabolic components of the tube along the "biisotropic equilateral hyperbola" (4.16). The yellow arcs are the umbilical curves, while the red arcs are the images of the dual maps.

fields, defined as in (4.6), and let $\gamma(s) = (x(s), y(s))$ and $\tilde\gamma(s) = (\tilde x(s), \tilde x(s))$ be the affine reductions of Γ and $\tilde\Gamma$ relative to $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$. Consider the adapted frames $F_\pm$ and $\tilde F_\pm$ along $f^\pm_\Gamma$ and $f^\pm_{\tilde\Gamma}$ relative to $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$. Let Φ and $\tilde\Phi$ be the smooth diffeomorphisms

$$\Phi : (-1,1)\times\mathbb{R} \ni (s,u) \mapsto \Big(s, u - \int_0^s \frac{\mu^2 + (xx' - yy')^2}{\upsilon^4} dt\Big) \in (0,1)\times\mathbb{R},$$
$$\tilde\Phi : (-1,1)\times\mathbb{R} \ni (s,u) \mapsto \Big(s, u - \int_0^s \frac{\tilde\mu^2 + (\tilde x\tilde x' - \tilde y\tilde y')^2}{\upsilon^4} dt\Big) \in (0,1)\times\mathbb{R}.$$

Then, $G = F_\pm \circ \Phi$ and $\tilde G = \tilde F_\pm \circ \tilde\Phi$ are adapted frames along $f^\pm_\Gamma \circ \Phi$ and $f^\pm_{\tilde\Gamma} \circ \tilde\Phi$, respectively. From (4.13) it follows that the 1-forms $\phi = G^{-1}dG$ and $\tilde\phi = \tilde G^{-1} d\tilde G$ are is as in (3.9), with

$$\begin{cases} \phi^2_1 = \tilde\phi^2_1 = 0, \\ \phi^1_0 = \tilde\phi^1_0 = \frac{1}{2}(du - ds), \\ \phi^2_0 = \tilde\phi^2_0 = -\frac{1}{2}(du + ds), \end{cases}$$

and $r = \tilde r = 0$, $\alpha = h ds$, $\tilde\alpha = \tilde h ds$, where h and $\tilde h$ are the affine curvatures of Γ and $\tilde\Gamma$. Let η be the closed 1-form defined as in (3.12), with $\ell = \tilde h - h$. Then, $\delta = Ad_G(\eta)$ is an infinitesimal second order deformation of $f^\pm_\Gamma \circ \Phi$. It is easily seen that $\delta = \Delta^{-1}\cdot d\Delta$, where $\Delta = \tilde G G^{-1}$. By construction, $f^\pm_{\tilde\Gamma} \circ \tilde\Phi = \Delta \cdot f^\pm_\Gamma \circ \Phi$. This implies that $f^\pm_{\tilde\Gamma}$ is a second order deformation of $f^\pm_\Gamma$, that $\Delta \circ \tilde\Phi^{-1}$ is a map realizing the deformation, and that $\Phi \circ \tilde\Phi^{-1}$ is the change of parameters. □

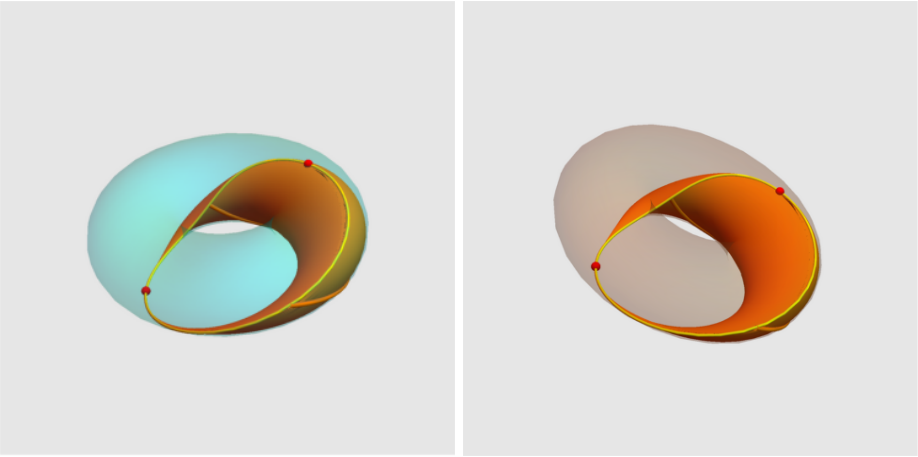

FIGURE 7. The positive (on the left) and the negative (on the right) parabolic components of the tube along the "biisotropic parabola" (4.17). The yellow arcs are the umbilical curves. The red conical points of the parabolic components are the images of the dual maps.

4.3. **The normal tubes of conics and homogeneous quasi-umbilical immersions.** An affine plane curve with constant affine curvature h is a conic. By our convention (cf. Remark 4.4), for constant negative values of h the curve is an ellipse, for constant positive values of h it is a connected component of a hyperbola, and for $h = 0$ it is a parabola. As for the construction of the normal tubes it suffices to consider the cases $h = -1$, $h = 1$, and $h = 0$.

