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A_p weights on nonhomogeneous trees equipped with measures of exponential growth

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Abstract

This paper aims to study A_p weights in the context of a class of metric measure spaces with exponential volume growth, namely infinite trees with root at infinity equipped with the geodesic distance and flow measures. Our main result is a Muckenhoupt Theorem, which is a characterization of the weights for which a suitable Hardy–Littlewood maximal operator is bounded on the corresponding weighted L^p spaces. We emphasise that this result does not require any geometric assumption on the tree or any condition on the flow measure. We also prove a reverse Hölder inequality in the case when the flow measure is locally doubling. We finally show that the logarithm of an A_p weight is in BMO and discuss the connection between A_p weights and quasisymmetric mappings.

Keywords A_p weights · Trees · Exponential growth · Muckenhoupt theorem

Mathematics Subject Classification 05C05 · 05C21 · 43A99

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1 Introduction

The study of A_p weights originated with Muckenhoupt's seminal work [22], which initially characterized these weights in $\mathbb{R}^n$ in terms of the boundedness of the Hardy-Littlewood maximal operator from the weighted L^p space to itself. Subsequently, Coifman and Fefferman introduced the fundamental concept of the reverse Hölder inequality [5] and Calderón extended these results to spaces of homogeneous type [4]. Moreover, A_p weights were further characterized in $\mathbb{R}$ by linking them to the boundedness of the Hilbert transform in weighted L^p spaces, as shown in [14]. This line of inquiry led to a comprehensive theory regarding the boundedness of Calderón-Zygmund operators in weighted L^p spaces; we refer to [7, 8, 11, 12, 24] and the references therein for more information on the topic.

This work focuses on trees, which are infinite connected graphs with no loops. It is important to note that a tree, when equipped with the usual geodesic distance and the counting measure, is a nondoubling metric measure space in most cases. In this context, Muckenhoupt's characterization of A_p weights fails. Indeed, on the one hand, the A_p condition may be too weak, given that on some trees the Hardy–Littlewood maximal operator can be unbounded on every L^p for $p < \infty$ [17]. On the other hand, it can be too strong since there are other examples of trees on which the A_p condition is not necessary for the boundedness of the maximal operator [26]. For this reason, we introduce different measures on trees, the *flow measures*, where we will be able to prove a Muckenhoupt's characterization of A_p weights. Before providing the details of our results, some preliminaries are in order.

Let T be a tree and d denote the geodesic distance. We say that two vertices $x, y \in T$ are neighbours if and only if $d(x, y) = 1$. In this case, we write $x \sim y$. Let ∂T be the boundary of T defined as the set of all half-infinite geodesics starting from a fixed reference vertex (see, e.g., [9, Section I.1]). We fix a reference point ω_* in the boundary of T and think of T hanging down from ω_*. This choice induces a partial order $\leq$ on T, namely, $x \leq y$ if and only if $y \in [x, \omega_*)$, where $[x, \omega_*)$ denotes the semi-infinite geodesic with endpoints x and ω_*. Fix a reference vertex $o \in T$ and let $\{x_j\}_{j\in\mathbb{N}}$ be an enumeration of $[o, \omega_*)$ such that $x_0 = o$ and $d(x_j, x_k) = |j - k|$. We define the level function

$$\ell(x) = \lim_{j\to\infty} j - d(x, x_j), \qquad \forall x \in T. \tag{1.1}$$

For every $x \in T$, let $s(x)$ denote the set of *successors* of x, i.e,

$$s(x) := \{y \sim x \, \ell(y) = \ell(x) - 1\}.$$

We call a positive function μ on T a flow measure if it satisfies the flow condition

$$\mu(x) = \sum_{y\in s(x)} \mu(y). \tag{1.2}$$

We point out that, for a general flow measure μ, (T, d, μ) is not a doubling metric measure space and the measure of balls grows at least exponentially (see [18, Theorem 2.5 and Proposition 2.8]). For instance, let T be a homogeneous tree of order $q+1$ equipped with the canical flow measure $\mu(\cdot) = q^{\ell(\cdot)}$. Then, for every vertex x and positive integer r

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} = \frac{q^{2r+1} + q^{2r} - 2}{q^{r+1} + q^r - 2},$$

which tends to infinity as $r \to +\infty$. See [3, Proposition 1].

Since the doubling condition fails in this setting one has to identify a suitable definition of A_p weights.

Let a flow measure μ on T be fixed. Inspired by a similar definition given in the setting of homogeneous trees in [13], in [18] the authors introduced a family $\mathcal{R}$ of *admissible trapezoids* in (T, d, μ) (see Sect. 2 for their precise definition) and used those sets to replace balls and develop a Calderón–Zygmund theory in this setting, at least when μ is locally doubling. For this reason, it is quite natural to give a definition of $A_p(\mu)$, $p \in [1, \infty)$, on (T, d, μ) which is similar to the classical one, where balls are replaced by admissible trapezoids. In Sect. 2 we study the properties of such weights and prove a result in the spirit of Muckenhoupt Theorem, showing that the Hardy–Littlewood maximal operator $\mathcal{M}_\mu$ associated to the family of admissible trapezoids and the flow measure μ is bounded on the weighted $L^p(\mu w)$ space if and only if $w \in A_p(\mu)$ when $p \in (1, \infty)$. Similarly, $\mathcal{M}_\mu$ is bounded from $L^1(\mu w)$ to $L^{1,\infty}(\mu w)$ if and only if $w \in A_1(\mu)$ (Theorem 2.9). It is remarkable that for this result we do not require any condition on the geometry of the tree, which may have unbounded degree (i.e., the number of neighbours of a vertex is not uniformly bounded on T) and even not be locally finite, or on the measure μ which does not need to be locally doubling. A key ingredient to study the boundedness properties of the operator $\mathcal{M}_\mu$ is the fact that every admissible trapezoid R admits an envelope set $\widetilde{R}$ which plays a role in a covering lemma and has flow measure comparable with the flow measure of R. Unfortunately, $\widetilde{R}$ is not admissible any longer; this makes quite involved to show that also the weighted measure of $\widetilde{R}$ and R are comparable for every $A_p(\mu)$ weight (see Theorem 2.7).

In Sect. 3 we restrict our attention to the case when μ is locally doubling, which implies that the tree has uniformly bounded degree. Under these conditions, we prove a reverse Hölder inequality for $A_p(\mu)$ weights, and we provide a characterization of the class $A_\infty(\mu)$, which we define as the union of all $A_p(\mu)$ spaces as $p > 1$. One difficulty that we have to face is that in our setting the measure $w\mu$ is not a flow and is nondoubling for a general weight $w \in A_p(\mu)$, so that a Calderón–Zygmund decomposition for such a measure is not available. We tackle this challenge in Proposition 3.5 where, despite the lack of a Calderón–Zygmund theory for $w\mu$, we are able to control the averages of a function on admissible trapezoids with respect to such measure. As in the classical setting, we also prove that the space $BMO(\mu)$, defined in this context in [18], coincides with the space of multiples of logarithms of $A_\infty(\mu)$ weights.

In Sect. 4, we discuss the connection between $A_p(\mu)$ weights and quasisymmetric mappings. A classical theorem of Reimann [29] states that the Jacobian determinant of a quasiconformal mapping on $\mathbb{R}^n$ is an A_p weight, which in turn implies

that its logarithm is in BMO($\mathbb{R}^n$). Since Reimann's result was published, several attempts to generalise his results to different metric measure spaces were made, see, e.g., [15]. Recently, in [25], the authors proved that the logarithm of the generalised Jacobian of a quasisymmetric mapping on spaces of homogeneous type is always in BMO. Following Reimann's strategy, they first prove that the generalised Jacobian of a quasisymmetric mapping is an A_p weight. The proof relies heavily on the doubling property. It turns out that with the geodesic distance on the tree, the (suitably defined) Jacobian of a quasisymmetric mapping is not an $A_\infty(\mu)$ weight, implying that Reimann's result is false in this context. In fact, we will show (Theorem 4.1) that for a homogeneous tree equipped with the geodesic distance and the canonical flow measure (see Sect. 4 for the precise definition), the Jacobian of an isometry is not an $A_\infty(\mu)$ weight. However, if we change the metric to the Gromov distance, we can establish a positive result: the Jacobian of a bilipschitz mapping and its reciprocal belong to L^∞ (see Theorem 4.2). Moreover, every such mapping induces $(1, C)$-quasi-isometries with respect to the geodesic distance on the tree. It then makes sense to argue that the link between BMO functions and $A_p(\mu)$ weights in the setting of homogeneous trees, endowed with the geodesic distance, is better represented by (a subset of) quasi-isometric bijections, rather than quasisymmetric mappings. We plan to investigate this question in the future.

Let us discuss some other open problems related to the theory of A_p weights that will be the object of further investigation. It would be interesting to study a suitable version of a Fefferman-Stein inequality adapted to our setting in the spirit of [1, 10, 27] and weighted estimates for integral operators, such as Riesz transforms and spectral multipliers of the flow Laplacian, for which unweighted L^p estimates were obtained in [13, 16, 20, 21]. Another challenging problem is the study of two-weight inequalities in this context (see [2] for a recent contribution in this direction).

