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A monotonicity theorem for subharmonic functions on manifolds ☆

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A B S T R A C T

We prove a sharp monotonicity theorem about the distribution of subharmonic functions on manifolds, which can be regarded as a new, measure theoretic form of the uncertainty principle. As an illustration of the scope of this result, we deduce contractivity estimates for analytic functions on the Riemann sphere, the complex plane and the Poincaré disc, with a complete description of the extremal functions, hence providing a unified and illuminating perspective on a number of results and conjectures on this subject, in particular on the Wehrl entropy conjecture of Lieb and Solovej. In this connection, we completely prove that conjecture for the group $SU(2)$, by showing that the corresponding extremals are only

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Shape optimization

the coherent states. Also, we show that the above (global) estimates admit a local counterpart and in all cases we characterize also the extremal subsets, among those of fixed assigned measure.

1. Main results and applications

In the recent works of the second and fourth authors [15] in the Euclidean case and of the first author [10] in the hyperbolic case, a new method was discovered for studying the distribution of analytic functions. In this paper we single out key properties required for this approach to work, in the process generalizing it to wider classes of functions and to new geometries, in particular the spherical geometry.

To state the main result, we need to introduce some notation first. Let M be a smooth n-dimensional Riemannian manifold without boundary. We assume that it satisfies an isoperimetric inequality, that is for all open sets $A \subset M$ with compact closure and smooth boundary we have

$$|\partial A|_m^2 \geq H(|A|_M), \tag{1.1}$$

where $|\cdot|_m$ is the $n-1$-dimensional Hausdorff measure on M, $|\cdot|_M$ is the n-dimensional volume on M associated to the metric, and $H : (0, |M|_M) \to (0, +\infty)$ is a C^1 function (if $|M|_M$ is finite, we extend it to $H(|M|_M) = 0$).

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold satisfying* (1.1) *and let $u : M \to \mathbb{R}$ be a Morse function on M, $u \in C^2(M)$, such that for all $t \in \mathbb{R}$ the superlevel sets $u^{-1}([t, +\infty))$ are compact and $\Delta_M u \geq -c$, for some constant $c > 0$ where Δ_M is the Laplace–Beltrami operator on M. Put $\mu(t) = |u^{-1}([t, +\infty))|_M$ and $t_0 = \sup_{p \in M} u(p)$. Then $\mu(t)$ is locally absolutely continuous and*

$$\mu'(t) \leq -\frac{H(\mu(t))}{c\mu(t)} \tag{1.2}$$

for almost all $t \in (-\infty, t_0)$.

Roughly speaking, this result tells us that u cannot be too concentrated in the measure theoretic sense, which can be regarded as a new form of the uncertainty principle.

The assumption that u is a Morse function, unlike every other one, is purely technical for this theorem to hold. For a general function u satisfying all the other conditions, we can get an almost equivalent result. To state it, it is convenient to define, for $t_1 < t_2 < t_0$ and $\mu > 0$, $D(t_1, t_2, \mu) := g(t_1)$, where $g(t)$ is the solution, on the interval $[t_1, t_2]$, of the (backward) differential equation

$$g'(t) = -\frac{H(g(t))}{cg(t)} \tag{1.3}$$

with initial condition $g(t_2) = \mu$, provided that such a solution exists.

Theorem 1.2. *Let M be an n-dimensional Riemannian manifold satisfying (1.1) and let $u : M \to \mathbb{R}$ be a function in $C^2(M)$ such that for all $t \in \mathbb{R}$ the sets $u^{-1}([t, +\infty))$ are compact and $\Delta_M u \geq -c$ for some constant $c > 0$, where Δ_M is the Laplace-Beltrami operator on M. Put $\mu(t) = |u^{-1}([t, +\infty))|_M$ and $t_0 = \sup_{p \in M} u(p)$. Then for all $t_1 < t_2 < t_0$ we have*

$$D(t_1, t_2, \mu(t_2)) \leq \mu(t_1) \tag{1.4}$$

Part of the result is that the solution $D(t, t_2, \mu(t_2))$ exists for every $t < t_2$. This a consequence of the above a priori bound, which prevents blow-up in finite time in the (backward) Cauchy problem. If u is a Morse function, then this theorem is a direct consequence of Theorem 1.1 and a basic comparison principle for first-order ODE. If the function u is not Morse, then we can approximate it by Morse functions while preserving all the other assumptions and use the continuity of the solution to the differential equation on the initial conditions. For the reader's convenience we put the deduction of the Theorem 1.2 from the Theorem 1.1 in the Appendix.

A version of this result was used in [15] and [10], the difference being that in these papers the authors worked with weighted analytic functions of the form $f(z) = g(z)e^{-\varphi(z)}$, with g holomorphic and φ having constant Laplacian, for which we have a lower bound of $\Delta \log |f(z)|$ instead of $\Delta|f(z)|$, which amount just to a simple change of variables. The advantages of Theorem 1.2 are first of all that we can consider many more different manifolds than just Euclidean and hyperbolic spaces, in particular we can also work in the spherical geometry, but more generally on simply-connected two-dimensional manifolds of bounded curvature. Another advantage is that the functions that we work with are no longer analytic, thus vastly enlarging the domain of applicability of this theorem.

As a consequence of the monotonicity result in Theorem 1.2 we will prove the following sharp functional inequality.

