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Nonlinear Dyn (2025) 113:24889–24915
https://doi.org/10.1007/s11071-025-11407-6

RESEARCH

Barrier Lyapunov Functions and Model Reference Adaptive Control with Uncertainty in Native Spaces

Giorgio Antonio Orlando · Haoran Wang · Andrea L'Afflitto · Andrew J. Kurdila

Received: 27 November 2024 / Accepted: 19 May 2025 / Published online: 8 June 2025

Abstract Nonparametric model reference adaptive control (MRAC) theory studies a novel class of nonlinear control systems able to counter functional uncertainties for which no parameterization is selected *a priori*, and which are contained in a reproducing kernel Hilbert space (RKHS). This paper presents, for the first time, robust nonparametric MRAC systems that, employing barrier Lyapunov functions, enforce user-defined time-varying constraints, or feasible approximations thereof, on the tracking error at all times. The proposed results generalize parametric MRAC systems, that is, MRAC systems that require a regressor vector or an equivalent representation of the matched uncertainties, and existing nonparametric MRAC systems that, presently, rely on the strong assumption of *a priori* knowledge of some bounded region in which the closed-loop trajectory lies.

Keywords Reproducing Kernel Hilbert Spaces · Model reference adaptive control · Barrier functions

Mathematics Subject Classification 93C40 · 46E22 · 93D21

This paper was partly supported by NSF through Grant no. 2137159 and the project PNRR-NGEU, which received funding from the MUR - DM 118/2023.

G. A. Orlando
Department of Mechanical and Aerospace Engineering, Polytechnic of Turin, Turin, Italy
e-mail: giorgio.orlando@polito.it

H. Wang · A. J. Kurdila
Department of Mechanical Engineering, Virginia Tech, Virginia, USA
e-mail: haoran9@vt.edu

A. J. Kurdila
e-mail: kurdila@vt.edu

A. L'Afflitto (✉)
Grado Department of Industrial and Systems Engineering, Virginia Tech, Virginia, USA
e-mail: a.lafflitto@vt.edu

1 Introduction

1.1 Motivation of this Paper

The expectation of autonomous robotic systems to operate precisely in unknown dynamic environments despite substantial uncertainties and modeling disturbances is challenging the state of the art in control theory. Data-driven methods provide a tool to attain this expectation for their ability to set control policies according to measured data without relying solely on plant models chosen *a priori* by the control system designer. Numerous data-driven methods require sampling the state space, possibly in a neighborhood of the closed-loop plant trajectory. A challenge in this class of data-driven methods is the need to sample large portions of the state spaces while keeping the computational burden acceptable for modern single-board computers such as those installed on autonomous aerial, marine, and terrestrial vehicles [1]. If the control system constrained the closed-loop plant trajectory within user-defined bounded sets, then this challenge would be overcome. However, in general, the tighter the constraints on the closed-loop trajectory, the larger the control effort to enforce such constraints. To mitigate this additional challenge, the diameter of the constraint set should be chosen dynamically according to some user-defined criteria, trading off between computational power and control effort.

The authors in [2–4] presented a systematic deterministic framework to design model reference adaptive control (MRAC) systems assuming that the functional uncertainties affecting the plant lie in a user-defined reproducing kernel Hilbert space (RKHS, also known as native space). Despite classical MRAC systems, which require a regressor vector or an equivalent tool to parameterize the functional uncertainties, these results do not assume any parameterization of the functional uncertainty. In [2–4], the native space of functional uncertainties, also known as *hypothesis space*, is defined by a user-defined kernel function and sampling points in the state space, known as kernel centers. As discussed in Chapters 6 and 7 of [2], the kernel centers can be chosen employing data-driven methods.

Methods based on explicit parameterizations of the functional uncertainties, also known as *parametric methods*, unrealistically assume that the same parameterization (*e.g.*, the same regressor vector) applies everywhere on the state space and yield best results in some specific regions of the state space where the closed-loop plant trajectory is expected to lie; see [5,6] for two classical textbooks on parametric adaptive control. Consistent with the existing works on parametric adaptive control, the *nonparametric control systems* in [2–4] rely on a critical assumption, that is, the *a priori* knowledge of a region of the state space in which the closed-loop trajectory is expected to dwell. Knowledge of this region is important for allocating the centers of the kernels that define the hypothesis space. The further the closed-loop trajectory lies from this region and, hence, from the centers of the kernels, the lower the performance of nonparametric adaptive laws. The naïve approach of increasing the number of centers may lead to overly high computational loads. The centers that define the hypothesis space can be judiciously chosen according to some data-driven methods [2, Ch. 6,7]. However, the computational cost of these methods grows with the sampled subset of the state space.

1.2 Main Contributions of this Paper

This paper presents the first nonparametric MRAC system that employs barrier Lyapunov functions to enforce user-defined constraints, or implementable approximations thereof, on the trajectory tracking error and guarantee that the user-defined closed-loop trajectory dwells in some region of the state space defined by the user *a priori*. Thus, the centers that characterize the hypothesis space can be placed in relatively small regions of the state space, where the closed-loop plant trajectory is known to lie at all times. This result is attained by considering both time-varying and time-invariant constraint sets. The diameter of time-varying constraint sets can be increased or reduced according to user-defined criteria such as for example, the available control authority.

The proposed approach explicitly uses a limiting distributed parameter system (DPS) to characterize the ideal performance of the nonparametric control strategy. Since DPSs are not implementable in practice, the limiting DPS is approximated by finite-dimensional realizations. The performance of the proposed controllers for the ideal limiting DPS, as well as for consistent approximations of the DPS that yield practical controllers, is discussed. Numerical examples show that the proposed results guarantee smaller tracking errors for comparable control efforts than their unconstrained counterparts [2–4].

1.3 Novelty of the Proposed Results

Several other authors explored a nonparametric approach to adaptive control; see[7–14] for some of the most relevant examples. These works, however, assume that independent and identically distributed (IID) measurements of the uncertainty are available and characterize the controller's performance employing classical tools from stochastic analysis. The results proposed in this paper and those presented in [2–4] rely on a purely deterministic formulation.

An appealing feature of deterministic nonparametric adaptive control systems based on the concept of a limiting DPS, such as the proposed MRAC systems and those in [2–4], is that they capture bounds on the trajectory tracking error in terms of some properties of the native space of functional uncertainties, such as the smoothness of the functional uncertainties, the number of centers of the kernels employed to define the hypothesis RKHS, and the distance between these centers in a given region. Of the nonparametric approaches based on stochastic analysis of the uncertainty, a similar notion of an underlying infinite-dimensional system can be found in [12], but the guarantee on performance bounds is stochastic and depends on IID measurements of the uncertainty.

Parametric control systems provide ultimate bounds on the closed-loop trajectory tracking error as a function of user-defined bounds on the unmatched uncertainties, that is, of those uncertainties not captured by a regressor vector. These classical results, however, are unable to provide ultimate bounds on the tracking error that depend on the dimension of the regressor vector or the class of functions collected by the regressor vector.

In the context of parametric MRAC, the use of barrier Lyapunov functions has been studied extensively. The authors in [15] employed barrier Lyapunov functions to enforce user-defined properties on the closed-loop plant trajectory. The authors in [16,17] studied the problem of constraining both the trajectory tracking error and the control input employing barrier Lyapunov functions. In [18], the authors designed MRAC systems with barrier Lyapunov functions to enforce user-defined constraints on the tracking error, while the reference model, which is user-defined as well, violates such constraints. Recently. the authors in [19] leveraged barrier Lyapunov functions and designed for the first time MRAC systems able to enforce time-varying constraints, hence improving the classical results in [20–22], where constraints on the tracking error are enforced by modifying the reference command input and, hence, depriving the user of their freedom to design the reference model and the reference trajectory. Whereas the results in this paper and [17,19] share some similarities, it is essential to remark on the fact that, in this paper, functional uncertainties are not parameterized using a regressor vector, and, in fact, may belong to an infinite-dimensional RKHS, whereas [17,19] fall in the narrower class of parametric control systems.

An additional point of novelty of this paper over the existing literature on nonparametric MRAC systems is that a new class of uncertainties is considered. Existing results account for both matched functional uncertainties and uncertainties in the linearized, uncontrolled plant dynamics. However, thus far, uncertainties in the ability of the control system to produce the desired input have not been considered. This paper is the first to address uncertainties in realizing the desired control input. Therefore, the proposed results address large classes of robust and adaptive control problems, wherein the plant actuators may be affected by disturbances and uncertainties. Within the context of fault-tolerant control, we note the recent works [23–26], which employ parametric fuzzy adaptive control schemes, [27], which employs an adaptive event-triggered strategy, [28], which employs an adaptive switching strategy, and [29], which employs neural networks. A comprehensive discussion on the state of the art in fault-tolerant control can be found at [30].

The proposed numerical simulations show how the proposed results can be employed both to enforce user-defined *hard* constraints on the closed-loop trajectory tracking error and *soft* saturation constraints. Specifically, time-varying constraints can be relaxed whenever the plant actuators approach their saturation limits and can be tightened whenever the actuators allow for stronger control efforts.

1.4 Outline of this Paper

This paper is structured as follows. In Section 2, we recall the fundamental notation and key notions needed in this paper. Section 3 summarizes key elements of native space theory needed for the scope of this paper. In Section 5, we define an MRAC system able to enforce user-defined time-varying constraints on the trajectory tracking error. In this MRAC system, the adaptive laws that correspond to the RKHS-embedded functional uncertainties are captured by a set of partial differential equations (PDEs), and the overall MRAC system is a DPS. Hence, this infinite-dimensional MRAC system is not implementable in practice. In Section 6, we present finite-dimensional, robust MRAC systems that approximate the ideal MRAC system evolving on the DPS that, for this reason, is called *limiting* DPS. The adaptive laws in Section 6 depend on the kernel matrix function underlying the chosen RKHS, and, hence, may not be directly implementable in problems of practical interest. In Section 7, we present an implementable realization of the proposed finite-dimensional, robust MRAC systems. This realization is expressed in terms of the state space coordinates and of the Grammian matrix underlying the finite-dimensional approximation of the hypothesis space. In Section 8, we present the results of numerical simulations that illustrate the applicability of the proposed results; computer codes implementing these numerical simulations are available at [31]. Finally, in Section 9, we draw conclusions and outline future work directions.

2 Notation

Let $\mathbb{N}$ denote the *set of positive integers*, $\mathbb{R}$ denote the *set of real numbers*, $\mathbb{R}^+$ denote the *set of nonnegative real numbers*, $\mathbb{R}^n$ the *set of* $n \times 1$ *real column vectors*, and $\mathbb{R}^{n\times m}$ the *set of* $n \times m$ *real matrices*. The *closed ball centered* at $x \in \mathbb{R}^n$ with *radius* ε is denoted by $\overline{\mathcal{B}}_\varepsilon(x)$. The *interior of the set* $\mathcal{E} \subset \mathbb{R}^n$ is denoted by $\mathring{\mathcal{E}}$ and the *boundary of* $\mathcal{E}$ is denoted by $\partial\mathcal{E}$.

In this paper, we use the symbols $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ to denote the state space and the space of control values, respectively. A generic Hilbert space of functions defined on $\mathbb{X}$ with values in $\mathbb{U}$ is denoted by $\mathbb{H}(\mathbb{X}, \mathbb{U})$. If the domain and codomain of functions in some Hilbert space are clear from the context, we write $\mathbb{H}$ instead of $\mathbb{H}(\mathbb{X}, \mathbb{U})$. If $\mathbb{H}(\mathbb{R}^n, \mathbb{R})$ is a native space, then we write $\mathcal{H}(\mathbb{R}^n, \mathbb{R})$ or $\mathcal{H}$. If $\mathbb{H}(\mathbb{R}^n, \mathbb{U})$ is a native space, then we write $\boldsymbol{\mathcal{H}}(\mathbb{R}^n, \mathbb{U})$ or $\boldsymbol{\mathcal{H}}$. The former native space is referred to as a scalar-valued RKHS, and the latter is referred to as a vector-valued RKHS.

The *real part of* $z \in \mathbb{C}$ is denoted by $\mathrm{Re}(z)$. The closure of the set $\mathcal{E}$ is denoted by $\overline{\mathcal{E}}$. The *zero vector* in $\mathbb{R}^n$ is denoted by 0_n or 0, the *zero matrix* in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$ or 0, the *zero vector* in $\mathbb{H}$ is denoted by $0_{\mathbb{H}}$ or 0, and the *identity matrix* in $\mathbb{R}^{n\times n}$ is denoted by I_n or I. The jth column of I_n, with $j \in \{1, \ldots, n\}$, is denoted by e_j as is known as *jth canonical basis of* $\mathbb{R}^n$. The block-diagonal matrix formed by $M_i \in \mathbb{R}^{n_i\times n_i}$, $i = 1, \ldots, p$, is denoted by $M = \text{block-diag}\left(M_1, \ldots, M_p\right)$. The *transpose of* $B \in \mathbb{R}^{n\times m}$ is denoted by B^{T}, the *trace of* $A \in \mathbb{R}^{n\times n}$ is denoted by $\mathrm{tr}(A)$, the *eigenvalues of* A *with maximum and minimum real parts* are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. We write $\|\cdot\|$ for the *Euclidean vector norm* and the corresponding *equi-induced matrix norm*, $\|\cdot\|_{\mathrm{F}}$ for the *Frobenius matrix norm* [32, pp. 18-19], and $\|\cdot\|_{\mathbb{V}}$ for the norm on the generic normed space $\mathbb{V}$. The inner product over on some inner product space H is denoted by $\langle\cdot,\cdot\rangle_H$. The inner product over $\mathbb{R}^{n\times m}$ generated by the trace operator is denoted by $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ so that $\langle A, A\rangle_{\mathrm{tr}} = \|A\|_{\mathrm{F}}^2$ for any $A \in \mathbb{R}^{n\times m}$.

Given two normed vector spaces $\mathbb{U}$, $\mathbb{V}$, $\mathcal{L}(\mathbb{U}, \mathbb{V})$ denotes the *space of linear bounded operators* from $\mathbb{U}$ to $\mathbb{V}$. If $\mathbb{U} = \mathbb{V}$, then we write $\mathcal{L}(\mathbb{U})$. The *Lebesgue space of p-th power integrable functions* from $\mathbb{X}$ to $\mathbb{U}$ is denoted by $L^p(\mathbb{X}, \mathbb{U})$.

Given $f, g \in \mathbb{R}$, if there exists $c_1 > 0$ such that $f \leq c_1 g$, then we write $f \lesssim g$. The converse inequality is readily defined similarly. Furthermore, if there exist $c_1, c_2 > 0$ such that $c_2 g \leq f \leq c_1 g$, then we write $f \approx g$.

3 Elements of Native Space Theory

In this section, we recall notions of native space theory that are essential to the comprehension of the proposed approach to robust MRAC systems design. For more detailed discussions, see [2, Ch. 3].

3.1 Scalar-Valued Native Spaces

In this section, we present some fundamental properties of scalar-valued RKHSs, that is, native spaces of scalar-valued functions defined on $\mathbb{X} \triangleq \mathbb{R}^n$.

Definition 1 (Scalar-Valued Symmetric Kernel) A kernel function $\mathfrak{K} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ is *symmetric* if $\mathfrak{K}(x, y) = \mathfrak{K}(y, x)$ for all $x, y \in \mathbb{X}$.

Definition 2 (Scalar-Valued Kernel of Positive Type) A kernel function $\mathfrak{K} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ is *of positive type* if, for any integer $N \in \mathbb{N}$, any collection of N *centers*

$$\Xi_N \triangleq \{\xi_i \in \Omega : 1 \leq i \leq N\} \subset \Omega \subseteq \mathbb{X}, \tag{1}$$

and any set of real coefficients $\{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{R}$, it holds that

$$\sum_{i,j=1}^{N} \mathfrak{K}(\xi_i, \xi_j)\alpha_i\alpha_j \geq 0. \tag{2}$$

If the inequality in (2) is strict for any choice of N distinct centers in Ξ_N, then the kernel is of *strictly positive type*.

Definition 3 (Admissible Kernels) A real-valued function $\mathfrak{K} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ is an *admissible kernel* if it is symmetric and of positive type. Additionally, if this kernel is continuous, then it is called a *Mercer kernel*.

Table 1 Compactly supported, minimal degree (Wendland) functions. In these expressions, $(x)_+ \triangleq x$ if $x \geq 0$, and $(x)_+ \triangleq 0$ if $x < 0$

Dimension	Function $\eta_{n,s}$	Smoothness
$n = 1$	$\eta_{1,0}(r) = (1-r)_+$	C^0
	$\eta_{1,1}(r) = (1-r)_+^3(3r+1)$	C^2
	$\eta_{1,2}(r) = (1-r)_+^5(8r^2+5r+1)$	C^4
$n \leq 3$	$\eta_{3,0}(r) = (1-r)_+^2$	C^0
	$\eta_{3,1}(r) = (1-r)_+^4(4r+1)$	C^2
	$\eta_{3,2}(r) = (1-r)_+^6(35r^2+18r+3)$	C^4
	$\eta_{3,3}(r) = (1-r)_+^8(32r^3+25r^2+8r+1)$	C^6
$n \leq 5$	$\eta_{5,0}(r) = (1-r)_+^3$	C^0
	$\eta_{5,1}(r) = (1-r)_+^5(5r+1)$	C^2
	$\eta_{5,2}(r) = (1-r)_+^7(16r^2+7r+1)$	C^4

Only Mercer kernels are used in this paper. Once an admissible kernel $\mathfrak{K}(\cdot,\cdot)$ is selected, we define the *kernel section located at* x as $\mathfrak{K}_x(\cdot) \triangleq \mathfrak{K}(x,\cdot)$. Often, a kernel section $\mathfrak{K}_x(\cdot)$ is used to define bases in building approximations, so we also refer to $\mathfrak{K}_x(\cdot)$ as the *kernel basis function centered at* x.

