Robust Adaptive Backstepping Control Over Native Spaces

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Giorgio A. Orlando, Andrea L'Afflitto, Andrew J. Kurdila

Abstract— **This paper presents the first robust adaptive backstepping control system for dynamical systems whose functional uncertainties are assumed to lie in a user-defined native space (also known as reproducing kernel Hilbert space). This work generalizes classical adaptive backstepping control systems and their non-adaptive counterparts by freeing the user from providing a parametric representation of the functional uncertainties. Such representations are usually in the form of regressor vectors, or some equivalent structure, to be provided *a priori* or reconstructed online, and without employing conservative upper bounds on the functional uncertainties. The adaptive laws for the proposed control system are shown to form a distributed parameter system (DPS) evolving over the native space. Finite-dimensional approximations of such adaptive laws enable their applications to problems of practical interest, as shown by the proposed numerical examples.**

Index Terms— **Native spaces, robust adaptive control, backstepping control, distributed parameter systems**

I. INTRODUCTION

This paper presents the first robust adaptive backstepping system for time-varying plants, whose dynamics are affected by parametric and nonparametric uncertainties. Similar to existing works on adaptive backstepping, parametric uncertainties are characterized *a priori* by a regressor vector. Nonparametric uncertainties are unmodeled and assumed to lie in a *reproducing kernel Hilbert space* (RKHS – also known as *native space*); to the authors' knowledge, this is the first work in the adaptive backstepping literature to account for nonparametric uncertainties modeled as elements of an RKHS.

Classical results on adaptive backstepping control parameterize uncertainties using a regressor vector provided *a priori* or reconstructed online. Hence, these methods implicitly assume that uncertainties lie in a finite-dimensional space or are best approximated by such a space. The use of native spaces to capture nonparametric uncertainties significantly expands the class of admissible uncertainties and reduces the user's burden of finding a parameterization of uncertainties. Furthermore, the proposed approach allows the user to increase the number of centers in those regions of the space of uncertainties, where additional care is needed in parameterizing uncertainties. Parametric methods characterize uncertainties uniformly in the chosen space of nonlinearities. However, the proposed work is not intended to be an alternative to classical methods, but to augment those methods. Indeed, the proposed approach allows for parametric uncertainties as well. An advantage of using an approach based on native spaces is the array of tools to compute ultimate bounds on the trajectory tracking error as explicit functions of the assumed smoothness of the functional uncertainties and the number of kernel centers employed [1, Ch. 3]. Alternative methods, such as those based on neural networks, provide bounds on the trajectory tracking error irrespective of these properties and, hence, in general, larger ultimate bounds.

As for the classical backstepping control architecture, we consider plant models in a cascaded form, that is, plant models that can be partitioned into two components. As in classical backstepping control, we assume the existence of a feedback control law and a robust control Lyapunov function (RCLF) such that if the state of the second partition of the plant model could be used as a control input for the first partition, then the state of the first partition of the plant model would reach an ultimate bound deemed satisfactory by the user. Hence, we design an adaptive control law for the second partition of the plant model that guarantees boundedness of all time-varying variables and enforces the ideal ultimate bounds on the trajectory tracking error for the first part of the plant model.

The authors pioneered the design of robust and adaptive control systems for nonlinear plant models affected by nonparametric uncertainties; examples of these works are provided in [1]–[4] to name a few. Before this series of efforts, robust adaptive control systems have been developed under the assumption of parametric uncertainties, that is, assuming a parameterization of uncertainties provided *a priori* or reconstructed online; for details, see [5], [6] and the numerous references therein. Worthy of mention is that in the literature on nonparametric control, this work is the first in which the nonparametrtic uncertainties are evaluated on any continuous signal, and not exclusively along the plant's trajectory.

In [2], the authors generalized the classical backstepping control architecture and proposed a robust, non-adaptive, backstepping control system for time-invariant plants affected by parametric and nonparametric uncertainties. The results presented in this paper significantly advance those in [2] for its adaptive features and the ability to steer also time-varying plant models. Adaptivitiy of the proposed control architecture allows countering perfectly the uncertainties lying in native spaces defined by the span of kernel functions at user-defined centers. Robustness of the proposed control

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architecture allows countering any other functional, possibly infinite-dimensional, uncertainty. The results in [2] are robust to functional uncertainties on native spaces but do not have any adaptive mechanism. The control system presented in this paper can be specialized to the non-adaptive backstepping controller for time-invariant systems for plants with non-parametric uncertainties shown in [2]. The proposed results can be also specialized to the classical backstepping architecture for plants with parametric uncertainties presented in [5, Ch. 2], and, hence, provide extensions of existing control solutions for cascaded systems.

For its use of native spaces to characterize the functional uncertainties, the proposed adaptive backstepping technique may be classified within the broad category of data-driven methods, for which RKHS theory tools have proven profitable. Within this context, worthy of mention are recent works leveraging Gaussian processes for adaptive backstepping, such as [7], [8], which ensure global uniform ultimate boundedness of the trajectory tracking error through offline and event-triggered online learning, respectively. Recently, a variation of classical backstepping control, namely incremental backstepping, and methods based on Gaussian processes have been employed in advanced applications, such as the design of fault-tolerant flight control systems [9]. It is important to remark on how the proposed results do not assume any stochastic disturbance and do not leverage tools from probability theory to guarantee uniform ultimate boundedness of the trajectory tracking error. As argued in [1, Ch. 7], such tools usually provide more conservative and more complicated results than those attained by applying the proposed framework.

The proposed work concerns plant models partitioned into two subsystems. The state of the second subsystem can be considered as a control input to the first subsystem. The state of the first subsystem does not affect the dynamics of the second one. We assume that a reference trajectory for the state of the second subsystem is provided; this reference trajectory serves as an ideal feedback control law for the first subsystem. In this regard, the proposed results contribute to the general area of backstepping control. A typical challenge in parametric backstepping control systems, whose uncertain parameters are estimated or assigned an adaptive gain, lies in the fact that the size of the parameters to be estimated grows at every step. The proposed results are not affected by this challenge.