Theorem 1.3. *Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that $\lim_{t \to -\infty} F(t) = 0$ and $G : [0, F(t_0)] \to \mathbb{R}$ be a continuous convex function with $G(0) = 0$, for some $t_0 \in \mathbb{R}$. Let M be an n-dimensional Riemannian manifold satisfying (1.1) and let $u : M \to \mathbb{R}$ be a $C^2(M)$ function such that for all $t \in \mathbb{R}$ the sets $u^{-1}([t, +\infty))$ are compact and $\Delta_M u \geq -c$ for some constant $c > 0$, where Δ_M is the Laplace-Beltrami operator on M, with*

$$\int_M F(u(p))\, d\mathrm{Vol}(p) = 1. \tag{1.5}$$

Let $\mu_0(t) > 0$ be a solution to the differential equation (1.3) on $(-\infty, t_0)$, such that

$$\int_{-\infty}^{t_0} F'(t)\mu_0(t)dt = 1$$

and $\lim_{t\to t_0^-} \mu_0(t) = 0$. *Then* $u(p) \le t_0$ *for all* $p \in M$ *and*

$$\int_M G(F(u(p)))\, d\mathrm{Vol}(p) \le \int_{-\infty}^{t_0} G'(F(t))F'(t)\mu_0(t)dt. \tag{1.6}$$

Moreover, if G *is not linear on* $[0, F(t_0)]$, *then equality in* (1.6) *is possible only if either both integrals are* $-\infty$ *or* $|u^{-1}([t, +\infty))|_M = \mu_0(t)$ *for all* $t < t_0$.

The function F corresponds to the change of variables, for example $F(t) = e^t$ if we want to consider log-subharmonic functions, while the function G corresponds to what we want to integrate, for example $G(t) = t^p$ if we want to consider L^p-norms.

Remark 1.4. Since we want to apply our theorem to logarithms of analytic functions, which can be 0 at some points, sometimes it is convenient for us to assume that $u : M \to [-\infty, \infty)$ is continuous and C^2 on $u^{-1}(\mathbb{R})$. This case follows from the above theorem since $u^{-1}(\mathbb{R})$ is still a manifold without boundary, $u^{-1}([t, +\infty))$ are compactly embedded into it and $F(-\infty) = G(0) = 0$ so the integrals do not change.

By applying Theorem 1.3 to some particular instances of manifolds M and functions u it is possible to prove that in many occasions the most concentrated (normalized) functions in a reproducing kernel Hilbert space are given by the normalized reproducing kernels. This is particularly clear when the space consists of holomorphic functions and there is a group acting on M which is compatible with the reproducing kernel structure. If we quantify the concentration of the functions in terms of the Wehrl entropy, this is essentially the content of Section 4. We prove a generalized version of the conjecture, that gives as a corollary the hypercontractive embeddings among spaces.

There is also an analogous local problem, where we inquire which is the domain of a given measure where a normalized function in a reproducing kernel Hilbert space is mostly concentrated. Again the extremal functions for such problems are reproducing kernels and the extremal domains are the corresponding super-level sets. This is the content of Section 5.

Rupert Frank [7] has independently and simultaneously obtained analogous results to those in Sections 4 and 5 in the three classical geometric models: sphere, Euclidean plane and hyperbolic disk. We have chosen to present a streamlined proof in a more general setting that covers as a particular instance the classical cases. Moreover, our approach allows us to identify the maximizers in the local estimates; see Section 5.

2. Proof of Theorem 1.1

Since u is a Morse function, we have $|\{p \in M : \nabla u(p) = 0\}|_M = 0$, which implies that $\mu(t)$ is locally absolutely continuous and the coarea formula holds in the following form

$$\mu'(t) = -\int_{\partial A_t} |\nabla u|^{-1} d\mathcal{H}^{n-1}$$

for almost all $t \in \mathbb{R}$, where $A_t = u^{-1}([t, +\infty))$, $|\nabla u|$ stands for the length of ∇u in the tangent space and $\mathcal{H}^{n-1} = |\cdot|_m$ is the $n-1$-dimensional Hausdorff measure on M (cf. [6, Sections 3.2.12 and 3.2.46]). By Sard's theorem, for almost all t we have $\nabla u \neq 0$ where $u = t$, so that $\partial A_t = u^{-1}(\{t\})$ is a smooth submanifold, which is compact by the assumption that $u^{-1}([t, +\infty))$ is compact for all $t \in \mathbb{R}$.

Next, we apply the Cauchy–Schwarz inequality on ∂A_t:

$$|\partial A_t|_m^2 = \left(\int_{\partial A_t} d\mathcal{H}^{n-1} \right)^2 \leq \int_{\partial A_t} |\nabla u|^{-1} \, d\mathcal{H}^{n-1} \int_{\partial A_t} |\nabla u| \, d\mathcal{H}^{n-1}.$$

Now, ∇u is orthogonal to ∂A_t and pointing inside A_t. Thus, denoting by ν the unit outward normal to ∂A_t, we have $|\nabla u| = -\nabla u \cdot \nu$. Plugging this in and using Gauss–Green's theorem, we have

$$\int_{\partial A_t} |\nabla u| \, d\mathcal{H}^{n-1} = -\int_{\partial A_t} \nabla u \cdot \nu \, d\mathcal{H}^{n-1} = -\int_{A_t} \Delta u \, d\mathrm{Vol} \leq c|A_t|_M = c\mu(t).$$

By the isoperimetric inequality, we have $|\partial A_t|_m^2 \geq H(\mu(t))$. Combining everything and dividing by $c\mu(t)$ (note that here we used that $t < t_0$, that is $\mu(t) > 0$), we get

$$-\mu'(t) \geq \frac{H(\mu(t))}{c\mu(t)}.$$

Multiplying this by -1 we get the desired result.

Remark 2.1. It turns out that the claim of Theorem 1.1 is of local nature, and the assumption that every superlevel sets of u is compact can be weakened. In fact, the previous proof yields (without changes) the following more general result: if the superlevel sets $\{u \geq t\}$ are compact for $t > \tau$ (for some $\tau < t_0$), and $\Delta_M u \geq -c$ in the open set where $u > \tau$, then $\mu(t)$ is locally absolutely continuous in (τ, t_0), and (1.2) holds true for a.e. $t \in (\tau, t_0)$.

3. Proof of Theorem 1.3

The following preliminary result will play a crucial role in the following.

