An Extension of Vector Fitting to Weakly Nonlinear Circuits

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***Abstract*—This paper describes a macromodeling algorithm to build behavioral models of nonlinear systems and components in a data-driven manner. The main tool is an extension of the Vector Fitting algorithm to perform rational approximation of multivariate transfer functions that describe nonlinear systems according to Volterra series theory. In addition to the rational model, a realization in terms of differential equations is provided so as to enable time-domain simulation. The raw data required by this algorithm can be extracted by means of Harmonic Balance analysis or from X-parameters, both available in commercial software.**

I. Introduction

Signal and Power Integrity (SI/PI) analyses are often conducted through time-domain simulation to assess and verify performance and design constraints. This is enabled by accurate modeling techniques to enable system-level verification in the design phase. The ever-increasing complexity of modern electronic systems, with technologically advanced packaging solutions and high data rates, implies that system-level simulations become exceedingly computationally costly due to the need to co-simulate several components at once, including electrical parasitics as extracted from electromagnetic (EM) simulation.

Many electrical structures of interest in SI/PI are linear and passive, which has motivated past efforts into developing linear macromodeling [1]. Using macromodeling techniques such as rational approximation, compact and efficient behavioral models can be constructed to represent system components in SPICE simulations. In particular, the Vector Fitting (VF) algorithm [2] has received much attention in SI/PI applications, because it allows building black-box rational models of packages and interconnects starting from S-parameter data extracted from EM solvers. This is essential to include the output of field solvers in SPICE-like circuit-level simulations. In addition, VF is also effective for building reduced models of large lumped networks to speed up time-domain integration.

However, systems including nonlinear components cannot be directly modeled using VF, for which samples of a transfer function are required. Although considering small-signal transfer functions (e.g., from AC analysis in SPICE) is an option, the resulting model would be unable to represent large-signal nonlinear effects as the input signal significantly departs from the bias point. Practical examples of nonlinear devices are voltage regulators, arising in PI, and I/O buffers used for signal transmission in the SI context.

This paper presents an extension of VF to build nonlinear behavioral models via rational approximation of generalized transfer functions, as defined by Volterra series theory. Higher-order nonlinear Volterra kernels are approximated in the frequency domain by a multivariate extension of the VF algorithm, whose poles and generalized residues are computed recursively. Finally, the resulting model is synthesized into a nonlinear (bilinear) state-space form for fast time-domain simulation.

In the following, Sec. II provides the background material that is necessary to develop the nonlinear macromodeling algorithm in Sec. III. The proposed approach is demonstrated in Sec. IV on a nonlinear system made up of an Low-DropOut (LDO) regulator loaded with a passive multiport representing a power delivery network. On this example, a $110\times$ speedup is achieved with respect to HSPICE with well-controlled accuracy.

II. Background

Volterra series theory provides a foundation for analyzing and representing nonlinear systems. Consider a generic nonlinear one-port electronic circuit or system, with the variables $v(t)$ and $i(t)$ denoting the port voltage and current, respectively. The dynamic relationship between these two variables can be seen as a mapping from either of the two, denoted as the *input* $u(t)$, and the *output* $y(t)$. Under technical conditions detailed in [3], this input-output (I-O) map admits the following expansion

$$y(t) = y_0 + \sum_{m=1}^{\infty} \int_0^t \cdots \int_0^t h_m(\tau_1, \ldots, \tau_m) \tilde{u}(t-\tau_1) \cdots \tilde{u}(t-\tau_m) \mathrm{d}\boldsymbol{\tau} \quad (1)$$