The applicability of the results presented in this paper is demonstrated through numerical simulations concerning the inner loop dynamics of a fixed-wing aircraft. The proposed robust adaptive control system is compared to the robust control system over native spaces proposed by the authors in [2, Th. 5.2] and a robust backstepping controller derived by extending [5, Lemma 2.28] to time-varying plants with vector control inputs and parametric uncertainties. Although it is well-known that aerodynamic forces and moments are functions of the aircraft's velocity relative to the wind, aerodynamic forces are usually complex to model. Existing analytical models apply primarily under the classical thin-body assumption, which may not hold for large and rapid rotations, and analytical blunt body models are generally less accurate. In this context, native spaces defined over the space of velocities relieve us from the burden of defining an analytical or numerical model of the aerodynamic forces to design robust control systems or allow us to capture effects not well described by parametric approximations. The proposed control system not only allows for improved tracking performance, but requires a smaller control effort and counters disturbances that are overly strong for the other two controllers while keeping the same tunable parameters and devoting comparable control efforts. Computer codes generating these numerical simulations are available at [10]. The proposed results are also summarized at [11].

II. Essential Notation

The set of n-dimensional real-valued vectors is denoted by $\mathbb{R}^n$. The closure of the set $\mathcal{C}$ is denoted by $\overline{\mathcal{C}}$. The boundary of $\mathcal{C}$ is denoted by $\partial\mathcal{C}$, and its interior is denoted by $\mathring{\mathcal{C}}$. Generic vector-valued Banach spaces are denoted by boldface letters, such as $\mathbb{Y}$; we use the same notation to denote a Banach space and the underlying set of vectors. Vector-valued RKHSs taking values from $\mathbb{Y}$ to $\mathbb{U}$ are denoted by $\boldsymbol{\mathcal{H}}(\mathbb{Y},\mathbb{U})$. Scalar-valued RKHSs are denoted by $\mathcal{H}(\mathbb{Y},\mathbb{U})$. A generic inner product over $\boldsymbol{\mathcal{H}}$ is denoted by $\langle\cdot,\cdot\rangle_{\boldsymbol{\mathcal{H}}}$, and the induced norm is denoted by $\|\cdot\|_{\boldsymbol{\mathcal{H}}}$. The inner product over $\mathbb{R}^{n\times m}$ is denoted by $\langle\cdot,\cdot\rangle_{\rm tr}$ and is such that $\langle A,B\rangle_{\rm tr} = {\rm tr}\left(A^{\rm T}B\right)$, where ${\rm tr}(\cdot)$ denotes the *trace* of its argument; recall that $\langle A,A\rangle_{\rm tr} = \|A\|_{\rm F}^2$, where $\|\cdot\|_{\rm F}$ denote the *Frobenius norm*. We denote by $\mathcal{L}(\mathbb{X},\mathbb{Y})$ the *normed linear space of bounded linear operators from* $\mathbb{X}$ *to* $\mathbb{Y}$ equipped with the equi-induced operator norm. For brevity, $\mathcal{L}(\mathbb{X})$ stands for $\mathcal{L}(\mathbb{X},\mathbb{X})$.

III. Problem Statement

Consider the nonlinear time-varying dynamical system

$$\dot{x}(t) = a(t,x(t)) + b(t,x(t))\xi(t) + E^Y_{y(t)}f, \qquad x(t_0)=x_0, \quad t\geq t_0, \tag{1a}$$

$$\dot{\xi}(t) = A\xi(t) + \Lambda\left(u(t) + \Theta^{\rm T}\Phi(t,\xi(t)) + E^Z_{z(t)}g\right), \qquad \xi(t_0)=\xi_0, \tag{1b}$$

where $x:[t_0,\infty)\to\Omega\subseteq\mathbb{X}$, $\mathbb{X}\triangleq\mathbb{R}^n$, $\xi:[t_0,\infty)\to\mathbb{R}^m$, $\left[x^{\rm T}(t),\xi^{\rm T}(t)\right]^{\rm T}$ denotes the *plant state*, $u:[t_0,\infty)\to\mathbb{U}\subseteq\mathbb{R}^m$ denotes the *control input*, $a:[t_0,\infty)\times\Omega\to\mathbb{R}^n$ and $b:[t_0,\infty)\times\Omega\to\mathbb{R}^{n\times m}$ are continuous and bounded in the first argument, Liptschitz continuous in the second argument, $A\in\mathbb{R}^{m\times m}$ is unknown, $\Lambda\in\mathbb{R}^{m\times m}$ is diagonal, positive-definite, and unknown, and $\Theta\in\mathbb{R}^{N\times m}$ is unknown.

The terms $f\in\boldsymbol{\mathcal{H}}_Y(\mathbb{Y},\mathbb{R}^n)$ and $g\in\boldsymbol{\mathcal{H}}_Z(\mathbb{Z},\mathbb{U})$, where $\mathbb{Y}\subseteq\mathbb{R}^{n_y}$ and $\mathbb{Z}\subseteq\mathbb{R}^{n_z}$ are compact sets, denote *functional nonparametric uncertainties*, that is, uncertainties not explicitly captured by any parametrization. It is important to note that neither $\mathbb{Y}$ nor $\mathbb{Z}$ must be subsets of the state space $\mathbb{X}$ and how both $E^Y_{y(t)}f$ and $E^Z_{z(t)}g$ must be considered as unknown time-varying terms through $y(\cdot)$ and $z(\cdot)$. The