Lemma 3.1. *With the same notation and assumptions as in Theorem 1.3, let $\mu(t) = |u^{-1}([t,+\infty))|_M$ and suppose $\mu \neq \mu_0$ at some point, where $\mu_0(t)$ in understood to be extended by 0 past t_0. There exist $t_1 < t_0$ such that $\mu(t) \geq \mu_0(t)$ if $t \leq t_1$ and $\mu(t) < \mu_0(t)$ if $t_1 < t < t_0$. In particular, $\mu(t) = 0$ for $t \geq t_0$. Moreover $\mu(t) > \mu_0(t)$ if $t_1 - t > 0$ is large enough.*

Note that this lemma already implies that $u(p) \leq t_0$ for all $p \in M$.

Proof. The condition (1.5) is equivalent to

$$\int_{-\infty}^{\infty} F'(t)\mu(t)dt = 1.$$

Hence

$$\int_{-\infty}^{\infty} F'(t)\mu(t)dt = \int_{-\infty}^{\infty} F'(t)\mu_0(t)dt.$$

Since $F'(t) \geq 0$ and F' is not identically zero on any subinterval, $\mu \neq \mu_0$ and μ, μ_0 are left-continuous, there should be t_2 and t_3 such that $\mu(t_2) > \mu_0(t_2)$ and $\mu(t_3) < \mu_0(t_3)$; in particular $t_3 < t_0$, because $\mu_0(t) = 0$ for $t \geq t_0$. By Theorem 1.2 for $t \leq t_2$ we have $\mu(t) > \mu_0(t)$ while for $t_3 \leq t < t_0$ we have $\mu(t) < \mu_0(t)$. We denote by t_1 the infimum of admissible t_3's. The conclusion is then clear. $\square$

Proof of Theorem 1.3. As in Lemma 3.1 we set $\mu(t) = |u^{-1}([t,+\infty))|_M$ for $t \in \mathbb{R}$.

The left-hand side of (1.6) is equal to

$$\int_{-\infty}^{t_0} G'(F(t))F'(t)\mu(t)dt,$$

because $\mu(t) = 0$ and $G(0) = 0$ for $t \geq t_0$ by Lemma 3.1.

Observe that this latter integral could be $-\infty$ but not $+\infty$. Indeed, the positive part of $G'(F(t))$ is bounded outside of the vicinity of t_0 and near t_0 we have that $\mu(t)$ is bounded while $\int_{-\infty}^{t_0} G'(F(t))F'(t)\,dt = G(F(t_0)) < \infty$ is convergent. The same can be said for the right-hand side of (1.6). Moreover it is clear that we have an equality in (1.6) if $\mu(t) = \mu_0(t)$ for $t < t_0$.

Now, if the right-hand side of (1.6) is $-\infty$ then $G'(x) < 0$ for $x > 0$ small enough and

$$\int_{-\infty}^{\bar{t}} G'(F(t))F'(t)\mu_0(t)dt = -\infty,$$

if $t_0 - \overline{t} > 0$ is large enough, because G' is bounded below on the compact subintervals of $(0, F(t_0)]$. By Lemma 3.1 the same holds with μ_0 replaced by μ, which implies that the left-hand side of (1.6) is $-\infty$ as well.

Suppose now that both sides of (1.6) are finite. Let t_1 be as in Lemma 3.1. We must have

$$\int_{-\infty}^{t_0} G'(F(t))F'(t)(\mu_0(t) - \mu(t))dt = \int_{-\infty}^{t_1} (G'(F(t)) - G'(F(t_1)))F'(t)(\mu_0(t) - \mu(t))dt$$
$$+ \int_{t_1}^{t_0} (G'(F(t)) - G'(F(t_1)))F'(t)(\mu_0(t) - \mu(t))dt \geq 0$$

where in the last step we used Lemma 3.1 and that G' is non-decreasing. To be precise, G' is in fact defined only almost everywhere, but the above formulas hold true if G' is understood e.g. as the left derivative, so that it is an everywhere defined non-decreasing function on $(0, +\infty)$ (and the pointwise value $G'(F(t_1))$ makes sense).

If we have equality in the latter estimate, we have

$$(G'(F(t)) - G'(F(t_1)))(\mu_0(t) - \mu(t)) = 0$$

for almost every $t < t_1$ and for almost every $t \in (t_1, t_0)$, hence for almost every $t < t_0$. Since $\mu(t) < \mu_0(t)$ if $t \in (t_1, t_0)$ we have $G'(F(t)) = G'(F(t_1))$ for almost every $t \in (t_1, t_0)$. On the other hand by Lemma 3.1 $\mu_0(t) < \mu(t)$ for $t < \overline{t}$, for some $\overline{t} < t_0$. Hence $G'(F(t)) = G'(F(t_1))$ for almost every $t < \overline{t}$. Since $G'(F(t)) - G'(F(t_1))$ is non-positive and non-decreasing for $t < t_1$ we deduce that $G'(F(t)) = G'(F(t_1))$ for every $t < t_1$. Summing up, $G'(F(t)) = G'(F(t_1))$ for almost every $t < t_0$ (in fact, for every $t < t_0$) and therefore $G(t)$ is affine on $(F(-\infty), F(t_0)) = (0, F(t_0))$, and therefore linear on $[0, F(t_0)]$, because G is continuous and $G(0) = 0$. $\square$

4. Applications, the generalized Wehrl conjecture

In [18], Wehrl conjectured that among all Glauber states, the coherent states minimize the Wehrl entropy. To be more precise, we recall the basic terminology.

We are given a locally compact group G and a unitary representation T of G on a Hilbert space $\mathcal{H}$. We fix a vector $\psi \in \mathcal{H}$ and denote by $H \subset G$ the subgroup that leaves ψ invariant by the action T on elements of H up to a unimodular factor, i.e., $T[h](\psi) = e^{i\theta_h}\psi$ for all $h \in H$. Let $X = G/H$. In many instances, the Haar measure on G induces a measure μ on X that is invariant under the action of G. Then for any coset $x \in X$ we take a representative $g(x)$ and define the state $v_x = T[g(x)](\psi)$. This vector is well-defined up to a unimodular factor that may change with the representative $g(x)$ that has been chosen. These vectors are the coherent states. For every $u \in \mathcal{H}$ and every $x \in X$

we may form $u(x) = \langle u, v_x \rangle$. In this way, we may think of $\mathcal{H}$ as a reproducing kernel Hilbert space of functions over X. We have that $\langle v_x, v_y \rangle = K(x, y)$ is the reproducing kernel for $\mathcal{H}$, i.e., for all $v \in \mathcal{H}$, $v(x) = \int_X K(x, y)v(y)\, d\mu(y)$.