Definition 4 (Scalar-Valued Reproducing Kernel Hilbert Space) Let $\mathfrak{K} : \mathbb{X}\times\mathbb{X} \to \mathbb{R}^+$ denote a continuous admissible Mercer kernel. A *scalar-valued reproducing kernel Hilbert space* (RKHS) is defined as

$$\mathcal{H} \triangleq \overline{\text{span}\{\mathfrak{K}_x : x \in \mathbb{X}\}}, \tag{3}$$

where the closure is taken with respect to the candidate inner product defined as

$$\left\langle \mathfrak{K}_{x_1}, \mathfrak{K}_{x_2} \right\rangle_{\mathcal{H}} \triangleq \mathfrak{K}(x_1, x_2), \quad \text{for any } x_1, x_2 \in \mathbb{X}. \tag{4}$$

3.2 Compactly Supported Kernels on $\mathbb{R}^n$

Numerous kernel functions are defined globally on $\mathbb{R}^n$. In this paper, however, we focus on the problem of characterizing uncertainties by means of RKHSs defined by kernels that have compact support in $\mathbb{R}^n \times \mathbb{R}^n$. Functions having compact support in a domain $\Omega \subset \mathbb{X}$ vanish on the boundary of the domain, and this property is highly relevant to the problems discussed in this paper, wherein the closed-loop plant trajectory is constrained to a user-defined compact set. The author in [33] derives a family of compactly supported radial functions $\eta_{n,s}(\cdot)$ that are indexed by a dimension n and a smoothness index s; see Table 1 for some examples of such radial functions. These functions are sometimes referred to as *Wendland functions*, or *compactly supported radial functions of minimal degree*. The Wendland function $\eta_{n,s}(\cdot)$ defines the kernel

$$\mathfrak{K}_{n,s}(x_1, x_2) \triangleq \eta_{n,s}(\|x_1 - x_2\|_2), \quad \text{for all } (x_1, x_2) \in \mathbb{R}^{\hat{n}} \times \mathbb{R}^{\hat{n}}, \tag{5}$$

that is positive definite on $\mathbb{R}^{\hat{n}}$ for all $\hat{n} \leq n$. This way, the radial function $\eta_{n,s}(\cdot)$ can define a positive-definite kernel on any Euclidean space with dimension $\hat{n} \leq n$. The index s describes the global smoothness of the radial function $\eta_{n,s}$ in the sense that $\eta_{n,s} \in C^{2s}(\mathbb{R}^+)$, where $C^k(\mathbb{R}^+)$ denotes the space of scalar-valued function defined on $\mathbb{R}^+$ that are continuous with their first k derivatives. In the following, we denote (5) by $\mathfrak{K}(\cdot,\cdot)$.

3.3 Vector-Valued Native Spaces Generated by $\mathbb{X} \triangleq \mathbb{R}^n$ and $\mathbb{U} \triangleq \mathbb{R}^m$

In this paper, we consider the class of vector-valued native spaces defined as the Cartesian product of scalar-valued RKHSs. Thus, we let

$$\boldsymbol{\mathcal{H}} \triangleq \mathcal{H} \times \cdots \times \mathcal{H} = \mathcal{H}^m, \tag{6}$$

where $\mathcal{H}(\mathbb{X}, \mathbb{R})$ denotes a scalar-valued RKHS. In this case, if the scalar-valued RKHS is generated by the kernel function $\mathfrak{K} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$, the corresponding vector-valued native space underlies a diagonal operator-valued kernel function defined as $\mathcal{K}$:

$\mathbb{X} \times \mathbb{X} \to \mathbb{R}^{m \times m}$ so that $\mathcal{K}(x, y) = \mathfrak{K}(x, y) I_m$ for all $(x, y) \in \mathbb{X} \times \mathbb{X}$, where $\mathcal{K}(x, y) \in \mathcal{L}(\mathbb{R}^m)$, which denotes the *space of bounded linear operators* from $\mathbb{R}^m$ to $\mathbb{R}^m$. Furthermore, $\mathcal{K}_x(\cdot) \triangleq \mathfrak{K}_x(\cdot) I$ for all $x \in \mathbb{X}$.

In general, vector-valued RKHSs are not reducible to the Cartesian product of scalar-valued RKHSs. This class of native spaces, though interesting, has been considerably less investigated in the literature, and fewer properties are available, compared to the scalar-valued RKHSs counterpart. Remarkably, it can be proven that any vector-valued RKHS underlying a diagonal operator-values kernel can be expressed as the Cartesian product of scalar-valued native spaces [2, Sec. 3.6].

Definition 5 (Evaluation Operator for a Vector-Valued Hilbert Spaces) Given the Hilbert space $\mathbb{H}(\mathbb{X}, \mathbb{R}^m)$ of vector-valued functions, the *evaluation operator* $E_x : \mathbb{H} \to \mathbb{R}^m$ for a point $x \in \mathbb{X}$ is such that

$$E_x f \triangleq f(x) \in \mathbb{R}^m, \quad \text{for all } f \in \mathbb{H}. \tag{7}$$

If $\mathbb{H}$ is selected to be the RKHS $\boldsymbol{\mathcal{H}}(\mathbb{X}, \mathbb{R}^m)$, E_x is a bounded linear operator, and $E_x \in \mathcal{L}(\boldsymbol{\mathcal{H}}, \mathbb{R}^m)$.

Next, we recall the following result linking evaluation operators and admissible operator kernel functions.

Theorem 1 ([2, Th. 3.17]) *The canonical kernel* $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{R}^m)$

$$\mathcal{K}(x, z) \triangleq E_x E_z^* = \mathcal{K}_x^* \mathcal{K}_z, \quad \textit{for all } (x, z) \in \mathbb{X} \times \mathbb{X}. \tag{8}$$

is admissible, where $\mathcal{K}_x(\cdot) \triangleq \mathcal{K}(\cdot, \cdot)$.

Theorem 2 ([34, pp. 380-381, Prop. 2.3]) *Let* $\mathcal{K} : \mathbb{X} \times \mathbb{X} \to \mathcal{L}(\mathbb{R}^m)$ *be an admissible operator-valued kernel over the set* $\mathbb{X}$*. Then, there is a unique vector-valued RKHS* $\boldsymbol{\mathcal{H}}$ *such that* $\mathcal{K}_x \in \mathcal{L}(\mathbb{R}^m, \boldsymbol{\mathcal{H}})$ *and the reproducing property*

$$\langle f(x), u \rangle_{\mathbb{R}^m} = \langle f, \mathcal{K}_x u \rangle_{\mathbb{H}}, \quad \textit{for all } f \in \mathbb{H}(\mathbb{X}, \mathbb{R}^m), \tag{9}$$

is verified. Furthermore, for any $x \in \mathbb{X}$ *and* $f \in \boldsymbol{\mathcal{H}}$*, it holds that*

$$\|f(x)\|_{\mathbb{R}^m} \leq \sqrt{\|\mathcal{K}(x, x)\|_{\mathcal{L}(\mathbb{R}^m)}} \|f\|_{\boldsymbol{\mathcal{H}}}. \tag{10}$$

3.4 Subspaces Generated by Restrictions to $\Omega \subset \mathbb{X}$

This paper is concerned with the design of MRAC systems that constrain the dynamics of nonlinear plants affected by uncertainties within user-defined compact subsets of $\mathbb{X}$. Thus, it is important to characterize RKHSs and approximations that are generated from samples contained in subsets of the entire state space $\mathbb{X}$. We define the closed subspace generated by $\Omega \subset \mathbb{X}$ as

$$\boldsymbol{\mathcal{H}}_\Omega \triangleq \overline{\text{span}\{\mathcal{K}_x u : u \in \mathbb{R}^m, x \in \Omega\}} \subset \boldsymbol{\mathcal{H}}, \tag{11}$$

where the closure is taken with respect to the norm that $\boldsymbol{\mathcal{H}}_\Omega$ inherits as a subspace of $\boldsymbol{\mathcal{H}}$. As for scalar-valued kernels, we emphasize that the functions in $f \in \boldsymbol{\mathcal{H}}_\Omega$ are mappings $f : \mathbb{X} \to \mathbb{R}^m$ defined on *all* of $\mathbb{X}$. The closed subspace $\boldsymbol{\mathcal{H}}_\Omega$ is an RKHS of vector-valued functions with the operator kernel

$$\mathcal{K}_{\Omega,x} \triangleq {}^{\boldsymbol{\cdot}}\boldsymbol{\Pi}_\Omega \mathcal{K}_x \in \boldsymbol{\mathcal{H}}_\Omega, \quad \text{for all } x \in \mathbb{X}, \tag{12}$$

where ${}^{\boldsymbol{\cdot}}$ denotes the $\boldsymbol{\mathcal{H}}$-orthogonal projection onto $\boldsymbol{\mathcal{H}}_\Omega \subset \boldsymbol{\mathcal{H}}$. This operator kernel satisfies the *reproducing property*

$$\begin{aligned}
\left\langle f, \mathcal{K}_{\Omega,x} u \right\rangle_{\boldsymbol{\mathcal{H}}} &= \langle f, \boldsymbol{\Pi}_\Omega \mathcal{K}_x u \rangle_{\boldsymbol{\mathcal{H}}} = \left\langle \boldsymbol{\Pi}_\Omega f, \mathcal{K}_{\Omega,x} u \right\rangle_{\boldsymbol{\mathcal{H}}} \\
&= \langle f, \mathcal{K}_x u \rangle_{\boldsymbol{\mathcal{H}}} = \langle E_x f, u \rangle_{\mathbb{R}^m} \\
&= \langle f(x), u \rangle_{\mathbb{R}^m}.
\end{aligned}$$

3.5 Approximations in Vector-Valued Native Spaces

Some key results presented in this paper, especially those concerned with the limiting DPS originating the proposed MRAC solutions, are not implementable in practice because the control relies on terms that are contained in infinite-dimensional native spaces. Thus, these feedback controls, and the corresponding MRAC systems, need to be approximated in finite dimensions. To define approximations of $\boldsymbol{\mathcal{H}}(\mathbb{X}, \mathbb{R}^m)$, we begin by building approximations of $f : \mathbb{X} \to \mathbb{R}^m$ with a selection of distinct points or centers Ξ_N given by (1), where $\Omega \subset \mathbb{X}$ is compact. The positivity of the matrix kernel $\mathcal{K}(\cdot, \cdot)$ is equivalent to the positive semidefiniteness of the *generalized Grammian matrix*

$$\mathbb{K}_N \triangleq \begin{bmatrix} \mathcal{K}(\xi_1,\xi_1) & \cdots & \mathcal{K}(\xi_1,\xi_N) \\ \vdots & \ddots & \vdots \\ \mathcal{K}(\xi_N,\xi_1) & \cdots & \mathcal{K}(\xi_N,\xi_N) \end{bmatrix} \in \mathbb{R}^{mN\times mN}. \tag{13}$$

Given a collection of centers Ξ_N, we define the *space of approximants*

$$\mathcal{H}_N \triangleq \operatorname{span}\left\{\mathcal{K}_{\xi_i} e_j : \xi_i \in \Xi_N \text{ for } 1 \le i \le N, 1 \le j \le m\right\} \tag{14a}$$
$$= \operatorname{span}\left\{\mathcal{K}_{\xi_i} u : \xi_i \in \Xi_N \text{ for } 1 \le i \le N, u \in \mathbb{U}\right\}, \tag{14b}$$

where e_j, $j \in \{1,\ldots,m\}$, denotes the jth canonical basis for $\mathbb{U} \triangleq \mathbb{R}^m$, and $\mathcal{K}_y(x) \triangleq \mathcal{K}(x,y)$ for any $x, y \in \mathbb{X}$. In this paper, we assume that the collection of all centers $\Xi \triangleq \bigcup_{N=1}^{\infty} \Xi_N$ is dense in the compact set $\Omega \subseteq \mathbb{X}$, that is, $\Omega \subseteq \overline{\bigcup_{N\in\mathbb{N}} \Xi_N} \subseteq \mathbb{X}$.

The kernel underlying $\mathcal{H}_N$ is given by [35]

$$\mathcal{K}_N(x_1,x_2) \triangleq \mathcal{K}_{\Xi_N}(x_1)\mathbb{K}_N^{-1}\mathcal{K}_{\Xi_N}(x_2), \quad \text{for all } x_1, x_2 \in \mathbb{X}, \tag{15}$$

where $\mathbb{K}_N^{-1} \triangleq (\mathbb{K}_N)^{-1}$ denotes the inverse of the generalized Grammian (13), and

$$\mathcal{K}_{\Xi_N}(x) \triangleq \left[\mathcal{K}_{\xi_1}(x),\ldots,\mathcal{K}_{\xi_N}(x)\right] \in \mathbb{R}^{m\times mN}, \quad \text{for all } x \in \mathbb{X}. \tag{16}$$

We define $\boldsymbol{\Pi}_N : \mathcal{H} \to \mathcal{H}_N$ to be *$\mathcal{H}$-orthogonal projection onto $\mathcal{H}_N \subseteq \mathcal{H}$.*
Since the kernel is of strictly positive type, the $\mathcal{H}$-orthogonal projection operator $\boldsymbol{\Pi}_N$ can be understood as an interpolation operator. Indeed, it can be proven that

$$\left\langle (I-\boldsymbol{\Pi}_N)f, \mathcal{K}_{\xi_i}\alpha\right\rangle_{\mathcal{H}} = 0 \tag{17}$$

for each $\xi_i \in \Xi_N$ and $\alpha \in \mathbb{R}^m$. Furthermore, it can be shown that

$$0 = \left\langle (I-\boldsymbol{\Pi}_N)f, E_{\xi_i}^*\alpha\right\rangle_{\mathcal{H}} = \left\langle E_{\xi_i}(I-\boldsymbol{\Pi}_N)f, \alpha\right\rangle_{\mathbb{R}^m}, \quad \text{for all } \xi_i \in \Xi_N, \alpha \in \mathbb{R}^m, \tag{18}$$

which implies that the projection operator interpolates f with $f(\xi_i) = (\boldsymbol{\Pi}_N f)(\xi_i)$ at each center $\xi_i \in \Xi_N$.

4 Problem Statement

We consider the problem of controlling nonlinear plant models in the same form as

$$\dot{x}(t) = Ax(t) + B\Lambda[u(t) + E_{x(t)}f], \quad x(t_0) = x_0, \quad t \ge t_0, \tag{19}$$

where $x : [t_0,\infty) \to \Omega$ denotes the *plant state* or *plant trajectory*, $\Omega \subset \mathbb{X}$ is a user-defined compact set, $u : [t_0,\infty) \to \mathbb{R}^m$ denotes the *control input*, $A \in \mathbb{R}^{n\times n}$ is unknown, $B \in \mathbb{R}^{n\times m}$, $\Lambda \in \mathbb{R}^{m\times m}$ is diagonal, positive-definite, and unknown, the pair $(A, B\Lambda)$ is controllable, $f \in \mathcal{H}(\Omega,\mathbb{R}^m)$ denotes the *matched uncertainty*, and E_x denotes the evaluation operator (see Definition 5). Remarkably, the classical MRAC design problem [6, Ch. 9] considers plants in the same form as (19) with a major difference. In the plant models addressed by MRAC, the matched uncertainty is assumed to lie in a finite-dimensional space spanned by a regressor vector. In this paper, the matched uncertainty is assumed to lie in an infinite-dimensional vector-valued RKHS. This is the first paper in the nonparametric MRAC literature that includes the unknown diagonal matrix Λ in the plant model to account for uncertainties, faults, and failures in the plant actuators.

Consider the *reference model*

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t), \quad x_r(t_0) = x_{r,0}, \quad t \ge t_0, \tag{20}$$

where $x_r : [t_0,\infty) \to \Omega$ denotes the *reference trajectory*, the *reference command input* $r : [t_0,\infty) \to \mathbb{R}^m$ is continuous and uniformly bounded, $A_r \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_r \in \mathbb{R}^{n\times m}$, and the pair (A_r, B_r) is controllable. Our goal is to design an MRAC system such that $\lim_{t\to\infty}\|x(t) - x_r(t)\| = 0$, and the closed-loop plant trajectory lies in a user-defined, time-varying, simply connected, and compact constraint set given by

$$\overline{\mathcal{E}}_{t,\text{user}} \triangleq \{e \in \mathbb{R}^n : H_{\text{user}}(t, e^{\mathrm{T}}Me) \ge 0\}, \quad \text{for each } t \in [t_0,\infty), \tag{21}$$

where $e(t) \triangleq x(t) - x_r(t)$, $t \ge t_0$, denotes the *trajectory tracking error*, $H_{\text{user}} : [t_0,\infty)\times\mathbb{R} \to \mathbb{R}$ is continuously differentiable in both arguments, $M \in \mathbb{R}^{n\times n}$ is symmetric, nonnegative-definite, user-defined, and such that $A_r^{\mathrm{T}}M + MA_r < 0$, and

$$H_{\text{user}}(t,0) > 0, \quad \text{for all } t \in [t_0,\infty) \tag{22a}$$

$$H_{\min}(e^{\mathrm{T}}Me) \le H_{\text{user}}(t, e^{\mathrm{T}}Me) \le H_{\max}(e^{\mathrm{T}}Me), \tag{22b}$$

$$\left.\frac{\partial H_{\text{user}}(t,\alpha)}{\partial\alpha}\right|_{\alpha=e^{\mathrm{T}}Me} \le 0, \quad \text{for all } (t,e) \in [t_0,\infty) \times \mathring{\mathcal{E}}_{t,\text{user}}, \tag{22c}$$

$$\left|\left.\frac{\partial H_{\text{user}}(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}Me}\right| \le k, \quad \text{for all } (t,e) \in [t_0,\infty) \times \overline{\mathcal{E}}_{t,\text{user}}, \tag{22d}$$

where $k > 0$ is a user-defined arbitrarily large constant, $H_{\min}, H_{\max} : \mathbb{R} \to \mathbb{R}$, $H_{\max}(e^{\mathrm{T}}Me) > H_{\min}(e^{\mathrm{T}}Me) > 0$ for all $(t,e) \in [t_0,\infty) \times \mathring{\mathcal{E}}_{t,\text{user}}$, and

$$\{e \in \mathbb{R}^n : H_{\max}(e^{\mathrm{T}}Me) \ge 0\} \subset \Omega. \tag{23}$$

An example of function $H_{\text{user}}(\cdot,\cdot)$ that meets the requirements captured by (22) is given by

$$H_{\text{user}}(t, e^{\mathrm{T}}Me) = e_{\max}^2 - e_{\text{bound}}^2(t) - e^{\mathrm{T}}Me, \quad (t,e) \in [t_0,\infty) \times \mathbb{R}^n, \tag{24}$$

where $e_{\text{bound}} : [t_0,\infty) \to [0,\bar{E})$, $\left|\frac{de_{\text{bound}}(t)}{dt}\right| \le k$ for all $t \ge t_0$, $e_{\max} > 0$, and $\bar{E} \in [0, e_{\max})$ are user-defined. Indeed, (22a) is verified since $H_{\text{user}}(t,0) \in [e_{\max}^2 - \bar{E}^2, e_{\max}^2] \subset (0,\infty)$, (22b) is verified by

$$H_{\max}(e^{\mathrm{T}}Me) = e_{\max}^2 - e^{\mathrm{T}}Me, \tag{25a}$$

$$H_{\min}(e^{\mathrm{T}}Me) = e_{\max}^2 - E^2 - e^{\mathrm{T}}Me, \tag{25b}$$

(22c) is verified since $\left.\frac{\partial H_{\text{user}}(t,\alpha)}{\partial\alpha}\right|_{\alpha=e^{\mathrm{T}}Me} = -1$, and (22d) is verified by assumption. The smaller $e_{\max}$, the smaller the diameter of the constraint set, and, in general, the larger the control effort to enforce such constraints.