where $\tilde{u}(t) = u(t) - u_0$ is the deviation from a DC input value u_0 that defines an operating point where the expansion is centered. In (1), y_0 is a constant, and each integral in the sum is a multidimensional convolution of the input with the degree-m multivariate Volterra kernel $h_m(\tau_1, \ldots, \tau_m)$. The notation $\mathrm{d}\boldsymbol{\tau} \triangleq \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_m$ indicates that the integral is multi-dimensional. Note that this series generalizes the well-known convolution formula that describes linear I-O relationships. Several different choices are admissible for $h_m(\tau_1, \ldots, \tau_m)$, as detailed in [4], with the preferred one being the *symmetric* kernel. Henceforth, each $h_m(\tau_1, \ldots, \tau_m)$ is assumed to be

a symmetric function, that is invariant under arbitrary permutation of its arguments. In practice, truncating the infinite series (1) to the first M terms leads to an approximation of the initial I-O map. If, for a given maximum amplitude of $|u(t)|$, truncating to a small number M of leading terms provides an accurate approximation, then the system is said to be *weakly nonlinear* [5]. The additive term corresponding to $m = 1$ in the series (1) is the response of a *small-signal* approximation of the nonlinear I-O map. Further terms are interpreted as higher-order nonlinear corrections to a local small-signal model.

Though by no means the only way to study nonlinear systems, the Volterra series conveniently gives a generalization of the concept of transfer function (TF) that plays a central role in electronic design. In fact, the multidimensional Laplace transform of h_m allows defining the degree-m Generalized Transfer Function (GTF) $H_m(s_1, \dots, s_m)$ [4],

$$H_m(s_1, \dots, s_m) \triangleq \mathcal{L}\{h_m(\tau_1, \dots, \tau_m\} \triangleq \int_0^t \cdots \int_0^t h_m(\tau_1, \dots, \tau_m) e^{-\sum_{i=1}^m s_i \tau_i} \mathrm{d}\boldsymbol{\tau}$$

For notational convenience we introduce the vector $\boldsymbol{s}_m \triangleq (s_1, \dots, s_m)$ and the shorthand $H_m(\boldsymbol{s}_m) = H_m(s_1, \dots, s_m)$.

The GTFs are symmetric transfer functions. In [6], it is shown that measured or simulated X-parameters of a nonlinear device can used as a starting point to extract values of GTFs at a discrete set of frequencies. In [6], [7], these sampled GTFs values constitute a nonparametric *Volterra model*, as they allow to find the time-domain system response $y(t)$ to a given $u(t)$ by means of the inverse Fourier transform. While nonparametric methods based on collections of GTF data are abundant in the literature, parametric modeling where a dynamical model is built starting from this data are more scarce. Fundamental work in this direction is presented in [8], which deals (among others) with bilinear identification in the Loewner framework using GTF data.

III. Vector Fitting of Generalized Transfer Functions

For linear devices, it is well known that rational macromodeling via VF [1], [2] is a valuable approach to enable fast and accurate signal and power integrity simulations. The core idea behind rational macromodeling consists in finding a rational function $H(s)$ that approximates the TF $\breve{H}(s)$ of the original device, starting from knowledge of sampled values $\breve{H}(\mathrm{j}\omega_k)$ at a discrete set of frequencies.

A. Problem statement

In this work, we assume that samples of GTFs up to some maximum degree M are available from measurement or simulation via commercial software. These constitute a dataset containing K_m frequency-value pairs for each degree $m = 1, \dots, M$,

$$\left\{ \left(\boldsymbol{s}_m^{(k)}, \breve{H}_m(\boldsymbol{s}_m^{(k)}) \right), k = 1, \dots, K_m, m = 1, \dots, M \right\}. \quad (2)$$

The main objective is to construct a nonlinear dynamical model that approximates the behavior of the original device as described by the GTF data samples. The goal is to use such a model, expressed in terms of nonlinear differential equations, to represent the original complex device in time-domain simulations with an advantage in computational efficiency.