Given a vector $v \in \mathcal{H}$ of norm one we define the Wehrl entropy as

$$\int_X -|v(x)|^2 \log |v(x)|^2 \, d\mu(x).$$

The conjecture is that this is minimized for the coherent states. Namely, the conjecture postulates that among the functions with unit norm the reproducing kernels are the most concentrated. Sometimes a more general conjecture is formulated, replacing the function $f(x) = x \log(1/x)$ in the Wehrl entropy definition by any other concave function $f : [0, 1] \to \mathbb{R}$.

This conjecture was originally formulated by Wehrl in [18] for Glauber states. In this original setting $\mathcal{H}$ is the Fock space of entire functions such that $\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty$, $X = \mathbb{C}$ and G is the Heisenberg group. It was proved by Lieb that the coherent states are minimizers in [11]. Later on, Carlen in [5] found a new proof that moreover confirmed that these are the unique minimizers.

In [11] Lieb extended the conjecture to the case of Bloch states. In this case, the group is $SU(2)$ and $\mathcal{H}$ is a space of holomorphic polynomials of degree up to j endowed with the Fubini–Study metric. The coherent states associated are the corresponding reproducing kernels, see Subsection 4.1 for details. Thirty-six years later, in [12] Lieb and Solovej proved that the reproducing kernels are minimizers for the Wehrl entropy. The fact that these are the only minimizers remained open. They expect that a similar result should hold for any semi-simple Lie group.

In [14] they formulated the analogous problem for the group $SU(1, 1)$ where the reproducing kernel Hilbert space is the Bergman space and X is the unit disk and proved some partial cases. The full conjecture in this case was proved in [10]. We will see now how all these cases and possibly many other instances of the Wehrl conjecture follow from our scheme. As a bonus, we will prove the uniqueness of the minimizers, thus setting the last piece of the Lieb conjecture for $SU(2)$.

4.1. Bloch coherent states

In [12] Lieb and Solovej proved the generalized Wehrl conjecture that states that the Wehrl entropy is minimized at the coherent states in the Hilbert spaces of the irreducible representations of $SU(2)$. They did not prove that the coherent states alone minimize the entropy. In [13] they extended their results to symmetric $SU(N)$ coherent states.

To define the space of functions that we will consider, we first introduce the spherical measure on $\mathbb{C}$, which corresponds to the metric inherited from the Euclidean metric restricted to the sphere of radius $\frac{1}{2\sqrt{\pi}}$ transported to $\mathbb{C}$ by the stereographical projection.

Namely, on $\mathbb{C} \ni z = x+iy$ we consider the Riemannian metric $\pi^{-1}(1+|z|^2)^{-2}(dx^2+dy^2)$ and the corresponding measure

$$dm(z) = \frac{1}{(1+|z|^2)^2}\frac{dxdy}{\pi}.$$

We will also sometimes denote the spherical measure of the set A by $|A|_M = m(A)$.

Definition 1. Let $j \in \mathbb{N}$. We define $\mathcal{P}_j$ as the finite dimensional space of polynomials:

$$z \mapsto \sum_{k=0}^{j} c_k z^k,$$

with inner product:

$$\langle f, g\rangle = (j+1)\int_{\mathbb{C}} \frac{f(z)\overline{g}(z)}{(1+|z|^2)^j}\, dm(z)$$

and reproducing kernel

$$K_j(z,w) = (1+z\overline{w})^j.$$

The functions $K_j(\cdot, w)$ are the coherent states.

For each $p > 0$ the p-(quasi-)norm of $f \in \mathcal{P}_j$ is defined as

$$\|f\|^p_{\mathcal{P}_j,p} := (pj/2+1)\int_{\mathbb{C}} \left|\frac{f(z)}{(1+|z|^2)^{j/2}}\right|^p dm(z).$$

The factor $(pj/2+1)$ is introduced to guarantee that $\|1\|_{\mathcal{P}_j,p} = 1$. Thus, the subharmonicity of $|f|^p$ and integration in polar coordinates yield $|f(0)| \le \|f\|_{\mathcal{P}_j,p}$. There is an invariance of the space $\mathcal{P}_j$ under a subgroup of the Möbius transformations that preserve the spherical metric, specifically, for any $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$ the map $T_{\alpha,\beta} f = (\beta z + \bar{\alpha})^j f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right)$ is an isometry of $\mathcal{P}_j$ for all p. This is, in fact, the unitary representation of $SU(2)$ on $\mathcal{P}_j$.

This entails that for all $f \in \mathcal{P}_j$ and all $p > 0$ we have

$$\sup_{z\in\mathbb{C}} \frac{|f(z)|}{(1+|z|^2)^{j/2}} \le \|f\|_{\mathcal{P}_j,p}. \tag{4.1}$$

Then the conjecture of Bodmann [3, Conjecture 3.5] is that

$$\|f\|_{\mathcal{P}_j,q} \le \|f\|_{\mathcal{P}_j,p} \tag{4.2}$$

when $1 \le p \le q$ and equality is achieved if and only if f is a multiple of the reproducing kernel. The generalized Wehrl conjecture in this context is that for any $f \in \mathcal{P}_j$ normalized such that $\|f\|_{\mathcal{P}_j,2} = 1$, we have

$$S_j(|f|^2) := -(j+1)\int\limits_{\mathbb{C}} \frac{|f(z)|^2}{(1+|z|^2)^j} \ln \frac{|f(z)|^2}{(1+|z|^2)^j} dm(z) \ge \frac{j}{j+1}.$$

This inequality follows from (4.2) by observing that $\frac{\partial \|f\|_p}{\partial p}(2) \le 0$ or directly from the theorem below.