4.3.1. *The normal tube of a unit circle.* Consider the "biisotropic circle"

$$\Gamma : \mathbb{R}/2\pi\mathbb{Z} \ni s \mapsto \cos(s)(E_1 - E_2) + E_3 + \sin(s)E_4 \in S^{2,2}. \tag{4.15}$$

Then

$$\mathrm{R}(s) = \sin(s)E_0 + \cos(s)E_1 + E_3 + \frac{1}{2}\sin(s)E_4$$

is an isotropic normal vector field along Γ. The timelike immersed torus originated by Γ is

$$f : \mathbb{T} = \mathbb{R}^2/2\pi\mathbb{Z}^2 \ni (s,\theta) \mapsto [\cos(\theta)\mathrm{R}(\mathrm{s}) + \frac{1}{2}\sin(\theta)\Gamma'(\mathrm{s})] \in \mathcal{E}^{1,2}.$$

Each one of the umbilical curves C_+, C_- is taken by f onto the null geodesic $\mathcal{C} = \{[\sum_{j=0}^4 x_j E_j] \in \mathcal{E}^{1,2} \mid x^2 = x^3 = 0\}$ lying in the AdS wall $\partial\mathcal{A}^{\pm}_{E_3}$ (see Figure 4). The parabolic components $\mathbb{T}^{\pm}$ are two cylinders and the exceptional quasi-umbilical immersions $f^{\pm}_{\Gamma}$ are embeddings. The parabolic component $f^{+}_{\Gamma}(\mathbb{T}^{+}_{\Gamma})$ is contained in the AdS chamber $\mathcal{A}^{+}_{E_3}$ and $f^{-}_{\Gamma}(\mathbb{T}^{-}_{\Gamma})$ is contained in $\mathcal{A}^{-}_{E_3}$. The central torus $\partial\mathcal{A}|_{(s_*,\theta_*)}$ is the AdS wall of $\Gamma|_s$. It is tangent to $f(\mathbb{T})$ along the coordinate line $s = s_*$. The image of the dual

maps $f^\sharp_\pm$ is the umbilical curve $\mathcal{C}$ and all central tori pass through $\mathcal{C}$ (see Figure 5).

4.3.2. *The normal tube of a equilateral hyperbola.* Consider the "biisotropic equilateral hyperbola"

$$\Gamma : \mathbb{R} \ni s \mapsto \sinh(s)(E_1 - E_2) + E_3 + \cosh(s)E_4 \in S^{2,2}. \tag{4.16}$$

Then

$$\mathrm{R}(s) = -\cosh(s)E_0 + \sinh(s)E_1 - E_3 - \frac{1}{2}E_4$$

is an isotropic normal vector field along Γ. So, Γ originates the timelike embedded cylinder

$$f : \mathbb{T} = \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} \ni (s, \theta) \mapsto [\cos(\theta)\mathrm{R(s)} + \frac{1}{2}\sin(\theta)\Gamma'(\mathrm{s})] \in \mathcal{E}^{1,2}.$$

The umbilical curves C_+ and C_- are mapped onto two disjoint arcs $\mathcal{C}_\pm$ of the null geodesic $\mathcal{C}$, the umbilical arcs of f. The parabolic component $f^+_\Gamma(\mathbb{T}^+)$ is contained in the negative AdS chamber $\mathcal{A}^-$, while the other one is contained in the positive AdS chamber. The images of the dual maps $f^\sharp_\pm$ are two disjoint open arcs $\mathcal{C}^\sharp_\pm$ contained in the null geodesic $\mathcal{C}$. They do not intersect the umbilical arcs. Taking the limits of $f(s, \theta)$ as $s \to \pm\infty$, we get two disjoint null geodesics $\mathcal{B}^+$ and $\mathcal{B}^-$ of the AdS wall $\partial\mathcal{A}_{E_3}$. They intersect $\mathcal{C}$ into four distinct points. The null geodesic $\mathcal{C}$ is the disjoint union of these four points with the arcs $\mathcal{C}_\pm$ and $\mathcal{C}^\sharp_\pm$. As in the previous case, all central tori of f pass through the null geodesic $\mathcal{C}$ (see Figure 6).

4.3.3. *The normal tube of a parabola.* Consider the "biisotropic parabola"

$$\Gamma : \mathbb{R} \ni s \mapsto s(E_1 - E_2) + E_3 + \frac{s^2}{2}E_4 \in S^{2,2}. \tag{4.17}$$

The map

$$\mathrm{R}(s) = -E_0 + \frac{s(2+s^2)^2}{8(1+s^2)}E_1 + \frac{s}{8}(4 - \frac{s^4}{1+s^2})E_2 + \frac{s^2}{2}E_3 - \frac{s^4}{8(1+s^2)}E_4$$

is an isotropic normal vector field along Γ and the timelike tautological immersion of the normal tube can be written as

$$f : \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} \ni (s, \theta) \mapsto [\cos(\theta)\mathrm{R(s)} + \frac{1}{2}\sin(\theta)\Gamma'(\mathrm{s})] \in \mathcal{E}^{1,2}.$$

The images of the umbilical curves $f(\mathrm{C}_+)$ and $f(\mathrm{C}_-)$ are two adjacent disjoint open arcs of $\mathcal{C}$. Their common boundary consits of the two points $[\pm E_4]$. The parabolic components $f^\pm_\Gamma(\mathbb{T}^\pm)$ are contained in the AdS chambers $\mathcal{A}^\pm_{E_3}$ and the dual maps are constants. Also in this case the central tori pass through the null geodesic $\mathcal{C}$ (see Figure 7).

4.3.4. *Homogeneous quasi-umbilical immersions.* Let $f : X \to \mathcal{E}^{1,2}$ be a timelike immersed surface. A *conformal symmetry* of f is a pair (Φ, B) consisting of a diffeomorphism $\Phi : X \to X$ and an element $B \in \mathrm{O}^{\uparrow}_{+}(2,3)$, such that $B \cdot (f \circ \Phi) = f$. The totality of conformal symmetries of f is a group G_f, the group of conformal symmetries of f, with multiplication given by $(\Phi, B) \star (\Phi', B') = (\Phi' \circ \Phi, B\,B')$. The group G_f acts on X on the left by $(\Phi, B)x = \Phi(x)$, for all $(\Phi, B) \in G_f$ and for all $x \in X$. If this action is transitive, the immersion is said to be *homogeneous*. For instance, the timelike totally umbilical tori of $\mathcal{E}^{1,2}$ are homogeneous and have a six-dimensional group of symmetries. Other examples are the parabolic components of the tubes of a biisotropic conic of $S^{2,2}$. In these cases, the symmetry group is isomorphic to a 2-dimensional abelian subgroup of $\mathrm{O}^{\uparrow}_{+}(2,3)$. In [46], The showed that a generic quasi-umbilical immersion cannot be homogeneous. It is not difficult to show that *an exceptional quasi-umbilical immersion is homogeneous if and only if it is* $\mathrm{O}^{\uparrow}_{+}(2,3)$*-equivalent to a parabolic component of the normal tube of a biisotropic conic.*