It is also worth mentioning that the discrete setting of trees equipped with flow measures has a natural continuous counterpart given by solvable extensions of stratified Lie groups equipped with flow measures (see [6, 13, 19, 31]). It is then natural to define A_p weights in this context and study a characterization of such weights in terms of the boundedness properties of a suitable maximal operator.

We will use the variable constant convention and write C, possibly with subscripts, to denote a positive constant that may vary from place to place and depend on any factor quantified (implicitly or explicitly) before its occurrence but not on factors quantified afterwards. If two positive quantities A and B are comparable (i.e., $\frac{1}{C}B \leq A \leq CB$ for some $C > 0$), we shall write $A \approx B$. We may add a subscript to $\approx$ to trace the dependence of C on a parameter.

2 Admissible trapezoids and A_p weights

Throughout this section, we consider an infinite tree T with root ω_* at infinity and we assume that

i) μ is a flow measure on T (possibly not locally doubling);
ii) w is a weight on T, i.e., a positive function defined on T.

Notice that there are no assumptions on the geometry of T. Indeed, (T, d) need not be a locally compact space.

Without loss of generality, we fix an integer $\beta \geq 12$ once and for all and we define for every $s \geq 0$

$$\lceil s \rceil := \min\{j \in \mathbb{N} : j \geq s\}, \qquad \lfloor s \rfloor := \max\{j \in \mathbb{N} : j \leq s\}.$$

Definition 2.1 A *trapezoid* with root $x \in T$ and heights $h_1, h_2 \in \mathbb{N}$, $h_1 < h_2$, is defined by

$$R_{h_1}^{h_2}(x) := \{y \in T \;\; y \leq x, h_1 \leq d(x, y) < h_2\}.$$

We point out that

$$\mu(R_{h_1}^{h_2}(x)) = (h_2 - h_1)\mu(x). \tag{2.1}$$

A trapezoid $R = R_{h_1}^{h_2}(x)$ is called admissible if $2 \leq \frac{h_2}{h_1} \leq \beta$, and in such case we define its *envelope* as the set

$$\widetilde{R} := R_{\lceil h_1/\beta \rceil}^{h_2\beta}(x). \tag{2.2}$$

We observe that $\mu(\widetilde{R}) \lesssim_\beta \mu(R)$.

We denote by $\mathcal{R}$ the family of all admissible trapezoids and singletons. We shall think of a singleton $\{x\}$ as a trapezoid with root x and we say that the envelope of a singleton is the singleton itself.

We now recall some useful facts about admissible trapezoids. For detailed proofs, we refer to [18].

Lemma [18, Lemma 3.2] *Let $R_1, R_2 \in \mathcal{R}$ with roots $x_1, x_2 \in T$, such that $R_1 \cap R_2 \neq \emptyset$ and assume that $\mu(x_1) \geq \mu(x_2)$. Then, $R_2 \subset \widetilde{R}_1$.*

For every positive function m on T we define the associated *noncentred* Hardy–Littlewood maximal operator $\mathcal{M}_m$, that is the operator that acts on a function f on T by

$$\mathcal{M}_m f(x) = \sup_{R \in \mathcal{R} : x \in R} \frac{1}{m(R)} \sum_{y \in R} |f(y)| m(y) \qquad \forall x \in T.$$

As a consequence of 2, $\mathcal{M}_\mu$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and on $L^p(\mu)$ for every $p > 1$. We remark that this was proved assuming that μ is locally doubling. A careful inspection of the proof of [18, Theorem 3.3] reveals that this property is not necessary to prove the weak type $(1, 1)$ boundedness of $\mathcal{M}_\mu$. Note that the aforementioned assumption implies a restriction on the geometry of a tree. Specifically, if T admits a locally doubling flow measure, then T has bounded degree (as discussed in [18, Corollary 2.3]). We emphasize that in the first part of this paper, there are no hypotheses regarding the geometry of the tree; it may not even be locally finite.

Inspired by the classical results obtained in the Euclidean setting, namely on $\mathbb{R}^n$ equipped with the Lebesgue measure (see [22]), we now focus on the study of the boundedness of $\mathcal{M}_\mu$ on $L^p(\mu w)$. In other words, the measure μ serves as the underlying measure, and our goal is to investigate the boundedness of $\mathcal{M}_\mu$ on the weighted L^p spaces. Clearly, $w\mu$ is a weight. We denote by χ_E the characteristic function supported on $E \subset T$.

Assume that $\mathcal{M}_\mu$ is bounded on $L^p(\mu w)$ for some $p \in (1, \infty)$. By testing $\mathcal{M}_\mu$ on $\chi_R w^{-p'/p}$ where $R \in \mathcal{R}$, it is easy to verify that necessarily

$$\sup_{R\in\mathcal{R}} \left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)\mu(y)\right)^{1/p} \left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)^{-1/(p-1)}\mu(y)\right)^{1/p'} < \infty, \quad (2.3)$$

where p' is such that $1/p + 1/p' = 1$. Similarly, if $\mathcal{M}_\mu$ is bounded from $L^1(\mu w)$ to $L^{1,\infty}(\mu w)$, then one can prove that

$$\mathcal{M}_\mu(w)(x) \le C w(x), \qquad \forall x \in T.$$

This motivates the following definition of $A_p(\mu)$ weights.

Definition 2.2 Let $p \in (1, \infty)$. We say that a weight w is an $A_p(\mu)$ weight if

$$[w]_{A_p(\mu)} := \sup_{R\in\mathcal{R}} \left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)\mu(y)\right)\left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)^{-1/(p-1)}\mu(y)\right)^{p-1} < \infty. \quad (2.4)$$

We say that w is an $A_1(\mu)$ weight if

$$[w]_{A_1(\mu)} := \sup_{R\in\mathcal{R}} \left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)\mu(y)\right) \|w^{-1}\|_{L^\infty(R)} < \infty.$$

For notational convenience we set $w_\mu(E) := \sum_{y\in E} w(y)\mu(y)$ for every $E \subset T$. Hence, for all $p \in (1, \infty)$ we can write

$$[w]_{A_p(\mu)} = \sup_{R\in\mathcal{R}} \frac{w_\mu(R)}{\mu(R)} \left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)^{-1/(p-1)}\mu(y)\right)^{p-1},$$

and similarly

$$[w]_{A_1(\mu)} = \sup_{R\in\mathcal{R}} \frac{w_\mu(R)}{\mu(R)} \|w^{-1}\|_{L^\infty(R)}.$$

The next proposition summarizes some properties of $A_p(\mu)$ weights.

Proposition 2.3 *Let $w \in A_p(\mu)$ and $p \in [1, \infty)$. Then,*

i) $[\lambda w]_{A_p(\mu)} = [w]_{A_p(\mu)}$ *for all* $\lambda > 0$;
ii) *if* $p \in (1, \infty)$, *then the function* $w^{-1/(p-1)}$ *is in* $A_{p'}(\mu)$, *where* $1/p' = 1 - 1/p$, *and*

$$[w^{-1/(p-1)}]_{A_{p'}(\mu)} = [w]^{1/(p-1)}_{A_p(\mu)};$$

iii) $[w]_{A_p(\mu)} \geq 1$ *and the equality holds if and only if* w *is a constant weight;*
vi) *the classes* $A_p(\mu)$ *are increasing with* p, *specifically*

$$[w]_{A_q(\mu)} \leq [w]_{A_p(\mu)},$$

whenever $1 < p \leq q < \infty$;
v) $\lim_{q \to 1^+} [w]_{A_q(\mu)} = [w]_{A_1(\mu)}$ *if* $w \in A_1(\mu)$;
vi) *we have the following equivalent characterization of* $A_p(\mu)$ *weights:*

$$[w]_{A_p(\mu)} = \sup_{R \in \mathcal{R}} \sup_{|f| \not\equiv 0 \text{ on } R} \frac{\left(\frac{1}{\mu(R)} \sum_{y \in R} |f(y)| \mu(y) \right)^p}{\frac{1}{w_\mu(R)} \sum_{y \in R} |f(y)|^p w(y) \mu(y)};$$

vii) *if* $R_1, R_2 \in \mathcal{R}$ *and* $\mu(R_1) \leq C\mu(R_1 \cap R_2)$, *then* $w_\mu(R_1) \leq C^p [w]_{A_p(\mu)} w_\mu(R_2)$;
viii) *for every* $\xi \in (0, 1)$ *and* $S \subset R \in \mathcal{R}$ *such that* $\mu(S) \leq \xi \mu(R)$, *we have that* $w_\mu(S) \leq \left(1 - \frac{(1-\xi)^p}{[w]_{A_p(\mu)}}\right) w_\mu(R)$.