Our aim is to prove the following theorem.

Theorem 4.1. *Let $G : [0,1] \to \mathbb{R}$ be a continuous convex function such that $G(0) = 0$, $j \in \mathbb{N}$, $p > 0$. Then the maximum value of*

$$\int\limits_{\mathbb{C}} G\left(\frac{|f(z)|^p}{(1+|z|^2)^{pj/2}}\right) dm(z) \tag{4.3}$$

subject to the condition that $f \in \mathcal{P}_j$ and $||f||_{\mathcal{P}_j,p} = 1$, is attained for $f(z) = (\beta z + \bar{\alpha})^j$, for any $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. If G is not linear on $[0,1]$, then these are the only maximizers.

The first part of the statement corresponds to Theorem 2.1 in [12]. The uniqueness of the maximizers is new. We observe that the expression in (4.3) is always finite, since $m(\mathbb{C}) = 1$ is finite.

Proof. Take M to be the sphere in $\mathbb{R}^3$ of radius $\frac{1}{2\sqrt{\pi}}$ with the Riemannian metric inherited from $\mathbb{R}^3$. Paul Levy's isoperimetric inequality for the sphere (see e.g. [16]) says that for any open set $A \subset M$ with smooth boundary we have

$$|\partial A|^2_{\mathcal{H}^1} \ge 4\pi |A|_M - 4\pi |A|^2_M.$$

Thus, we have (1.1) with $H(x) = 4\pi x(1-x)$. For any polynomial $f \in \mathcal{P}_j$, take $u = \log\left(\frac{|f(z)|}{(1+|z|^2)^{j/2}}\right)$, pull it back to the sphere via the stereographic projection and extend to the North Pole N by continuity $u(N) = \log(|c_j|)$, where c_j is the coefficient of z^j in $f(z)$. We will apply Theorem 1.3 (and Remark 1.4) to the function u and M.

The spherical Laplacian in stereographic coordinates is $\Delta_M = \pi(1+|z|^2)^2\Delta_e$, where Δ_e is the ordinary Euclidean Laplacian, thus

$$\Delta_M u = \pi(1+|z|^2)^2 \Delta_e \log(|f|) - \pi(1+|z|^2)^2 \Delta_e \log(1+|z|^2)^{j/2} \ge -2\pi j$$

and $F(t) = (pj/2+1)e^{pt}$. We assume that the polynomial is normalized, i.e.,

$$1 = \|f\|^p_{\mathcal{P}_j,p} = \int\limits_{\mathbb{C}} F(u(z))dm(z).$$

In order to apply Theorem 1.3, we identify μ_0. The function $\mu_0(t)$ is the solution to

$$g'(t) = 4\pi \frac{g(t) - g^2(t)}{-\pi 2jg(t)} = \frac{g(t) - 1}{j/2},$$

with the normalization

$$(pj/2 + 1) \int\limits_{-\infty}^{t_0} pe^{pt} \mu_0(t)\, dt = 1$$

and $\lim_{t \to t_0^-} \mu_0(t) = 0$. The solution is attained when $t_0 = 0$ and $\mu_0(t) = 1 - e^{2t/j}$ when $t \in (-\infty, 0)$. This is exactly

$$m\Big(\Big\{z \in \mathbb{C} : \log \frac{1}{(1 + |z|^2)^{j/2}} > t\Big\}\Big),$$

thus $f = 1$ attains the maximum. Any other coherent state $T_{\alpha,\beta}1 = (\beta z + \bar{\alpha})^j$, with $|\alpha|^2 + |\beta|^2 = 1$ has the same distribution function, thus it will also attain the maximum. Let us check that they are the only maximizers. Indeed, if $f \in \mathcal{P}_j$ is a maximizer with $\|f\|_{\mathcal{P}_j,p} = 1$ we may assume (after an application of $T_{\alpha,\beta}$) that $\sup_{z\in\mathbb{C}} \frac{|f(z)|}{(1+|z|^2)^{j/2}}$ is attained at $z = 0$ (note that it must be attained somewhere since the sphere is compact). Subharmonicity implies that $|f(0)| \le 1$ and the equality is attained only when f is constant, which can be seen by integrating in polar coordinates and applying subharmonicity to the function $|f(z)|^p$ on each circle $\{|z| = r\}$ and noting that subharmonicity is strict for large enough r unless f is constant. On the other hand by Theorem 1.3 if f is a maximizer then $|u^{-1}([t, +\infty))|_M = \mu_0(t) > 0$ if $t < 0$. Thus $\sup u = 0$, i.e., $\sup_{z\in\mathbb{C}} \frac{|f(z)|}{(1+|z|^2)^{j/2}} = 1$. □

4.2. *Glauber coherent states*

For $p > 0$, $\alpha > 0$, we consider the Bargmann–Fock space of entire functions f of one complex variable $z = x + iy$ with p-(quasi-)norm

$$\frac{p\alpha}{\pi} \int\limits_{\mathbb{C}} \left| f(z)e^{-\alpha|z|^2} \right|^p dA(z) < \infty,$$

with $dA(z) = dxdy$. For $p = 2$ we have a reproducing kernel Hilbert space and the coherent states are given by $e^{\alpha \bar{a} z - \alpha|a|^2/2}$, with $a \in \mathbb{C}$, see e.g. [19].

The result that follows from Theorem 1.3 is:

Theorem 4.2. *Let $G : [0,1] \to \mathbb{R}$ be a convex function such that $G(0) = 0$. Let $\alpha > 0$, $p > 0$. Then the supremum of the functional*

$$\int_{\mathbb{C}} G\left(\left|f(z)e^{-\alpha|z|^2/2}\right|^p\right) dA(z) \tag{4.4}$$

subject to the condition that $f \in \mathcal{H}(\mathbb{C})$ and

$$\frac{p\alpha}{\pi} \int_{\mathbb{C}} \left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z) = 1 \tag{4.5}$$

is attained at $f(z) = e^{\alpha\bar{a}z-\alpha|a|^2/2}$ for any $a \in \mathbb{C}$. If G is not linear on $[0,1]$, and this supremum is finite (i.e. $> -\infty$), then these are the only maximizers, up to a unimodular factor.