5. Quasi-umbilical immersions: the general case

5.1. Generic null curves in the neutral space form $S^{2,2}$.

Definition 5.1. An immersed null curve $\Gamma : \Lambda \to S^{2,2}$ is said to be *generic* if $\langle \Gamma'', \Gamma'' \rangle$ is either strictly positive or strictly negative. A *line element* is a nowhere zero 1-form $\zeta \in \Omega^1(\Lambda)$, such that

$$\langle \Gamma'', \Gamma'' \rangle = -2\eta, \quad \eta = -\operatorname{sgn}\langle \Gamma'', \Gamma'' \rangle = \pm 1.$$

The line element is unique up to the sign.

5.1.1. *The left and right normal tubes.* Let ζ be a line element. Differentiating the identities $\langle \Gamma, \Gamma \rangle = 1$, $\langle \Gamma', \Gamma' \rangle = 0$ and $\langle \Gamma'', \Gamma'' \rangle = -2\eta$, it follows that $\mathcal{R}_1|_\tau := \operatorname{span}\{\Gamma'|_\tau, \Gamma'''|_\tau\}$ is a plane of type $(1,1)$ and $\mathcal{R}_2|_\tau := \operatorname{span}\{\Gamma|_\tau, \Gamma'|_\tau, \Gamma'''|_\tau\}$ is a subspace of type $(1,2)$, for every $\tau \in \Lambda$. The polar space $\mathcal{R}_2^{\perp}|_\tau$ of $\mathcal{R}_2|_\tau$ is a plane of type $(1,1)$ such that $\Gamma''|_\tau \in \mathcal{R}_2^{\perp}|_\tau$, for every τ. The polar space $\mathcal{R}_2^{\perp}|_\tau$ has a natural orientation: a basis (V, W) of $\mathcal{R}_2^{\perp}|_\tau$ is positive if and only if $-\eta V \wedge \Gamma|_\tau \wedge \Gamma'|_\tau \wedge \Gamma'''|_\tau \wedge W > 0$. Let $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_2^{\perp}$ be the vector bundles

$$\begin{cases} \pi_1 : \mathcal{R}_1 = \{(\tau, V) \in \Lambda \times \mathbb{R}^{2,3} \mid V \in \mathcal{R}_1|_\tau\} \to \Lambda, \\ \pi_2 : \mathcal{R}_2 = \{(\tau, V) \in \Lambda \times \mathbb{R}^{2,3} \mid V \in \mathcal{R}_2|_\tau\} \to \Lambda, \\ \pi_2^{\perp} : \mathcal{R}_2^{\perp} = \{(\tau, V) \in \Lambda \times \mathbb{R}^{2,3} \mid V \in \mathcal{R}_2^{\perp}|_\tau\} \to \Lambda \end{cases} \tag{5.1}$$

The *dual* of a null vector $V \in \mathcal{R}_2^{\perp}|_\tau$ is a null vector $V_* \in \mathcal{R}_2^{\perp}|_\tau$, such that $\langle V, V_* \rangle = -1$. We say that a null vector $V \in \mathcal{R}_2^{\perp}|_\tau$ is *right-handed* if (V, V_*) is a positive basis of $\mathcal{R}_2^{\perp}|_\tau$. Right-handed null vectors generate a null line subbundle $\mathcal{P}$ of $\mathcal{R}_2^{\perp}$. The function $\mathcal{P} \setminus \{0\} \ni V \mapsto \langle V, \Gamma'' \rangle$ is nowhere vanishing. Then there exists a unique trivialization C_λ of $\mathcal{P}$, such that

$\langle \mathrm{C}_\lambda, \Gamma'' \rangle = 1$. We denote by C_ϱ the unique dual of C_λ, such that $\langle \mathrm{C}_\varrho, \Gamma'' \rangle = \eta$. By construction, $(\mathrm{C}_\lambda, \mathrm{C}_\varrho)$ is a positive null trivialization of $\mathcal{R}_2^\perp$, such that

$$-\eta \mathrm{C}_\lambda \wedge \Gamma \wedge \Gamma' \wedge \Gamma''' \wedge \mathrm{C}_\varrho > 0, \quad \Gamma'' = -(\eta \mathrm{C}_\lambda + \mathrm{C}_\varrho). \tag{5.2}$$

Definition 5.2. Let ζ be the unique line element, such that the null bivectors $\mathrm{C}_\varrho|_\tau \wedge \Gamma'|_\tau$ are positive, for every $\tau \in \Lambda$. We call ζ the *canonical line element* of Γ. The *left and right isotropic normal tubes* along Γ are the trivial circle bundles

$$\begin{cases} \pi_\lambda : \mathbb{T}_\lambda = \{(\tau, [V]) \in \Lambda \times \mathcal{E}^{1,2} \mid V \in [(\mathrm{C}_\lambda \wedge \Gamma')|_\tau]\} \to \Lambda, \\ \pi_\varrho : \mathbb{T}_\varrho = \{(\tau, [V]) \in \Lambda \times \mathcal{E}^{1,2} \mid V \in [(\mathrm{C}_\varrho \wedge \Gamma')|_\tau]\} \to \Lambda. \end{cases}$$