Consistent with the classical MRAC formulation, we assume that the *matching conditions*

$$A_r = A + B\Lambda\alpha^{\mathrm{T}}, \tag{26a}$$

$$B_r = B\Lambda\beta^{\mathrm{T}}, \tag{26b}$$

are verified by some $\alpha \in \mathbb{R}^{n\times m}$ and $\beta \in \mathbb{R}^{m\times m}$, which are unknown. The matching conditions imply that at least one control input exists that can track the reference trajectory of the closed-loop plant trajectory.

5 Limiting DPS Control and Adaptive Law Formulation

To meet the design specifications outlined in Section 4, first, we design an MRAC system whose adaptive gains corresponding to the matched uncertainty evolve in the vector-valued RKHS $\boldsymbol{\mathcal{H}}(\Omega,\mathbb{R}^m) = \mathcal{H}^m(\Omega,\mathbb{R})$, and, hence, form a *limiting DPS*. For the statement of this result, consider the functional uncertainty class

$$\mathcal{C}_R \triangleq \left\{f \in \boldsymbol{\mathcal{H}} : \|f\|_{\boldsymbol{\mathcal{H}}} \le R\right\} \tag{27}$$

of functions that lie in the RKHS $\boldsymbol{\mathcal{H}}(\Omega,\mathbb{R}^m)$ and are bounded by some user-defined constant $R > 0$. The assumption whereby the uncertainties are bounded is extremely mild since, in general, no control system can counter an infinitely large uncertainty. Furthermore, it is common practice in robust and adaptive control to assume that an upper bound on the uncertainties and disturbances is known. Approaches based on the existence of such bound, but not requiring knowledge of its value are left as future research topics.

Consider the control law

$$\phi(x,r,\hat{\alpha},\hat{\beta},\hat{f}) = \hat{\alpha}^{\mathrm{T}}x + \hat{\beta}^{\mathrm{T}}r + E_x\hat{f}, \quad \text{for all } (x,r,\hat{\alpha},\hat{\beta},\hat{f}) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathcal{H}^m(\Omega,\mathbb{R}), \tag{28}$$

where $\hat{\alpha} : [t_0,\infty) \to \mathbb{R}^{n\times n}$, $\hat{\beta} : [t_0,\infty) \to \mathbb{R}^{n\times m}$, and $\hat{f} : [t_0,\infty) \to \mathcal{H}^m(\Omega,\mathbb{R})$ denote the *adaptive gains*. The adaptive gains verify the *adaptive laws*

$$\dot{\hat{\alpha}}(t) = -\Gamma_\alpha x(t)e^{\mathrm{T}}(t)\tilde{P}(t,e(t))B, \quad \hat{\alpha}(t_0) = \hat{\alpha}_0, \quad t \ge t_0, \tag{29a}$$

$$\dot{\hat{\beta}}(t) = -\Gamma_\beta r(t)e^{\mathrm{T}}(t)\tilde{P}(t,e(t))B, \quad \hat{\beta}(t_0) = \hat{\beta}_0, \tag{29b}$$

$$\frac{\partial\hat{f}(t,\cdot)}{\partial t} = \Gamma_f\mathcal{K}_{x(t)}B^{\mathrm{T}}\tilde{P}(t,e(t))e(t), \quad \hat{f}(t_0,\cdot) = \hat{f}_0(\cdot), \tag{29c}$$

where $\Gamma_\alpha \in \mathbb{R}^{n\times n}$, $\Gamma_\beta \in \mathbb{R}^{m\times m}$, and $\Gamma_f \in \mathbb{R}^{m\times m}$ denote the user-defined, symmetric, and positive-definite *adaptive rate matrices*,

$$\tilde{P}(t,e) \triangleq H^{-1}(t, e^{\mathrm{T}}Me) \left(P - V_e(t,e)\frac{\partial H(t,\alpha)}{\partial \alpha}\bigg|_{\alpha=e^{\mathrm{T}}Me} M\right), \quad \text{for all } (t,e) \in [t_0,\infty) \times \mathring{\mathcal{E}}, \tag{30}$$

$$V_e(t,e) \triangleq \frac{\langle e, Pe\rangle_{\mathbb{R}^n}}{H(t,e^{\mathrm{T}}Me)}, \quad \text{for all } (t,e) \in [t_0,\infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right), \tag{31}$$

the symmetric, positive-definite matrix $P \in \mathbb{R}^{n\times n}$ verifies the *Lyapuov equation*

$$-Q = A_r^{\mathrm{T}}P + PA_r, \tag{32}$$

and the symmetric, positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ is user-defined. It is worth noting that the adaptive law in (29c) is not an ordinary differential equation (ODE) defined in Euclidean space. Indeed, it can be viewed either as a PDE or as an ODE taking values in Hilbert spaces, which justifies its classification as a DPS.

In (29), we employ the constraint function $H : [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$ defined so that

$$\frac{\partial}{\partial t}H(t,e^{\mathrm{T}}(t)Pe(t)) = \begin{cases} \frac{\partial}{\partial t}H_{\text{user}}(t,e^{\mathrm{T}}(t)Pe(t)), \text{ if } \langle e(t), Qe(t)\rangle_{\mathbb{R}^n} \\ \quad > -V_e(t,\alpha)\, \frac{\partial H(t,e^{\mathrm{T}}(t)Pe(t))}{\partial t}\Big|_{\alpha=e^{\mathrm{T}}(t)Me(t)} \\ 0, \quad \text{otherwise,} \end{cases}$$
$$H(t_0,\alpha(t_0)) = H_{\text{user}}(t_0,\alpha(t_0)), \quad t \geq t_0. \tag{33}$$

Remarkably, it follows from (33) and (22c) that

$$H_{\min}(e^{\mathrm{T}}Me) \leq H(t,e^{\mathrm{T}}Me) \leq H_{\max}(e^{\mathrm{T}}Me), \quad \text{for all } (t,e) \in [t_0,\infty)\times\mathbb{R}^n, \tag{34}$$

and

$$\overline{\mathcal{E}}_{t,\text{user}} \subseteq \overline{\mathcal{E}}_t \subset \Omega, \quad \text{for all } t \in [t_0,\infty), \tag{35}$$

where

$$\overline{\mathcal{E}}_t \triangleq \left\{e \in \mathbb{R}^n : H(t,e^{\mathrm{T}}Me) \geq 0\right\}. \tag{36}$$

The constraint function $H(\cdot,\cdot)$ approximates $H_{\text{user}}(\cdot,\cdot)$ compatibly with the controller's performance and the plant uncertainties. Indeed, as it will become apparent from the Lyapunov analysis in the proof of Theorem 3 below, $\langle e(t), Qe(t)\rangle_{\mathbb{R}^n}$ can be interpreted as a measure of the energy of the closed-loop system dissipated by the control system at time $t \in [t_0,\infty)$ and $-V_e(t,e^{\mathrm{T}}(t)Pe(t))\, \frac{\partial H(t,\alpha)}{\partial t}\Big|_{\alpha=e^{\mathrm{T}}(t)Me(t)}$ can be interpreted as a measure of the energy injected into the controlled system by contracting the constraint set. For this reason, less restrictive constraints, captured by $H(\cdot,\cdot)$, may be needed. However, the resulting closed and bounded constraint set $\overline{\mathcal{E}}_t$ is still contained in the user-defined compact set Ω.

The following result captures the effectiveness of the proposed MRAC system.

Theorem 3 *Consider the plant model* (19)*, the reference model* (20)*, the constraint set* (36) *with constraint function $H(\cdot,\cdot)$ such that* (33) *is verified, the control input* (28)*, and the adaptive laws* (29)*. Assume that the kernel $\mathfrak{K}(\cdot,\cdot)$ underlying $\mathcal{H}(\Omega,\mathbb{R})$ is uniformly bounded, that is, there exists $\overline{\mathfrak{K}}$ such that $\mathfrak{K}(x,y) \leq \overline{\mathfrak{K}}$ for all $(x,y) \in \Omega\times\Omega$. Assume also that there exist unique forward complete solutions of* (19) *with $u(t) = \phi(x(t), r(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{f}(t,\cdot))$ and* (29) *for all $t \in [t_0,\infty)$, and for all*

$$(x_0, \hat{\alpha}_0, \hat{\beta}_0, \hat{f}_0(\cdot)) \in \Omega \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \times C^0(\mathbb{R}^n,\mathbb{R}^m).$$

Finally, assume that $f \in \mathcal{C}_R$. If $e(t_0) \in \mathring{\mathcal{E}}_{t_0}$, then $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \in [t_0,\infty)$, $\lim_{t\to\infty} e(t) = 0$, and the adaptive gains $\hat{\alpha}(t)$, $\hat{\beta}(t)$, and $\hat{f}(t,\cdot)$ are uniformly bounded for all $t \in [t_0,\infty)$.

Proof It follows from (19) with control law (28), (20), and the matching conditions (26) that

$$\dot{e}(t) = A_{\text{ref}}e(t) + B\Lambda\left(\tilde{K}^{\mathrm{T}}(t)\pi(t) - E_{x(t)}\tilde{f}(t,\cdot)\right), \quad e(t_0) = e_0, \quad t \geq t_0, \tag{37}$$

where $\tilde{K}(t) \triangleq \hat{K}(t) - K$, $\hat{K}(t) \triangleq \left[\hat{\alpha}^{\mathrm{T}}(t), \hat{\beta}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{(n+m)\times m}$, $K \triangleq \left[\alpha^{\mathrm{T}}, \beta^{\mathrm{T}}\right]^{\mathrm{T}}$, $\pi(t) \triangleq \left[x^{\mathrm{T}}(t), r^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and $\tilde{f}(t,\cdot) \triangleq \hat{f}(t,\cdot) - f$. Thus, the proof is divided into two parts. First, we assume that $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$, and we prove that the equilibrium point $(e, \tilde{K}, \tilde{f}) =$

$(0_n, 0_{(n+m)\times m}, 0_{\boldsymbol{\mathcal{H}}})$ of the closed-loop system (37) and (29) is uniformly Lyapunov stable and $\lim_{t\to\infty} e(t) = 0$. Successively, we employ a contradiction argument to prove that if $e(t_0) \in \mathring{\mathcal{E}}_t$, then $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$.

Assume that $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$. To prove Lyapunov stability of the equilibrium point $(e, \tilde{K}, \tilde{f}) = (0_n, 0_{(n+m)\times m}, 0_{\boldsymbol{\mathcal{H}}}) \in \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \times \mathbb{R}^{(n+m)\times m} \times \boldsymbol{\mathcal{H}}$ for the closed-loop system, let

$$\begin{aligned} V(t, e, \tilde{K}, \tilde{f}) = V_e(t, e) &+ \left\langle \tilde{K}, \Gamma_K^{-1} \tilde{K} \Lambda \right\rangle_{\mathrm{tr}} \\ &+ \left\langle \tilde{f}, \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f} \right\rangle_{\boldsymbol{\mathcal{H}}}, \\ &(t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \mathring{\mathcal{E}}_t \times \mathbb{R}^{(n+m)\times m} \\ &\times \boldsymbol{\mathcal{H}}, \end{aligned} \tag{38}$$

where $\Gamma_K \triangleq \text{block-diag}(K_\alpha, K_\beta)$, the diagonal operator $\mathcal{L}_\Lambda : \boldsymbol{\mathcal{H}} \mapsto \boldsymbol{\mathcal{H}}_{\mathcal{L}_\Lambda}$ is defined so that

$$\Lambda E_x f = E_{x,\mathcal{L}_\Lambda} \mathcal{L}_\Lambda f, \quad \text{for all } (x, f) \in \mathbb{X} \times \boldsymbol{\mathcal{H}}, \tag{39}$$

and $E_{x,\mathcal{L}_\Lambda}$ denotes the evaluation operator defined on the RKHS

$$\begin{aligned} &\boldsymbol{\mathcal{H}}_{\mathcal{L}_\Lambda}(\Omega, \mathbb{R}^m) \\ &\quad \triangleq \{g = \mathcal{L}_\Lambda f : f \in \boldsymbol{\mathcal{H}}(\Omega, \mathbb{R}^m), \mathcal{L}_\Lambda \text{ is diagonal}\}. \end{aligned} \tag{40}$$

Next, we prove that, if $\boldsymbol{\mathcal{H}}(\Omega, \mathbb{R}^m) = \mathcal{H}^m(\Omega, \mathbb{R})$, then $\boldsymbol{\mathcal{H}}_{\mathcal{L}_\Lambda}(\Omega, \mathbb{R}^m) = \boldsymbol{\mathcal{H}}(\Omega, \mathbb{R}^m)$ and, hence, $E_{x,\mathcal{L}_\Lambda} = E_x$ for all $x \in \mathbb{X}$. By the reproducing property (9), for any $f \in \boldsymbol{\mathcal{H}}$ and $\alpha \in \mathbb{R}^m$, it holds that

$$\begin{aligned} \alpha^{\mathrm{T}} \left(\Lambda^{\mathrm{T}} (E_x f)\right) &= \langle E_{x,\mathcal{L}_\Lambda} \mathcal{L}_\Lambda^* f, \alpha \rangle_{\mathbb{U}} \\ &= \left\langle \mathcal{L}_\Lambda^* f, E_{x,\mathcal{L}_\Lambda}^* \alpha \right\rangle_{\boldsymbol{\mathcal{H}}_{\mathcal{L}_\Lambda}} \\ &= \left\langle f, \mathcal{L}_\Lambda E_{x,\mathcal{L}_\Lambda}^* \alpha \right\rangle_{\boldsymbol{\mathcal{H}}}, \end{aligned} \tag{41}$$

$$(\Lambda\alpha)^{\mathrm{T}} E_x f = \langle E_x f, \Lambda\alpha \rangle_{\mathbb{U}} = \langle f, E_x^* \Lambda\alpha \rangle_{\boldsymbol{\mathcal{H}}}. \tag{42}$$

Hence, $\mathcal{K}_{x,\mathcal{L}_\Lambda}(\cdot) = \mathcal{L}_\Lambda^{-1} K_x(\cdot) \mathcal{L}_\Lambda$, and since $\boldsymbol{\mathcal{H}} = \mathcal{H}^m$, it holds that

$$\begin{aligned} \mathcal{K}_{x,\mathcal{L}_\Lambda}(\cdot) &= \mathcal{L}_\Lambda^{-1} \text{diag}\left(\mathfrak{K}_\xi(\cdot), \ldots, \mathfrak{K}_\xi(\cdot)\right) \mathcal{L}_\Lambda \\ &= \text{diag}\left(\mathfrak{K}_\xi(\cdot), \ldots, \mathfrak{K}_\xi(\cdot)\right) \mathcal{L}_\Lambda^{-1} \mathcal{L}_\Lambda = \mathcal{K}_x(\cdot) \end{aligned} \tag{43}$$

Since $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}_{\mathcal{L}_\Lambda}$ share the same kernel, by the Moore-Aronszajn theorem [36], these spaces coincide.

Now, we prove that (38) is positive-definite and, hence, is a Lyapunov function candidate for the system given by (37) and (29). It follows from (22b) that

$$\begin{aligned} &\frac{\langle e, Pe \rangle_{\mathbb{R}^n}}{H_{\max}(e^{\mathrm{T}} M e)} + \left\langle \tilde{K}, \Gamma_K^{-1} \tilde{K} \Lambda \right\rangle_{\mathrm{tr}} + \left\langle \tilde{f}, \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f} \right\rangle_{\boldsymbol{\mathcal{H}}} \\ &\quad \leq V(t, e, \tilde{K}, \tilde{f}) \leq \frac{\langle e, Pe \rangle_{\mathbb{R}^n}}{H_{\min}(e^{\mathrm{T}} M e)} + \left\langle \tilde{K}, \Gamma_K^{-1} \tilde{K} \Lambda \right\rangle_{\mathrm{tr}} \\ &\quad + \left\langle \tilde{f}, \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f} \right\rangle_{\boldsymbol{\mathcal{H}}}, \\ &\quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t \right) \times \mathbb{R}^{(n+m)\times m} \\ &\quad \times \boldsymbol{\mathcal{H}}. \end{aligned} \tag{44}$$

Hence, $V_e(t, e)$ is bounded from above and below by class $\mathcal{K}_\infty$ functions of $\|e\|$ for all $t \in [t_0, \infty)$. Furthermore, $\left\langle \tilde{K}, \Gamma_K^{-1} \tilde{K} \Lambda \right\rangle_{\mathrm{tr}}$ is equivalent to the squared Frobenius norm of $\Gamma_K^{-1/2} \tilde{K} \Lambda^{1/2}$, where $\Gamma_K^{-1/2} = \left(\Gamma_K^{1/2}\right)^{-1}$, $\Gamma_K^{1/2} \in \mathbb{R}^{(n+m)\times(n+m)}$ is symmetric and positive-definite, $\Gamma_K = \Gamma_K^{1/2} \Gamma_K^{1/2}$, $\Lambda^{1/2} \in \mathbb{R}^{m\times m}$ is symmetric and positive-definite, and $\Lambda = \Lambda^{1/2} \Lambda^{1/2}$. Thus, the term $\left\langle \tilde{K}, \Gamma_K^{-1} \tilde{K} \Lambda \right\rangle_{\mathrm{tr}}$ is positive-definite in $\tilde{K}$. Finally, it follows from (39) that $\mathcal{L}_\Lambda = E_x^* \Lambda E_x$ for any $x \in \mathbb{X}$, and

$$\left\langle \tilde{f}, \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f} \right\rangle_{\boldsymbol{\mathcal{H}}} = \left\langle E_x \tilde{f}, \Gamma_f^{-1} \Lambda E_x \tilde{f} \right\rangle_{\mathbb{R}^m}, \tag{45}$$

which is positive-definite since Λ is symmetric and positive-definite. Thus, $\left\langle \tilde{f}, \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f} \right\rangle_{\boldsymbol{\mathcal{H}}}$ is positive-definite in $\tilde{f}$.