Proof Property $vii)$ follows easily from $vi)$, and all the other properties can be proved by using well-known arguments as in [12, Proposition 9.1.5, Lemma 9.2.1]; we omit the details. □

Recall that, as a particular case of the theory of A_p weights developed in the context of spaces of homogeneous type in [4], one sees that $w : \mathbb{Z} \to (0, \infty)$ is an A_p weight on $\mathbb{Z}$ with $p \in (1, \infty)$ if

$$[w]_{A_p(\mathbb{Z})} := \sup_{I \text{ intervals in } \mathbb{Z}} \left(\frac{1}{|I|} \sum_{x \in I} w(x) \right) \left(\frac{1}{|I|} \sum_{x \in I} w(x)^{-1/(p-1)} \right)^{p-1} < \infty,$$

where $|\cdot|$ denotes the counting measure on $\mathbb{Z}$. Similarly,

$$[w]_{A_1(\mathbb{Z})} := \sup_{I \text{ intervals in } \mathbb{Z}} \left(\frac{1}{|I|} \sum_{x \in I} w(x) \right) \|w^{-1}\|_{L^\infty(I)} < \infty.$$

One may inquire whether $A_p(\mu)$ is trivial. In this regard, we now show that it is possible to construct plenty of nontrivial $A_p(\mu)$ weights.

We denote by $\mathcal{L}(T)$ the family of weights w on T that only depend on the level, i.e., $w(x) = W(\ell(x))$ for some $W : \mathbb{Z} \to (0, \infty)$ and for every $x \in T$. In the next theorem we characterize the $A_p(\mu)$ weights in $\mathcal{L}(T)$. Remarkably, it turns out that such weights do not depend on the particular choice of μ.

Theorem 2.4 *For every $p \in [1,\infty)$, $w \in \mathcal{L}(T)$ is an $A_p(\mu)$ weight on T if and only if $W \in A_p(\mathbb{Z})$. Moreover, we have that $[w]_{A_p(\mu)} = [W]_{A_p(\mathbb{Z})}$.*

Proof Assume that $p \in (1,\infty)$. Observe that for every $R = R_{h_1}^{h_2}(x_0) \in \mathcal{R}$,

$$\begin{aligned} I_R &:= \left(\frac{1}{\mu(R)}\sum_{x\in R} w(x)\mu(x)\right)\left(\frac{1}{\mu(R)}\sum_{x\in R} w(x)^{-1/(p-1)}\mu(x)\right)^{p-1} \\ &= \left(\frac{1}{\mu(x_0)(h_2-h_1)}\sum_{\ell=\ell(x_0)-h_2+1}^{\ell(x_0)-h_1} W(\ell)\sum_{x\le x_0,\ell(x)=\ell}\mu(x)\right) \\ &\quad\times\left(\frac{1}{\mu(x_0)(h_2-h_1)}\sum_{\ell=\ell(x_0)-h_2+1}^{\ell(x_0)-h_1} W(\ell)^{-1/(p-1)}\sum_{x\le x_0,\ell(x)=\ell}\mu(x)\right)^{p-1} \end{aligned}$$

and since μ is a flow measure, $\sum_{x\le x_0,\ell(x)=\ell}\mu(x) = \mu(x_0)$ for every $\ell \le \ell(x_0)$. We conclude that

$$I_R = \left(\frac{1}{h_2-h_1}\sum_{\ell=\ell(x_0)-h_2+1}^{\ell(x_0)-h_1} W(\ell)\right)\left(\frac{1}{h_2-h_1}\sum_{\ell=\ell(x_0)-h_2+1}^{\ell(x_0)-h_1} W(\ell)^{-1/(p-1)}\right)^{p-1}.$$

Since $|[\ell(x_0)-h_2+1,\ell(x_0)-h_1]| = h_2-h_1$, this implies that $[w]_{A_p(\mu)} \le [W]_{A_p(\mathbb{Z})}$. For the converse inequality, it suffices to prove that given an interval $I = [a,b] \subset \mathbb{Z}$ there exists an admissible trapezoid $R_{h_1}^{h_2}(x_0)$ (where h_1, h_2 and x_0 depend on a and b) such that

$$\ell(x_0) - h_1 = b, \quad \ell(x_0) - h_2 + 1 = a \quad \text{and} \quad 2 \le \frac{h_2}{h_1} \le \beta.$$

Hence, we need to show that there exists x_0 such that

$$\begin{cases} 1 \le h_1 = \ell(x_0) - b \\ h_2 = \ell(x_0) + 1 - a \\ 2 \le \frac{h_2}{h_1} \le \beta. \end{cases}$$

For example, set $h_1 = b-a+1$ and $h_2 = 2h_1$ so the third condition above is satisfied. Next, $\ell(x_0) = 2b - a + 1$ satisfies the first two conditions above and it is consistent with $h_2 = 2h_1$.

Finally, if $p = 1$ the result follows by taking the limit for $p \to 1^+$, in view of Proposition 2.3 *v*) and the corresponding property for A_p weights on $\mathbb{Z}$. □

Remark 2.5 Observe that $\mathbb{Z}$ can be seen as a tree in an obvious way and the counting measure on $\mathbb{Z}$ is a flow measure. Theorem 2.4 applied to $T = \mathbb{Z}$ gives $[w]_{A_p(|\cdot|)} = [w]_{A_p(\mathbb{Z})}$. This shows that the notion of $A_p(\mu)$ weight introduced in Definition 2.2 is consistent with the one of $A_p(\mathbb{Z})$ weight. Indeed, the fact that all integer intervals of

$\mathbb{Z}$ are admissible trapezoids (else said, admissible trapezoids coincide with balls) is shown in the proof of Theorem 2.4.

In a doubling metric measure space (X, ρ, ν), one can prove that the measure $w\nu$ is doubling (see e.g. [4, Lemma 4]), where w is an A_p weight defined as in (2.4) but using balls instead of trapezoids. However, this does not hold true in our context due to the nondoubling nature of a flow measure.

By drawing a parallel with the doubling setting, we aim to show that $w_\mu(\widetilde{R})$ is uniformly bounded by a multiple of $w_\mu(R)$ for every admissible trapezoid R and $w \in A_p(\mu)$. Notice that the envelope of an admissible trapezoid, as defined in (2.2), never meets the criteria for admissibility. This fact prevents us from following the classical proof. Indeed, if $\widetilde{R}$ were admissible, choosing $f = \chi_R$ and $\widetilde{R}$ in place of R in Proposition 2.3 *vi*) we would directly deduce that $w_\mu(\widetilde{R}) \leq C w_\mu(R)[w]_{A_p(\mu)}$ for every $R \in \mathcal{R}$. For this reason, we shall use a different strategy based on the following geometric lemma.

Lemma 2.6 *Let R be an admissible trapezoid. Then $\widetilde{R}$ is contained in the union of at most 4 admissible trapezoids $Q_1, .., Q_4$, with $\mu(Q_i) \approx_\beta \mu(R)$, and such that, either Q_i intersects R, and in this case $\mu(Q_i \cap R) \approx_\beta \mu(R)$, or it intersects some Q_j intersecting R, and in this case $\mu(Q_i \cap Q_j) \approx_\beta \mu(R)$.*

Proof Let $R = R^{h_2}_{h_1}(x) \in \mathcal{R}$.

We shall prove the following:

i) if $h_1 \geq 3$, set

$$\begin{aligned} Q_1 &:= R^{h_1}_{\lceil h_1/\beta \rceil}(x), \\ Q_2 &:= R^{\lfloor (h_1+h_2)/2 \rfloor}_{\lceil (h_1+h_2)/(2\beta) \rceil}(x), \\ Q_3 &:= R^{\lfloor (h_1+h_2)/2 \rfloor \beta}_{\lfloor (h_1+h_2)/2 \rfloor}(x), \\ Q_4 &:= R^{\lceil (h_1+h_2)/2 \rceil \lfloor \beta/2 \rfloor \beta}_{\lceil (h_1+h_2)/2 \rceil \lfloor \beta/2 \rfloor}(x). \end{aligned}$$

Then,

$$\widetilde{R} \subset \bigcup_{j=1}^{4} Q_j, \tag{2.5}$$

$Q_j \in \mathcal{R}$, $\mu(R) \approx_\beta \mu(Q_j)$ for $j = 1, 2, 3, 4$ and

$$\mu(R) \approx_\beta \mu(Q_1 \cap Q_2) \approx_\beta \mu(Q_2 \cap R) \approx_\beta \mu(R \cap Q_3) \approx_\beta \mu(Q_4 \cap Q_3); \tag{2.6}$$

ii) if $h_1 = 1$ and $h_2 \geq 3$ or $h_1 = 2$, set

$$\begin{aligned} Q_1 &:= R^{\beta}_{1}(x), \\ Q_2 &:= R^{\lfloor \beta/2 \rfloor \beta}_{\lfloor \beta/2 \rfloor}(x), \end{aligned}$$

$$Q_3 := R^{\lfloor (h_1+h_2)/2 \rfloor \beta}_{\lfloor (h_1+h_2)/2 \rfloor}(x),$$
$$Q_4 := R^{\lceil (h_1+h_2)/2 \rceil \lfloor \beta/2 \rfloor \beta}_{\lceil (h_1+h_2)/2 \rceil \lfloor \beta/2 \rfloor}(x).$$