RKHSs $\boldsymbol{\mathcal{H}}_Y(\mathbb{Y},\mathbb{R}^n)$ and $\boldsymbol{\mathcal{H}}_Z(\mathbb{Z},\mathbb{U})$ are induced by the operator kernels $\mathcal{K}^Y : \mathbb{Y}\times\mathbb{Y}\to\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{K}^Z : \mathbb{Z}\times\mathbb{Z}\to\mathcal{L}(\mathbb{U})$, respectively. These RKHSs can be constructed as restrictions of $\boldsymbol{\mathcal{H}}_Y(\mathbb{R}^{n_\mathcal{Y}},\mathbb{R}^n)$ and $\boldsymbol{\mathcal{H}}_Z(\mathbb{R}^{n_\mathcal{Z}},\mathbb{U})$, respectively [1, Ch. 3]. If it is clear from the context, we write $\boldsymbol{\mathcal{H}}_Y$ and $\boldsymbol{\mathcal{H}}_Z$ for $\boldsymbol{\mathcal{H}}_Y(\mathbb{Y},\mathbb{R}^n)$ and $\boldsymbol{\mathcal{H}}_Z(\mathbb{Z},\mathbb{U})$, respectively. These functional uncertainties affect the plant dynamics through the bounded linear *evaluation operators* $E^Y_y : \boldsymbol{\mathcal{H}}_Y\to\mathbb{R}^n$ and $E^Z_z : \boldsymbol{\mathcal{H}}_Z\to\mathbb{R}^m$ [1, Def. 3.13]. The functions $y : [t_0,\infty)\to\mathbb{Y}$ and $z : [t_0,\infty)\to\mathbb{Z}$ are continuous and observable. The *regressor vector* $\Phi : [t_0,\infty)\times\mathbb{R}^m\to\mathbb{R}^N$ is continuous and bounded in the first argument, Liptschitz continuous in the second argument, and is known. Thus, the term $\Theta^{\mathrm{T}}\Phi(t,\xi)$ captures matched parametric uncertainties for all $(t,\xi)\in[t_0,\infty)\times\mathbb{R}^m$. Note that (1b) is in the form of plants controllable by nonparametric model reference adaptive control (MRAC) systems [1, Ch. 5]. Considering the cascaded nature of the plant model, the proposed approach will generalize the classical non-parametric adaptive backstepping framework [5, Ch. 2].

Consistent with the literature on classical backstepping control, we assume that there exists a continuously differentiable feedback law $\alpha : [t_0,\infty)\times\Omega\to\mathbb{R}^m$ such that if $\xi(t)=\alpha(t,x(t))$ for all $t\in[t_0,\infty)$, then $x(\cdot)$ is (globally) uniformly ultimately bounded or (globally) asymptotically convergent to zero; techniques for constructing such adaptive laws are discussed in [1, Ch. 5]. In this paper, our objective is to define a feedback control law $\pi(\cdot)$ for $u(\cdot)$ such that all signals are (globally) uniformly bounded, and both $x(\cdot)$ and $s(\cdot,x(\cdot),\xi(\cdot))$ are (globally) uniformly ultimately bounded, where

$$s(t,x,\xi)\triangleq\xi-\alpha(t,x),\quad \text{for all } (t,x,\xi)\in[t_0,\infty)\times\Omega\times\mathbb{R}^m; \tag{2}$$

global boundedness is proven only in the case $\Omega=\mathbb{X}\triangleq\mathbb{R}^n$.

Remark 3.1: Although the backstepping control problem in the presence of functional uncertainties has been addressed by the authors in [1, Ch. 5], to the authors' knowledge, this is the first paper presenting an *adaptive* backstepping control system in the presence of uncertainties that are assumed to be generic elements of a native space.

IV. Adaptive Backstepping: A DPS Formulation

In this paper, we assume that the native spaces of the functional uncertainties f and g are given by the Cartesian product of scalar-valued native spaces, that is, $\boldsymbol{\mathcal{H}}_Y(\mathbb{Y},\mathbb{R}^n)=\mathcal{H}^n_Y(\mathbb{Y},\mathbb{R})$ and $\boldsymbol{\mathcal{H}}_Z(\mathbb{Z},\mathbb{U})=\mathcal{H}^m_Z(\mathbb{Z},\mathbb{R})$. Furthermore, we assume that the kernels $\mathfrak{K}_Y(\cdot,\cdot)$ and $\mathfrak{K}_Z(\cdot,\cdot)$ underlying $\mathcal{H}^n_Y(\mathbb{Y},\mathbb{R})$ and $\mathcal{H}^m_Z(\mathbb{Z},\mathbb{R})$, respectively, are of the Mercer kind [1, Def. 3.6] and uniformly bounded, that is, there exist $\overline{\mathfrak{K}}_Y,\overline{\mathfrak{K}}_Z>0$ such that $\mathfrak{K}_Y(y_1,y_2)\le\overline{\mathfrak{K}}^2_Y$ for all $(y_1,y_2)\in\mathbb{Y}$, and $\mathfrak{K}_Z(z_1,z_2)\le\overline{\mathfrak{K}}^2_Z$ for all $(z_1,z_2)\in\mathbb{Z}$. Finally, we assume that both f and g are bounded, that is,

$$f\in\overline{\mathcal{C}}_{R_Y}\triangleq\{f\in\boldsymbol{\mathcal{H}}_Y:\|f\|_{\boldsymbol{\mathcal{H}}_Y}\le R_Y\}, \tag{3a}$$

$$g\in\overline{\mathcal{C}}_{R_Z}\triangleq\{g\in\boldsymbol{\mathcal{H}}_Z:\|g\|_{\boldsymbol{\mathcal{H}}_Z}\le R_Z\}, \tag{3b}$$

and $R_Y,R_Z>0$. This boundedness assumption is reasonable since, in practice, no control system can effectively counter an infinitely large disturbance.