Taking the time derivative of (38) along the trajectories of (37) and (29) yields

$$\begin{aligned} &\dot{V}(t, e, \tilde{K}, \tilde{f}) \\ &\quad = (\langle \dot{e}(t), Pe \rangle_{\mathbb{R}^n} + \langle e, P\dot{e}(t) \rangle_{\mathbb{R}^n}) \, H(t, e^{\mathrm{T}} M e) H^{-2}(t, e^{\mathrm{T}} M e) \\ &\quad - \langle e, Pe \rangle_{\mathbb{R}^n} \left(\left. \frac{\partial H(t, \alpha)}{\partial \alpha} \right|_{e^{\mathrm{T}} M e} (\langle \dot{e}(t), Me \rangle_{\mathbb{R}^n} \right. \\ &\quad + \langle e, M\dot{e}(t) \rangle_{\mathbb{R}^n}) \\ &\quad \left. + \left. \frac{\partial H(t, \alpha)}{\partial t} \right|_{e^{\mathrm{T}} M e} \right) H^{-2}(t, e^{\mathrm{T}} M e) \\ &\quad + 2 \left\langle \dot{\tilde{K}}(t), \Gamma_K^{-1} \tilde{K} \Lambda \right\rangle_{\mathrm{tr}} + 2 \left\langle \dot{\tilde{f}}(t), \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f} \right\rangle_{\boldsymbol{\mathcal{H}}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{H(t, e^{\mathrm{T}}Me)}\Bigg(\Big\langle A_r e - B\Lambda \tilde{K}^{\mathrm{T}}\pi(t) + BE_x\mathcal{L}_\Lambda \tilde{f}, Pe\Big\rangle_{\mathbb{R}^n} \\
&\quad + \Big\langle e, P\left(A_r e - B\Lambda\tilde{K}^{\mathrm{T}}\pi(t) + BE_x\mathcal{L}_\Lambda \tilde{f}\right)\Big\rangle_{\mathbb{R}^n}\Bigg) \\
&\quad - \frac{V_e(t,e)}{H(t, e^{\mathrm{T}}Me)} \left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me} \\
&\quad \Big(\Big\langle A_r e - B\Lambda\tilde{K}^{\mathrm{T}}\pi(t) + BE_x\mathcal{L}_\Lambda\tilde{f}, Me\Big\rangle_{\mathbb{R}^n} \\
&\quad + \Big\langle e, M\left(A_r e - B\Lambda\tilde{K}^{\mathrm{T}}\pi(t) + BE_x\mathcal{L}_\Lambda\tilde{f}\right)\Big\rangle_{\mathbb{R}^n}\Big) \\
&\quad - \frac{V_e(t,e)}{H(t, e^{\mathrm{T}}Me)} \left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me} \\
&\quad + 2\Big\langle \dot{\hat{K}}(t), \Gamma_K^{-1}\tilde{K}\Lambda\Big\rangle_{\mathrm{tr}} + 2\Big\langle \dot{\hat{f}}(t), \Gamma_f^{-1}\mathcal{L}_\Lambda\tilde{f}\Big\rangle_{\boldsymbol{\mathcal{H}}} \\
&= \frac{1}{H(t, e^{\mathrm{T}}Me)}\Bigg(\langle A_r e, Pe\rangle_{\mathbb{R}^n} + \langle e, PA_r e\rangle_{\mathbb{R}^n} \\
&\quad - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}(\langle A_r e, Me\rangle_{\mathbb{R}^n} + \langle e, MA_r e\rangle_{\mathbb{R}^n})\Bigg) \\
&\quad - \frac{2}{H(t, e^{\mathrm{T}}Me)}\left(\Big\langle PB\Lambda\tilde{K}^{\mathrm{T}}\pi(t), e\Big\rangle_{\mathbb{R}^n} - \Big\langle PBE_x\mathcal{L}_\Lambda\tilde{f}, e\Big\rangle_{\mathbb{R}^n}\right) \\
&\quad + \frac{2V_e(t,e)}{H(t, e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}\Big(\Big\langle MB\Lambda\tilde{K}^{\mathrm{T}}\pi(t), e\Big\rangle_{\mathbb{R}^n} \\
&\quad - \Big\langle MBE_x\mathcal{L}_\Lambda\tilde{f}, e\Big\rangle_{\mathbb{R}^n}\Big) \\
&\quad - \frac{V_e(t,e)}{H(t, e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me} \\
&\quad + 2\Big\langle \dot{\hat{K}}(t), \Gamma_K^{-1}\tilde{K}\Lambda\Big\rangle_{\mathrm{tr}} + 2\Big\langle \dot{\hat{f}}(t), \Gamma_f^{-1}\mathcal{L}_\Lambda\tilde{f}\Big\rangle_{\boldsymbol{\mathcal{H}}}, \\
&\quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \times \mathbb{R}^{(n+m)\times m} \times \boldsymbol{\mathcal{H}}.
\end{aligned} \tag{46}
$$

The series of arguments in (46) relies on the identity

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\Big\langle\tilde{f}(t,\cdot), \tilde{f}(t,\cdot)\Big\rangle_{\boldsymbol{\mathcal{H}}}\right) = \left\langle\frac{\partial\tilde{f}(t,\cdot)}{\partial t}, \tilde{f}(t,\cdot)\right\rangle_{\boldsymbol{\mathcal{H}}}, \quad \text{for all } t \in [t_0, \infty). \tag{47}
$$

As pointed out in [2, Th. 5.3], this identity holds since $\tilde{f} \in C^1(\mathbb{R}^+, \boldsymbol{\mathcal{H}})$ implies that $t \mapsto \frac{\partial\hat{f}(t,\cdot)}{\partial t} \in C^0(\mathbb{R}^+, \boldsymbol{\mathcal{H}})$. Since $H(t, e^{\mathrm{T}}(t)Me(t))$ and $e(t)$ are solutions of the ODEs in (33) and (37), respectively, (47) is verified.

From (32), the nonnegative-definiteness of M, and (22), it follows that

$$
\begin{aligned}
&\dot{V}(t, e, \tilde{K}, \tilde{f}) \\
&\le -\frac{1}{H(t, e^{\mathrm{T}}Me)}\left(\langle e, Qe\rangle_{\mathbb{R}^n} + V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right) \\
&\quad - \frac{2}{H(t, e^{\mathrm{T}}Me)}\left\langle\left(P - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}M\right)B\Lambda\tilde{K}^{\mathrm{T}}\pi(t), e\right\rangle_{\mathbb{R}^n} \\
&\quad + \frac{2}{H(t, e^{\mathrm{T}}Me)}\left\langle\left(P - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}M\right)BE_x\mathcal{L}_\Lambda\tilde{f}, e\right\rangle_{\mathbb{R}^n} \\
&\quad + 2\Big\langle\dot{\hat{K}}(t), \Gamma_K^{-1}\tilde{K}\Lambda\Big\rangle_{\mathrm{tr}} + 2\Big\langle\dot{\hat{f}}(t), \Gamma_f^{-1}\mathcal{L}_\Lambda\tilde{f}\Big\rangle_{\boldsymbol{\mathcal{H}}} \\
&= -\frac{1}{H(t, e^{\mathrm{T}}Me)}\left(\langle e, Qe\rangle_{\mathbb{R}^n} + V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right) \\
&\quad - 2\left(\left\langle\pi(t)e^{\mathrm{T}}H^{-1}(t, e^{\mathrm{T}}Me)\left(P - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}M\right)B, \tilde{K}\Lambda\right\rangle_{\mathrm{tr}}\right. \\
&\quad - \left\langle\left(H^{-1}(t, e^{\mathrm{T}}Me)\left(P - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}M\right)B\right)^* e, \right\rangle_{\mathbb{R}^n} E_x\mathcal{L}_\Lambda\tilde{f} \\
&\quad \left. - \Big\langle\Gamma_K\dot{\hat{K}}(t), \tilde{K}\Lambda\Big\rangle_{\mathrm{tr}} - \Big\langle\Gamma_f\dot{\hat{f}}(t), \mathcal{L}_\Lambda\tilde{f}\Big\rangle_{\boldsymbol{\mathcal{H}}}\right) \\
&= -\frac{1}{H(t, e^{\mathrm{T}}Me)}\left(\langle e, Qe\rangle_{\mathbb{R}^n} + V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right) \\
&\quad - 2\left(\left\langle\pi(t)e^{\mathrm{T}}H^{-1}(t, e^{\mathrm{T}}Me)\left(P - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}M\right)B, \tilde{K}\Lambda\right\rangle_{\mathrm{tr}}\right. \\
&\quad - \left\langle\mathcal{K}_x\left(H^{-1}(t, e^{\mathrm{T}}Me)\left(P - V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial\alpha}\right|_{e^{\mathrm{T}}Me}M\right)B\right)^* e, \right\rangle_{\boldsymbol{\mathcal{H}}}\mathcal{L}_\Lambda\tilde{f} \\
&\quad \left. - \Big\langle\Gamma_K\dot{\hat{K}}(t), \tilde{K}\Lambda\Big\rangle_{\mathrm{tr}} - \Big\langle\Gamma_f\dot{\hat{f}}(t), \mathcal{L}_\Lambda\tilde{f}\Big\rangle_{\boldsymbol{\mathcal{H}}}\right), \\
&\quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t\in[t_0,\infty)}\mathring{\mathcal{E}}_t\right) \\
&\quad \times \mathbb{R}^{(n+m)\times m} \times \boldsymbol{\mathcal{H}}.
\end{aligned} \tag{48}
$$

Thus, applying the adaptive laws (29), it holds that

$$
\begin{aligned}
&\dot{V}(t, e, \tilde{K}, \tilde{f}) \le -H^{-1}(t, e^{\mathrm{T}}Me) \\
&\quad \left(\langle e, Qe\rangle_{\mathbb{R}^n} + V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right), \\
&\quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t\in[t_0,\infty)}\mathring{\mathcal{E}}_t\right) \times \mathbb{R}^{(n+m)\times m} \\
&\quad \times \boldsymbol{\mathcal{H}}.
\end{aligned} \tag{49}
$$

It follows from (33) that, for all $e \neq 0$,

$$
\begin{aligned}
&\langle e, Qe\rangle_{\mathbb{R}^n} + V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me} \\
&= \left\langle e, \left(Q + \frac{P}{H(t, e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right)e\right\rangle_{\mathbb{R}^n} \\
&> 0,
\end{aligned} \tag{50}
$$

and, hence,

$$
\begin{aligned}
&\dot{V}(t, e, \tilde{K}, \tilde{f}) \\
&\le -\frac{1}{H_{\max}(e^{\mathrm{T}}Me)} \\
&\quad \left\langle e, \left(Q + \frac{P}{H(t, e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right)e\right\rangle_{\mathbb{R}^n}
\end{aligned}
$$

$$\leq 0, \quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t \in [t_0, \infty)} \mathring{\mathcal{E}}_t \right) \times \mathbb{R}^{(n+m)\times m} \times \mathcal{H}. \tag{51}$$

Since $\dot{V}(t, e(t), \tilde{K}(t), \tilde{f}(t)) \leq 0$ for all $t \geq t_0$, $V(t, e(t), \tilde{K}(t), \tilde{f}(t))$ is non-increasing, and since $V(t, e(t), \tilde{K}(t), \tilde{f}(t))$ is also positive-definite, it follows from the monotone convergence theorem that

$$\lim_{t\to\infty} V(t, e(t), \tilde{K}(t), \tilde{f}(t)) \triangleq V_\infty \leq V(t_0, e(t_0), \tilde{K}(t_0), \tilde{f}(t_0)). \tag{52}$$

Finally, $V(t, e(t), \tilde{K}(t), \tilde{f}(t))$ is bounded for all $t \geq t_0$ and the equilibrium point $(e, \tilde{K}, \tilde{f}) = (0_n, 0_{(n+m)\times m}, 0_{\mathcal{H}})$ of (29) and (37) is uniformly Lyapunov stable. Hence, all the closed-loop signals are bounded uniformly in $t_0 \geq 0$, that is, $e(\cdot) \in L^\infty([t_0, \infty), \mathbb{R}^n)$, $\tilde{K}(\cdot) \in L^\infty([t_0, \infty), \mathbb{R}^{(n+m)\times m})$, and $\tilde{f}(\cdot) \in L^\infty([t_0, \infty), \mathcal{H})$.

Next, we consider two alternative cases, namely if the constraint set contracts, and if the constraint set does not contract. If the constraint set contracts, it follows from (33) that

$$\left\langle e, \left(Q + \frac{P}{H(t, e^{\mathrm{T}} M e)} \left. \frac{\partial H(t, \alpha)}{\partial t} \right|_{\alpha = e^{\mathrm{T}} M e} \right) e \right\rangle_{\mathbb{R}^n} \geq c \|e\|^2_{\mathbb{R}^n} > 0, \quad \text{for all } e \neq 0_n, \tag{53}$$

where $c > 0$. Moreover, it follows from (22) that

$$0 < H(t, e^{\mathrm{T}} M e) \leq H_{\max}(e^{\mathrm{T}} M e) \leq H_{\max}(0), \quad \text{for all } (t, e) \in [t_0, \infty) \times \mathring{\mathcal{E}}_t. \tag{54}$$

Therefore,

$$\dot{V}(t, e, \tilde{K}, \tilde{f}) \leq -\frac{c}{H_{\max}(0)} \|e\|^2_{\mathbb{R}^n}, \quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t \in [t_0, \infty)} \mathring{\mathcal{E}}_t \right) \times \mathbb{R}^{(n+m)\times m} \times \mathcal{H}. \tag{55}$$

If the constraint set does not contract, then it follows from (22) that

$$\dot{V}(t, e, \tilde{K}, \tilde{f}) \leq -H^{-1}_{\max}(0) \left\langle e, \right\rangle_{\mathbb{R}^n} \left(Q + \frac{Pk}{H_{\max}(0)} \right) e, \quad (t, e, \tilde{K}, \tilde{f}) \in [t_0, \infty) \times \left(\bigcup_{t \in [t_0, \infty)} \mathring{\mathcal{E}}_t \right) \times \mathbb{R}^{(n+m)\times m} \times \mathcal{H}. \tag{56}$$

Thus, integrating (55) or (56) according to needs, we deduce that $e(\cdot) \in L^2(\mathbb{R}^+, \mathbb{R}^n)$. Recalling that $\|\tilde{f}(t, \cdot)\|_{\mathcal{H}} \leq \|\tilde{f}(t, \cdot)\|_{L^\infty(\mathbb{R}^+, \mathcal{H})}$ for all $t \geq t_0$ (see [2, Sec. 2.4.6]) and that, for a diagonally bounded kernel operator, the operator norm has ultimate bound $\|E_{x(t)}\|_{\mathcal{L}(\mathcal{H}, \mathbb{U})} = \|E^*_{x(t)}\|_{\mathcal{L}(\mathbb{U}, \mathcal{H})} \leq \overline{\mathfrak{K}}$, we deduce that

$$\begin{aligned}
\|\dot{e}(t)\|_{\mathbb{R}^n} &\leq \|A_r\| \|e\|_{L^\infty([t_0,\infty),\mathbb{R}^n)} \\
&\quad + \|B\Lambda\| \|\tilde{K}\|_{L^\infty([t_0,\infty),\mathbb{R}^{(n+m)\times m})} \|\pi(t)\|_{L^\infty([t_0,\infty),\mathbb{R}^{n+m})} \\
&\quad + \|B\Lambda\| \|E_{x(t)}\|_{L^\infty(\mathcal{H},\mathbb{U})} \|\tilde{f}(t,\cdot)\|_{\mathcal{H}} \\
&\leq \|A_r\| \|e\|_{L^\infty([t_0,\infty),\mathbb{R}^n)} \\
&\quad + \|B\Lambda\| \|\tilde{K}\|_{L^\infty([t_0,\infty),\mathbb{R}^{(n+m)\times m})} \|\pi(t)\|_{L^\infty([t_0,\infty),\mathbb{R}^{n+m})} \\
&\quad + \|B\Lambda\| \overline{\mathfrak{K}} \|\tilde{f}(t,\cdot)\|_{L^\infty([t_0,\infty),\mathcal{H})}, \quad t \geq t_0.
\end{aligned} \tag{57}$$

Notably, $\pi(\cdot) \in L^\infty([t_0, \infty), \mathbb{R}^{n+m})$ since the user-defined reference signal $r(t)$, and, consequently $x_r(t)$ solution to (20), are both uniformly bounded functions of time. Hence, since $e(t) = x(t) - x_r(t)$, $t \geq t_0$, it holds that $x(\cdot) \in L^\infty([t_0, \infty), \mathbb{R}^n)$. It follows from (57) proves that $\dot{e}(t) \in L^\infty([t_0, \infty), \mathbb{R}^n)$. Finally, since $e(\cdot) \in L^\infty([t_0, \infty), \mathbb{R}^n) \cap L^2([t_0, \infty), \mathbb{R}^n)$ and $\dot{e}(\cdot) \in L^\infty([t_0, \infty), \mathbb{R}^n)$, it follows from Barbalat's lemma that $\lim_{t\to\infty} e(t) = 0$ uniformly in $t_0 \geq 0$. This concludes the first part of this proof.

Next, assuming that $e(t_0) \in \mathring{\mathcal{E}}_{t_0}$, we prove that $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$. Assume, *ad absurdum*, that there exists $T > t_0$ such that $e(T) \in \partial\mathcal{E}_T$, that is, $H(T, e^{\mathrm{T}}(T) M e(T)) = 0$. In this case,

$$\begin{aligned}
&\lim_{t\to T^-} V(t, e(t), \tilde{K}(t), \tilde{f}(t,\cdot)) \\
&\quad = \lim_{t\to T^-} \left(\frac{\langle e(t), Pe(t)\rangle_{\mathbb{R}^n}}{H(t, e^{\mathrm{T}}(t) M e(t))} + \left\langle \tilde{K}(t), \Gamma_K^{-1} \tilde{K}(t) \Lambda \right\rangle_{\mathrm{tr}} \right. \\
&\qquad \left. + \left\langle \tilde{f}(t,\cdot), \Gamma_f^{-1} \mathcal{L}_\Lambda \tilde{f}(t,\cdot) \right\rangle_{\mathcal{H}} \right) = \infty;
\end{aligned} \tag{58}$$

note that $\lim_{t\to T^-}\left\langle \tilde{f}(t,\cdot), \Gamma_f^{-1}\mathcal{L}_\Lambda \tilde{f}(t,\cdot)\right\rangle_{\mathcal{H}}$ exists due to the forward completeness of the limiting DPS. However, it follows from (51) that $e(t)$, $\tilde{K}(t)$, and $\tilde{f}(t,\cdot)$ are bounded for all $t \geq t_0$, which contradicts (58), and the result is proven. □

The popular kernel functions presented in Section 3.2 are bounded, and, hence, the first assumption of Theorem 3 can be easily verified. The dynamical system given by (29c) is a DPS. Hence, the system given by (19) with $u(t) = \phi(x(t), r(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{f}(t,\cdot))$, for all $t \in [t_0, \infty)$, and (29c) is a DPS. Sufficient conditions for the existence of unique forward complete solutions of this kind of DPSs, and hence, to verify the second assumption of Theorem 3, are provided by [2, Th. 5.4].