Then,

$$\widetilde{R} \subset \bigcup_{j=1}^{4} Q_j, \tag{2.7}$$

$Q_j \in \mathcal{R}$ for every $j = 1, ..., 4$ and

$$\mu(R) \approx_\beta \mu(Q_1 \cap R) \approx_\beta \mu(R \cap Q_3) \approx_\beta \mu(Q_2 \cap Q_3) \approx_\beta \mu(Q_4 \cap Q_3) \approx_\beta \mu(Q_4); \tag{2.8}$$

iii) if $h_1 = 1$ and $h_2 = 2$, then we define

$$Q_1 := R^{\beta}_{1}(x),$$
$$Q_2 := R^{\lfloor \beta/2 \rfloor \beta}_{\lfloor \beta/2 \rfloor}(x),$$

and similarly we have that

$$\widetilde{R} \subset \bigcup_{j=1}^{2} Q_j, \tag{2.9}$$

$Q_j \in \mathcal{R}$ for every $j = 1, 2$, and

$$\mu(R) = \mu(R \cap Q_1) \approx_\beta \mu(Q_1 \cap Q_2) \approx_\beta \mu(Q_1) \approx_\beta \mu(Q_2).$$

We prove i). Suppose $h_1 \geq 3$. Then, it is clear that Q_1, Q_3, $Q_4 \in \mathcal{R}$. To show that $Q_2 \in \mathcal{R}$ we first observe that

$$\frac{\lfloor \frac{h_1+h_2}{2} \rfloor}{\lceil \frac{h_1+h_2}{2\beta} \rceil} \leq \frac{h_1+h_2}{2} \frac{2\beta}{h_1+h_2} = \beta.$$

On the one hand, if $h_1 + h_2 < 2\beta$ we get

$$2 \leq h_1 \leq \frac{\lfloor \frac{h_1+h_2}{2} \rfloor}{1},$$

on the other hand, if $h_1 + h_2 \geq 2\beta$ then

$$2 \leq \frac{(\beta-1)}{2} = \frac{(\beta-1)(h_1+h_2)}{2(h_1+h_2)} \leq \frac{\beta(h_1+h_2)-2\beta}{h_1+h_2+2\beta} \leq \frac{\lfloor \frac{h_1+h_2}{2} \rfloor}{\lceil \frac{h_1+h_2}{2\beta} \rceil}.$$

We now prove (2.5). For this purpose, we first notice that the smaller heights of Q_1 and $\widetilde{R}$ coincide by definition, and by the fact that $\beta \geq 12$ and $h_2 \leq \beta h_1$, we deduce that

$$h_1 > \frac{h_1}{2}(1/\beta + 1) + 1 > \frac{h_1 + h_2}{2\beta} + 1 > \lceil (h_1 + h_2)/(2\beta) \rceil,$$

which implies $Q_1 \cap Q_2 \neq \emptyset$. Moreover, Q_4 entirely covers the lower part of $\widetilde{R}$ since $\lceil (h_1 + h_2)/2 \rceil \lfloor \beta/2 \rfloor \beta \geq h_2 \beta$.

It remains to prove (2.6). Formula (2.1) yields

$$\mu(R \cap Q_2) \approx \mu(R \cap Q_3) \approx \mu(x)(h_2 - h_1) = \mu(R),$$
$$\mu(Q_4 \cap Q_3) = \mu(x)\big(\beta - \lfloor \beta/2 \rfloor\big)\lfloor (h_1 + h_2)/2 \rfloor \approx \mu(Q_4) \approx_\beta \mu(R).$$

Moreover,

$$\mu(Q_1 \cap Q_2) = \mu(x)\big(h_1 - \lceil (h_1 + h_2)/(2\beta) \rceil\big).$$

By the admissibility condition $h_2 \leq \beta h_1$, the fact that $\beta \geq 12$, and the assumption $h_1 \geq 3$,

$$h_1 - \lceil (h_1 + h_2)/(2\beta) \rceil \geq h_1 - \frac{h_1 + h_2}{2\beta} - 1 \geq h_1\Big(1 - \frac{1}{2\beta} - \frac{1}{2}\Big) - 1 \geq \frac{h_1}{\beta}. \tag{2.10}$$

This implies that

$$\mu(Q_1 \cap Q_2) \geq \mu(x)\frac{h_1}{\beta} \approx_\beta \mu(R),$$

as required to conclude the proof of $i)$.

The remaining assertions are can be proved similarly, so we only sketch their proof. Assume that R and Q_j, $j = 1, ..., 4$ are as in $ii)$. Then, Q_1, Q_2, Q_3, and Q_4 belong to $\mathcal{R}$ because the ratio of their corresponding heights is equal to β. Observe that

$$Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = \{y \in T \; y \leq x, 1 \leq d(x, y) < \lceil (h_1 + h_2)/2 \rceil \lfloor \beta/2 \rfloor \beta\},$$

and the set in the right-hand side contains $\widetilde{R}$, so (2.7) follows. Observe that in this case $h_2 \approx_\beta h_1 \approx_\beta 1$ and it is not hard to prove that the measure of every element appearing in (2.8) is comparable to $\mu(x)$, with constants depending on β.

Finally, $iii)$ consists of one case and its proof is straightforward. □

Theorem 2.7 *Let $w \in A_p(\mu)$ for some $p \in [1, \infty)$. Then, there exists a constant $C_{\beta,p} > 0$ such that $w_\mu(\widetilde{R}) \leq C_{\beta,p}[w]^2_{A_p(\mu)} w_\mu(R)$ for every $R \in \mathcal{R}$.*

Proof Let R be an admissible trapezoid and $Q_1, ..., Q_4$ be the associated admissible trapezoids prescribed by Lemma 2.6. We know that for each $i = 1, .., 4$ there exists $j = j(i)$ such that $R \cap Q_j$, $Q_i \cap Q_j \neq \emptyset$. Then, by Lemma 2.6 and Proposition 2.3 $iii)$ and $vii)$ we have,

$$\frac{w_\mu(\widetilde{R})}{w_\mu(R)} \leq \sum_{i=1}^{4} \frac{w_\mu(Q_i)}{w_\mu(R)} = \sum_{i=1}^{4} \frac{w_\mu(Q_i)}{w_\mu(Q_j)} \frac{w_\mu(Q_j)}{w_\mu(R)} \lesssim_{\beta,p} [w]^2_{A_p(\mu)}.$$

□

We now prove that $A_p(\mu)$ weights are those for which the maximal operator $\mathcal{M}_\mu$ is bounded on $L^p(\mu w)$.

The next result is an immediate consequence of Theorem 2.7.

Corollary 2.8 *Let w be an $A_q(\mu)$ weight for some $q \in [1, \infty)$. Then, $\mathcal{M}_{\mu w}$ is bounded from $L^1(\mu w)$ to $L^{1,\infty}(\mu w)$ and on $L^p(\mu w)$ for every $p \in (1, \infty]$.*

Proof It suffices to prove that $\mathcal{M}_{\mu w}$ is of weak type $(1, 1)$ and interpolate with the obvious L^∞ bound.

Let $f \in L^1(\mu w)$, fix $\lambda > 0$ and set $E_\lambda = \{\mathcal{M}_{\mu w} f > \lambda\}$. We observe that there exists a sequence $\{R_j\}_{j\in J} \subset \mathcal{R}$ such that $\cup_{j\in J} R_j = E_\lambda$ and

$$w_\mu(R_j) < \frac{1}{\lambda} \sum_{y\in R_j} |f(y)|w(y)\mu(y), \qquad \forall j \in J.$$

Reasoning as in the proof of [18, Theorem 3.3], one shows that there exists a subsequence $\{R_{j_k}\}_k \subset \{R_j\}_{j\in J}$ of mutually disjoint sets such that $E_\lambda \subset \cup_k \widetilde{R}_{j_k}$. Thus, by Theorem 2.7

$$w_\mu(E_\lambda) \le \sum_k w_\mu(\widetilde{R}_{j_k}) \le C_{\beta,q,w} \sum_k w_\mu(R_{j_k}) \le \frac{C_{\beta,q,w}}{\lambda} \|f\|_{L^1(\mu w)},$$

as desired. □

The next theorem is the analogue of Muckenhoupt's characterisation in our setting.

Theorem 2.9 *Let $w \in A_p(\mu)$ for some $p \in (1, \infty)$. Then, $\mathcal{M}_\mu$ is bounded on $L^p(\mu w)$. Moreover, if $w \in A_1(\mu)$, then $\mathcal{M}_\mu$ is bounded from $L^1(\mu w)$ to $L^{1,\infty}(\mu w)$.*

Proof Corollary 2.8 allows us to follow the proof used in the Euclidean setting [12, Theorem 9.1.9]; we provide the details for completeness. Assume that $p \in (1, \infty)$ and let w be an $A_p(\mu)$ weight and $f \in L^p(\mu w)$. Define $\sigma := w^{-\frac{1}{p-1}}$. Observe that for every $R \in \mathcal{R}$

$$\frac{1}{\mu(R)} \sum_{y\in R} |f(y)|\mu(y) = \frac{w_\mu(R)^{\frac{1}{p-1}} \sigma_\mu(R)}{\mu(R)^{p/(p-1)}} \left(\frac{\mu(R)}{w_\mu(R)} \left(\frac{1}{\sigma_\mu(R)} \sum_{y\in R} |f(y)|\mu(y) \right)^{p-1} \right)^{\frac{1}{p-1}}. \tag{2.11}$$