Following the classical backstepping approach, and considering $\xi(\cdot)$ as a control input for (1a), we assume that there exist a continuously differentiable control law $\alpha : [t_0,\infty)\times\Omega\to\mathbb{R}^m$ for $\xi(\cdot)$ and an RCLF $V_1 : [t_0,\infty)\times\Omega\to\mathbb{R}$ such that

$$W_1(x)\le V_1(t,x)\le W_2(x),\quad \text{for all } (t,x,f)\in[t_0,\infty)\times\Omega\times\boldsymbol{\mathcal{H}}_Y, \tag{4a}$$

$$\frac{\partial V_1(t,x)}{\partial t}+\left\langle\left(\frac{\partial V_1(t,x)}{\partial x}\right)^{\mathrm{T}},\gamma(t,x,\alpha(t,x),f)\right\rangle_{\mathbb{R}^n}\le -W(x)+k, \tag{4b}$$

for some $k>0$, where $W_1,W_2,W:\Omega\to\mathbb{R}$ are positive-definite, and

$$\gamma(t,x,\alpha(t,x),f)\triangleq a(t,x)+b(t,x)\alpha(t,x)+E^Y_{y(t)}f; \tag{5}$$

if $\Omega=\mathbb{R}^n$, then we assume that both $W_1(\cdot)$ and $,W_2(\cdot)$ are radially unbounded. Such a virtual control law can be found by proceeding, for instance, as in [1, Ch. 5].

We propose the *control law* for $u(\cdot)$ in (1b) given by

$$\begin{aligned}&\pi(t,x,\xi,\hat f,\hat g,\hat\Theta_{\mathrm{aug}})\\&\quad=-Ks(t,x,\xi)-\hat\Theta^{\mathrm{T}}_{\mathrm{aug}}\Phi_{\mathrm{aug}}(t,x,\xi,\hat f)-E^Z_{z(t)}\hat g,\\&\quad\text{for all } (t,x,\xi,\hat f,\hat g,\hat\Theta_{\mathrm{aug}})\\&\quad\in[t_0,\infty)\times\Omega\times\mathbb{R}^m\times\boldsymbol{\mathcal{H}}_Y\times\boldsymbol{\mathcal{H}}_Z\times\mathbb{R}^{(N+2m)\times m},\end{aligned} \tag{6}$$

where $K\in\mathbb{R}^{n\times n}$ denotes a user-defined, diagonal, positive-definite *gain matrix*, $\Phi_{\mathrm{aug}}(t,x,\xi,\hat f)\triangleq\left[\Phi^{\mathrm{T}}(t,\xi),\xi^{\mathrm{T}},h^{\mathrm{T}}(t,x,\xi,\hat f)\right]^{\mathrm{T}}\in\mathbb{R}^{N+2m}$, $\hat\Theta_{\mathrm{aug}} : [t_0,\infty)\to\mathbb{R}^{(N+2m)\times m}$ denotes the *parametric adaptive gain matrix*, $\hat f : [t_0,\infty)\to\boldsymbol{\mathcal{H}}_Y$ and $\hat g : [t_0,\infty)\to\boldsymbol{\mathcal{H}}_Z$ denote the *nonparametric adaptive gains*, and

$$\begin{aligned}h(t,x,\xi,\hat f)\triangleq&-\frac{\partial\alpha(t,x)}{\partial x}\left(a(t,x)+b(t,x)\xi+E^Y_{y(t)}\hat f\right)\\&-\frac{\partial\alpha(t,x)}{\partial t}+\left(\frac{\partial V_1(t,x)}{\partial x}b(t,x)\right)^{\mathrm{T}},\\&\text{for all } (t,x,\xi,\hat f)\in[t_0,\infty)\times\Omega\times\mathbb{R}^m\times\boldsymbol{\mathcal{H}}_Y;\end{aligned} \tag{7}$$

note that the uncertain term $A\xi$ in (1b) is accounted for by the augmented regressor vector $\Phi_{\mathrm{aug}}(\cdot,\cdot,\cdot,\cdot)$.

The adaptive gains verify the *adaptive laws*

$$\frac{\partial\hat{f}(t,\cdot)}{\partial t}=\Gamma_f\left(-\mathcal{K}^{Y}_{y(t)}(\cdot)\frac{\partial\alpha^{\mathrm{T}}(t,x(t))}{\partial x}s(t)-\sigma_{\tilde{f}}\hat{f}(t,\cdot)\right),\quad \hat{f}(t_0,\cdot)=\hat{f}_0(\cdot),\quad t\geq t_0,\tag{8a}$$

$$\frac{\partial\hat{g}(t,\cdot)}{\partial t}=\Gamma_g\left(\mathcal{K}^{Z}_{z(t)}(\cdot)s(t)-\sigma_{\tilde{g}}\hat{g}(t,\cdot)\right),\quad \hat{g}(t_0,\cdot)=\hat{g}_0(\cdot),\tag{8b}$$

$$\dot{\hat{\Theta}}_{\mathrm{aug}}(t)=\Gamma_\Theta\left(\Phi_{\mathrm{aug}}\left(t,x(t),\xi(t),\hat{f}(t,y(t)\right)s^{\mathrm{T}}(t)-\sigma_{\tilde{\Theta}}\hat{\Theta}_{\mathrm{aug}}(t)\right),\quad \hat{\Theta}_{\mathrm{aug}}(t_0)=\hat{\Theta}_{\mathrm{aug},0},\tag{8c}$$

where $\Gamma_f\in\mathbb{R}^{n\times n}$, $\Gamma_g\in\mathbb{R}^{m\times m}$, and $\Gamma_\Theta\in\mathbb{R}^{(N+2m)\times(N+2m)}$ denote user-defined, symmetric, and positive-definite *adaptive rate matrices*, $\mathcal{K}^Y_y(\cdot)\triangleq\mathcal{K}(y,\cdot)$ for all $y\in\mathbb{R}^n$ and $\mathcal{K}^Z_z(\cdot)\triangleq\mathcal{K}(z,\cdot)$ for all $z\in\mathbb{R}^m$ denote the *kernel sections* at y and z, respectively, and $\sigma_{\tilde{f}},\sigma_{\tilde{g}},\sigma_{\tilde{\Theta}}>0$ are arbitrarily small.