The larger the adaptive rate matrices Γ_α, Γ_β, and Γ_f in some consistent matrix norm, the larger the variations of the adaptive gains to variations in the norm of the tracking error $e(\cdot)$ and the proximity of the tracking error to the boundary of the constraint set, which is measured by $H(\cdot,\cdot)$. The user-defined matrix Q that characterizes the Lyapunov equation (32) is usually set equal to the identity matrix in $\mathbb{R}^n$ to maximize the ratio $\lambda_{\max}(P)/\lambda_{\min}(Q)$ and, hence, maximize the variations of the adaptive gains to variations in the norm of the tracking error and the proximity of the tracking error to the boundary of the constraint set.

If the closed-loop plant trajectory $x(t)$ intersects a kernel center for some $t \geq t_0$, then the kernel section $\mathcal{K}_{x(t)}$ attains its maximum. Thus, the term $H^{-1}(t, e^{\mathrm{T}}(t)Me(t))\mathcal{K}_{x(t)}$ in (29c) captures a trade-off between the ability to capture uncertainties through a native spaces representation and the proximity of $x(t)$ to the boundary of the constraint set at time $t \geq t_0$.

Theorem 3 employs the constraint set $\overline{\mathcal{E}}_t$ defined for all $t \geq t_0$ through the function $H(\cdot,\cdot)$ such that (33) is verified. This function either follows the same rate of contraction of the constraint set provided by the user or stops the contraction of $\overline{\mathcal{E}}_t$ altogether. In the following, we present an alternative to Theorem 3, wherein the approximating constraint set $\overline{\mathcal{E}}_t$ is either contracted at the user-defined rate or at a lower rate. In particular, for all $t \geq t_0$, $\overline{\mathcal{E}}_t$ is characterized by $H : [t_0,\infty)\times\mathbb{R} \to \mathbb{R}$ such that

$$\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}(t)Me(t)} = \max\left\{ \left.\frac{\partial H_{\text{user}}(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}(t)Me(t)}, - H(t, e^{\mathrm{T}}(t)Me(t))\left(\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) - a\right)\right\},$$

$$H(t_0, \alpha(t_0)) = H_{\text{user}}(t_0, \alpha(t_0)), \quad t \geq t_0, \tag{59}$$

where $a > 0$ is user-defined and arbitrarily small. It is worthwhile noting that, in this case,

$$H(t, e^{\mathrm{T}}(t)Me(t)) = \max\left\{ H_{\text{user}}(t), H_{\text{user}}(t_0, \alpha(t_0))e^{-\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})+a}\right\}, \quad \text{for all } t \geq t_0, \tag{60}$$

and, hence, the constraint set $\overline{\mathcal{E}}_t$ contracts to the singleton $\{0_n\}$ exponentially fast at a rate bounded by $\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) - a$.

Theorem 4 *Consider the plant model (19), the reference model (20), the constraint set (36) with constraint function $H(\cdot,\cdot)$ such that (59) is verified, the control input (28), and the adaptive laws (29). Assume that the kernel $\mathfrak{K}(\cdot,\cdot)$ underlying $\mathcal{H}(\Omega,\mathbb{R})$ is uniformly bounded, that is, there exists $\overline{\mathfrak{K}}$ such that $\mathfrak{K}(x,y) \leq \overline{\mathfrak{K}}$ for all $(x,y) \in \Omega\times\Omega$. Assume also that there exist unique forward complete solutions of (19) with $u(t) = \phi(x(t), r(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{f}(t,\cdot))$ and (29) for all $t \in [t_0,\infty)$, and for all*

$$(x_0, \hat{\alpha}_0, \hat{\beta}_0, \hat{f}_0(\cdot)) \in \Omega\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times m}\times C^0(\mathbb{X},\mathbb{U}).$$

Finally, assume that $f \in \mathcal{C}_R$. If $e(t_0) \in \mathring{\mathcal{E}}_{t_0}$, then $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \in [t_0,\infty)$, $\lim_{t\to\infty} e(t) = 0$, and the adaptive gains $\hat{\alpha}(t)$, $\hat{\beta}(t)$, and $\hat{f}(t,\cdot)$ are uniformly bounded for all $t \in [t_0,\infty)$.

Proof This result follows as in the proof of Theorem 4. The key difference between these proofs lies in showing that

$$\begin{aligned}
&\dot{V}(t, e, \tilde{K}, \tilde{f}) \\
&\quad\leq \frac{2}{H_{\max}(e^{\mathrm{T}}Me)} \left\langle e, P\left(A_r - \frac{1}{2H(t,e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me} I\right)e\right\rangle_{\mathbb{R}^n} \\
&\quad\leq 0, \\
&\quad (t, e, \tilde{K}, \tilde{f}) \in [t_0,\infty)\times\left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \times\mathbb{R}^{(n+m)\times m}\times\mathcal{H}.
\end{aligned} \tag{61}$$

The rest of the proof is omitted for brevity. □

If the user-defined constraints are time-invariant, that is, $\frac{\partial}{\partial t}H_{\text{user}}(t, e^{\mathrm{T}}Me) = 0$ for all $(t,e) \in [t_0,\infty)\times\mathbb{R}^n$, then (22) reduces to

$$H_{\text{user}}(0) > 0, \tag{62a}$$

$$\left.\frac{\partial H_{\text{user}}(\alpha)}{\partial \alpha}\right|_{\alpha=e^{\mathrm{T}}Me} \le 0, \quad e \in \mathring{\mathcal{E}}_{\text{user}}, \tag{62b}$$

where the dependence of $H_{\text{user}}(\cdot)$ on t is omitted. Furthermore, the adaptive laws (29) reduce to

$$\dot{\hat{\alpha}}(t) = -\Gamma_\alpha x(t) e^{\mathrm{T}}(t) \tilde{P}_{\text{user}}(e(t)) B,$$

$$\hat{\alpha}(t_0) = \hat{\alpha}_0, \quad t \ge t_0, \tag{63a}$$

$$\dot{\hat{\beta}}(t) = -\Gamma_\beta r(t) e^{\mathrm{T}}(t) \tilde{P}_{\text{user}}(e(t)) B, \quad \hat{\beta}(t_0) = \hat{\beta}_0, \tag{63b}$$

$$\frac{\partial \hat{f}(t,\cdot)}{\partial t} = \Gamma_f \mathcal{K}_{x(t)} B^{\mathrm{T}} \tilde{P}_{\text{user}}(e(t)) e(t), \quad \hat{f}(t_0,\cdot) = \hat{f}_0(\cdot), \tag{63c}$$

where

$$\tilde{P}_{\text{user}}(e) \triangleq H_{\text{user}}^{-1}(e^{\mathrm{T}}Me) \left(P - V_e(e) \left.\frac{\partial H_{\text{user}}(\alpha)}{\partial \alpha}\right|_{\alpha=e^{\mathrm{T}}Me} M \right), \tag{64}$$

$$V_e(e) \triangleq \frac{\langle e, Pe \rangle_{\mathbb{R}^n}}{H_{\text{user}}(e^{\mathrm{T}}Me)} \tag{65}$$

for all $e \in \mathring{\mathcal{E}}_{\text{user}}$, and the compact, time-invariant constraint set is given by

$$\overline{\mathcal{E}}_{\text{user}} \triangleq \left\{ e \in \mathbb{R}^n : H_{\text{user}}(e^{\mathrm{T}}Me) \ge 0 \right\}. \tag{66}$$

In this case, the statement of Theorem 3 remains substantially identical and is omitted for brevity. Finally, it is worth noting that, if $H_{\text{user}}(t, e^{\mathrm{T}}Me) \equiv 1$ for all $(t, e) \in [t_0, \infty) \times \mathbb{R}^n$, that is, if no constraint is imposed, then the proposed MRAC system reduces to the nonparametric MRAC system presented in [2, Th. 5.3].

6 Robust Finite-Dimensional Implementations

The adaptive laws (29c) and (63c) are DPSs, and, hence, these systems are not realizable in practice. This section discusses robust finite-dimensional implementations of these PDEs and, hence, of the proposed MRAC architecture. To this goal, we consider the plant model

$$\dot{x}(t) = Ax(t) + B\Lambda\left(u(t) + E_{x(t)} f\right) + d(t),$$

$$x(t_0) = x_0, \quad t \ge t_0, \tag{67}$$

which is in the same form as (19), except for the continuous, bounded *unmatched uncertainty* $d : [t_0, \infty) \to \mathbb{R}^n$. The bound on $d(\cdot)$ in (67) is denoted by $d_{\max} \ge 0$ so that $\|d(t)\| \le d_{\max}$ for all $t \in [t_0, \infty)$. Furthermore, we assume that $f \in \mathcal{C}_{R,\eta,N}$, where

$$\mathcal{C}_{R,\eta,N} \triangleq \left\{ f \in \boldsymbol{\mathcal{H}} : \|f\|_{\boldsymbol{\mathcal{H}}} \le R, \|(I - \dot{}_N) f\|_{\boldsymbol{\mathcal{H}}} \le \eta \right\}, \tag{68}$$

that is, we assume that the functional uncertainty $f \in \boldsymbol{\mathcal{H}}(\Omega, \mathbb{R}^m) = \mathcal{H}^m(\Omega, \mathbb{R})$ is bounded and its component orthogonal to the space of approximants $\boldsymbol{\mathcal{H}}_N$, which is given by (14), is bounded by some known constant $\eta \ge 0$. The mildness of the assumption whereby $\|f\| \le R$ has been discussed in Section 5. Furthermore, it is reasonable to provide a conservative bound $\eta \ge 0$ on the orthogonal component of f relative to $\boldsymbol{\mathcal{H}}_N$. Indeed, η in (68) is a bound on the error for capturing matched uncertainties using the finite-dimensional RKHS $\boldsymbol{\mathcal{H}}_N$ instead of the infinite-dimensional RKHS $\boldsymbol{\mathcal{H}}$.

The term $d(\cdot)$ in (67) accounts for uncertainties that affect the plant dynamics through channels different from those of the control input. If the matched uncertainty lies in a native space that is not the m-times Cartesian product of the same scalar-valued RKHS, in a generic Hilbert space, in a Banach space, or in a metric space, then f in (67) captures the element of $\boldsymbol{\mathcal{H}}(\Omega, \mathbb{R}^m)$ closest to this uncertainty, and $d(\cdot)$ captures the complement to $E_{x(\cdot)} f$. Finally, the unmatched uncertainty allows accounting for bounded matched uncertainties whose norm on $\boldsymbol{\mathcal{H}}(\Omega, \mathbb{R}^m)$ is comprised in the interval $(R, d_{\max}]$.

The next theorem is the main result of this section. For its statement, newly consider the user-defined, time-varying constraint set (21) and the constraint set given by (36) designed to approximate (21) compatibly with the closed-loop plant dynamics. Furthermore, in addition to (22), we assume that

$$\left.\frac{\partial H_{\text{user}}(t, e^{\mathrm{T}}(t_0)Me(t_0))}{\partial t}\right|_{t=t_0} > -H_{\text{user}}(t_0, e^{\mathrm{T}}(t_0)Me(t_0)) \lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}). \tag{69}$$

Finally, in this section, the constraint set (36) is defined through $H : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial}{\partial t} H(t, \alpha(t)) = \begin{cases} \max\left\{ \frac{\partial H_{\mathrm{user}}(t, \alpha(t))}{\partial t}, H(t, \alpha(t)) \left(\frac{2\tilde{d}_{\max}}{\varepsilon(t)} - \lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}) \right) \right\}, \\ \quad \text{if} \quad H_{\max}(\alpha(t)) \geq H(t, \alpha(t)) > H_{\min}(\alpha(t)) > 0, \\ \max\left\{ \frac{\partial H_{\mathrm{user}}(t, \alpha(t))}{\partial t}, 0 \right\}, \\ \quad \text{if} \quad H(t, \alpha(t)) = H_{\min}(\alpha(t)) > 0, \end{cases}$$
$$H(t_0, \alpha(t_0)) = H_{\mathrm{user}}(t_0, \alpha(t_0)), \quad t \geq t_0, \tag{70}$$

where $\overline{\varpi} \triangleq \frac{2\tilde{d}_{\max}}{\lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}})}$, $\tilde{d}_{\max} \geq d_{\max} + \eta \overline{\mathfrak{K}} \| B\Lambda \|_{\mathrm{F}}$ is user-defined and captures the user's ability to estimate uncertainties,

$$\varepsilon(t) \triangleq \rho(t) - \delta, \tag{71}$$

$\delta > 0$ denotes an arbitrarily small user-defined constant, and

$$\rho(t) \triangleq \min_{e \in \partial \mathcal{E}_t} \|e\|, \quad \text{for all } t \geq t_0; \tag{72}$$

note that $\partial \mathcal{E}_t \triangleq \left\{ e \in \mathbb{R}^n : H(t, e^{\mathrm{T}} M e) = 0 \right\}$, $t \geq t_0$, and, hence, $H_{\min}(\alpha(t)) = H(t, \alpha(t))|_{\varepsilon(t) = \overline{\varpi}}$. Figure 1 provides a graphical representation of the key elements needed to define the constraint set $\overline{\mathcal{E}}_t$ for any $t \geq t_0$.

Employing (70) to define (36), condition (35) is verified. As already noted in Section 5, the user-defined requirements on the constraint set are mitigated to account for the control system's ability to dissipate the energy injected into the controlled system by the contraction of the constraint set over time. Furthermore, it follows from the first term on the right-hand side of the multi-part definition (70) that $\varepsilon(t) > E(t)$ for all $t \geq t_0$, where

$$E(t) \triangleq \max \left\{ \overline{\varpi}, \left(\frac{H^{-1}(t, e^{\mathrm{T}}(t) M e(t)) \left. \frac{\partial H(t, \alpha)}{\partial t} \right|_{\alpha = e^{\mathrm{T}}(t) M e(t)}}{\tilde{d}_{\max}} + \overline{\varpi}^{-1} \right)^{-1} \right\}. \tag{73}$$

Thus, for all $t \in [t_0, \infty)$, the constraint set $\overline{\mathcal{E}}_t$ contains the closed ball $\overline{\mathcal{B}}_{\overline{\varpi}}(0_n)$, which captures the effect of unmatched uncertainties and errors in approximating $\boldsymbol{\mathcal{H}}$ through $\boldsymbol{\mathcal{H}}_N$. In particular, it follows from (70) that if

$$\frac{\partial}{\partial t} H_{\mathrm{user}}(t, e^{\mathrm{T}}(t) M e(t)) \geq 0$$

for some $t \geq t_0$, that is, if the user wishes not to contract the constraint set, then $H(t, \alpha) = H_{\mathrm{user}}(t, \alpha)$ for any $\alpha \in \mathbb{R}$. Alternatively, if

$$\frac{\partial}{\partial t} H_{\mathrm{user}}(t, e^{\mathrm{T}}(t) M e(t)) < 0$$

for some $t \geq t_0$, then the constraint set $\overline{\mathcal{E}}_t$ defined by (36) may not contract as fast as $\overline{\mathcal{E}}_{t,\mathrm{user}}$.

In particular, it follows from (73) and (69) that

$$H(t, e^{\mathrm{T}}(t) M e(t)) \geq H(t_0, e^{\mathrm{T}}(t_0) M e(t_0)) e^{-\lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}) \int_{t_0}^{t} \Phi(\tau) \mathrm{d}\tau}, \tag{74}$$

for all $t \geq t_0$, where $\Phi(t) \triangleq \varepsilon^{-1}(t) (\varepsilon(t) - \overline{\varpi})$; per definition, $\varepsilon(t) > 0$ for all $t \geq t_0$. Since $\Phi(t) \in [0, 1]$ for all $t \geq t_0$, we deduce that

$$H(t, e^{\mathrm{T}}(t) M e(t)) \gtrsim e^{-\lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}})(t - t_0)} \quad \text{for all } t \geq t_0. \tag{75}$$

This relationship implies that the constraint set cannot converge at a rate greater than $\lambda_{\min}(P^{-\frac{1}{2}} Q \lambda_{\min} P^{-\frac{1}{2}})$, which is slower than twice the decay rate of the closed-loop trajectory tracking error given by $|\mathrm{Re}(\lambda_{\max}(A_r))|$ [17]. Lastly, we note that if $f \in \boldsymbol{\mathcal{H}}_N$ and $\tilde{d}_{\max} = 0$, then $\overline{\mathcal{B}}_{\overline{\varpi}}(0_n) = \{0_n\}$ and asymptotic convergence of the tracking error to zero can be attained.

For the statement of the next result, consider the control law (28) and the adaptive laws

$$\dot{\hat{\alpha}}(t) = -\Gamma_\alpha x(t) e^{\mathrm{T}}(t) \tilde{P}(t, e(t)) B - \sigma_\alpha \hat{\alpha}(t),$$
$$\hat{\alpha}(t_0) = \hat{\alpha}_0, \quad t \geq t_0, \tag{76a}$$

$$\dot{\hat{\beta}}(t) = -\Gamma_\beta r(t) e^{\mathrm{T}}(t) \tilde{P}(t, e(t)) B - \sigma_\beta \hat{\beta}(t),$$
$$\hat{\beta}(t_0) = \hat{\beta}_0, \tag{76b}$$

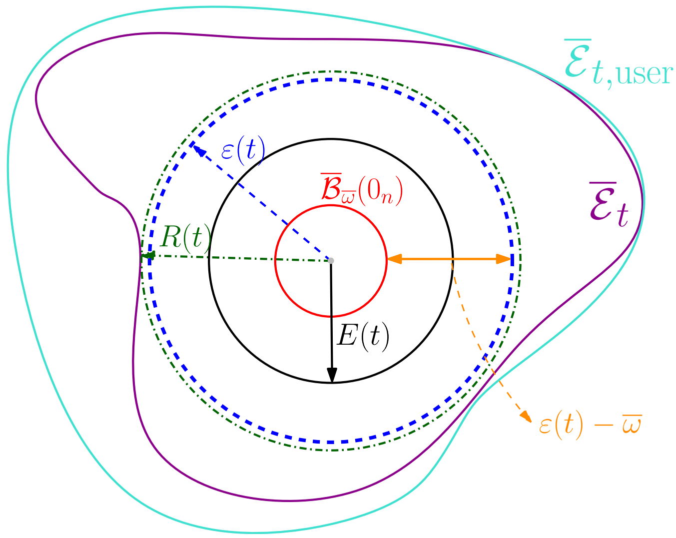


Fig. 1 Schematic representation of the user-defined constraint set $\overline{\mathcal{E}}_{t,\text{user}}$, of the approximating constraint set $\overline{\mathcal{E}}_t$, and of the closed ball $\overline{\mathcal{B}}_{\overline{\omega}}(0_n)$ that needs to be contained in $\overline{\mathcal{E}}_t$ at each $t \geq t_0$

$$\dot{\hat{f}}_N(t) = \Gamma_f \cdot_N \mathcal{K}_{x(t)} B^{\mathrm{T}} \tilde{P}(t, e(t)) e(t) - \sigma_f \hat{f}_N(t),$$
$$\hat{f}_N(t_0) = \hat{f}_0, \tag{76c}$$

where $\sigma_\alpha, \sigma_\beta, \sigma_f > 0$ are user-defined and $\tilde{P}(\cdot,\cdot)$ is given by (30). Remarkably, the adaptive law that corresponds to the matched uncertainty, namely (76c), is an ODE. Furthermore, comparing (29) and (76), we observe that a dissipative term has been introduced in each adaptive law.