It is clear that

$$\frac{1}{\sigma_\mu(R)} \sum_{y\in R} |f(y)|\mu(y) \le \mathcal{M}_{\mu\sigma}(f\sigma^{-1})(x), \qquad \forall x \in R, \tag{2.12}$$

thus

$$\frac{1}{\sigma_\mu(R)} \sum_{y\in R} |f(y)|\mu(y) \le \inf_{x\in R} \mathcal{M}_{\mu\sigma}(f\sigma^{-1})(x)$$

and

$$\mu(R)\left(\frac{1}{\sigma_\mu(R)}\sum_{y\in R}|f(y)|\mu(y)\right)^{p-1}\leq\sum_{y\in R}\left[\mathcal{M}_{\mu\sigma}(f\sigma^{-1})(y)\right]^{p-1}\mu(y).$$

Moreover, by definition of $[w]_{A_p(\mu)}$

$$\frac{w_\mu(R)\sigma_\mu(R)^{p-1}}{\mu(R)^p}\leq[w]_{A_p(\mu)}. \tag{2.13}$$

By (2.11), (2.12), and (2.13)

$$\frac{1}{\mu(R)}\sum_{y\in R}|f(y)|\mu(y)\leq[w]^{1/(p-1)}_{A_p(\mu)}\left[\mathcal{M}_{\mu w}\left(\mathcal{M}_{\mu\sigma}(|f|\sigma^{-1})^{p-1}w^{-1}\right)(x)\right]^{1/(p-1)},\qquad\forall x\in R.$$

We deduce that $\mathcal{M}_\mu f\leq[w]^{1/(p-1)}_{A_p(\mu)}\left[\mathcal{M}_{\mu w}\left(\mathcal{M}_{\mu\sigma}(|f|\sigma^{-1})^{p-1}w^{-1}\right)\right]^{1/(p-1)}$. By computing the $L^p_\mu(w)$ norm, we obtain that

$$\begin{aligned}\|\mathcal{M}_\mu f\|_{L^p_\mu(w)}&\leq[w]^{1/(p-1)}_{A_p(\mu)}\|\mathcal{M}_{\mu w}\big(\mathcal{M}_{\mu\sigma}(|f|\sigma^{-1})^{p-1}w^{-1}\big)\|^{1/(p-1)}_{L^{p'}_\mu(w)}\\&\leq[w]^{1/(p-1)}_{A_p(\mu)}\|\mathcal{M}_{\mu w}\|^{1/(p-1)}_{L^{p'}_\mu(w)\to L^{p'}_\mu(w)}\|\mathcal{M}_{\mu\sigma}(|f|\sigma^{-1})^{p-1}w^{-1}\|^{1/(p-1)}_{L^{p'}_\mu(w)}\\&=[w]^{1/(p-1)}_{A_p(\mu)}\|\mathcal{M}_{\mu w}\|^{1/(p-1)}_{L^{p'}_\mu(w)\to L^{p'}_\mu(w)}\|\mathcal{M}_{\mu\sigma}(|f|\sigma^{-1})\|_{L^p_\mu(\sigma)}\\&\leq[w]^{1/(p-1)}_{A_p(\mu)}\|\mathcal{M}_{\mu w}\|^{1/(p-1)}_{L^{p'}_\mu(w)\to L^{p'}_\mu(w)}\|\mathcal{M}_{\mu\sigma}\|_{L^p_\mu(\sigma)\to L^p_\mu(\sigma)}\|f\|_{L^p_\mu(w)}.\end{aligned}$$

The conclusion now follows by Corollary 2.8 and the fact that $\sigma\in A_{p'}(\mu)$ by Proposition 2.3 *ii*).

If $p=1$ and $w\in A_1(\mu)$, we have that, for every $f\in L^1(\mu w)$

$$\begin{aligned}\frac{w_\mu(R)}{\mu(R)}\sum_{y\in R}|f(y)|\mu(y)&=\frac{w_\mu(R)}{\mu(R)}\sum_{y\in R}|f(y)|\frac{w(y)}{w(y)}\mu(y)\\&\leq\frac{w_\mu(R)}{\mu(R)}\|w^{-1}\|_{L^\infty(R)}\sum_{y\in R}|f(y)|w(y)\mu(y)\\&\leq[w]_{A_1(\mu)}\sum_{y\in R}|f(y)|w(y)\mu(y).\end{aligned}$$

This implies that

$$\mathcal{M}_\mu f(x)\leq[w]_{A_1(\mu)}\mathcal{M}_{\mu w}f(x),\qquad\forall x\in T.$$

Again, an application of Corollary 2.8 concludes the proof. □

In summary, we have proved the following version of the Muckenhoupt Theorem.

Corollary 2.10 *Let w be a weight on T. Then,*

i) *$\mathcal{M}_\mu$ is bounded on $L^p(\mu w)$ if and only if $w \in A_p(\mu)$;*
ii) *$\mathcal{M}_\mu$ is bounded from $L^1(\mu w)$ to $L^{1,\infty}(\mu w)$ if and only if $w \in A_1(\mu)$.*

3 Reverse Hölder inequality and A_∞ weights

In this section we assume that μ is a locally doubling flow measure. Under this additional assumption, which in turn implies that T has bounded degree, a Calderón-Zygmund theory was developed on (T, d, μ) using admissible trapezoids instead of metric balls (see [18, Section 3]). In particular, there exist a constant $C_D \in (0, 1)$ and an integer N such that $R \in \mathcal{R}$ admits a partition of at most N admissible trapezoids Q such that

$$\frac{\mu(Q)}{\mu(R)} \geq C_D \qquad \text{for every } Q \text{ in the partition.} \tag{3.1}$$

We point out that the quantities C_D and N depend on the locally doubling constant. The previous fact and a stopping-time argument provide the following useful result (see [18, Lemma 3.5]).

Lemma 3.1 *Let f be a function on T, $\lambda > 0$, $R \in \mathcal{R}$ and ν be a measure satisfying (3.1) such that*

$$\frac{1}{\nu(R)} \sum_{y \in R} |f(y)|\nu(y) < \lambda.$$

Then, there exist a (possibly empty) family $\mathcal{F}$ of disjoint admissible trapezoids and a constant D_{CZ} such that for each $E \in \mathcal{F}$ the following hold:

i) $\frac{1}{\nu(E)} \sum_{y \in E} |f(y)|\nu(y) \geq \lambda$;
i) $\frac{1}{\nu(E)} \sum_{y \in E} |f(y)|\nu(y) < D_{CZ}\lambda$;
iii) *if $x \in R \setminus \cup_{E \in \mathcal{F}} E$, then $|f(x)| < \lambda$.*

We aim to prove a reverse Hölder inequality. Now we can prove the reverse Hölder inequality in our setting.

Theorem 3.2 (Reverse Hölder inequality) *Let $w \in A_p(\mu)$ for some $p \in [1, \infty)$. Then, there are $C, \varepsilon > 0$ such that for every $R \in \mathcal{R}$*

$$\left(\frac{1}{\mu(R)} \sum_{y \in R} w(y)^{1+\varepsilon} \mu(y) \right)^{1/(1+\varepsilon)} \leq \frac{C}{\mu(R)} \sum_{y \in R} w(y)\mu(y).$$

Proof Pick $R \in \mathcal{R}$ and define

$$\gamma_0 = \frac{1}{\mu(R)} \sum_{y \in R} w(y)\mu(y).$$

We fix $\gamma \in (0,1)$ and we define $\gamma_k = (D_{CZ}\gamma^{-1})^k\gamma_0$. Since $\gamma_0 < \gamma_k$ we can apply Lemma 3.1 with $\lambda = \gamma_k$ and $f = w$ and we call $\mathcal{F}_k$ the obtained family of mutually disjoint admissible trapezoids. Then, for every $E \in \mathcal{F}_k$, the following properties are satisfied

a) $\gamma_k < \frac{1}{\mu(E)} \sum_{y\in E} w(y)\mu(y) \le D_{CZ}\gamma_k$;
b) $w \le \gamma_k$ on $R \setminus U_k$ where $U_k := \cup_{E\in\mathcal{F}_k} E$;
c) each $E' \in \mathcal{F}_{k+1}$ is contained in some $E \in \mathcal{F}_k$.

Indeed, $a), b)$ are immediate consequences of Lemma 3.1 $i), ii)$, and $c)$ follows by recalling the stopping-time nature of the proof for Lemma 3.1. Observe that, if $w = \gamma_0$ on R, then the reverse Hölder inequality is trivially satisfied. So we can assume that w is not constant on R, so that $U_0 \neq \emptyset$. Let $\overline{k} = \overline{k}(R, w)$ be the smallest positive integer such that $U_{\overline{k}} = \emptyset$, which exists because $w \in L^\infty(R)$ and $\gamma_k \to \infty$. We show that for every $k < \overline{k}$ and $E \in \mathcal{F}_k$,

$$\mu(E \cap U_{k+1}) < \gamma\mu(E). \tag{3.2}$$

This is trivial when $k = \overline{k} - 1$, whereas for every $k < \overline{k} - 1$,

$$\begin{aligned}
D_{CZ}\gamma_k &\ge \frac{1}{\mu(E)} \sum_{y\in E} w(y)\mu(y) \\
&\ge \frac{1}{\mu(E)} \sum_{y\in E\cap U_{k+1}} w(y)\mu(y) \\
&= \frac{1}{\mu(E)} \sum_{\substack{E'\in\mathcal{F}_{k+1} \\ E'\subset E}} \frac{\mu(E')}{\mu(E')} \sum_{y\in E'} w(y)\mu(y) \\
&> \frac{\mu(E\cap U_{k+1})}{\mu(E)}\gamma_{k+1} \\
&= \frac{\mu(E\cap U_{k+1})}{\mu(E)} D_{CZ}\gamma^{-1}\gamma_k.
\end{aligned}$$

By (3.2) and Proposition 2.3 $viii)$, we conclude that

$$\frac{w_\mu(E\cap U_{k+1})}{w_\mu(E)} \le \eta = 1 - \frac{(1-\gamma)^p}{[w]_{A_p(\mu)}}.$$

Taking the sum over all $E \in \mathcal{F}_k$,

$$w_\mu(U_{k+1}) \le \eta w_\mu(U_k), \qquad \forall k < \overline{k}.$$

The proof can be completed following the argument in the proof of [12, Theorem 9.2.2]. □

We observe that the proof of Theorem 3.2 implies the following more general result.