Distributed parameter systems (DPSs) [1, pp. 128-132], such as the adaptive laws (8) underlying the proposed results, are not realizable in practice. Finite-dimensional implementations of (8) can be obtained by proceeding as in [1, Th. 5.7]; although this step is omitted for brevity, its practical implementation is shown in [10], [11]. A relevant problem for future investigation is the propagation of errors induced by finite-dimensional approximations of (8); an approach to this problem can be the one proposed in [12, Sec. 11.4].

Theorem 4.1: Consider the plant model (1) with control law (6) and the adaptive laws (8). If there exist a feedback control law $\alpha(\cdot,\cdot)$ and an RCLF $V_1(\cdot,\cdot)$ such that (4) is verified, then, for all $(f,g)\in\overline{\mathcal{C}}_{R_Y}\times\overline{\mathcal{C}}_{R_Z}$, $x(t)$, $s(t)$, $\tilde{f}(t,\cdot)$, $\tilde{g}(t,\cdot)$, and $\hat{\Theta}_{\mathrm{aug}}(t)$ are uniformly bounded for all $t\geq t_0$, $x(\cdot)$ is uniformly ultimately bounded, and $s(\cdot,x(\cdot),\xi(\cdot))$ is globally uniformly ultimately bounded. If $\Omega=\mathbb{X}\triangleq\mathbb{R}^n$, then, for all $(f,g)\in\overline{\mathcal{C}}_{R_Y}\times\overline{\mathcal{C}}_{R_Z}$, $x(t)$, $\tilde{f}(t,\cdot)$, $\tilde{g}(t,\cdot)$, and $\hat{\Theta}_{\mathrm{aug}}(t)$ are globally uniformly bounded for all $t\geq t_0$, and $x(\cdot)$ is globally uniformly ultimately bounded.

The proof of this result is omitted for brevity and is presented in [13].

Remark 4.1: Theorem 4.1 provides sufficient conditions for the convergence to the solution of (1a) to a neighborhood of $x=0$ while applying the control input $u(t)=\pi(t,x(t),\xi(t),\hat{f}(t,\cdot),\hat{g}(t,\cdot),\hat{\Theta}_{\mathrm{aug}}(t))$ for all $t\geq t_0$. The proof of Theorem 4.1 shows how the diameter of this neighborhood is irrespective of the uncertainties Λ, $\tilde{g}$, and Θ in (1b). Furthermore, as shown in the proof of this theorem, which is presented in [13], the same (global) ultimate bound on $x(\cdot)$ is attained in the case $s(t)=0$ for all $t\geq T$, that is, in the ideal case, whereby $\xi(t)=\alpha(t,x(t))$. In practice, according to Theorem 4.1, the control law (6) only guarantees convergence of the solution of (1b) to a neighborhood of $s=0$. □

Remark 4.2: Classic parametric backstepping control implies that the size of the parameters to be estimated grows at every step of the iterative process. This can be considered a limitation of classical backstepping control. Interpreting the adaptive gains as estimates of their unknown counterparts, it appears from (8) that the proposed non-parametric approach reduces to a single-step estimation process.

Remark 4.3: The architecture described by Theorem 4.1 is a *direct* adaptive control system, that is, the adaptive gains $\hat{f}(\cdot,\cdot)$, $\hat{g}(\cdot,\cdot)$, and $\hat{\Theta}(\cdot)$ are proven to be bounded, but are not designed to converge to their unknown counterparts. Future work directions may involve the design of an indirect method that utilizes persistently exciting signals.

The proof of Theorem 4.1 shows that the trajectories of $x(\cdot)$, $s(\cdot)$, $\hat{f}(\cdot,\cdot)$, $\hat{g}(\cdot,\cdot)$, and $\hat{\Theta}(\cdot)$ enter closed and bounded compact sets in some finite time. The diameter of these compact sets is an explicit function of the user-defined terms R_Y and R_Z that characterize the functional uncertainty classes in (3), the user-defined parameter k in (4b), and unknown matrix Λ. The larger the uncertainties, that is, the larger R_Y, R_Z, and Λ in some consistent norm, the larger these closed and bounded compact sets and the larger the ultimate bounds on $x(\cdot)$ and $s(\cdot,x(\cdot),\xi(\cdot))$. For additional details, see [13].

Theorem 4.1 assumes the existence of a feedback control law $\alpha(\cdot,\cdot)$ that guarantees (global) uniform ultimate boundedness of the trajectories of (1a) with $\xi(t)=\alpha(t,x(t))$ for all $t\geq t_0$. In the following, we assume that there exists $\alpha:[t_0,\infty)\times\Omega\to\mathbb{R}^m$ that guarantees (global) uniform asymptotic stability of the equilibrium point $x=0$ of (1a) with $\xi(t)=\alpha(t,x(t))$ for all $t\geq t_0$. We assume that this result is certified by the control Lyapunov function (CLF) $V_1:[t_0,\infty)\times\Omega\to\mathbb{R}$ such that (4) is verified with $k=0$; if $\Omega=\mathbb{R}^n$, then we assume that both $W_1(\cdot)$ and $W_2(\cdot)$ are radially unbounded. Thus, we consider the control law (6) for $u(\cdot)$ in (1b) and the *adaptive laws*

$$\frac{\partial\hat{f}(t,\cdot)}{\partial t}=-\Gamma_f\mathcal{K}^{Y}_{y(t)}(\cdot)\frac{\partial\alpha^{\mathrm{T}}(t,x(t))}{\partial x}s(t),\quad \hat{f}(t_0,\cdot)=\hat{f}_0(\cdot),\quad t\geq t_0,\tag{9a}$$

$$\frac{\partial\hat{g}(t,\cdot)}{\partial t}=\Gamma_g\mathcal{K}^{Z}_{z(t)}(\cdot)s(t),\quad \hat{g}(t_0,\cdot)=\hat{g}_0(\cdot),\tag{9b}$$

$$\dot{\hat{\Theta}}_{\mathrm{aug}}(t)=\Gamma_\Theta\Phi_{\mathrm{aug}}\left(t,x(t),\xi(t),\hat{f}(t,y(t)\right)s^{\mathrm{T}}(t),\quad \hat{\Theta}_{\mathrm{aug}}(t_0)=\Theta_{\mathrm{aug},0}.\tag{9c}$$