Theorem 5 *Consider the plant model* (67)*, the reference model* (20)*, the constraint set* (36) *with constraint function* $H(\cdot,\cdot)$ *such that* (70) *is verified, the control input* (28)*, and the adaptive laws* (76)*. Assume that* $\mathfrak{K}(x, y) \leq \overline{\mathfrak{K}}$ *for all* $(x, y) \in \Omega \times \Omega$*. Assume also that* $f \in \mathcal{C}_{R,\eta,N}$*. If* $e(t_0) \in \mathring{\mathcal{E}}_{t_0}$*, then* $e(t) \in \mathring{\mathcal{E}}_t$ *for all* $t \in [t_0, \infty)$*, and the adaptive gains* $\hat{\alpha}(t)$*,* $\hat{\beta}(t)$*, and* $\hat{f}_N(t)$ *are uniformly bounded for all* $t \in [t_0, \infty)$.

Proof It follows from (67) with control law (28), (20), and the matching conditions (26) that

$$\dot{e}(t) = A_{\text{ref}} e(t) + B\Lambda\left(\tilde{K}^{\mathrm{T}}(t)\pi(t) - E_{x(t)}\tilde{f}_N(t,\cdot)\right) + d(t),$$
$$e(t_0) = e_0, \quad t \geq t_0, \tag{77}$$

where $\tilde{K}(t) \triangleq \hat{K}(t) - K$, $\hat{K}(t) \triangleq \left[\hat{\alpha}^{\mathrm{T}}(t), \hat{\beta}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{(n+m)\times m}$, $K \triangleq \left[\alpha^{\mathrm{T}}, \beta^{\mathrm{T}}\right]^{\mathrm{T}}$, $\pi(t) \triangleq \left[x^{\mathrm{T}}(t), r^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and $\tilde{f}_N(t,\cdot) \triangleq \hat{f}_N(t,\cdot) - \cdot_N f(t,\cdot)$. Thus, the proof is divided into two parts. First, we assume that $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$ and show that the trajectories of (76) are globally bounded and those of (77) are uniformly ultimately bounded. Successively, we prove that if $e(t_0) \in \mathring{\mathcal{E}}_{t_0}$, then $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$.

Assume that $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \geq t_0$, and consider the Lyapunov function candidate

$$V(t, e, \tilde{K}, \tilde{f}_N) = V_e(t, e(t)) + \left\langle \tilde{K}, \Gamma_K^{-1}\tilde{K}\Lambda\right\rangle_{\text{tr}}$$
$$+ \left\langle \tilde{f}_N, \Gamma_f^{-1}\mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\boldsymbol{\mathcal{H}}},$$
$$(t, e, \tilde{K}, \tilde{f}_N) \in [t_0, \infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \times \mathbb{R}^{(n+m)\times m} \times \boldsymbol{\mathcal{H}}_N. \tag{78}$$

The time derivative of (78) along the trajectories of (77) and (76) is given by

$$\dot{V}(t, e, \tilde{K}, \tilde{f}_N)$$
$$= \frac{1}{H(t, e^{\mathrm{T}}Me)}\left(2\left\langle e, PA_r e\right\rangle_{\mathbb{R}^n} - 2V_e(t, e)\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right.$$
$$\left.\left\langle e, MA_r e\right\rangle_{\mathbb{R}^n}\right)$$
$$+ \frac{1}{H(t, e^{\mathrm{T}}Me)}\left(2\left\langle e, Pd(t)\right\rangle_{\mathbb{R}^n} - 2V_e(t, e)\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right.$$
$$\left.\left\langle e, Md(t)\right\rangle_{\mathbb{R}^n}\right)$$
$$- \frac{2}{H(t, e^{\mathrm{T}}Me)}\left(\left\langle PB\Lambda\tilde{K}^{\mathrm{T}}\pi(t), e\right\rangle_{\mathbb{R}^n}\right.$$
$$\left.-\left\langle PBE_x\mathcal{L}_\Lambda\tilde{f}_N, e\right\rangle_{\mathbb{R}^n}\right)$$
$$+ \frac{2V_e(t, e)}{H(t, e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}$$
$$\cdot\left(\left\langle MB\Lambda\tilde{K}^{\mathrm{T}}\pi(t), e\right\rangle_{\mathbb{R}^n} - \left\langle MBE_x\mathcal{L}_\Lambda\tilde{f}_N, e\right\rangle_{\mathbb{R}^n}\right)$$

$$
\begin{aligned}
&- \frac{V_e(t,e)}{H(t,e^{\mathrm{T}}Me)} \left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me} + 2\left\langle \dot{\hat{K}}(t), \Gamma_K^{-1}\tilde{K}\Lambda\right\rangle_{\mathrm{tr}} \\
&+ 2\left\langle \dot{\hat{f}}_N(t), \Gamma_f^{-1}\mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\mathcal{H}} \\
&\leq \frac{1}{H(t,e^{\mathrm{T}}Me)} \left(2\langle e, PA_r e\rangle_{\mathbb{R}^n} - V_e(t,e) \left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me}\right. \\
&+2\langle e, Pd(t)\rangle_{\mathbb{R}^n}) \\
&+ \frac{V_e(t,e)}{H(t,e^{\mathrm{T}}Me)} \left| \left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right| \\
&(2\langle e, MA_r e\rangle_{\mathbb{R}^n} + 2\langle e, Md(t)\rangle_{\mathbb{R}^n}) \\
&- 2\sigma\left(\left\langle \tilde{K}, \tilde{K}\Lambda\right\rangle_{\mathrm{tr}} + \left\langle K, \tilde{K}\Lambda\right\rangle_{\mathrm{tr}} + \left\langle \tilde{f}_N, \mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\mathcal{H}}\right. \\
&\left.+ \left\langle {}^{\cdot}{}_N f, \mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\mathcal{H}}\right) \\
&(t, e, \tilde{K}, \tilde{f}_N) \in [t_0,\infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \\
&\times \mathbb{R}^{(n+m)\times m} \times \mathcal{H}_N.
\end{aligned} \tag{79}
$$

It follows from (27) and the Cauchy-Schwartz inequality that

$$
\begin{aligned}
\left|\left\langle {}^{\cdot}{}_N f, \mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\mathcal{H}}\right| &\leq \|\Lambda\|_F \left\|\tilde{f}_N\right\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \|{}^{\cdot}{}_N\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_N)} \\
&\leq \|\Lambda\|_F \left\|\tilde{f}_N\right\|_{\mathcal{H}} R,
\end{aligned} \tag{80}
$$

which implies that

$$
-2\sigma\left\langle {}^{\cdot}{}_N f, \mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\mathcal{H}} \leq 2\sigma\|\Lambda\|_F\|\tilde{f}_N\|_{\mathcal{H}} R, \tag{81}
$$

and applying the Rayleigh quotient yields

$$
-2\sigma\left\langle \tilde{f}_N, \mathcal{L}_\Lambda \tilde{f}_N\right\rangle_{\mathcal{H}} \leq -2\sigma\lambda_{\min}(\Lambda)\|\tilde{f}_N\|^2_{\mathcal{H}}. \tag{82}
$$

Similarly,

$$
-2\sigma\left\langle K, \tilde{K}(t)\Lambda\right\rangle_{\mathrm{tr}} \leq 2\sigma\|\Lambda\|_F\|K\|_F\|\tilde{K}\|_F, \tag{83}
$$

$$
-2\sigma\left\langle \tilde{K}, \tilde{K}\Lambda\right\rangle_{\mathrm{tr}} \leq -2\sigma\lambda_{\min}(\Lambda)\|\tilde{K}\|^2_F. \tag{84}
$$

Hence,

$$
\begin{aligned}
&\dot{V}(t,e,\tilde{K},\tilde{f}_N) \\
&\leq \frac{1}{H(t,e^{\mathrm{T}}Me)}(2\langle e, PA_r e\rangle_{\mathbb{R}^n} \\
&\quad\left. -V_e(t,e)\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{e^{\mathrm{T}}Me} + 2\langle e, Pd(t)\rangle_{\mathbb{R}^n}\right) \\
&\quad + \frac{V_e(t,e)}{H(t,e^{\mathrm{T}}Me)} \left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right| (2\langle e, MA_r e\rangle_{\mathbb{R}^n} \\
&\quad +2\langle e, Md(t)\rangle_{\mathbb{R}^n}) \\
&\quad + 2\sigma\left[\|\tilde{K}\|_F\left(\|\Lambda\|_F\|K\|_F - \lambda_{\min}(\Lambda)\|\tilde{K}\|_F\right)\right. \\
&\quad \left.+ \left\|\tilde{f}_N\right\|_{\mathcal{H}}\left(\|\Lambda\|_F R - \lambda_{\min}(\Lambda)\left\|\tilde{f}_N\right\|_{\mathcal{H}}\right)\right] \\
&\quad (t, e, \tilde{K}, \tilde{f}_N) \in [t_0,\infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \\
&\quad \times \mathbb{R}^{(n+m)\times m} \times \mathcal{H}_N,
\end{aligned} \tag{85}
$$

and

$$
\begin{aligned}
&\dot{V}(t,e,\tilde{K},\tilde{f}_N) \\
&\leq \frac{2}{H(t,e^{\mathrm{T}}Me)}\left(\langle e, PA_r e\rangle_{\mathbb{R}^n}\right. \\
&+ \left\langle e, V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right| MA_r e\right\rangle_{\mathbb{R}^n} \\
&+ \langle e, Pd(t)\rangle_{\mathbb{R}^n} + \left\langle e, V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right| Md(t)\right\rangle_{\mathbb{R}^n} \\
&\left.- \frac{V_e(t,e)}{2}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}Me}\right) \\
&(t, e, \tilde{K}, \tilde{f}_N) \in [t_0,\infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \\
&\times \left(\mathbb{R}^{(n+m)\times m} \setminus \overline{\mathcal{F}}_K\right) \times (\mathcal{H}_N \setminus \overline{\mathcal{F}}_{f_N}),
\end{aligned} \tag{86}
$$

where

$$
\overline{\mathcal{F}}_K \triangleq \left\{\tilde{K} \in \mathbb{R}^{(n+m)\times m} : \|\tilde{K}\|_F \leq \frac{\|\Lambda\|_F\|K\|_F}{\lambda_{\min}(\Lambda)}\right\}, \tag{87a}
$$

$$
\overline{\mathcal{F}}_{f_N} \triangleq \left\{\tilde{f}_N \in \mathcal{H}_N : \left\|\tilde{f}_N\right\|_{\mathcal{H}} \leq \frac{\|\Lambda\|_F R}{\lambda_{\min}(\Lambda)}\right\} \tag{87b}
$$

are compact sets, and it follows from (31) that

$$
\begin{aligned}
&\dot{V}(t,e,\tilde{K},\tilde{f}_N) \\
&\leq \frac{2}{H(t,e^{\mathrm{T}}Me)} \\
&\left(\left\langle e, P\left(A_r e + d(t) - \frac{e}{2H(t,e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}Me}\right)\right\rangle_{\mathbb{R}^n}\right. \\
&\left.+\left\langle e, V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right| M(A_r e + d(t))\right\rangle_{\mathbb{R}^n}\right) \\
&(t, e, \tilde{K}, \tilde{f}_N) \in [t_0,\infty) \times \left(\bigcup_{t\in[t_0,\infty)} \mathring{\mathcal{E}}_t\right) \\
&\times \left(\mathbb{R}^{(n+m)\times m} \setminus \overline{\mathcal{F}}_K\right) \times (\mathcal{H}_N \setminus \overline{\mathcal{F}}_{f_N}).
\end{aligned} \tag{88}
$$

Since $P \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite, there exists $P^{\frac{1}{2}} \in \mathbb{R}^{n\times n}$, known as *positive square-root*, such that $P = P^{\frac{1}{2}}P^{\frac{1}{2}}$. Similarly, since $\left(V_e(t,e)\left|\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right|M\right)$ is nonnegative-definite for all $(t,e) \in [t_0,\infty)\times\mathbb{R}^n$, its *principal nonnegative square root* exists. Hence,

$$\begin{aligned}
&\dot{V}(t,e,\tilde{K},\tilde{f}_N)\\
&\quad\le \frac{2}{H(t,e^{\mathrm{T}}Me)}\\
&\left(\left\langle P^{\frac{1}{2}}e, P^{\frac{1}{2}}\left(A_r e + d(t) - \frac{e}{2H(t,e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}Me}\right\rangle_{\mathbb{R}^n}\right.\right.\\
&\quad+\left\langle\left(V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right|M\right)^{\frac{1}{2}}e,\right.\\
&\quad\left.\left.\left(V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right|M\right)^{\frac{1}{2}}(A_r e + d(t))\right\rangle_{\mathbb{R}^n}\right),\\
&\quad(t,e,\tilde{K},\tilde{f}_N)\in[t_0,\infty)\times\left(\bigcup_{t\in[t_0,\infty)}\mathring{\mathcal{E}}_t\right)\\
&\quad\times\left(\mathbb{R}^{(n+m)\times m}\setminus\overline{\mathcal{F}}_K\right)\times\left(\mathcal{H}_N\setminus\overline{\mathcal{F}}_{f_N}\right).
\end{aligned}\tag{89}$$

Now, let $e = \|e\|_{\mathbb{R}^n}\hat{e}$, where $\hat{e} \in \partial\mathcal{B}_1(0)$; if $\|e\| = 0$, then any $\hat{e} \in \partial\mathcal{B}_1(0)$ can be employed to represent e. In this case,

$$\begin{aligned}
&\dot{V}(t,e,\tilde{K},\tilde{f}_N)\\
&\quad\le \frac{2}{H(t,e^{\mathrm{T}}Me)}\\
&\quad\left(\left\langle P^{\frac{1}{2}}e, P\right\rangle_{\mathbb{R}^n}^{\frac{1}{2}}\left(A_r\|e\|_{\mathbb{R}^n} + \tilde{d}_{\max}I\right.\right.\\
&\quad\left.-\frac{\|e\|_{\mathbb{R}^n}}{2H(t,e^{\mathrm{T}}Me)}\left.\frac{\partial H(t,\alpha)}{\partial t}\right|_{\alpha=e^{\mathrm{T}}Me}I\right)\hat{e}\\
&\quad+\left\langle\left(V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right|M\right)^{\frac{1}{2}}e,\right.\\
&\quad\left(V_e(t,e)\left|\left.\frac{\partial H(t,\alpha)}{\partial \alpha}\right|_{e^{\mathrm{T}}Me}\right|M\right)^{\frac{1}{2}}\\
&\quad\left.\left.\left(A_r\|e\|_{\mathbb{R}^n} + \tilde{d}_{\max}I\right)\hat{e}\right\rangle_{\mathbb{R}^n}\right),\\
&\quad(t,e,\tilde{K},\tilde{f}_N)\in[t_0,\infty)\times\left(\bigcup_{t\in[t_0,\infty)}\mathring{\mathcal{E}}_t\right)\\
&\quad\times\left(\mathbb{R}^{(n+m)\times m}\setminus\overline{\mathcal{F}}_K\right)\times\left(\mathcal{H}_N\setminus\overline{\mathcal{F}}_{f_N}\right),
\end{aligned}\tag{90}$$

and it follows from (70) that

$$\begin{aligned}
&\dot{V}(t,e,\tilde{K},\tilde{f}_N) < 0,\\
&(t,e,\tilde{K},\tilde{f}_N)\in[t_0,\infty)\times\left(\bigcup_{t\in[t_0,\infty)}\left(\mathring{\mathcal{E}}_t\setminus\overline{\mathcal{B}}_{E(t)}(0_n)\right)\right)\\
&\times\left(\mathbb{R}^{(n+m)\times m}\setminus\overline{\mathcal{F}}_K\right)\times\left(\mathcal{H}_N\setminus\overline{\mathcal{F}}_{f_N}\right).
\end{aligned}\tag{91}$$

Hence, uniform boundedness of (77) and (76) is proven. The second part of this proof, that is, the proof by contradiction whereby if $e_0 \in \mathring{\mathcal{E}}_{t_0}$, then $e(t) \in \mathring{\mathcal{E}}_t$ for all $t \ge t_0$ follows by proceeding as in the second part of Theorem 3. □

Theorem 5 provides a practical implementation of the result on infinite-dimensional spaces captured by Theorem 3. Furthermore, it follows from the proof of Theorem 5 that there exists $T \ge t_0$ such that $e(t) \in \mathring{\mathcal{B}}_{\overline{\omega}}(0_n)$ for all $t > T$. Therefore, despite any existing adaptive control systems over native spaces, Theorem 5 ensures that the closed-loop trajectory tracking error, and, hence, the closed-loop plant trajectory, lie in a user-defined constraint set at all times. This result was attained by employing for the first time in an RKHS setting the notion of barrier Lyapunov functions; Figure 2 provides a graphical representation of (78).

To employ native spaces to capture uncertainties, the centers of the kernel functions that characterize this space need to be distributed in the state space. These centers need to be distributed so that sufficient rich information on the functional uncertainty can be deduced. As discussed in [2, Ch. 6, 7], the kernel centers can be defined *a priori* by the user or through some data-driven method. However, the larger the number of centers, the larger the computational cost of a non-parametric control law [2, Ch. 6,7]. For each $t \ge t_0$, Theorem 5 allows users to choose in what regions of the state space, namely $\overline{\mathcal{E}}_t$, centers should be located. Future work directions concern kernel center pruning as the constraint set contracts.

The larger the user-defined upper bound η that characterizes the uncertainty class (68), the larger the parameter $\tilde{d}_{\max}$. It follows from (70) and (73) that if the user-defined constraint set is required to contract, then the larger $\tilde{d}_{\max}$, the sooner the user-defined constraint function $H_{\mathrm{user}}(\cdot,\cdot)$ is approximated by $H(\cdot,\cdot)$, which captures a constraint set contracting less rapidly than desired. Furthermore, the larger $\tilde{d}_{\max}$, the sooner the constraint set $\overline{\mathcal{E}}_t$ stops contracting. Finally, the larger $\tilde{d}_{\max}$, the larger the diameter of the constraint set after it stops contracting.