B. Model structure

A simple model structure to represent nonlinear I-O maps is the bilinear state-space system defined by the following equations

$$\begin{cases} \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{N}\boldsymbol{x}u + \boldsymbol{b}u \\ y = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \end{cases} \quad (3)$$

where $\boldsymbol{x} \in \mathbb{R}^\nu$ is the hidden state vector. The state-space coefficients are $\mathbf{A}, \mathbf{N} \in \mathbb{R}^{\nu \times \nu}$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^\nu$. The bilinear model structure is adopted in this work because it can be shown that many nonlinear systems can be approximated with a model of the form (3) [4]. The symmetric GTFs $H_m(\boldsymbol{s}_m)$ of (3) are derived via the *growing exponential* approach [4] in terms of the realization coefficients. As stated in [8], these also admit a pole-residue form as a combination of multivariate *partial fractions* [9]. For $m \geq 1$, we can write

$$H_m(\boldsymbol{s}_m) = \sum_{i_1=1}^{\nu_1} \cdots \sum_{i_m=1}^{\nu_m} r_{i_1,\dots,i_m} \varphi_{i_1,\dots,i_m}(s_1, \dots, s_m), \quad (4)$$

where the coefficients $r_{i_1,\dots,i_m} \in \mathbb{C}$ are interpreted as *generalized residues*. For each degree m, the basis functions $\varphi_{i_1,\dots,i_m}(s_1, \dots, s_m)$ are defined in terms of a set of ν_m basis poles $\mathcal{P}_m = \{p_i^{(m)}\}_{i=1}^{\nu_m}$ as

$$\varphi_{i_1}(s_1) = \left(s_1 - p_{i_1}^{(1)} \right)^{-1} \quad (5)$$

for degree $m = 1$, and

$$\varphi_{i_1,\dots,i_m}(s_1, \dots, s_m) = \frac{1}{m!} \left(s_1 + \cdots + s_m - p_{i_m}^{(m)} \right)^{-1} \cdot \cdot \sum_{\sigma \in \mathbb{P}_m} \varphi_{i_1,\dots,i_{m-1}}(s_{\sigma(1)}, \dots, s_{\sigma(m-1)}) \quad (6)$$

for $m \geq 2$. In (6), the sum is on the set $\mathbb{P}_m$ of all permutations σ of the indices $\{1, \dots, m\}$.

C. Rational approximation

In a rational approximation context, building a nonlinear macromodel starting from the data in (2) translates into finding poles and residues of (4) for all m up to the maximum degree M where data is available. As in the standard VF algorithm, this task is broken down in two steps: an iterative *pole-finding* phase, followed by a least-squares optimization to find residues.

Starting with the first set of poles $\mathcal{P}_1$, we see that finding a set of poles to define an approximant $H_1(s_1)$ of the data $\breve{H}_1(s_1)$ is a standard univariate rational fitting problem that can be addressed via VF. Hence, running VF to find residues r_{i_1} and poles $p_{i_1}^{(1)}$ to minimize the least-squares residual of

$$\sum_{i_1=1}^{\nu_1} r_{i_1} \left(s_1^{(k)} - p_{i_1}^{(1)} \right)^{-1} \approx \breve{H}_1(s_1^{(k)}), \quad k = 1, \dots K_1$$

yields the first pole set $\mathcal{P}_1$.

We proceed by induction to higher degrees $m \geq 2$ using the observation that, based on (4) and (6), $H_m(\boldsymbol{s}_m)$ can written as the product of two vectors

$$H_m(\boldsymbol{s}_m) = \boldsymbol{F}_m(s_1 + \cdots + s_m)^{\mathrm{T}} \boldsymbol{G}_m(\boldsymbol{s}_m)$$

where $\boldsymbol{F}_m \in \mathbb{C}^{\nu_1 \cdots \nu_{m-1}}$ is a rational function of $(s_1 + \cdots + s_m)$ with poles in the set $\mathcal{P}_m$, and $\boldsymbol{G}_m(\boldsymbol{s}_m)$ only depends on lower-degree pole sets $\mathcal{P}_i$ with $1 \leq i \leq m-1$. Hence, once $\mathcal{P}_1, \ldots, \mathcal{P}_{m-1}$ are known, the vectors $\boldsymbol{G}_m(\boldsymbol{s}_m)$ can be regarded as known quantities. This suggests an iterative approach to determine $\mathcal{P}_m$ after pole sets up to $m-1$ are known. In fact, considering the approximation problem for degree-m GTFs,