Corollary 4.1: Consider the plant model (1) with control law (6) and the adaptive laws (8). If there exist a feedback control law $\alpha(\cdot,\cdot)$ and a CLF $V_1(\cdot,\cdot)$ such that (4) is verified with $k=0$, then, for all $(f,g)\in\overline{\mathcal{C}}_{R_Y}\times\overline{\mathcal{C}}_{R_Z}$, $x(t)$, $s(t)$, $\tilde{f}(t,\cdot)$, $\tilde{g}(t,\cdot)$, and $\hat{\Theta}_{\mathrm{aug}}(t)$ are uniformly bounded for all $t\geq t_0$ and $\lim_{t\to\infty}x(t)=0$ for all $x_0\in\Omega$ and $\lim_{t\to\infty}s(t,x(t),\xi(t))=0$ all $(t_0x_0,\xi_0)\in[0,\infty)\times\Omega\times\mathbb{R}^m$. If $\Omega=\mathbb{X}\triangleq\mathbb{R}^n$, then, for all $(f,g)\in\overline{\mathcal{C}}_{R_Y}\times\overline{\mathcal{C}}_{R_Z}$, $x(t)$, $s(t)$, $\tilde{f}(t,\cdot)$, $\tilde{g}(t,\cdot)$, and $\hat{\Theta}_{\mathrm{aug}}(t)$ are globally uniformly bounded for all $t\geq t_0$.

Proof: This result substantially follows as in the proof of Theorem 4.1 and, hence, is omitted for brevity. ■

V. Numerical Simulations

In this section, we present the results of two sets of numerical simulations to showcase the effectiveness of the

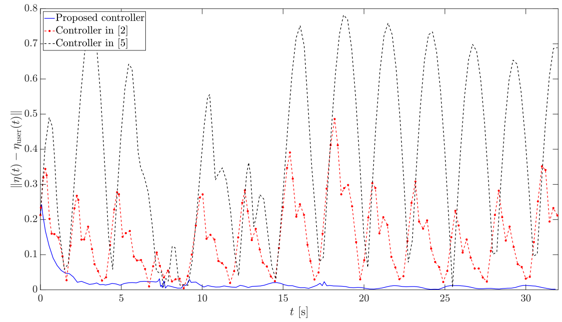

Fig. 1. l_2-norm of the tracking error $e(t) \triangleq \eta(t) - \eta_{\text{user}}(t)$, $t \in [0, 35]$ s, for the plant model (10) affected by a functional uncertainty in $\operatorname{span}\left\{|\phi|\omega_1, |\omega_1|\omega_1, \phi^3\right\}$ as shown in [14, Ex. 9.3]

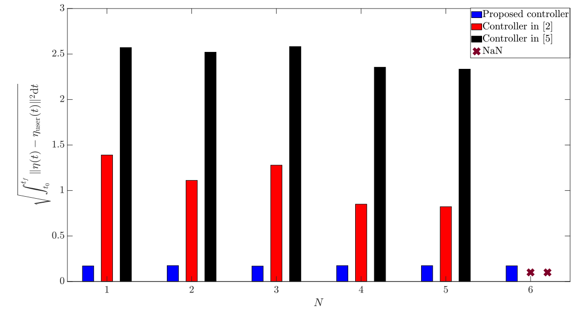

Fig. 2. l_2-norm of the trajectory tracking error $e(t) \triangleq \eta(t) - \eta_{\text{user}}(t)$, $t \in [0, 35]$ s, assuming that the functional uncertainty in (10) lies in the span of Legendre polynomials of increasing order up to a user-defined degree $N \in \left\{N \in \mathbb{N} : g \in \operatorname{span}\{l_i(\phi), l_i(\omega_1)\}_{i=1}^N\right\}$

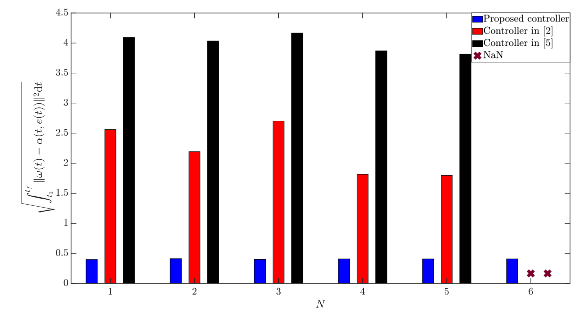

Fig. 3. l_2-norm of $s(t, e(t), \omega(t)) \triangleq \omega(t) - \alpha(t, e(t))$, $t \in [0, 35]$ s, for the same simulations employed to generate Figure 2. The proposed controller produces a significantly smaller tracking error

proposed robust adaptive backstepping control system captured by Theorem 4.1. These numerical simulations concern the control of the inner loop dynamics of a delta-wing aircraft at a high angle of attack; in this configuration, these aircraft are subject to an unstable behavior in roll known as wing-rock phenomenon [14, Ex. 9.3]. The proposed results are compared with those obtained by applying the robust, non-adaptive, backstepping controller over native spaces given by Theorem 5.2 in [2] and a classical robust backstepping controller obtained by extending Lemma 2.28 in [5] to time-varying plants with $m \geq 1$. For fairness of comparison, both the controller in [2, Th. 5.2] and the one in [5, Lemma 2.28] have been designed to augment the same parametric adaptive controller as in the formulation proposed in Theorem 4.1.