The elements of $\mathbb{T}_\lambda$ can be written as $(\tau, [V_\lambda(\tau, \theta)])$, where

$$V_\lambda(\tau, \theta) = \cos(\theta) \mathrm{C}_\lambda|_\tau + \sin(\theta) \Gamma'|_\tau, \tag{5.3}$$

while the elements of $\mathbb{T}_\varrho$ are of the form $(\tau, [V_\varrho(\tau, \theta)])$, where

$$V_\varrho(\tau, \theta) = \cos(\theta) \mathrm{C}_\varrho|_\tau + \sin(\theta) \Gamma'|_\tau. \tag{5.4}$$

The normal tubes do intersect along the curves

$$\mathrm{C}_\pm = \{(\tau, [\pm \Gamma'|_\tau]), \tau \in \Lambda\},$$

called *umbilical lines*. The complementary set of $\mathrm{C}_+ \cup \mathrm{C}_-$ has two connected components,

$$\mathbb{T}_\lambda^\pm = \{(\tau, [V_\lambda(\tau, \theta)]) \mid \mathrm{sgn}(\cos\theta) = \pm 1\}$$

and

$$\mathbb{T}_\varrho^\pm = \{(\tau, [V_\varrho(\tau, \theta)]) \mid \mathrm{sgn}(\cos\theta) = \pm 1\}.$$

Definition 5.3. We call $\mathbb{T}_\lambda^\pm$ and $\mathbb{T}_\varrho^\pm$ the *parabolic components* of the normal tubes. The map

$$J_\lambda^\varrho : \mathbb{T}_\lambda \ni (\tau, [V_\lambda(\tau, \theta)]) \mapsto (\tau, [V_\varrho(\tau, \theta)]) \in \mathbb{T}_\varrho,$$

is the *intertwining diffeomorphism* between the left and right normal tubes. The inverse diffeomorphism $(J_\lambda^\varrho)^{-1}$ will be denoted by J_ϱ^λ.

Definition 5.4. The left and right normal tubes can be mapped into the Einstein universe by means of the *left and right tautological maps*

$$f_\lambda : \mathbb{T}_\lambda \ni (\tau, [V]) \mapsto [V] \in \mathcal{E}^{1,2}, \quad f_\varrho : \mathbb{T}_\varrho \ni (\tau, [V]) \mapsto [V] \in \mathcal{E}^{1,2}.$$

The restrictions of the tautological maps to the parabolic components are denoted by $f_\lambda^\pm$ and by $f_\varrho^\pm$, respectively.

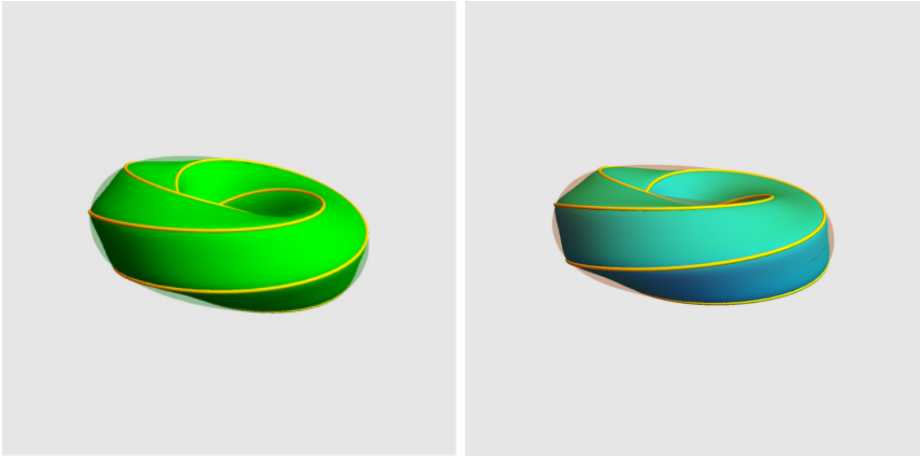

FIGURE 8. The left normal tube along a generic null curve with constant curvatures ($m = 3$ and $n = 5$). On the left, the positive parabolic component; on the right, the negative parabolic component. The yellow line is the umbilical curve of the surface. The umbilical points are double points. The parabolic components are embedded cylinders.

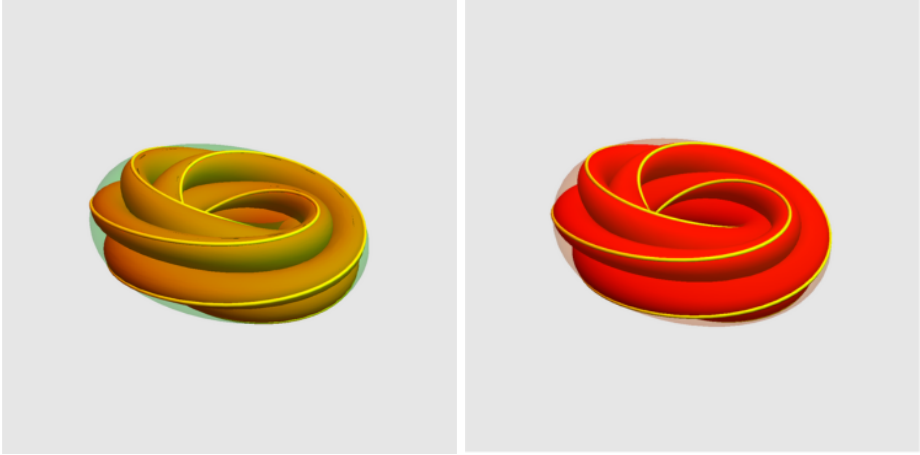

FIGURE 9. The right normal tube along a generic null curve with constant curvatures ($m = 3$ and $n = 5$). On the left, the positive parabolic component; on the right, the negative parabolic component. The yellow line is the umbilical curve of the surface. In this case the parabolic components are immersed but not embedded cylinders.