$$H_m(\boldsymbol{s}_m^{(k)}) = \boldsymbol{F}_m(s_1^{(k)} + \cdots + s_m^{(k)})^{\mathrm{T}} \boldsymbol{G}_m(\boldsymbol{s}_m^{(k)}) \approx \breve{H}_m^{(k)} \quad (7)$$

the vectors $\boldsymbol{G}_m(\boldsymbol{s}_m^{(k)})$ are completely determined by lower-degree pole sets, and the univariate rational function $\boldsymbol{F}_m(s)$ is the only unknown. As detailed in [9], poles of $\boldsymbol{F}_m$ are estimated via the *weighted* VF algorithm to obtain the pole set $\mathcal{P}_m$. This is achieved by iteratively solving a *linearized* version of the optimization problem (7).

Once the pole sets $\mathcal{P}_1, \ldots, \mathcal{P}_M$ are found via iterations of the weighted VF algorithm, the basis functions φ are known. At this stage, the generalized residues $r_{i_1,\ldots,i_m}$ in (4) follow from least-squares optimization (see [9] for details) using the format (4) to minimize the residual of $H_m(\boldsymbol{s}_m^{(k)}) \approx \breve{H}_m^{(k)}$.

D. Bilinear realization

The pole-residue model in (4) is a Laplace-domain expression. For time-domain simulation, it is convenient to translate it into a state-space realization made up of differential equations. The definition of the basis functions (5)-(6) is specifically chosen to make it possible to provide a bilinear realization of (4). As discussed in [9], this model admits the following realization

$$\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \mathbf{A}_1 \boldsymbol{x}_1 + \mathbf{1} u, \\
\dot{\boldsymbol{x}}_m &= (\mathbf{A}_m \otimes \mathbb{I}_{N_{m-1}}) \boldsymbol{x}_m + (\mathbf{1} \otimes \boldsymbol{x}_{m-1}) u, \quad 2 \leq m \leq M \\
y &= \sum_{m=1}^{M} \boldsymbol{r}_m^{\mathrm{T}} \boldsymbol{x}_m, \qquad\qquad 2 \leq m \leq M
\end{aligned} \quad (8)$$

where $\mathbf{A}_m = \mathrm{diag}\{p_i^{(m)}\}_{i=1}^{\nu_m}$, $\mathbf{1}$ is the column vector of ones, $N_m = \nu_1 \cdots \nu_m$ is the size of $\boldsymbol{x}_m$, and $\otimes$ denotes the Kronecker matrix product. The residue vectors $\boldsymbol{r}_m$ collect $r_{i_1,\ldots,i_m}$, i.e. the generalized residues of degree m.

The state-space form (8) can be solved in time-domain to find the output y, possibly after a change of coordinate to make the coefficients real-valued, as explained in [9, Sec. IV, Remark 1].

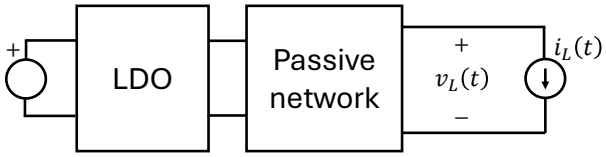

Fig. 1. Block diagram of the test device made up of an LDO loaded with a passive network.