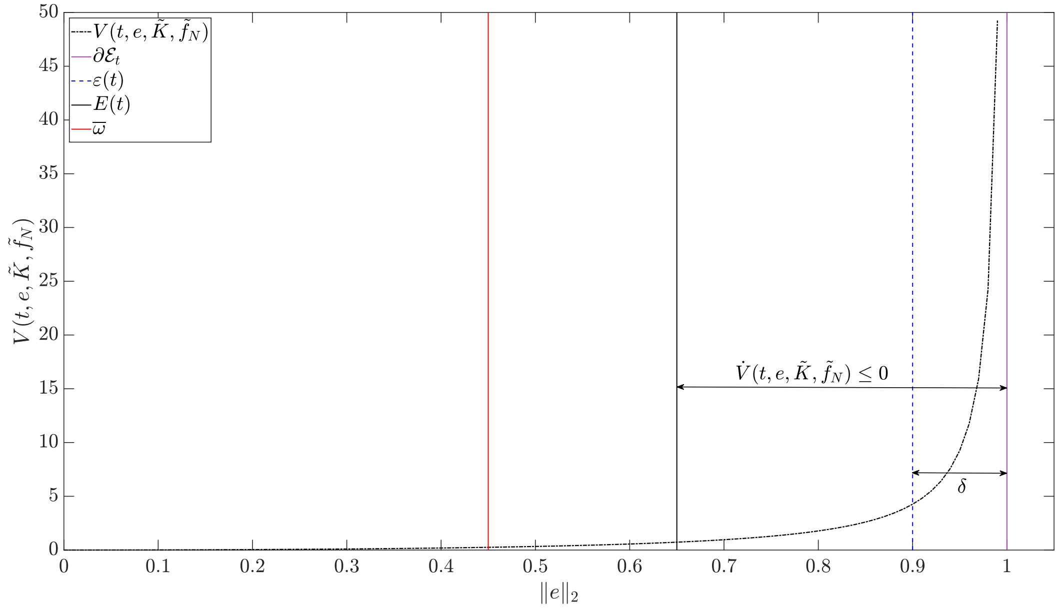


Fig. 2 Graphical representation of a barrier Lyapunov as a function of $\|e\|_2$ at some time $t \in [t_0, \infty)$ with $\max_{e\in\partial\mathcal{E}_t} \|e\| = 1$. The boundaries of the sets shown in Figure 1 are displayed as vertical lines

The power function [2, Def. 3.8] can be employed to correlate η to some measure of the distance between centers, such as the fill distance [2, p. 45]; for details, see Theorem 3.12 of [2]. For a proper subset of $\mathbb{R}^n$, this upper bound can be calculated numerically. Intuitively, the largest approximation error is likely to occur at points in the domain that are furthest from the kernel centers. A practical approach to identify these points in the state space is to use Delaunay triangulation. Since the vertices of these triangles correspond to the kernel centers, it follows that the points furthest from the vertices, and hence the points with the highest approximation error, can be reasonably estimated by the centroids of the triangles.

If the user-defined constraints are captured by the time-invariant set (66), then Theorem 5 specializes to the following corollary. For the statement of this result, we consider the adaptive laws

$$\dot{\hat{\alpha}}(t) = -\Gamma_\alpha x(t)e^{\mathrm{T}}(t)\tilde{P}_{\mathrm{user}}(e(t))B - \sigma_\alpha\hat{\alpha}(t),\quad \hat{\alpha}(t_0) = \hat{\alpha}_0, \quad t \geq t_0, \tag{92a}$$

$$\dot{\hat{\beta}}(t) = -\Gamma_\beta r(t)e^{\mathrm{T}}(t)\tilde{P}_{\mathrm{user}}(e(t))B - \sigma_\beta\hat{\beta}(t),\quad \hat{\beta}(t_0) = \hat{\beta}_0, \tag{92b}$$

$$\dot{\hat{f}}_N(t) = \Gamma_f {}^{\boldsymbol{\cdot}}{}_N\mathcal{K}_{x(t)}B^{\mathrm{T}}\tilde{P}_{\mathrm{user}}(e(t))e(t) - \sigma_f\hat{f}_N(t),\quad \hat{f}_N(t_0) = \hat{f}_0, \tag{92c}$$

where $\tilde{P}_{\mathrm{user}}(\cdot)$ is given by (64).

Corollary 1 *Consider the plant model (67), the reference model (20), the constraint set (66), the control input (28), and the adaptive laws (92). Assume that $\mathfrak{K}(x, y) \leq \overline{\mathfrak{K}}$ for all $(x, y) \in \Omega \times \Omega$, $f \in \mathcal{C}_{R,\eta,N}$, and $\overline{\mathcal{B}}_{\overline{\omega}} \subset \overline{\mathcal{E}}_{\mathrm{user}}$. If $e(t_0) \in \mathring{\mathcal{E}}_{\mathrm{user}}$, then $e(t) \in \mathring{\mathcal{E}}_{\mathrm{user}}$ for all $t \in [t_0, \infty)$, and the adaptive gains $\hat{\alpha}(t)$, $\hat{\beta}(t)$, and $\hat{f}_N(t)$ are uniformly bounded for all $t \in [t_0, \infty)$.*

The proof of this result is omitted for brevity. Both Theorem 5 and Corollary 1 extend the σ-modification of MRAC [37] to a nonparametric setting in the presence of constraints. The user-defined parameters σ_α, σ_β, and σ_f in (76) and (92) should be chosen sufficiently small to mitigate the dissipative effect in the respective adaptive laws. σ-modifications of MRAC laws, such as those in (76) and (92), are known to suffer from the "unlearning" effect, whereby the adaptive gains converge to zero if the tracking error converges to zero. This undesired effect can be avoided by lifting the classical e-modifications of the MRAC [38] or the convex projection operator [39] to a native space setting as shown in [2, Ch. 5]. These results are omitted for brevity.

As a concluding remark, note that if $H_{\mathrm{user}}(t, e^{\mathrm{T}} M e) \equiv 1$ for all $(t, e) \in [t_0, \infty) \times \mathbb{R}^n$, that is, if no constraint is enforced, then it follows from Theorem 5.7 of [2] that there exists $T > t_0$ such that

$$\|x(t) - x_r(t)\|_{\mathbb{X}}^2 \leq \frac{2\lambda_{\max}^2(P)}{\lambda_{\min}^2(Q)} \tilde{d}_{\max}^2 \sup_{\eta \in \Omega} \mathcal{P}_N^2(\eta) + \left\langle \alpha, \Gamma_\alpha^{-1} \alpha \right\rangle_{\mathrm{tr}} + \left\langle \beta, \Gamma_\beta^{-1} \beta \right\rangle_{\mathrm{tr}} + \left\langle f, \Gamma_f^{-1} f \right\rangle_{\boldsymbol{\mathcal{H}}} \tag{93}$$

for all $t \geq T$, where $\eta_N \triangleq \|(I - {}^{\cdot}{}_N) f\|_{\boldsymbol{\mathcal{H}}}$. Thus, Theorem 5 guarantees tighter ultimate bounds on the tracking error than Theorem 5.7 of [2]. Furthermore, in the case of time-invariant constraint sets, by applying Corollary 1, the diameter of the smallest user-defined constraint set on the tracking error can be designed to be smaller than the diameter of the set ultimately attained by the tracking error while employing the nonparametric MRAC system for unconstrained problems captured by Theorem 5.7 of [2].

7 Coordinates Implementation

This section details how to obtain operative adaptive laws for a diagonal, positive-definite, adaptive rate matrix $\Gamma_f \triangleq \mathrm{diag}(\gamma_f^1, \ldots, \gamma_f^m)$, given the finite-dimensional approximation $\boldsymbol{\mathcal{H}}_N$ of the vector-valued RKHS $\boldsymbol{\mathcal{H}} = \mathcal{H} \times \cdots \times \mathcal{H}$. Proceeding as in [2, Sec. 5.4.1.3], we introduce the approximating adaptive gain

$$\hat{f}_N(t, \cdot) = \sum_{j=1}^{N} \left[\hat{\theta}_{N,j}^1(t) \mathfrak{K}_{\xi_j}, \ldots, \hat{\theta}_{N,j}^m(t) \mathfrak{K}_{\xi_j}\right]^{\mathrm{T}} \tag{94}$$

where $\mathfrak{K}_{\xi_j}(\cdot)$ denotes the kernel basis function associated with $\mathcal{H}$ centered at ξ_j, and $\hat{\theta}_{N,j}^l(\cdot)$ denotes the l-th component of the adaptive gain in the Hammel basis $\left\{\sum_{j=1}^N \mathfrak{K}_{\xi_j}, \ldots, \sum_{j=1}^N \mathfrak{K}_{\xi_j}\right\} \subset \mathbb{U}$. Thus, the adaptive laws can be written as

$$\begin{aligned} \frac{\partial \hat{f}_N^l}{\partial t}(t, \cdot) &\triangleq \sum_{j=1}^{N} \dot{\hat{\theta}}_{N,j}^l(t) \mathfrak{K}_{\xi_j} \\ &= \gamma_f^l \Pi_N \mathfrak{K}_{x(t)} \left[B^{\mathrm{T}} \tilde{P}(t, e(t)) e(t)\right]_l \\ &\quad - \sigma \sum_{j=1}^{N} \hat{\theta}_{N,j}^l(t) \mathfrak{K}_{\xi_j}, \\ &\quad \text{for all } l \in \{1, \ldots, m\} \text{ and for all } t \geq t_0, \end{aligned} \tag{95, 96}$$

where Π_N denotes the projection operator from $\mathcal{H}$ to $\mathcal{H}_N$ and $[\cdot]_l$ denotes the l-th component of its argument. Such a projection can be computed by taking the inner product of adaptive laws in (96) with an arbitrary $\mathfrak{K}_{\xi_i}$ for $i \in \{1, \ldots, N\}$, yielding

$$\begin{aligned} \sum_{j=1}^{N} \langle \mathfrak{K}_{\xi_i}, \mathfrak{K}_{\xi_j} \rangle_{\mathcal{H}} \dot{\hat{\theta}}_{N,j}^l(t) &= \gamma_f^l \langle \mathfrak{K}_{\xi_i}, \Pi_N \mathfrak{K}_{x(t)} \rangle_{\mathcal{H}} \left[B^{\mathrm{T}} \tilde{P}(t, e(t)) e(t)\right]_l \\ &\quad - \sigma \sum_{j=1}^{N} \langle \mathfrak{K}_{\xi_i}, \mathfrak{K}_{\xi_j} \rangle_{\mathcal{H}} \hat{\theta}_{N,j}^l(t), \text{ for all } l \in \{1, \ldots, m\}, \\ &\quad \text{for all } t \geq t_0, \text{ and for each } i \in \{1, \ldots, N\}. \end{aligned} \tag{97}$$

From the definition of Grammian matrix in (13), it follows that

$$\begin{aligned} \dot{\hat{\theta}}_N^l(t) &= \gamma_f^l \mathbb{K}_N^{-1} \begin{bmatrix} \mathfrak{K}(\xi_1, x(t)) \\ \vdots \\ \mathfrak{K}(\xi_N, x(t)) \end{bmatrix} \left[B^{\mathrm{T}} \tilde{P}(t, e(t)) e(t)\right]_l - \sigma \hat{\theta}_N^l(t), \\ &\quad \text{for all } l \in \{1, \ldots, m\} \text{ and for all } t \geq t_0. \end{aligned} \tag{98}$$

Therefore, it follows from (98) that

$$\begin{aligned} E_{x(t)} \hat{f}_N(t, \cdot) &= \sum_{j=1}^{N} \left[\hat{\theta}_{N,j}^1(t) \mathfrak{K}_{\xi_j}(x(t)), \ldots, \hat{\theta}_{N,j}^m(t) \mathfrak{K}_{\xi_j}(x(t))\right]^{\mathrm{T}}, \\ &\quad \text{for all } t \geq t_0. \end{aligned} \tag{99}$$

Figure 4 provides a representation of the control system captured by Theorem 5 with coordinate implementation given by (98) and (99). The computer codes in [31] provide an implementation of this scheme for the problem discussed in Section 8 below. To apply Corollary 1, the dotted box in Figure 4 can be replaced with $H_{\mathrm{user}}(e^{\mathrm{T}}(t) M e(t)), t \geq t_0$.

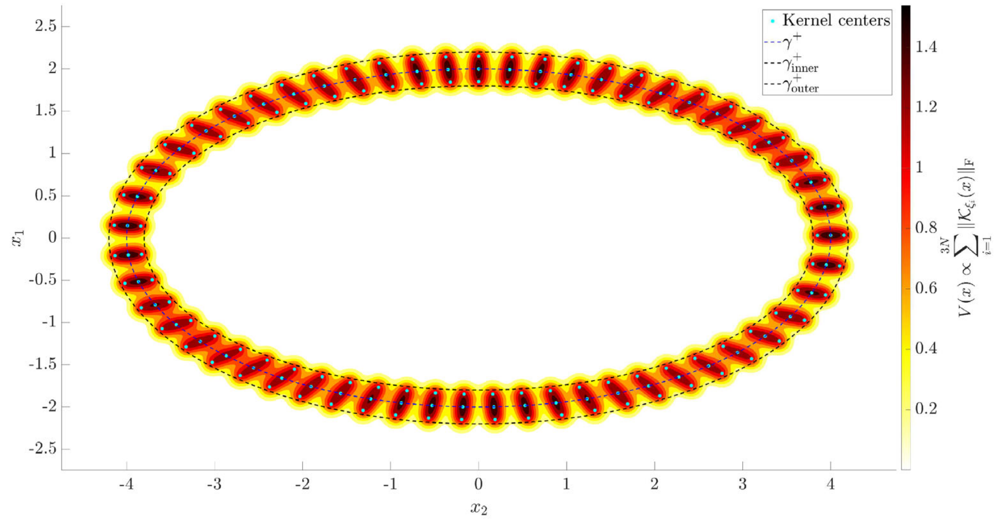


Fig. 3 Limit cycle of the reference trajectory and sets γ^+_{inner} and γ^+_{outer} needed to define the RKHS $\boldsymbol{\mathcal{H}}_\gamma$ and the set of kernel centers $\Xi_{\gamma,N}$

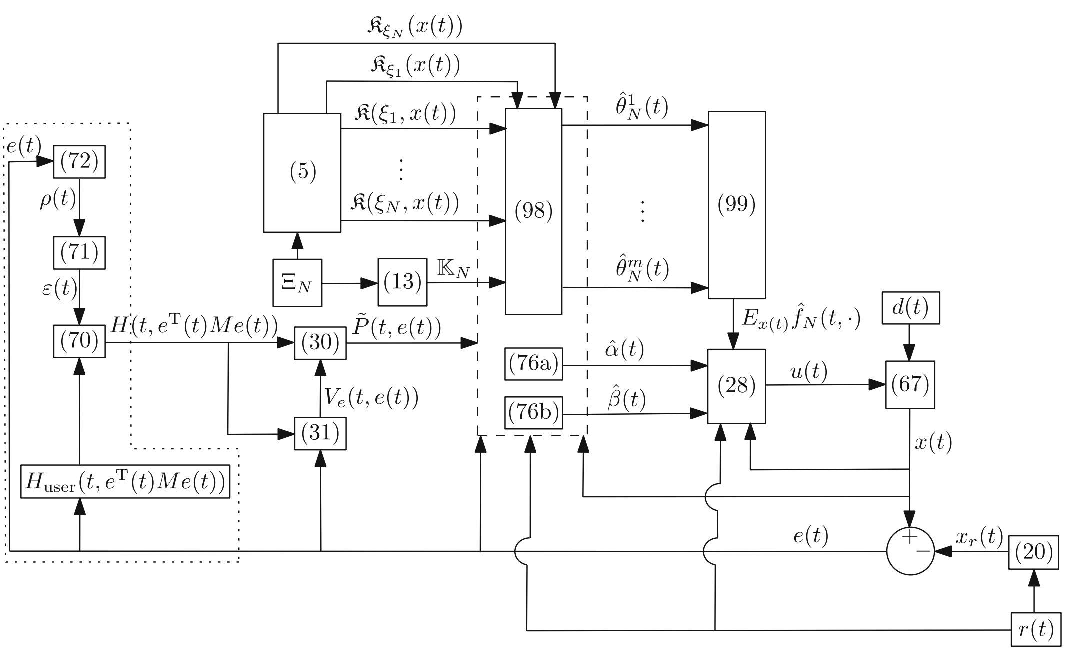


Fig. 4 Representation of the control system captured by Theorem 5 with coordinate implementation given by (98) and (99). The reference command input $r(\cdot)$, the set of centers Ξ_N, and the constraint function $H_{\text{user}}(\cdot,\cdot)$ are user-defined. The dashed box captures the adaptive laws. To apply Corollary 1, the dotted box on the must be replaced by $H_{\text{user}}(e^{\text{T}}(t)Me(t))$, $t \geq t_0$. The computer codes in [31] provide an implementation of this scheme for the problem discussed in Section 8 below

8 Numerical Examples

In this section, we discuss numerical implementations of some key results of this paper. In particular, we consider the plant model

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} x(t) + \Lambda \left(u(t) + \underbrace{\frac{x_1(t)}{2\|x(t)\|} x(t) - \|x(t)\|^2 x(t)}_{f(x)} \right) + d(t),$$

$$x(0) = x_0, \quad t \geq 0, \tag{100}$$

which is in the same form as (67) with $n = m = 2$, $\mathbb{X} = \mathbb{R}^2$, $\mathbb{U} = \mathbb{R}^2$, $t_0 = 0$, and $B = I_2$. The matrix A is considered unknown. To challenge our control systems, we set $\Lambda = I_2$ for all $t \in [t_0, t_{\text{switch}}]$ s, and we set $\Lambda = \text{diag}(0.1, 0.3)$ for all $t \in (t_{\text{switch}}, \infty)$ s. Although the proposed theoretical framework does not allow for switches in the plant dynamics, the scope of this additional challenge is to show the efficiency of the proposed results. Furthermore, we consider the reference model

$$\dot{x}_{\text{r}}(t) = \underbrace{\begin{bmatrix} -3.5 & -0.1 \\ 0.1 & -3.5 \end{bmatrix}}_{A_r} x_r(t) + \underbrace{I_2}_{B_r} \underbrace{\begin{bmatrix} 2\cos(0.1\pi t) \\ 4\sin(0.1\pi t) \end{bmatrix}}_{r(t)},$$

$$x_r(0) = x_{r,0}, \quad t \geq 0, \tag{101}$$

so that the matching conditions (26) are verified.