Proposition 3.3 *Let w be a weight such that there exist $\xi, \eta \in (0,1)$ satisfying*

$$\mu(S) < \xi\mu(R) \Longrightarrow w_\mu(S) < \eta w_\mu(R), \tag{3.3}$$

for every $R \in \mathcal{R}$ and $S \subset R$. Then, there are $C, \varepsilon > 0$ such that for every $R \in \mathcal{R}$

$$\left(\frac{1}{\mu(R)}\sum_{y\in R} w(y)^{1+\varepsilon}\mu(y)\right)^{1/(1+\varepsilon)} \leq \frac{C}{\mu(R)}\sum_{y\in R} w(y)\mu(y).$$

Observe that Proposition 3.3 holds true more generally if μ is a measure satisfying (3.1). The following are easy consequences of Theorem 3.2.

Corollary 3.4 *The following hold:*

i) *let $w \in A_p(\mu)$ for some $p \in [1,\infty)$. Then, there exists $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_p(\mu)$ and $[w^{1+\varepsilon}]_{A_p(\mu)} \leq C_{p,\varepsilon}[w]^{1+\varepsilon}_{A_p(\mu)}$;*
ii) *if $p \in (1,\infty)$, then*

$$A_p(\mu) = \bigcup_{s\in(1,p)} A_s(\mu).$$

Proof To prove *i*), given the reverse Hölder inequality, one can follow [12, Theorem 9.2.5] with obvious substitutions of cubes with admissible trapezoids and the Lebesgue measure with μ.

Next, given $w \in A_p(\mu)$ and $\varepsilon \in (0,1)$ as in *i*), we set $\delta = \frac{1}{1+\varepsilon}$ and $s = p\delta + 1 - \delta = \frac{p+\varepsilon}{1+\varepsilon}$. We claim that

$$[w^\delta]_{A_s(\mu)} \leq [w]^\delta_{A_p(\mu)}. \tag{3.4}$$

This easily follows by definition of $A_p(\mu)$. Indeed,

$$\begin{aligned}
[w^\delta]_{A_s(\mu)} &= \sup_{R\in\mathcal{R}} \left(\frac{1}{\mu(R)}\sum_{y\in R} w(y)^\delta\mu(y)\right)\left(\frac{1}{\mu(R)}\sum_{y\in R} w(y)^{-\delta/(s-1)}\mu(y)\right)^{s-1} \\
&\leq \sup_{R\in\mathcal{R}} \left(\frac{1}{\mu(R)}\sum_{y\in R} w(y)\mu(y)\right)^{\delta}\left(\frac{1}{\mu(R)}\sum_{y\in R} w(y)^{-1/(p-1)}\mu(y)\right)^{\delta(p-1)} \\
&= [w]^\delta_{A_p(\mu)},
\end{aligned}$$

where in the above inequality we have used Hölder's inequality. By (3.4) applied to the weight $w^{1/\delta} \in A_p(\mu)$ instead of w and *i*), we conclude that

$$[w]_{A_s(\mu)} \leq [w^{1+\varepsilon}]^{1/(1+\varepsilon)}_{A_p(\mu)} \leq C_{p,\varepsilon}[w]_{A_p(\mu)}.$$

Since $1 < s < p$, the desired result follows. □

We underline that if $w \in A_p(\mu)$, then $w\mu$ is not, in general, a flow measure or a doubling measure, so a variant of Lemma 3.1 for the measure $w\mu$ is needed. To do so, we will work with weights satisfying the following assumption.

Assumption 1 w is a weight and $\eta \in (0, 1)$ is such that for every $R \in \mathcal{R}$ and $S \subset R$

$$\mu(S) \leq (1 - C_D)\mu(R) \Longrightarrow w_\mu(S) \leq \eta w_\mu(R), \tag{3.5}$$

where C_D is as in (3.1).

Observe that Assumption 1 is automatically satisfied for $A_p(\mu)$ weights by Proposition 2.3 $viii$).

Moreover, by applying (3.5) to $R \setminus S$ and by measure additivity, Assumption 1 is equivalent to

$$w_\mu(S) < (1 - \eta)w_\mu(R) \Longrightarrow \mu(S) < C_D\mu(R),$$

where $S \subset R$ and η and C_D are as in (3.5). Under Assumption 1, we can prove the following variant of Lemma 3.1 for the measure $w\mu$.

Proposition 3.5 *Assume that w satisfies Assumption 1. Then, for every function f on T, $\lambda > 0$ and $R \in \mathcal{R}$ such that $\frac{1}{w_\mu(R)} \sum_{y\in R} |f(y)|w(y)\mu(y) < \lambda$ there exist a (possibly empty) family $\mathcal{F}$ of disjoint admissible trapezoids and a constant D'_{CZ} such that for each $E \in \mathcal{F}$ the following hold:*

i) $\frac{1}{w_\mu(E)} \sum_{y\in E} |f(y)|w(y)\mu(y) \geq \lambda$;
ii) $\frac{1}{w_\mu(E)} \sum_{y\in E} |f(y)|w(y)\mu(y) < D'_{CZ}\lambda$;
iii) *if $x \in R \setminus \cup_{E\in\mathcal{F}} E$, then $|f(x)| < \lambda$.*

Proof Let $R \in \mathcal{R}$ and $S \subset R$. By replacing S with $R \setminus S$ in (3.5), it is clear that Assumption 1 is equivalent to the following: there exists $\delta \in (0, 1)$ such that

$$\mu(S) \geq C_D\mu(R) \Longrightarrow w_\mu(S) \geq \delta w_\mu(R). \tag{3.6}$$

Let Q be any set in the partition of R introduced in (3.1). Combining (3.1) and (3.6), it follows that

$$\frac{w_\mu(Q)}{w_\mu(R)} \geq \delta,$$

where δ is a constant independent of R. Hence, w_μ satisfies the assumptions of Lemma 3.1 and the proof follows. □

The next result is a consequence of the above proposition.

Corollary 3.6 *If w is a weight which satisfies Assumption 1, then there exist $\varepsilon, C > 0$ such that the following reverse Hölder inequality holds*

$$\left(\frac{1}{w_\mu(R)} \sum_{y\in R} w^{-1-\varepsilon}(y)w(y)\mu(y)\right)^{1/(1+\varepsilon)} \leq \frac{C}{w_\mu(R)} \sum_{y\in R} \mu(y), \qquad \forall R \in \mathcal{R}.$$

Proof The proof follows the scheme of the one of Theorem 3.2 by replacing μ with $w\mu$ and w by w^{-1}. □

We now introduce the class of $A_\infty(\mu)$ weights.

Definition 3.7 We define $A_\infty(\mu) = \bigcup_{p>1} A_p(\mu)$ and we set

$$[w]_{A_\infty(\mu)} = \sup_{R\in\mathcal{R}} \frac{1}{\mu(R)} \sum_{y\in\mathcal{R}} w(y)\mu(y) \exp\left[\frac{1}{\mu(R)} \sum_{y\in R} \log(w(y)^{-1})\mu(y)\right].$$

We establish below the equivalence between $A_\infty(\mu)$ and the weights satisfying $[w]_{A_\infty(\mu)} < \infty$. Our main result is based on Corollary 3.6. We refer to [5, 11, 23] for related results in the Euclidean setting and to [28] for an example in a nondoubling setting.

Theorem 3.8 *Suppose that w is a weight on T. The following are equivalent:*

i) $w \in A_\infty(\mu)$;
ii) $[w]_{A_\infty(\mu)} < \infty$;
iii) *there is $\gamma_0 \in (0,1)$ such that for all $\gamma \in (0,\gamma_0)$ there exists $\delta = \delta(\gamma) \in (0,1)$ such that for all $R \in \mathcal{R}$*

$$\mu\left(\left\{x \in R : w(x) \le \frac{\gamma}{\mu(R)} \sum_{y\in R} w(y)\mu(y)\right\}\right) \le \delta\mu(R).$$

Furthermore, for every $\varepsilon > 0$ we can choose $\gamma \in (0,\gamma_0)$ such that $\delta(\gamma) < \varepsilon$;
iv) *for every $\xi \in (0,1)$ there is $\eta = \eta(\xi) \in (0,1)$ such that for all $R \in \mathcal{R}$ and $S \subset R$*

$$\mu(S) \le \xi\mu(R) \Longrightarrow w_\mu(S) \le \eta w_\mu(R).$$

Proof Assume that $w \in A_\infty(\mu)$. Then, since the classes $A_p(\mu)$ are increasing with p, $w \in A_p(\mu)$ for every $p \ge p_0$ for some $p_0 \in (1,\infty)$. Then,

$$\frac{1}{\mu(R)} \sum_{y\in R} w(y)\mu(y) \left(\frac{1}{\mu(R)} \sum_{y\in R} w(y)^{-1/(p-1)}\mu(y)\right)^{p-1} \le C < \infty,$$

for every $p \ge p_0$ and passing to the limit as $p \to \infty$ we get that $[w]_{A_\infty(\mu)} < \infty$ (see [30, Chapter 3, Exercise 5]).