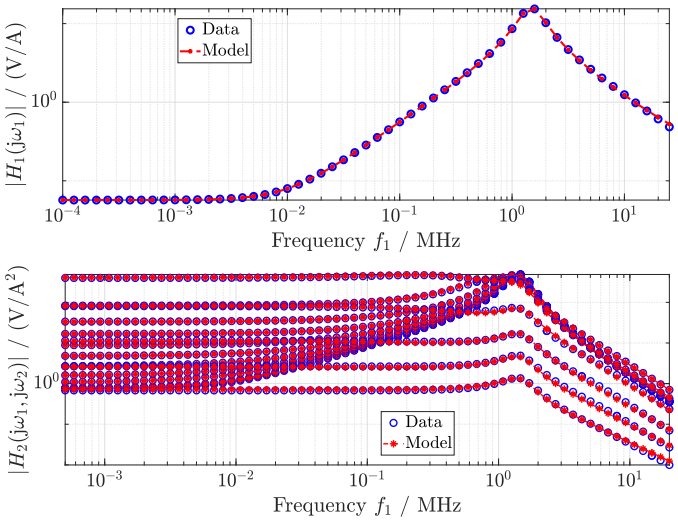

Fig. 2. Comparison of GTF data samples with the fitted nonlinear model.

IV. Numerical Verification

In this section, the proposed approach is validated on a nonlinear electronic circuit including an LDO regulator. The LDO circuit, reproduced from [10], is designed in a 350 nm technology for low-power applications to supply power to other analog blocks. This is connected to a voltage supply and to a passive component on the output side as depicted in Fig. 1. This passive component is a power delivery network represented through a linear S-parameters macromodel. In this example, we model the I-O relationship between the load current $i_L(t)$ and the load voltage $v_L(t)$, respectively regarded as the input $u(t) \triangleq i_L(t)$ and the output $y(t) \triangleq v_L(t)$. A nonlinear model is constructed around the operating point corresponding to a load current $i_{L,0} = u_0 = 20\,\mathrm{mA}$.

Through HSPICE Harmonic Balance (HB) analysis, it is possible to analyse the device behavior in steady-state conditions, with the input $u(t)$ set to a superposition of tones at different frequencies. Samples of Volterra GTFs are extracted via HB using the *harmonic probing* method described in [5]. In practice, the value $\breve{H}_m^{(k)}$ corresponding to the point $\boldsymbol{s}_m^{(k)} = (\mathrm{j}\omega_1^{(k)}, \ldots, \mathrm{j}\omega_m^{(k)})$ is extracted from HB analysis with m tones at frequencies $\omega_1^{(k)}, \ldots, \omega_m^{(k)}$.

A nonlinear model of maximum degree $M = 3$ is built via the proposed algorithm using samples $\breve{H}_1^{(k)}, \breve{H}_2^{(k)}, \breve{H}_3^{(k)}$. For degree $m = 1$, standard AC analysis was used to

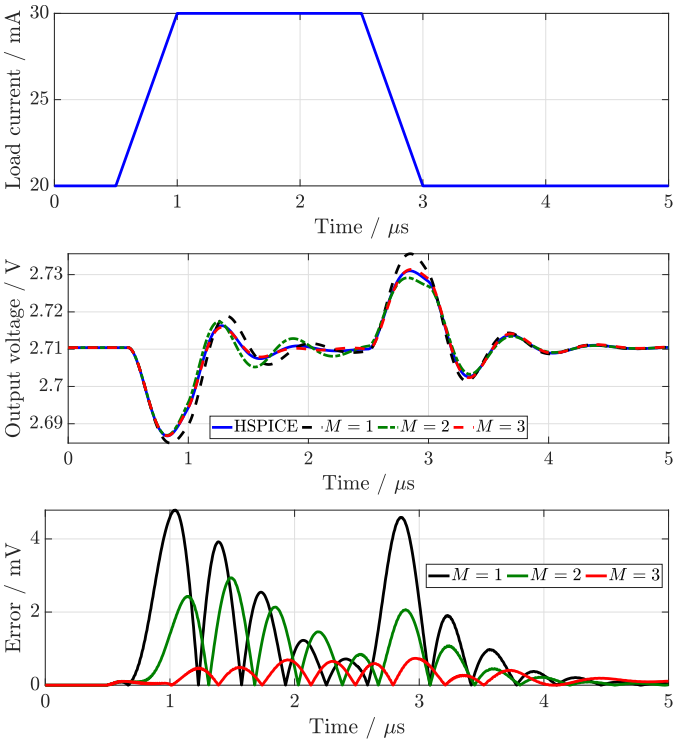

Fig. 3. Time-domain simulation of the nonlinear device in Fig.1. Top: load current signal (input). Middle: Load voltage response. Bottom: Instantaneous error.