We emphasize that the above functional takes values in $[-\infty, +\infty)$ and its supremum can be finite or $-\infty$, depending on G. This theorem for a general convex function G was proved by Lieb and Solovej in [12]. The fact that the minimizers are unique for a general convex function is new, as far as we know. For the classical Wehrt entropy, the uniqueness was proved by Carlen in [5].

Proof. The operators

$$T_a f(z) = e^{\alpha\bar{a}z-\alpha|a|^2/2} f(z-a),$$

with $a \in \mathbb{C}$, are an isometry in the Fock spaces. Moreover, under the assumption (4.5), $|f(z)e^{-\alpha|z|^2/2}| \leq 1$ for all $z \in \mathbb{C}$ (see e.g. [19]). The result follows by applying Theorem 1.3 using the classical isoperimetric inequality in the plane, that is (1.1) with $H(x) = 4\pi x$ (see e.g. [16]), and taking $u = \log(|f(z)|e^{-\alpha|z|^2/2})$, hence $\Delta u = -2\alpha$, and $F(t) = \frac{p\alpha}{2\pi}e^{pt}$. Here $\mu_0(t) = -2\pi t/\alpha$ for $t \in (-\infty, 0)$; hence $t_0 = 0$. The uniqueness of the maximizers follows as in Subsection 4.1 (here the supremum $\sup_{z\in\mathbb{C}} u$ is attained because $\lim_{z\to\infty} |f(z)|e^{-\alpha|z|^2/2} = 0$, cf. [19]). $\square$

4.3. SU(1,1) coherent states

Now we consider, for $\alpha > 0$, $p > 0$, the weighted Bergman space of analytic functions f in the unit disk $\mathbb{D} \subset \mathbb{C}$, with p-(quasi-)norm

$$\int_{\mathbb{D}} (\alpha - 1)|f(z)|^p (1 - |z|^2)^\alpha \, dm(z) < \infty,$$

where

$$dm(z) = \frac{dxdy}{\pi(1-|z|^2)^2}$$

is the area element, for $z = x + iy$. When $p = 2$ we obtain a reproducing kernel Hilbert space, and the coherent states are given by $(1 - z\bar{a})^{-2\alpha/p}$, $a \in \mathbb{D}$; see [8].

The issue addressed in the previous subsections was reformulated for these spaces as a function theory problem in [14]. This problem had been considered, and some partial solutions found in [4] and [1]. Finally, the following theorem was proved in [10, Theorem 1.2 and Remark 4.3].

Theorem 4.3. *Let $G : [0,1] \to \mathbb{R}$ be a continuous convex function such that $G(0) = 0$. Let $\alpha > 1$, $p > 0$. The supremum of the functional*

$$\int_{\mathbb{D}} G\left(|f(z)|^p(1-|z|^2)^\alpha\right) dm(z) \tag{4.6}$$

subject to the condition that $f \in \mathcal{H}(\mathbb{D})$ and

$$\int_{\mathbb{D}} (\alpha-1)|f(z)|^p(1-|z|^2)^\alpha \, dm(z) = 1$$

is attained at $f(z) = \frac{(1-|a|^2)^{\alpha/p}}{(1-z\bar{a})^{2\alpha/p}}$ for any $a \in \mathbb{D}$. If G is not linear on $[0,1]$, and this supremum is finite (i.e. $> -\infty$), then these are the only maximizers, up to a unimodular factor.

Proof. Again this is now a corollary of Theorem 1.3 where the manifold is $\mathbb{D}$ endowed with the hyperbolic metric, the function $u(z) = \log(|f(z)|(1-|z|^2)^{\alpha/p})$, the function $F(t) = (\alpha-1)\exp(pt)$ and we have the isoperimetric inequality (1.1) in the hyperbolic space with $H(x) = 4\pi(x+x^2)$ (see e.g. [16]). The Laplace–Beltrami operator now is given by $\Delta_{\mathbb{D}} = \pi(1+|z|^2)^2\Delta_e$, where Δ_e is the Euclidean Laplace operator. A straitforward computation shows that

$$\Delta_{\mathbb{D}} u \geq -4\pi\alpha/p,$$

which yields $\mu_0(t) = e^{-pt/\alpha} - 1$ for $t \in (-\infty, 0)$; hence $t_0 = 0$. The uniqueness of the maximizers follows as in Subsection 4.1. □

5. Local estimates: Faber–Krahn inequalities

In this section we prove a local counterpart of the estimate in Theorem 1.3. A similar result had first appeared in the special case of Glauber coherent states in [15]. The following theorem provides a far-reaching generalization of that result, and the proof is even simpler.

Theorem 5.1. *Under the same assumption and notation of Theorem 1.3, suppose in addition that $G(x) > 0$ for $x > 0$. Then for every set $\Omega \subset M$ and every u as in Theorem 1.3 we have*

$$\int_{\Omega} G(F(u(p)))\, d\mathrm{Vol}(p) \leq \int_{0}^{|\Omega|_M} G(F(\mu_0^{-1}(s)))ds. \tag{5.1}$$

Moreover equality in (5.1) is possible for some u as above and Ω with $|\Omega|_M > 0$ if and only if $|u^{-1}([t, +\infty))|_M = \mu_0(t)$ for all $t < t_0$ and $\Omega = u^{-1}([t, +\infty))$, with $t = \mu_0^{-1}(|\Omega|_M)$ (up to null sets) if $|\Omega|_M < |M|_M$, or $\Omega = M$ if $|\Omega|_M = |M|_M$.

Observe that the additional assumption $G(x) > 0$ for $x > 0$ implies that $G : [0, +\infty) \to \mathbb{R}$ is strictly increasing. Also, we have the characterization of the maximizers without any further assumption on G (such as non-linearity).