We now prove that $ii)$ implies $iii)$. Indeed, assume that $[w]_{A_\infty(\mu)} < \infty$ and let $R \in \mathcal{R}$. We can suppose without loss of generality that $\sum_{y\in R} \log w(y)\mu(y) = 0$ (otherwise we can divide w by a suitable constant) so that

$$\frac{1}{\mu(R)} \sum_{y\in R} w(y)\mu(y) \le [w]_{A_\infty(\mu)}. \tag{3.7}$$

Much like in [12, Theorem 9.3.3], we can prove that, given $\gamma > 0$ small enough to be chosen later

$$\begin{aligned}\mu\Big(\Big\{x \in R : w(x) \le \frac{\gamma}{\mu(R)} \sum_{y\in R} w(y)\mu(y)\Big\}\Big) &\le \frac{1}{\log(1 + (\gamma[w]_{A_\infty(\mu)})^{-1})} \sum_{y\in R} \log(1 + w(y))\mu(y)\\ &=: I.\end{aligned}$$

Next, we need a more precise estimate of I than the one obtained in the classical proof. To achieve this, we multiply and divide I by $\mu(R)$ and therefore an application of Jensen's inequality yields

$$\begin{aligned} I &\le \frac{\mu(R)}{\log(1 + (\gamma[w]_{A_\infty(\mu)})^{-1})} \log\left(\frac{1}{\mu(R)} \sum_{y\in R}(1 + w(y))\mu(y)\right)\\ &\le \frac{\mu(R)\log(1 + [w]_{A_\infty(\mu)})}{\log(1 + (\gamma[w]_{A_\infty(\mu)})^{-1})}\\ &= \delta(\gamma)\mu(R),\end{aligned}$$

where in the last inequality we used (3.7) and we chose $\gamma_0 = [w]^{-2}_{A_\infty(\mu)}$, $\gamma \in (0, \gamma_0)$, and

$$\delta(\gamma) := \frac{\log(1 + [w]_{A_\infty(\mu)})}{\log(1 + (\gamma[w]_{A_\infty(\mu)})^{-1})} < 1.$$

Observe that when $\gamma \to 0^+$ we have that $\delta(\gamma) \to 0$.

Next, assume that $iii)$ holds. Pick $R \in \mathcal{R}$, fix $\xi \in (0, 1)$ and suppose that $w_\mu(S) > \eta w_\mu(R)$ for some $\eta \in (0, 1)$ to be chosen.

Similarly to [12, Theorem 9.3.3] one can prove that $\mu(R \setminus S) < \left(\delta + \frac{1-\eta}{\gamma}\right)\mu(R)$. We now choose $\gamma \in (0, \gamma_0)$ small enough such that $1 - \delta(\gamma) > \xi$ (this is possible because $\delta(\gamma) \to 0$ as $\gamma \to 0^+$). Observe that $1 - \delta > \xi$ if and only if $1 - \xi > \delta$ thus we choose $\eta \in (0, 1)$ such that $\delta + \frac{1-\eta}{\gamma} = 1 - \xi$. Such a η exists because $\delta < 1 - \xi$ and $\delta + 1/\gamma > 1 > 1 - \xi$. Summarizing, we have proved that $\mu(E) < (1 - \xi)\mu(R)$, namely, that $\mu(S) > \xi\mu(R)$ from which follows $iv)$.

Next, $iv)$ implies Assumption 1, thus, by Corollary 3.6 we get

$$\left(\frac{1}{w_\mu(R)} \sum_{y\in R} w^{-(1+\varepsilon)} w(y)\mu(y)\right)^{1/(1+\varepsilon)} \le C \frac{\mu(R)}{w_\mu(R)}, \qquad \forall R \in \mathcal{R},$$

which is equivalent to

$$\frac{1}{\mu(R)} w_\mu(R)^{\varepsilon/(1+\varepsilon)} \left(\sum_{y\in R} w^{-\varepsilon}(y)\mu(y)\right)^{1/(1+\varepsilon)} \le C, \qquad \forall R \in \mathcal{R},$$

that is the $A_p(\mu)$ condition for w when $\varepsilon = 1/(p-1)$. This implies $i)$ and concludes the proof. □

In [18, Section 4], the space $BMO(\mu)$ is defined by

$$BMO(\mu) := \left\{ f : T \to \mathbb{R} \quad \sup_{R \in \mathcal{R}} \frac{1}{\mu(R)} \sum_{y \in R} |f(y) - f_R| \mu(y) < \infty \right\},$$

where f_R denotes the average of f on R with respect to the measure μ. This space exhibits several good properties. For instance, it is isomorphic to the dual of a suitable Hardy space and it also interpolates with $L^p(\mu)$, $p \in (1, \infty)$ (see [18, Section 4]).

In the Euclidean setting, it is known that BMO functions coincide with (multiples) of logarithms of A_p weights (see e.g. [11, Corollary 2.19]). In the next proposition, we prove that the same phenomenon occurs in our setting.

Proposition 3.9 *Let $w \in A_p(\mu)$ for some $p \in (1, \infty]$ and define $g = \log w$. Then $g \in BMO(\mu)$. Conversely, if $f \in BMO(\mu)$ then $f = \lambda \log w$ for some $w \in A_p(\mu)$ and λ large enough. In other words, $BMO(\mu) = \{\lambda \log w, w \in A_\infty(\mu), \lambda \in \mathbb{R}\}$.*

Proof It suffices to prove the first implication when $w \in A_2(\mu)$. Indeed, if $p < 2$ we have that $w \in A_2(\mu)$ and if $p > 2$ then $w^{-1/(p-1)} \in A_{p'}(\mu) \subset A_2(\mu)$ and replacing w by $w^{-1/(p-1)}$ we obtain that $-(p-1)\log w \in BMO(\mu)$.

For every $R \in \mathcal{R}$ define

$$R^+ = R \cap \left\{ x \in R \;\; g(x) - g_R \geq 0 \right\}, \quad R^- = R \setminus R^+,$$

where $g = \log w$. By Jensen's inequality, we have that

$$\exp\left[\frac{1}{\mu(R)} \sum_{y \in R} \chi_{R^+}(y)(g(y) - g_R)\mu(y) \right] \leq \frac{1}{\mu(R)} \left(\sum_{y \in R^+} (w(y)e^{-g_R})\mu(y) + \sum_{y \in R^-} \mu(y) \right).$$

Moreover, since $x \mapsto e^{-x}$ is convex, again by Jensen's inequality $e^{-g_R} \leq (w^{-1})_R$ and

$$1 = \left(\frac{1}{\mu(R)} \sum_{y \in R} w^{-1/2}(y) w^{1/2}(y) \mu(y) \right)^2 \leq w_R (w^{-1})_R$$

by Cauchy-Schwarz inequality. It follows that

$$\frac{1}{\mu(R)} \sum_{y \in R} \chi_{R^+}(y)(g(y) - g_R)\mu(y) \leq \log[2 w_R (w^{-1})_R].$$

A similar argument shows that

$$\exp\left[\frac{1}{\mu(R)} \sum_{y \in R} \chi_{R^-}(y)(g_R - g(y))\mu(y) \right] \leq \frac{1}{\mu(R)} \left(\sum_{y \in R^-} (e^{g_R} w(y)^{-1})\mu(y) + \sum_{y \in R^+} \mu(y) \right)$$

$$\leq 2w_R(w^{-1})_R.$$

Therefore,

$$\frac{1}{\mu(R)}\sum_{y\in R}|\log w(y)-(\log w)_R|\mu(y)\leq 2\log[2w_R(w^{-1})_R]\leq 2\log(2[w]_{A_2(\mu)}).$$

This shows that $g\in BMO(\mu)$.