obtain TF values at $K_1 = 55$ frequencies in the bandwidth $[500\,\text{Hz}, 20\,\text{MHz}]$. A total of $K_2 = 9940$ and $K_3 = 1200$ samples of higher-order GTFs were collected in the same frequency band. The number of poles chosen for model fitting are $\nu_1 = 7$, $\nu_2 = 5$, $\nu_3 = 5$. These were selected to obtain a satisfactory model-data error in the frequency domain, with the results reported in Fig. 2.

Time-domain validation is carried out using the load voltage signal depicted in Fig. 3 (top), that is a pulse signal of amplitude $10\,\text{mA}$ and $0.5\,\mu\text{s}$ rise/fall time. The exact load voltage response resulting from the reference HSPICE simulation is reported in Fig. 3 (middle) and compared with models of different nonlinear degrees (obtained by truncating the degree-3 macromodel). For each time instant, the error between the nonlinear macromodels and HSPICE is reported in Fig. 3 (bottom), suggesting increased accuracy of higher-order models over a simple small-signal approximation (black). The reference HSPICE simulation includes $2 \cdot 10^4$ timesteps that are solved in 99 s. The bilinear macromodel ($M = 3$) solved in MATLAB for the same number of timesteps takes only 0.89 s, resulting in a $110\times$ speedup.

A second test signal is depicted in Fig. 4 (top), obtained by low-pass filtering a white noise signal with a bandwidth of 5 MHz. The system response evaluated with HSPICE is compared with the nonlinear model in Fig. 4 (bottom) in terms of instantaneous error. This simulation requires 29.7 seconds in HSPICE, which produces 4568 simulation timesteps. The nonlinear model solved in MATLAB over 10^4 timesteps takes

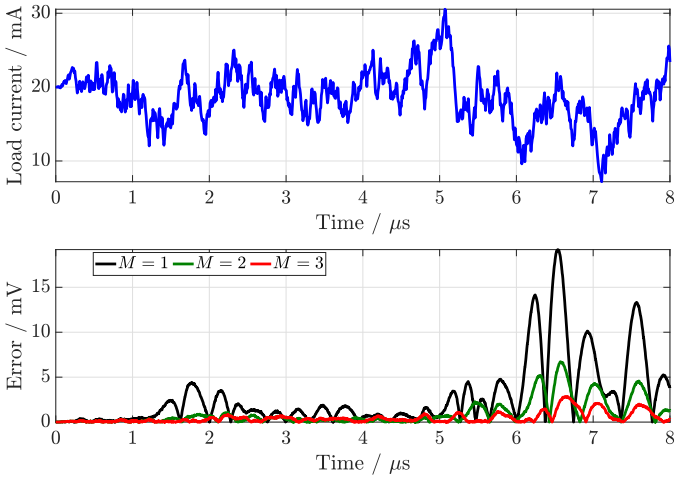

Fig. 4. Time-domain simulation of the nonlinear device in Fig.1 using a noisy signal. Top: Load current signal (input). Bottom: Instantaneous error.

0.16 s, resulting in a speedup of $180\times$.

V. Conclusions

This paper described an extension of Vector Fitting that allows modeling weakly nonlinear devices through rational approximation. The proposed algorithm provides a bilinear state-space model in terms of differential equations, that can be integrated in time through time-stepping. Numerical examples demonstrated a major speedup with respect to reference HSPICE simulations. The adopted model format allows translation into an equivalent behavioral circuit to be embedded in SPICE environments, a topic that is left for future work.

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