Proof. For u as in the statement, let $\mu(t) = \mu_u(t) = |u^{-1}([t, +\infty))|_M$, $t \in \mathbb{R}$, be its distribution function and $u^*(s) = \inf\{t : \mu(t) < s\}$, for $0 \leq s < |M|_M$, its non-increasing rearrangement. If $|\Omega|_M < |M|_M$, let $\tilde{\Omega} \subset M$ be any subset with $|\tilde{\Omega}|_M = |\Omega|_M$ and $u^{-1}((t, +\infty)) \subset \tilde{\Omega} \subset u^{-1}([t, +\infty))$, with $t = u^*(|\Omega|_M)$ (up to null sets). If $|\Omega|_M = |M|_M$, let $\tilde{\Omega} = M$. Then it is easy to check that

$$\int_{\Omega} G(F(u(p)))\, d\mathrm{Vol}(p) \leq \int_{\tilde{\Omega}} G(F(u(p)))\, d\mathrm{Vol}(p) = \int_{0}^{|\Omega|_M} G(F(u^*(s)))ds \tag{5.2}$$

where the equality follows from the fact that u and u^* are equi-measurable and the Fubini theorem. Hence we are going to prove that

$$\int_{0}^{s} G(F(u^*(\tau)))d\tau \leq \int_{0}^{s} G(F(\mu_0^{-1}(\tau)))d\tau \tag{5.3}$$

for $0 \leq s \leq |M|_M$. This is clear if $\mu(t) = \mu_0(t)$ for $t < t_0$. Suppose then that $\mu \neq \mu_0$ at some point. Consider the function

$$\varphi(s) := \int_{0}^{s} G(F(\mu_0^{-1}(\tau)))d\tau - \int_{0}^{s} G(F(u^*(\tau)))d\tau$$

for $0 \leq s < |M|_M$. Clearly φ is continuous and $\varphi(0) = 0$. As a consequence of Lemma 3.1, φ is strictly increasing on $[0, \mu_0(t_1)]$; indeed, for $t_1 < t < t_0$ we have $\mu(t) < \mu_0(t)$ which implies that $\mu_0^{-1}(s) > u^*(s)$ for $0 < s < \mu_0(t_1)$. Similarly, on $[\mu_0(t_1), |M|_M)$ φ is non-increasing (in fact strictly decreasing for $s < |M|_M$ large enough). Finally,

$$\lim_{s\to|M|_M^-}\varphi(s)=\int_0^{|M|_M}G(F(\mu_0^{-1}(\tau)))d\tau-\int_0^{|M|_M}G(F(u^*(\tau)))ds$$
$$=\int_0^{t_0}G'(F(t))F'(t)\mu_0(t)dt-\int_M G(F(u(p)))\,d\mathrm{Vol}(p)>0$$

where the latter inequality follows from Theorem 1.3. As a consequence, $\varphi(s)>0$ for $0<s<|M|_M$. This concludes the proof of (5.1).

The characterization of the cases of equality also follows from the above discussion. The claim about Ω follows from (5.2), since in that case equally occurs in (5.2) and the level sets $u^{-1}(\{t\})$ have zero measure, because $\mu_0(t)$ is continuous (if $|\Omega|_M=|M|_M$, hence $\tilde{\Omega}=\Omega$, we also use the fact that $G(F(u(p)))$ is continuous and strictly positive on M). $\square$

We now specialize the above result to the three geometries (spherical, Euclidean, hyperbolic). We begin with the spherical case and we use the notation of Subsection 4.1; in particular, for $z=x+iy\in\mathbb{C}$, $dm(z)=\frac{1}{(1+|z|^2)^2}\frac{dxdy}{\pi}$. Recall that $m(\mathbb{C})=1$.

Theorem 5.2. *Let $G:[0,1]\to\mathbb{R}$ be a continuous convex function such that $G(0)=0$ and $G(x)>0$ for $x>0$. Let $j\in\mathbb{N}$, $p>0$. Then for every $f\in\mathcal{P}_j$ with $||f||_{\mathcal{P}_j,p}=1$ and $\Omega\subset\mathbb{C}$,*

$$\int_\Omega G\left(\frac{|f(z)|^p}{(1+|z|^2)^{pj/2}}\right)dm(z)\le\int_0^{m(\Omega)}G((1-s)^{pj/2})ds. \tag{5.4}$$

Equality occurs in (5.4) for some f and Ω with $m(\Omega)>0$ if and only if $f(z)=(\beta z+\bar{\alpha})^j$ for some $\alpha,\beta\in\mathbb{C}$, $|\alpha|^2+|\beta|^2=1$ and Ω is (up to a null set) a superlevel set of $|f(z)|/(1+|z|^2)^{j/2}$ (which is a disk in $\mathbb{C}$) if $m(\Omega)<1$, or $\Omega=\mathbb{C}$ if $m(\Omega)=1$.

Proof. We apply Theorem 5.1, by arguing as in the proof of Theorem 4.1. In particular we have $\mu_0(t)=1-e^{2t/j}$ for $t\in(-\infty,0)$, hence if $u(z):=\log(|f(z)|/(1+|z|^2)^{j/2})$ is an extremal function, its maximum value is $t_0=0$, and the maximum of $|f(z)|/(1+|z|^2)^{j/2}$ is 1. This gives the desired extremal functions. $\square$

Similarly we obtain the following results for Glauber and $SU(1,1)$ coherent states, that we state without proof.

The following result generalizes [15, Theorem 3.1], which corresponds to the special case $p=2,\alpha=\pi$ and $G(x)=x$. Here $\mathbb{C}$ is endowed with the Lebesgue measure $dA(z)=dxdy$, with $z=x+iy$. We also write $|\Omega|=A(\Omega)$.