We employ the Wendland kernel $\eta_{3,2}(\cdot)$ shown in Table 1 to define an RKHS and capture the matched uncertainty. To define the native space and capture matched uncertainties, we proceed as follows. The reference trajectory exhibits a limit cycle, which we denote as

$$\gamma^+ = \bigcup_{t \geq 0} \begin{bmatrix} 2\cos(0.1\pi t) \\ 4\sin(0.1\pi t) \end{bmatrix}. \tag{102}$$

Thus, we define $\overline{\mathcal{D}}_{\text{inner}}, \overline{\mathcal{D}}_{\text{outer}} \subset \mathbb{X}$ such that $\partial \mathcal{D}_{\text{inner}} = \gamma^+_{\text{inner}}$ and $\partial \mathcal{D}_{\text{outer}} = \gamma^+_{\text{outer}}$, where

$$\gamma^+_{\text{inner}} \triangleq \bigcup_{t \geq t_0} \left(\begin{bmatrix} 2\cos(0.1\pi t) \\ 4\sin(0.1\pi t) \end{bmatrix} - (e_{\max} - \nu) \left(\gamma^+\right)^{\perp}(t) \right) \tag{103a}$$

$$\gamma^+_{\text{outer}} \triangleq \bigcup_{t \geq t_0} \left(\begin{bmatrix} 2\cos(0.1\pi t) \\ 4\sin(0.1\pi t) \end{bmatrix} + (e_{\max} - \nu) \left(\gamma^+\right)^{\perp}(t) \right), \tag{103b}$$

$e_{\max} > 0$ is user-defined and denotes the maximum tracking error to be enforced by the barrier function, $\nu > 0$, and $\left(\gamma^+\right)^{\perp}(t) \in \partial \mathcal{B}_1(0)$ denotes the direction normal to $\begin{bmatrix} 2\cos(0.1\pi\tau) \\ 4\sin(0.1\pi\tau) \end{bmatrix}$. Thus, we define the RKHS

$$\mathcal{H}_\gamma \triangleq \overline{\text{span}\left\{ \mathcal{K}_\xi u : \xi \in \left(\overline{\mathcal{D}}_{\gamma^+_{\text{outer}}} \setminus \mathring{\mathcal{D}}_{\gamma^+_{\text{inner}}} \right), u \in \mathbb{U} \right\}}. \tag{104}$$

Finally, we consider the set of centers of the kernel functions

$$\Xi_N = \Xi_{\gamma,N} \cup \Xi_{\gamma_{\text{outer}},N} \cup \Xi_{\gamma_{\text{inner}},N}, \tag{105}$$

where

$$\Xi_{\gamma,N} \triangleq \bigcup_{k=1}^{N} \begin{bmatrix} 2\cos(0.1\pi k\tau) \\ 4\sin(0.1\pi k\tau) \end{bmatrix}, \tag{106a}$$

$$\Xi_{\gamma_{\text{inner}},N} \triangleq \bigcup_{k=1}^{N} \left(\begin{bmatrix} 2\cos(0.1\pi k\tau) \\ 4\sin(0.1\pi k\tau) \end{bmatrix} - (e_{\max} - \nu) \left(\gamma^+\right)^{\perp}(k\tau) \right), \tag{106b}$$

$$\Xi_{\gamma_{\text{outer}},N} \triangleq \bigcup_{k=1}^{N} \left(\begin{bmatrix} 2\cos(0.1\pi k\tau) \\ 4\sin(0.1\pi k\tau) \end{bmatrix} + (e_{\max} - \nu) \left(\gamma^+\right)^{\perp}(k\tau) \right), \tag{106c}$$

for some $\tau > 0$ and $N \in \mathbb{N}$ user-defined.

Finally, we consider two alternative constraint sets, namely a time-varying constraint set and a time-invariant constraint set. The time-varying constraint set is characterized by (24) with $M = I_2$,

$$\dot{e}_{\text{bound,user}}(t) = \gamma e_{\text{bound}}(t) \frac{u_{\max} - \|u(t)\|_\infty}{\max(\|u(t)\|_1, u_{\min})},$$

$$e_{\text{bound,user}}(t_0) = e_{\text{bound,user},0}, \quad t \geq t_0, \tag{107}$$

$\gamma > 0$, $e_{\text{bound,user},0} > 0$, and $u_{\max} > u_{\min} > 0$ user-defined. The time-invariant constraint set considered in the proposed set of numerical examples is characterized by (24) with $e_{\text{bound}}(t) \equiv 0$ for all $t \geq t_0$. As discussed in [19], (107) captures *soft* saturation constraints on the

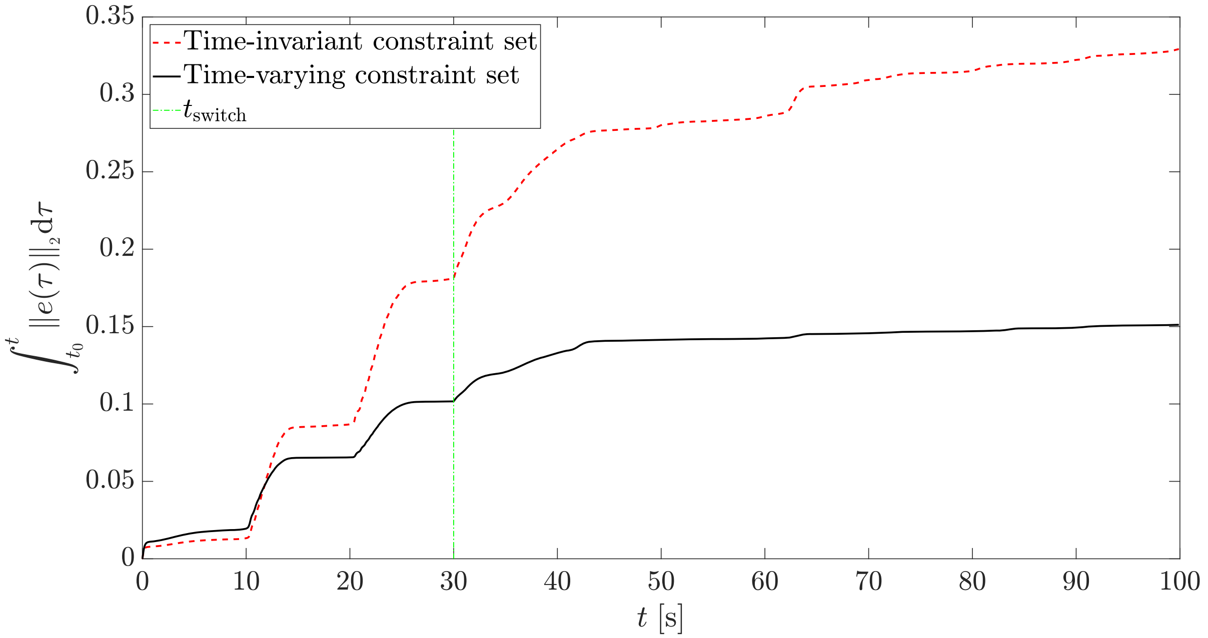
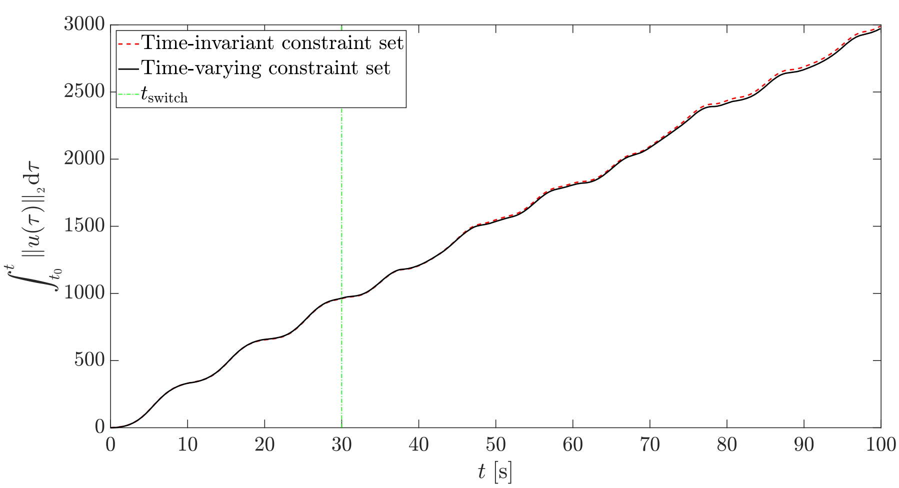


Fig. 5 $\mathcal{L}_1$-norm of the trajectory tracking error applying the results of Theorem 5 (dashed red line) and Corollary 1 (solid black line). Applying Theorem 5, a considerably smaller cumulative tracking error is attained, and the variations in the tracking error experience considerably smaller variations. At $t_{\text{switch}} = 30$ s, the matrix Λ switched from I_2 to $\text{diag}(0.1, 0.3)$ and the actuators deliver a smaller control input than expected. The matrix Λ is unknown to the control system

Fig. 6 $\mathcal{L}_1$-norm of the control input applying the results of Theorem 5(dashed red line) and Corollary 1 (solid black line). Both formulations show very similar cumulative control effort. However, as highlighted by Figure 6, Theorem 5 guarantees a considerably smaller tracking error



control input, that is, the function $e_{\text{bound}}(\cdot)$ such that (107) is verified allows contracting the constraint set whenever the 1-norm of the control input is sufficiently close to the user-defined lower saturation bound $u_{\min}$ and expanding the constraint set whenever the ∞-norm of the control input is sufficiently close to the user-defined upper saturation bound $u_{\max}$.

Figure 5 shows the $\mathcal{L}_1$-norm of the tracking error, Figure 6 shows the $\mathcal{L}_1$-norm of the control input, and Figure 7 shows the tracking error as a function of time. It is apparent how the framework captured by Theorem 5 allows for a smaller cumulative tracking error with a similar cumulative control effort as Corollary 1. Furthermore, Theorem 5 allows imposing smaller tracking errors and tune the hard constraints on the tracking

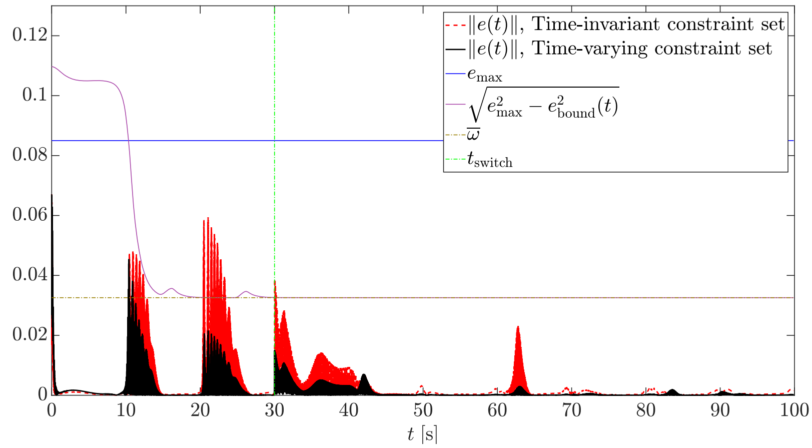


Fig. 7 Norm of the tracking error applying Theorem 5 (solid black line) and Corollary 1 (dashed red line) and user-defined bounds on the tracking error. Applying Theorem 5, the tracking error can meet a tighter constraint given by $\sqrt{e_{\max}^2 - e_{\text{bound}}^2(t)}$ for all $t \in [0, 100]$ s. Applying Corollary 1, the tracking error is required to be smaller than $e_{\max}$ at all times. It is apparent how applying Theorem 5, the tracking error never violates the time-varying constraint, which, in general, tends to tighten. The term $e_{\text{bound}}(\cdot)$ allows accounting for soft constraints on the control input. These soft constraints are met by imposing hard constraints on the tracking error. Applying Corollary 1, the tracking error does not violate the user-defined, time-invariant constraints given by (24) with $e_{\text{bound}}(t) \equiv 0$. The high-frequency oscillations in the tracking error are due to the high adaptive rates employed in the proposed simulations.

error according to the soft saturation constraints on the control input captured by (107).

Figures 8 and 9 show the $\mathcal{L}_1$-norms of the tracking error and of the control input by applying Corollary 1 and Theorem 5.7 of [2]. Corollary 1 guarantees smaller control tracking errors for similar control efforts than its unconstrained counterpart. Illustrative videos of the proposed results are available at [40]. Computer codes implementing the proposed results are available at [31].

9 Conclusion

This paper proposed for the first time the use of barrier Lyapunov functions to enforce user-defined constraints on the tracking error, and, hence, on the closed-loop plant trajectory, within the context of nonparametric MRAC systems; nonparametric MRAC systems counter parametric, matched, and unmatched uncertainties in the plant dynamics. Assuming that the parametric uncertainties lie in a native space, nonparametric control systems do not require that a regressor vector is provided *a priori* or constructed online, as it occurs for the overwhelming majority of existing adaptive control systems.

The proposed results allow enforcing both time-varying and time-invariant constraints. In the case of time-varying constraints, the proposed results mitigate user-defined requirements that are overly strong for the given plant dynamics. The proposed results are the first of this kind since, thus far, the literature on nonparametric control assumes *a priori* knowledge of the region in which the closed-loop plant trajectory is assumed to lie. This assumption, though verifiable in practice, may imply high computational costs for the need to cover large regions of the state space with kernel functions, and may lead to poor tracking error performance whenever it is not verified.

Numerical simulations verify the applicability of the proposed results and show how the proposed results lead to considerably smaller tracking errors than unconstrained nonparametric MRAC systems while retaining strikingly similar levels of control effort. Furthermore, these simulations show how the proposed framework can enforce both *hard* constraints on the tracking error and *soft* constraints on the required control. In particular, user-defined constraints can be relaxed whenever actuators approach their saturation limits and tightened whenever the actuators allow additional control effort.

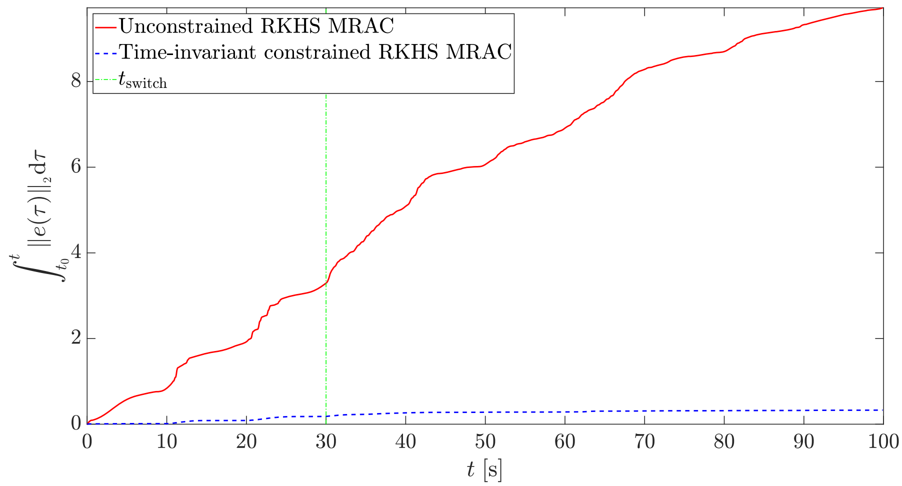


Fig. 8 $\mathcal{L}_1$-norm of the tracking error applying Corollary 1 (dashed blue line), and $\mathcal{L}_1$-norm of the tracking error applying Theorem 5.7 of [2] (solid red line), that is, nonparametric MRAC without constraints on the tracking error. Corollary 1 allows enforcing considerably smaller cumulative tracking errors

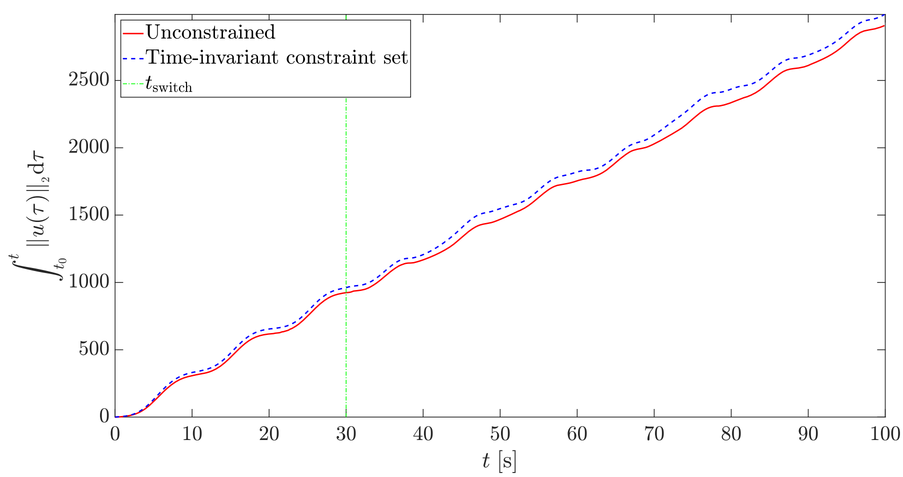


Fig. 9 $\mathcal{L}_1$-norm of the control input applying Corollary 1 (dashed blue line), and $\mathcal{L}_1$-norm of the control input applying Theorem 5.7 of [2] (solid red line). These cumulative control efforts are substantially comparable. However, as shown in Figure 8, Corollary 1 produces smaller tracking errors

Future work directions involve the use of the convex projection operator [2, Ch. 5] to prevent parameter drift of MRAC as an alternative to the σ-modification proposed in Section 6. Since using the convex projection operator allows setting explicit bounds on the adaptive gains, the resulting formulation would enable a more comprehensive framework to enforce user-defined constraints. Additional work directions involve applying the proposed MRAC systems to multi-rotor uncrewed aerial vehicles (UAVs) in simulation and flight tests, using the framework presented in [41].

Acknowledgements This publication is partially funded by the MUR - DM 118/2023, as part of the project PNRR-NGEU.

Finanziato dall'Unione europea NextGenerationEU | Ministero dell'Università e della Ricerca | Italiadomani

Author contributions Mr. Giorgio Orlando was responsible for writing the paper, producing theoretical results, and performing numerical simulations. Mr. Wang was responsible for producing part of the theoretical results in Section 5. Dr. L'Afflitto was responsible for proofreading the paper, verifying its technical correctness, setting up the theoretical problem, and seeking funding. Dr. Kurdila was responsible for proofreading the paper and verifying its technical correctness.

Data Availability Statement No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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