The rotational dynamics of an aircraft are captured by

$$\dot{\eta}(t) = H(\eta(t))\omega(t), \quad \eta(t_0) = \eta_0, \quad t \geq t_0, \tag{10a}$$
$$\dot{\omega}(t) = J^{-1}\big(u(t) - \omega^{\times}(t)J\omega(t) + (E_{(\phi,\omega_1)}g)\mathbf{e}_1\big), \quad \omega(t_0) = \omega_0, \tag{10b}$$

where $J \triangleq \operatorname{diag}(J_1, J_2, J_3) \in \mathbb{R}^{3\times 3}$ denotes the unknown *principal matrix of inertia matrix*, $\eta : [t_0, \infty) \to [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [0, 2\pi)$ denotes the set of *Euler angles* so that $\eta(t) \triangleq [\phi(t), \theta, \psi(t)]^{\mathrm{T}}$, $\omega : [t_0, \infty) \to \mathbb{R}^3$ denotes the *angular velocity*, $\omega(t) \triangleq [\omega_1(t), \omega_2(t), \omega_3(t)]^{\mathrm{T}}$,

$$H(\eta) \triangleq \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix}, \quad \eta \in [0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [0, 2\pi), \tag{11}$$

denotes the *Jacobian matrix*, and $\mathbf{e}_1 \triangleq [1, 0, 0]^{\mathrm{T}}$.

The plant model (10) is in the same form as (1) with $n = 3$, $m = 3$, $x = e \triangleq \eta - \eta_{\text{user}}$, $\xi = \omega$, $a(t, x) = 0$, $b(t, x) = H(\eta(t))$, $f = 0$, $A = 0$, $\Lambda = J^{-1}$, $\Theta = \operatorname{diag}(J_2 - J_3, J_3 - J_1, J_1 - J_2)$, and $\Phi(t, \xi) = [\omega_2\omega_3, \omega_1\omega_3, \omega_1\omega_2]^{\mathrm{T}}$.

To follow the continuously differentiable user-defined trajectory $\eta_{\text{user}} : [t_0, \infty) \to \mathbb{R}^3$, we let

$$\alpha(t, e(t)) = H^{-1}(\eta(t))\Big[-K_{\mathrm{P}}e(t) + \dot{\eta}_{\text{user}}(t)\Big], \tag{12}$$

for all $t \geq t_0$ where $K_{\mathrm{P}} \in \mathbb{R}^{3\times 3}$ are symmetric, positive-definite, and user-defined matrices. The associated CLF is such that $V_1(e) = e^{\mathrm{T}}K_{\mathrm{P}}e$, where $e(t) \triangleq \eta(t) - \eta_{\text{user}}(t)$.

As a first set of simulation scenarios, we let $g \in \operatorname{span}\left\{|\phi|\omega_1, |\omega_1|\omega_1, \phi^3\right\}$ so that $E_{(\phi,\omega_1)}g\mathbf{e}_1 = \left(\alpha_1|\phi|\omega_1 + \alpha_2|\omega_1|\omega_1 + \alpha_3\phi^3\right)[1, 0, 0]^{\mathrm{T}}$ for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ [14, Ex. 9.3]. Figure 1 shows how the proposed controller guarantees small tracking errors despite the functional uncertainties. The controllers in [2, Th. 5.2] and [5, Lemma 2.28] underperform significantly.

As a second set of simulation scenarios, we let $g \in \operatorname{span}\left\{\ell_i(\phi), \ell_i(\omega_1)\right\}_{i=1}^N$, where $\ell_i : [-1, 1] \to \mathbb{R}$ denotes the Legendre polynomial of degree i, so that $E_{(\phi,\omega_1)}g\mathbf{e}_1 = \sum_{i=1}^N \left(\alpha_i\ell_i(\phi) + \beta_i\ell_i(\omega_1)\right)[1, 0, 0]^{\mathrm{T}}$ for some $\{\alpha_i, \beta_i\}_{i=1}^N \subset \mathbb{R}$. Figure 2 shows how the proposed results provide an improved attitude tracking performance for $i \in \{1, \dots, 6\}$. For $i = 6$, the robust non-adaptive backstepping control system in [2] and the classical backstepping control system are unable to guarantee the boundedness of the closed-loop system. Notably, the user-defined tunable parameters are not adjusted for different functional uncertainties, proving the ability of the proposed method to capture uncertainties whose functional shape is completely unknown to the controller. Figures 3 and 4 show the l_2-norm of $s(t, e(t), \omega(t)) \triangleq \omega(t) - \alpha(t, e(t))$, $t \in [0, 35]$ s, and the l_2-norm of the associated control efforts, respectively. These plots show how the proposed results provide significantly

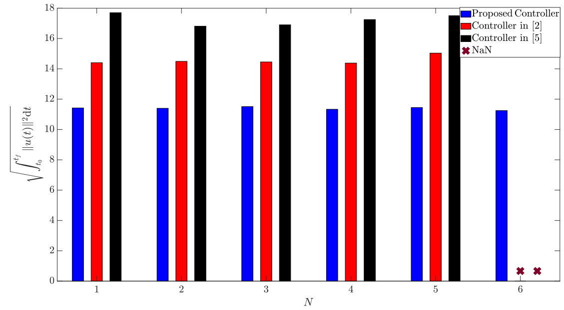

Fig. 4. l_2-norm of the control effort for the same simulations employed to generate Figure 2. The proposed controller requires a smaller control effort

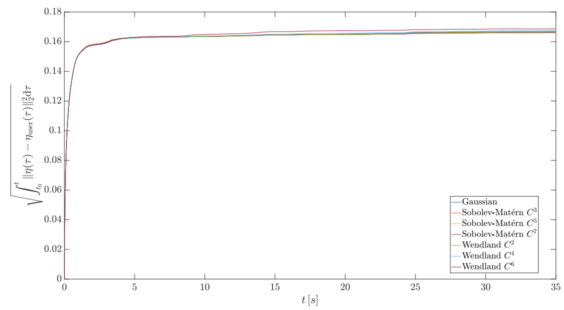

Fig. 5. $\mathcal{L}_2$-norm of the trajectory tracking error for multiple classes of kernel functions

improved performance compared to the results proposed in [2, Th. 5.2] and [5, Lemma 2.28].