Theorem 5.3. *Let $G:[0,1]\to\mathbb{R}$ be a continuous convex function such that $G(0)=0$ and $G(x)>0$ for $x>0$. Let $\alpha>0$, $p>0$. Then for every $f\in\mathcal{H}(\mathbb{C})$ satisfying*

$$\frac{p\alpha}{\pi}\int_{\mathbb{C}} \left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z) = 1$$

and $\Omega \subset \mathbb{C}$, *we have*

$$\int_{\Omega} G\left(\left|f(z)e^{-\alpha|z|^2/2}\right|^p\right) dA(z) \leq \int_0^{|\Omega|} G\left(e^{-p\alpha s/(2\pi)}\right) ds. \tag{5.5}$$

Equality occurs in (5.5) *for some* f *and* Ω *with* $|\Omega| > 0$ *if and only if* $f(z) = e^{\alpha\bar{a}z-\alpha|a|^2/2}$, *for some* $a \in \mathbb{C}$, *up a unimodular factor, and* Ω *is (up to a null set) a superlevel set of* $|f(z)|e^{-\alpha|z|^2/2}$ *(which is a disk in* $\mathbb{C}$*) if* $|\Omega| < \infty$, *or* $\Omega = \mathbb{C}$ *if* $|\Omega| = \infty$.

Finally, the following result generalizes [17, Theorem 3.1], which corresponds to the particular case $p = 2$, $G(x) = x$. Here the unit disk $\mathbb{D}$ is endowed with the measure $dm(z) = \frac{1}{(1-|z|^2)^2}\frac{dxdy}{\pi}$, for $z = x + iy \in \mathbb{D}$.

Theorem 5.4. *Let* $G : [0, 1] \to \mathbb{R}$ *be a continuous convex function such that* $G(0) = 0$ *and* $G(x) > 0$ *for* $x > 0$. *Let* $\alpha > 1$, $p > 0$. *Then for every* $f \in \mathcal{H}(\mathbb{D})$ *satisfying*

$$\int_{\mathbb{D}} (\alpha-1)|f(z)|^p(1-|z|^2)^\alpha \, dm(z) = 1$$

and $\Omega \subset \mathbb{C}$, *we have*

$$\int_{\mathbb{D}} G\left(|f(z)|^p(1-|z|^2)^\alpha\right) dm(z) \leq \int_0^{m(\Omega)} G((1+s)^{-\alpha})ds. \tag{5.6}$$

Equality occurs in (5.6) *for some* f *and* Ω *with* $m(\Omega) > 0$ *if and only if* $f(z) = \frac{(1-|a|^2)^{\alpha/p}}{(1-z\bar{a})^{2\alpha/p}}$ *for some* $a \in \mathbb{D}$, *up to a unimodular factor, and* Ω *is (up to a null set) a superlevel set of* $|f(z)|(1+|z|^2)^{\alpha/p}$ *(which is a disk in* $\mathbb{D}$*) if* $m(\Omega) < \infty$, *or* $\Omega = \mathbb{D}$ *if* $m(\Omega) = \infty$.

6. Appendix, proof of Theorem 1.2

First we establish the Theorem in the case when u is a Morse function (we may assume that u is not constant, so that $\mu(t) > 0$ for every $t < t_0$). In this case, by Theorem 1.1, u is locally absolutely continuous on $(-\infty, t_0)$ and satisfies the differential inequality (1.2) for a.e. $t < t_0$. Now consider an arbitrary $t_2 < t_0$, and let $g(t)$ be the solution of the (backward) Cauchy problem

$$g(t_2) = \mu(t_2), \qquad g'(t) = -\frac{H(g(t))}{cg(t)}, \quad t \leq t_2, \tag{6.1}$$

whose existence on some interval $(a, t_2]$ is guaranteed by the smoothness of $H > 0$ and the fact that $\mu(t_2) > 0$. On this interval, combining (1.2) and (6.1), by a standard comparison theorem for ODEs (see e.g. [2, Chapter 1]) we obtain that $g(t) \leq \mu(t)$ and, since μ is locally bounded due to the assumption that the level sets $\{u \geq t\}$ are compact, this prevents blow up in finite time for $g(t)$. Therefore, the existence of the solution $g(t)$ (together with the bound $g(t) \leq \mu(t)$) propagates $t \leq a$, and eventually one obtains (1.4) as claimed.

Now let u be as in Theorem 1.2. Since Morse functions are dense in the strong C^2 topology on M (see e.g. [9, Chapter 6, Theorem 1.2]), we can pick a sequence u_n of Morse functions such that $|u - u_n| < \frac{1}{n}$, $|\Delta u - \Delta u_n| < \frac{1}{n}$.

Let $A_n(t) = u_n^{-1}([t, +\infty))$, $\mu_n(t) = |A_n(t)|_M$ and $A(t) = u^{-1}([t, +\infty))$. Fix numbers $t_1 < t_2 < t_0$. Note that for all $t < t_0 - \frac{1}{n}$ we have $A_n(t + \frac{1}{n}) \subset A(t) \subset A_n(t - \frac{1}{n})$. Applying Morse version of the theorem to the function u_n we get for big enough n

$$D_n\left(t_1 + \frac{1}{n}, t_2 - \frac{1}{n}, \mu_n\left(t_2 - \frac{1}{n}\right)\right) \leq \mu_n\left(t_1 + \frac{1}{n}\right),$$

where $D_n(t_3, t_4, \mu)$ is the solution to the differential equation

$$g'(t) = -\frac{H(g(t))}{(c + \frac{1}{n})g(t)}$$

at t_3 with initial condition $g(t_4) = \mu$. Number n should be so big that $\min(c, t_0 - t_2, t_2 - t_1) > \frac{2}{n}$.

By the inclusions for the sets $A_n(t)$, $A(t)$ we have $\mu_n(t_2 - \frac{1}{n}) \geq \mu(t_2)$ and $\mu_n(t_1 + \frac{1}{n}) \leq \mu(t_1)$. Applying continuity to the solution of the differential equation on the parameters and the initial conditions and the fact that $D(t_3, t_4, \mu) \leq D(t_3, t_4, \nu)$ if $\mu \leq \nu$ we get

$$D(t_1, t_2, \mu(t_2)) \leq \mu(t_1),$$

as required.

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