5.2. **Structure of quasi-umbilical immersions of general type.** With reference to the constructions on generic null curves in $S^{2,2}$ developed in

Section 5.1, we can now describe the geometric structure of general quasi-umbilical immersions. The following is the main result of this section.

Theorem C. *Let $\Gamma : \Lambda \to S^{2,2}$ be a generic null immersion.*

(1) *The tautological map f_λ is a timelike immersion with umbilic locus $\mathrm{C}_+ \cup \mathrm{C}_-$ and conformal Gauss map $\mathcal{N}_{f_\lambda} = \Gamma \circ \pi_\lambda$. The restrictions $f_\lambda^\pm$ of f_λ to the parabolic components $\mathbb{T}_\lambda^\pm$ of the left normal tube $\mathbb{T}_\lambda$ are quasi-umbilical immersions of general type with helicity $\varepsilon = \pm 1$ and dual maps $(f_\lambda^\pm)^\sharp = f_\varrho^\pm \circ J_\lambda^\varrho$.*

(2) *The tautological map f_ϱ is a timelike immersion with umbilic locus $\mathrm{C}_+ \cup \mathrm{C}_-$ and conformal Gauss map $\mathcal{N}_{f_\varrho} = \Gamma \circ \pi_\varrho$. The restrictions $f_\varrho^\pm$ of f_ϱ to the parabolic components $\mathbb{T}_\varrho^\pm$ of the right normal tube $\mathbb{T}_\varrho$ are quasi-umbilical immersions of general type with helicity $\varepsilon = \pm 1$ and dual maps $(f_\varrho^\pm)^\sharp = f_\lambda^\pm \circ J_\varrho^\lambda$.*

Conversely, let $f : X \to \mathcal{E}^{1,2}$ be a regular quasi-umbilical immersion of general type. Then $f(X)$ is contained in one of the two parabolic components $f_\lambda^\pm(\mathbb{T}_\lambda^\pm)$ of the tautological immersion originated by the directrix curve of f.

Proof. Let $\Gamma : \Lambda \to S^{2,2}$ be a generic null immersion, ζ be its canonical line element, and let $B : \Lambda \to \mathrm{O}^\uparrow_+(2,3)$ be the map defined by

$$B_0 = \mathrm{C}_\lambda, \quad B_1 - B_2 = \Gamma', \quad 2B_1 \wedge B_2 = -\eta\Gamma' \wedge \Gamma''', \quad B_3 = \Gamma, \quad B_4 = \mathrm{C}_\varrho.$$

First, we claim that B is a solution of the linear system

$$B^{-1}B' = \begin{pmatrix} 0 & -\frac{\eta}{2} + \kappa_\varrho & \frac{\eta}{2} + \kappa_\varrho & 0 & 0 \\ \frac{1}{2} - \kappa_\lambda & 0 & 0 & 1 & \frac{\eta}{2} - \kappa_\varrho \\ \frac{1}{2} + \kappa_\lambda & 0 & 0 & -1 & \frac{\eta}{2} + \kappa_\varrho \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} + \kappa_\lambda & \frac{1}{2} + \kappa_\lambda & 0 & 0 \end{pmatrix}, \tag{5.5}$$

where κ_λ and κ_ϱ are smooth functions, the *left and right curvatures* of Γ. Denote by b^i_j, $0 \le i, j \le 4$, the components of the $\mathfrak{o}(2,3)$-valued map $B^{-1}B'$. Differentiating $B_3 = \Gamma$, recalling that $\Gamma' = B_1 - B_2$, we have

$$b^1_3 = -b^2_3 = 1, \quad b^0_3 = b^4_3 = 0, \tag{5.6}$$

and hence $b^3_1 = b^3_2 = 1$, $b^3_4 = b^3_0 = 0$, by the symmetry relations between the b^i_j. Observe that, since $B_0 = \mathrm{C}_\lambda$ and $B_4 = \mathrm{C}_\varrho$ are orthogonal to Γ and Γ', differentiating $\langle B_0, \Gamma\rangle = \langle B_4, \Gamma\rangle = 0$, we obtain $\langle B_0', \Gamma\rangle = \langle B_4', \Gamma\rangle = 0$, i.e., $b^3_0 = b^3_4 = 0$. Differentiating $B_1 - B_2 = \Gamma'$, taking into account that

$$\Gamma'' = -(\eta \mathrm{C}_\lambda + \mathrm{C}_\varrho) = -(\eta B_0 + B_4),$$

yields

$$b^2_1 = 0, \quad b^1_4 + b^2_4 = \eta, \quad b^1_0 + b^2_0 = 1. \tag{5.7}$$

This implies the existence of two smooth functions κ_λ and κ_ϱ, such that

$$b^1_0 = \frac{1}{2} - \kappa_\lambda, \quad b^2_0 = \frac{1}{2} + \kappa_\lambda, \quad b^1_4 = \frac{\eta}{2} - \kappa_\varrho, \quad b^2_4 = \frac{\eta}{2} + \kappa_\varrho. \tag{5.8}$$

The third derivative of Γ is a cross section of the bundle spanned by B_1 and B_2. Then, differentiating $\langle \Gamma'', \mathrm{C}_\lambda\rangle = 1$, we find $\langle \Gamma'', B_0'\rangle = 0$, and hence, $b_0^0 = 0$. This together with (5.6), (5.7) and (5.8) proves the claim.