Next, pick $f\in BMO(\mu)$ and fix $p\in(1,\infty)$. Define $\psi=e^{\lambda f}$ for some $\lambda=\lambda(p)>0$ to be chosen. Our goal is to show that $\psi\in A_p(\mu)$. Indeed,

$$\begin{aligned}
&\frac{1}{\mu(R)}\sum_{y\in R}\psi(y)\mu(y)\left(\frac{1}{\mu(R)}\sum_{y\in R}\psi(y)^{-1/(p-1)}\mu(y)\right)^{p-1}\\
&=\frac{1}{\mu(R)}\sum_{y\in R}e^{(f(y)-f_R)\lambda}\mu(y)\left(\frac{1}{\mu(R)}\sum_{y\in R}e^{-\lambda(f(y)-f_R)/(p-1)}\mu(y)\right)^{p-1}\\
&\leq\left(\frac{1}{\mu(R)}\sum_{y\in R}e^{\eta|(f(y)-f_R|}\mu(y)\right)^{p},
\end{aligned}$$

where $\eta:=\lambda\max\{1,1/(p-1)\}$. The last expression is uniformly bounded by a constant when λ is small enough by John-Nirenberg's inequality, see [18, Proposition 4.2. (i)]. □

4 $A_\infty(\mu)$ and quasisymmetric mappings

In this section, we assume that $T=\mathbb{T}_q$ is the homogeneous tree of order $q+1$, namely the tree such that every vertex has exactly $q+1$ neighbours. The associated canonical flow measure is $\mu(\cdot)=q^{\ell(\cdot)}$ where ℓ is the level function on T defined in (1.1). Let f be a bijection from T onto itself and J_f be the weight on T defined by

$$J_f(x)=\frac{\mu(f(x))}{\mu(x)},\qquad\forall x\in T.$$

Observe that, for every finite $E\subset T$, since f is a bijection, we have

$$(J_f)_\mu(E):=\sum_{y\in E}J_f(y)\mu(y)=\sum_{y\in E}\mu(f(y))=\mu(f(E)),$$

so J_f can be thought of as the discrete version of the *Jacobian* of f. We remark that, by Theorem 3.8, $J_f\in A_\infty(\mu)$ if and only if for every $\xi\in(0,1)$ there exists $\eta\in(0,1)$ such that for every $R\in\mathcal{R}$ and $S\subset R$

$$\frac{\mu(S)}{\mu(R)}\leq\xi\Longrightarrow\frac{\mu(f(S))}{\mu(f(R))}=\frac{(J_f)_\mu(S)}{(J_f)_\mu(R)}\leq\eta. \tag{4.1}$$

In the next proposition, we shall prove that isometries with respect to the geodesic distance do not generally have Jacobian in $A_\infty(\mu)$. We refer to [9, Chapter 1] for more information about the action of the group of isometries on the homogeneous tree.

Proposition 4.1 *Let f be an isometry on T with respect to d that exchanges ω_* with another boundary point $\omega_- \in \partial T$. Then, J_f does not belong to $A_\infty(\mu)$.*

Proof Without loss of generality assume that $o \in (\omega_-, \omega_*)$ and that $f(o) = o$. Let $\{x_n\}_{n\in\mathbb{Z}}$ denote an enumeration of the infinite geodesic (ω_-, ω_*) such that $\ell(x_n) = n$. Then, we have that $f(x_n) = x_{-n}$. Let $\{R_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ be defined by $R_n = R_n^{2n}(o)$ and set $E_n := \{x \le x_{-n}\} \cap R_n$. Then,

$$\frac{\mu(E_n)}{\mu(R_n)} = q^{-n}.$$

It suffices to disprove (4.1) showing that

$$\frac{\mu(f(E_n))}{\mu(f(R_n))} \to 1 \quad \text{as } n \to \infty. \tag{4.2}$$

Notice that $f(E_n)$ contains $f(x_{-2n+1}) = x_{2n-1}$, thus

$$\mu(f(E_n)) \ge \mu(x_{2n-1}) = q^{2n-1}.$$

Since f is an isometry, $\max_{x\in R_n\setminus E_n} d(x_{-n+1}, x) \le 3n - 1$ and $f(x_{-n+1}) = x_{n-1}$, one can see that

$$f(R_n \setminus E_n) \subset \{y \in T \; y \le x_{n-1}, d(x_{n-1}, y) \le 3n - 1\}.$$

Hence $\mu(f(R_n \setminus E_n)) \le 3nq^{n-1}$ and

$$\frac{\mu(f(R_n \setminus E_n))}{\mu(f(E_n))} \le \frac{3n}{q^n} \to 0, \qquad \text{as } n \to \infty.$$

It follows that

$$\frac{\mu(f(E_n))}{\mu(f(R_n))} = \frac{\mu(f(E_n))}{\mu(f(E_n)) + \mu(f(R_n \setminus E_n))} \to 1, \qquad \text{as } n \to \infty,$$

and (4.2) is proved. □

Intuitively, the problem with these isometries is that they do not consistently preserve the order relation $\le$ between vertices, because the relationship between their images under the isometry is not guaranteed in general.

We now introduce a different distance on T which is also natural when we deal with admissible trapezoids. We define the *Gromov* distance $\rho : T \times T \to [0, \infty)$ by

$$\rho(x, y) = \begin{cases} e^{\ell(x\wedge y)} & x \ne y, \\ 0 & x = y, \end{cases}$$

where $x \wedge y$ denotes the *confluent* of $x, y \in T$, namely, the vertex of minimum level which is above both x and y. It is easy to verify that ρ is a metric.

Recall that a map f from T to itself is bilipschitz with respect to ρ if there exists a positive constant C such that

$$e^{-C}\rho(f(x), f(y)) \leq \rho(x, y) \leq e^{C}\rho(f(x), f(y)), \qquad \forall x, y \in T. \tag{4.3}$$

Proposition 4.2 *Suppose that f is a surjective bilipschitz map on T with respect to ρ. Then,*

$$-C \leq \ell(x) - \ell(f(x)) \leq C \qquad \forall x \in T,$$

where C is as in (4.3). In particular, it follows that $J_f, 1/J_f \in L^{\infty}(T)$ and thus $J_f \in A_p(\mu)$ for every $p \in [1, \infty]$.

Proof Observe that (4.3) readily implies that for every $x, y \in T$ the following holds

$$-C \leq \ell(x \wedge y) - \ell(f(x) \wedge f(y)) \leq C. \tag{4.4}$$

Since $\ell(a \wedge b) \geq \max\{\ell(a), \ell(b)\}$ for every $a, b \in T$, and $\ell(a \wedge b) = \ell(a)$ for $b \leq a$, choosing $y \leq x$ in (4.4) one obtains $\ell(x) - \ell(f(x)) \leq C$, and choosing y such that $f(y) \leq f(x)$, which exists since f is surjective, one gets $\ell(x) - \ell(f(x)) \geq -C$. This concludes the proof. □

In the next proposition, we study some properties of isometries on T with respect to the Gromov distance ρ.

Proposition 4.3 *Let f be a surjective isometry on T with respect to ρ. Then, f is a level and order-preserving map, namely,*

i) $\ell(x) = \ell(f(x))$, *for every* $x \in T$;
ii) $x \leq y \iff f(x) \leq f(y)$, *for every* $x, y \in T$.

Proof Let f be a surjective isometry with respect to ρ. Then, $i)$ directly follows by Proposition 4.2 when $C = 0$. Next, we prove $ii)$.

Let $y < x$. Then $\rho(x, y) \leq \rho(x, z)$, for any $z \neq x$, with equality holding if and only if $z < x$. Choose a point $\xi < f(x)$ and let $z = f^{-1}(\xi)$. Then, if f is an isometry,

$$\rho(x, y) \leq \rho(x, z) = \rho(f(x), \xi) \leq \rho(f(x), f(y)) = \rho(x, y).$$

It follows that $\rho(f(x), f(y)) = \rho(f(x), \xi)$ which implies $f(y) < f(x)$, that is $ii)$.□

Remark 4.4 One may prove that any isometry on T with respect to ρ is in fact surjective. We omit the details because the proof is quite involved.

It turns out that bilipschitz maps with respect to ρ are quasi-isometries with respect to d.

Proposition 4.5 *Suppose that f is bilipschitz on T with respect to ρ. Then f is a $(1, 4C)$-quasi-isometry bijection with respect to d, namely*

$$d(f(x), f(y)) - 4C \leq d(x, y) \leq d(f(x), f(y)) + 4C,$$

for all $x, y \in T$. In particular, every isometry with respect to ρ is an isometry with respect to d.

Proof Proposition 4.2 implies that

$$-C \leq \ell(x) - \ell(f(x)) \leq C.$$

Moreover, (4.3) readily implies that

$$-C \leq \ell(x \wedge y) - \ell(f(x) \wedge f(y)) \leq C.$$

On the one hand, for every $x, y \in T$,

$$\begin{aligned} d(x, y) &= 2\ell(x \wedge y) - \ell(x) - \ell(y) \\ &\leq 2\ell(x \wedge y) + C - \ell(f(x)) + C - \ell(f(y)) + 2\ell(f(x) \wedge f(y)) \\ &\quad - 2\ell(f(x) \wedge f(y)) \\ &= 4C + d(f(x), f(y)). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(f(x), f(y)) &= 2\ell(f(x) \wedge f(y)) - \ell(f(x)) - \ell(f(y)) \\ &\leq 2\ell(f(x) \wedge f(y)) + C - \ell(x) + C - \ell(y) + 2\ell(x \wedge y) - 2\ell(x \wedge y) \\ &= 4C + d(x, y). \end{aligned}$$

The statement about the isometries follows by setting $C = 0$. □

The latter two propositions identify a class of quasi-isometric bijections on (T, d, μ) whose Jacobian is in L^∞. In a subsequent paper, we plan to investigate the statement that quasisymmetric mappings on (T, ρ, μ) are a subset of (L, C)-quasi-isometric bijections on (T, d, μ), and that these quasi-isometries have Jacobian in $A_p(\mu)$ for some p.

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