Figures 1–4 are obtained by employing Wendland C^2 kernel functions [1, p. 59]. Figures 5 and 6 compare the l_2-norms of the tracking error and the control effort, respectively, employing alternative forms of Wendland functions [1, p. 59], the Gaussian kernel, and Sobolev-Matérn kernel [1, p. 57]. Kernels smoother than the functional uncertainty, such as the Gaussian kernel, imply larger control efforts.

VI. Conclusion and Future Work

This paper introduced the first adaptive backstepping control law for systems with deterministic parametric and nonparametric uncertainties in native spaces. The approach reduces modeling effort and outperforms classical and non-adaptive backstepping methods. Numerical simulations and open-source code, available at [10], showcase its applicability. Future work includes deriving control laws for entire cascaded subsystems without assuming a pseudo-input controller, particularly with matched disturbances in native spaces. We will also apply the method to multi-rotor UAVs with unknown sling payloads [15], address the assumption that plant trajectories remain close to kernel centers [1, Ch. 3] using barrier Lyapunov functions and constrained kernel distributions, and investigate explicit bounds on trajectory tracking error as functions of kernel centers and uncertainty smoothness [1, Ch. 5].

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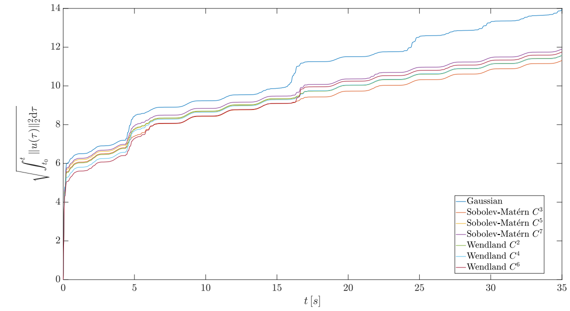

Fig. 6. $\mathcal{L}_2$-norm of the control effort for multiple classes of kernel functions. Kernel functions smoother than the actual functional uncertainty lead to larger control efforts for similar tracking performances

References

[1] A. J. Kurdila, A. L'Afflitto, and J. A. Burns, *Data-Driven, Nonparametric, Adaptive Control Theory*. London, UK: Springer, 2025.

[2] H. Wang, B. Scurlock, A. Kurdila, and A. L'Afflitto, "Robust, nonparametric backstepping control over reproducing kernel Hilbert spaces," in *Conference on Decision and Control*. Milan, Italy: IEEE, 2024.

[3] A. J. Kurdila, A. L'Afflitto, J. A. Burns, and H. Wang, "Nonparametric adaptive control in native spaces: A DPS framework (Part I)," *Annual Reviews in Control*, vol. 58, p. 100969, 2024.

[4] ——, "Nonparametric adaptive control in native spaces: Finite-dimensional implementations (Part II)," *Annual Reviews in Control*, vol. 58, p. 100968, 2024.

[5] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York, NY: John Wiley & Sons, 1995.

[6] J. Zhou and C. Wen, *Adaptive Backstepping Control of Uncertain Systems: Nonsmooth Nonlinearities, Interactions or Time-Variations*, ser. Lecture Notes in Control and Information Sciences. Berlin, Germany: Springer, 2008.

[7] A. Capone and S. Hirche, "Backstepping for partially unknown nonlinear systems using Gaussian processes," *IEEE Control Systems Letters*, vol. 3, no. 2, pp. 416–421, 2019.

[8] J. Jiao, A. Capone, and S. Hirche, "Backstepping tracking control using gaussian processes with event-triggered online learning," *IEEE Control Systems Letters*, vol. 6, pp. 3176–3181, 2022.

[9] D. I. Ignatyev, H.-S. Shin, and A. Tsourdos, "Sparse online Gaussian process adaptation for incremental backstepping flight control," *Aerospace Science and Technology*, vol. 136, p. 108157, 2023.

[10] G. A. Orlando and A. L'Afflitto, "Robust adaptive backstepping control over native spaces." [Online]. Available: http://lafflitto.com/Codes/RKHS_Adaptive_Backstepping_Code.zip

[11] G. A. Orlando, A. L'Afflitto, and A. J. Kurdila, "Robust adaptive backstepping control over native spaces." [Online]. Available: https://youtu.be/xfU1z2dpba8

[12] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*. Priceton, NJ: Princeton University Press, 2010.

[13] G. A. Orlando and A. J. L'Afflitto, Andrea Kurdila, "Robust adaptive backstepping control over native spaces," 2025. [Online]. Available: https://lafflitto.com/Documents/Orlando_Robust_Adaptive_Backstepping_Full.pdf

[14] E. Lavretsky and K. Wise, *Robust and Adaptive Control: With Aerospace Applications*, 2nd ed. London, UK: Springer, 2024.

[15] J. A. Marshall, W. Sun, and A. L'Afflitto, "A survey of guidance, navigation, and control systems for autonomous multi-rotor small unmanned aerial systems," *Annual Reviews in Control*, vol. 52, pp. 390–427, 2021.