Next, we identify the normal tubes with $\Lambda \times \mathbb{R}/2\pi\mathbb{Z}$ as in (5.3) and (5.4). Given a map $\Psi : \Lambda \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^k$, we set $d\Psi = \partial_\zeta \Psi \zeta + \partial_\theta \Psi d\theta$. Let us consider the lift of f_λ defined by $\mathrm{T}_\lambda = (\cos\theta)B_0 + (\sin\theta)(B_1 - B_2)$. From (5.5), we compute

$$\langle d\mathrm{T}_\lambda, d\mathrm{T}_\lambda\rangle = -2\zeta\big(d\theta + (\eta \sin^2\theta - \kappa_\lambda \cos^2\theta)\zeta\big).$$

This shows that f_λ is a timelike immersion. Let $F^\pm : \mathbb{T}^\pm_\lambda \to \mathrm{O}^\uparrow_+(2,3)$ be the frame fields along f_λ defined by

$$\begin{cases} F_0^\pm = \pm\big(B_0 + (\tan\theta)(B_1 - B_2)\big), \\ F_1^\pm = \pm\big((\tan\theta)(\frac{\eta}{3}B_0 - B_4) + B_1 + \frac{\eta}{3}(\tan\theta)^2(B_1 - B_2)\big), \\ F_2^\pm = \pm\big((\tan\theta)(\frac{\eta}{3}B_0 - B_4) + B_2 + \frac{\eta}{3}(\tan\theta)^2(B_1 - B_2)\big), \\ F_3^\pm = B_3 = \Gamma, \\ F_4^\pm = \pm\big(B_4 - \frac{\eta}{3}(\tan\theta)(B_1 - B_2)\big). \end{cases}$$

Using (5.5) one computes the $\mathfrak{o}(2,3)$-valued 1-forms $(F^\pm)^{-1}dF^\pm$, which yield

$$\begin{cases} \phi_0^3 = \phi_3^0 = \phi_3^4 = \phi_4^3 = 0, \\ \phi_0^0 = 2\phi_1^2 = -\frac{4\eta}{3}(\tan\theta)\zeta, \quad \phi_1^3 = \phi_2^3 = \pm\zeta \\ \phi_0^1 = (\frac{1}{2} - \kappa_\lambda + \eta(\tan\theta)^2)\zeta + (\sec\theta)^2 d\theta, \\ \phi_0^2 = (\frac{1}{2} + \kappa_\lambda - \eta(\tan\theta)^2)\zeta - (\sec\theta)^2 d\theta, \\ \phi_4^1 = \frac{\eta}{18}(9 - 18\eta\kappa_\varrho + 2\eta(\tan\theta)^2)\zeta - \frac{\eta}{3}(\sec\theta)^2 d\theta, \\ \phi_4^2 = \frac{\eta}{18}(9 + 18\eta\kappa_\varrho - 2\eta(\tan\theta)^2)\zeta + \frac{\eta}{3}(\sec\theta)^2 d\theta. \end{cases} \tag{5.9}$$

Then, $F^\pm$ are adapted frame fields along $f^\pm_\lambda$. This implies

$$\mathcal{N}_{f^\pm_\lambda}(\tau,\theta) = \Gamma(\theta), \quad (f^\pm_\lambda)^\sharp(\tau,\theta) = [F_4^\pm(\tau,\theta)] = f^\pm_\varrho \circ J^\varrho_\lambda(\tau,\theta).$$

Since $\langle d\mathcal{N}_{f^\pm_\lambda}, dF_4^\pm\rangle = \mp\eta\zeta^2$, we infer that $f^\pm_\lambda$ are of general type, with conformal Gauss map $\Gamma\circ\pi_\Lambda$, helicity $\varepsilon = \pm 1$ and dual map $(f^\pm_\lambda)^\sharp = f^\pm_\varrho \circ J^\lambda_\varrho$. By continuity, $\mathcal{N}_{f_\lambda} = \Gamma\circ\pi_\Lambda$. From (5.5), we have

$$d\mathcal{N}_{f_\lambda} = \zeta(B_1 - B_2)$$

and

$$\begin{aligned} d\mathrm{T}_\lambda = \Big(\cos(\theta)\big(\frac{1}{2}(B_1 + B_2) - \kappa_\lambda(B_1 - B_2) - \sin(\theta)(\eta B_0 + B_4)\big)\Big)\zeta \\ + \big(-\sin(\theta)B_0 + \cos(\theta)(B_1 - B_2)\big)d\theta. \end{aligned}$$

Then,

$$\langle d\mathcal{N}_{f_\lambda}, d\mathrm{T}_\lambda\rangle = -\cos(\theta)\zeta^2.$$

This proves that $\mathrm{C}_+ \cup \mathrm{C}_-$ is the umbilic locus of f_λ. This concludes the proof of assertion (1). Similarly, one can prove assertion (2).

Conversely, let $f : X \to \mathcal{E}^{1,2}$ be a regular quasi-umbilical immersion of general type and let $\Gamma : \Lambda \to S^{2,2}$ be its directrix curve. Since f is of general type, there exists a unique adapted frame field $F : X \to \mathrm{O}^{\uparrow}_{+}(2,3)$ along f, such that $\phi^1_4 + \phi^2_4 = \frac{\eta}{2}(\phi^1_0 + \phi^2_0)$, $\eta = \pm 1$. Then, the 1-form $\phi = F^{-1}dF$ is as in (3.9) with $r = \eta$ and $\tilde{\zeta} = \phi^1_0 - \phi^2_0$. Let ζ be a nowhere zero 1-form on Λ. Then $\phi^1_0 + \phi^2_0 = \pm e^u \pi^*_\Lambda(\zeta)$, where $u : X \to \mathbb{R}$ is a smooth function. From

$$F_3 = \Gamma \circ \pi_\Lambda \quad dF_3 = \varepsilon(\phi^1_0 + \phi^2_0)(F_1 - F_2) \tag{5.10}$$

we see that Γ is a null curve. Differentiating (5.10), using (3.9), we obtain

$$\pi^*_\Lambda(\Gamma''\zeta) = \pm\varepsilon\Big(e^u(du - \phi^2_1)(F_1 - F_2) - e^{2u}(\eta F_0 + F_4)\zeta\Big). \tag{5.11}$$

