The Virtual Element Method: Stability Issues and Mixed-Dimensional Applications in Soil Domains

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This dissertation investigates the application and enhancement of the Virtual Element Method (VEM) in the context of non-linear and geometrically complex problems, with a particular focus on 3D-1D coupled models for root-soil interaction. The flexibility of VEM in handling polytopal meshes proves crucial in efficiently meshing heterogeneous soils, especially in the presence of layered structures or arbitrarily shaped impermeable obstacles.

A central part of this work is devoted to addressing the challenges that arise when applying VEM to such demanding scenarios. Specifically, we analyze the impact of badly-shaped elements and high-order discretizations on the conditioning of local and global system matrices. To mitigate these effects, we propose several improvements: (i) the use of orthogonal polynomial bases for internal degrees of freedom and projection operators; (ii) geometric mappings that transform badly-shaped elements into better-shaped reference elements. Finally, a detailed study of how boundary degrees of freedom and stabilization terms influence the conditioning of such matrices is conducted particularly in the context of highly anisotropic problems.

Another key contribution is the development of a stabilization-free variant of the VEM, based on a new self-stabilized discrete bilinear form. This formulation avoids the need for ad hoc stabilization terms, improving robustness and simplifying implementation. In addition, we explore a novel approach based on neural networks to reduce the heavy usage of polynomial projections in the discretization process of non-linear problems. Indeed, these projections, though essential to access virtual functions, may slow down the convergence of iterative strategies employed to address non-linearities and pose challenges in the post-processing of the solution.

The numerical experiments confirm the effectiveness of the proposed techniques in improving conditioning and maintaining accuracy. The work contributes to expanding the applicability of VEM to real-world multiphysics problems involving complex geometries and strong non-linearities.