Then, $\langle \Gamma'' \circ \pi_\Lambda, \Gamma'' \circ \pi_\Lambda\rangle = -2\eta e^{4u}$. This implies that Γ is a generic null curve. So, we may assume that ζ is the canonical line element of Γ. Then, $\langle \Gamma'', \Gamma''\rangle = -2\eta$. Hence, $u = 0$ and, using (5.11), we infer that $\phi^2_1 = \mathrm{p}\pi^*_\Lambda(\zeta)$, where $\mathrm{p} : X \to \mathbb{R}$ is a smooth function. Hence the equation (5.11) can be written as

$$\Gamma'' \circ \pi_\Lambda = \mp\varepsilon\left(\mathrm{p}\,\Gamma' \circ \pi_\Lambda + (\eta F_0 + F_4)\right). \tag{5.12}$$

Differentiating (5.12), we obtain

$$\Gamma''' \circ \pi_\Lambda = \mp\varepsilon\big(\mathrm{p}(\eta F_0 \mp 3F_4) + \frac{\eta}{2}(3F_1 + F_2)\big). \tag{5.13}$$

Then, retaining the notation of (5.1) and using (5.13), we deduce that the vector bundle $\pi^*_\Lambda(\mathcal{R}^\perp_2)$ is spanned by the null vector fields

$$A = F_0 \pm \frac{3}{2}\eta\mathrm{p}(F_1 - F_2), \quad B = F_4 - \frac{1}{2}\eta\mathrm{p}(F_1 - F_2). \tag{5.14}$$

From (5.13) it follows that

$$-\eta A \wedge (\Gamma \wedge \Gamma' \wedge \Gamma''') \circ \pi_\Lambda \wedge B = 2F_0 \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4 > 0. \tag{5.15}$$

By (5.14), we have $\langle A, B\rangle = -1$ and $\langle A, A\rangle = \langle B, B\rangle = 0$. Then, (5.15) implies that A is a global trivialization of $\pi^*_\Lambda(\mathcal{P})$, where $\mathcal{P}$ is the line bundle generated by the right-handed null vectors of $\mathcal{R}^\perp_2$. From (5.12) and (5.14), we obtain $\langle A, \Gamma'' \circ \pi_\Lambda\rangle = \pm\varepsilon$. This implies $A = \pm\varepsilon \mathrm{C}_\lambda \circ \pi_\Lambda$. By (5.14), we have

$$F_0 = \pm\varepsilon \mathrm{C}_\lambda \circ \pi_\Lambda \mp \frac{3}{2}\eta\mathrm{p}\Gamma' \circ \pi_\Lambda.$$

Then, $f(X) \subseteq f^+_\lambda(\mathbb{T}^+)$ if $\pm\varepsilon = 1$ and $f(X) \subseteq f^-_\lambda(\mathbb{T}^-)$ if $\pm\varepsilon = -1$. This concludes the proof. □

Remark 5.5. Let Γ, $\tilde{\Gamma} : \Lambda \to S^{2,2}$ be two generic null curves, defined on the same open interval Λ. Suppose that Γ, $\tilde{\Gamma}$ have the same canonical line elements. Applying similar arguments as those in the proof of Proposition 4.9, we can prove that $f^\pm_\lambda$ and $f^\pm_{\tilde{\lambda}}$ (respectively, $f^\pm_\varrho$ and $f^\pm_{\tilde{\varrho}}$) are second order conformal deformations of each other (with fixed parameters) if and only if the left curvatures (respectively, the right curvatures) of Γ and $\tilde{\Gamma}$ coincide.

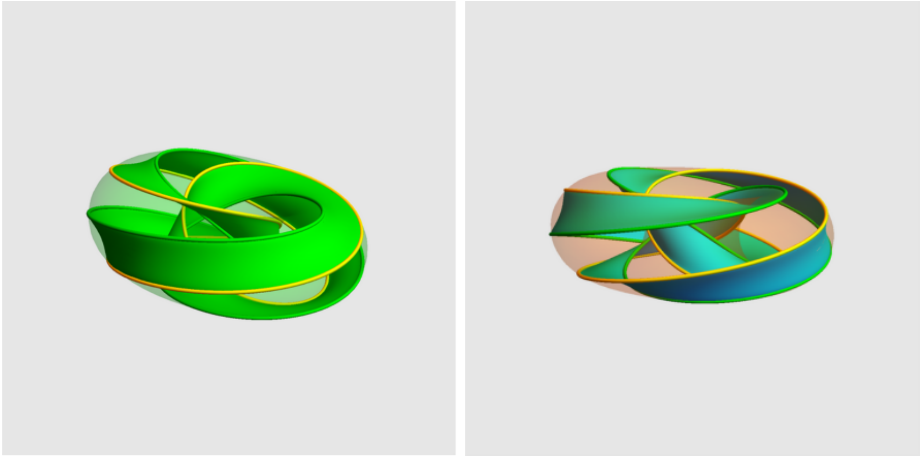

FIGURE 10. The left normal tube along a generic null curve with constant curvatures ($m = 2$ and $n = 3$). On the left, the positive parabolic component; on the right, the negative parabolic component. The yellow lines are the umbilical curves of the surface. In this example the tautological map is an embedding.

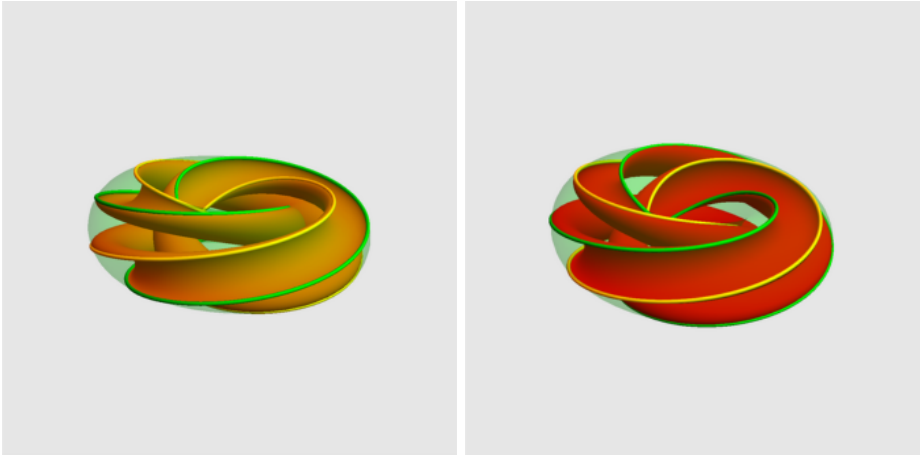

FIGURE 11. The right normal tube along a generic null curve with constant curvatures ($m = 2$ and $n = 3$). On the left, the positive parabolic component; on the right, the negative parabolic component. The yellow lines are the umbilical curves of the surface. In this example the tautological map is an embedding.

5.3. **Examples.** We illustrate our construction on a class of simple examples. The computations are carried out in the coordinates of $\mathbb{R}^{2,3}$ relative to the standard pseudo-orthogonal basis $\mathfrak{P} = (P_0, \ldots, P_4)$ of $\mathbb{R}^{2,3}$. The details and the intermediate steps will be omitted either because elementary or because they can be carried out with the help of symbolic computational

systems such as Mathematica, Maple or Matlab. Let $r \in (0,1)$ and let m, n be two relatively prime integers such that $n > m > 0$. Consider

$$W(s) = \left(m\cos(ns), m\sin(ns), n\cos(ms), n\sin(ms), \frac{nr}{\sqrt{1-r^2}} \right).$$

Then

$$\Gamma(s) := \frac{W(s)}{\sqrt{\langle W(s), W(s)\rangle}} \in S^{2,2}$$

defines a generic null curve of negative type (i.e. $\langle \Gamma'', \Gamma''\rangle < 0$) with constant left and right curvatures. Thus Γ is a homogeneous null curve of $S^{2,2}$. The isotropic normal vector fields along Γ take the form

$$\begin{aligned} \mathrm{C}_\lambda &= \upsilon_\lambda \left(c^0_\lambda \cos(ns), c^1_\lambda \sin(ns), c^2_\lambda \cos(ms), c^3_\lambda \sin(ms), c^4_\lambda \right), \\ \mathrm{C}_\varrho &= \upsilon_\varrho \left(c^0_\varrho \cos(ns), c^1_\varrho \sin(ns), c^2_\varrho \cos(ms), c^3_\varrho \sin(ms), c^4_\varrho \right), \end{aligned}$$

where υ_λ, υ_ϱ, c^j_λ, and c^j_ϱ, $j = 0, \dots, 4$, are constants which depend on the parameters m, n, and r. The tautological maps f_λ and f_ϱ arising from Γ are the timelike immersions of the torus $\mathbb{T} = \mathbb{R}^2/2\pi\mathbb{Z}^2$ given by

$$\begin{cases} f_\lambda : \mathbb{T} \ni (s,\theta) \mapsto [\cos(\theta)\mathrm{C}_\lambda(s) + \sin(\theta)\Gamma'(s)] \in \mathcal{E}^{1,2}, \\ f_\varrho : \mathbb{T} \ni (s,\theta) \mapsto [\cos(\theta)\mathrm{C}_\varrho(s) + \sin(\theta)\Gamma'(s)] \in \mathcal{E}^{1,2}. \end{cases}$$

Since $f_\lambda(s,\theta) = -f_\lambda(s,\theta+\pi)$ and $f_\varrho(s,\theta) = -f_\varrho(s,\theta+\pi)$, the immersed surfaces $f_\lambda(\mathbb{T})$ and $f_\varrho(\mathbb{T})$ are symmetrical with respect to the wall of the AdS chambers $\mathcal{A}^{\pm}_{P_4}$. The umbilical arcs $\mathcal{C}_\pm = f_\lambda(\mathrm{C}_\pm) = f_\varrho(\mathrm{C}_\pm)$ are the intersections of the surfaces with the AdS wall. They are standard torus knots of type (m,n) parametrized by

$$\mathbb{R}/2\pi\mathbb{Z} \ni s \mapsto [\pm(-\sin(ns), \cos(ns), -\sin(ms), \cos(ms), 0)] \in \mathcal{C}_\pm.$$

If both m and n are odd, then $\mathcal{C}_+ = \mathcal{C}_-$. Consequently, the umbilical points are double points of the tautological immersions. In this case, f_λ and f_ϱ cannot be embeddings (see Figures 8 and 9). If one among m and n is even, the umbilical arcs are disjoint. The parabolic components $f_\lambda(\mathbb{T}^+)$ and $f_\varrho(\mathbb{T}^-)$ belong to the positive AdS chamber, while $f_\lambda(\mathbb{T}^-)$ and $f_\varrho(\mathbb{T}^+)$ belong to the negative one. Due to the symmetry of the surfaces with respect to the wall, $f_\lambda(\mathbb{T}^-)$ and $f_\varrho(\mathbb{T}^+)$ are mirror images of $f_\lambda(\mathbb{T}^+)$ and $f_\varrho(\mathbb{T}^-)$ (see Figures 10 and 11). Although not homogeneous, these immersions are of cohomogeneity one. Since the directrix curves have constant curvatures, all these immersions are second order local deformations of each other.

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