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ABSTRACT

Stochastic orders have been widely used in reliability literature to compare the performances of coherent systems, and various criteria have been provided on this purpose. In particular, sufficient conditions have been found for the lifetime of a system to be stochastically larger than that of another system having the same components with identically distributed lifetimes but a different structure function. Known results of this kind concern some of the most relevant stochastic orders, but in the literature no conditions have been provided for the well-known increasing convex order (icx). Here we describe conditions such that two lifetimes of coherent systems are comparable in this stochastic sense when conditions for other stronger orders do not apply. Illustrative examples are also given.

1 | Introduction

Conditions to compare lifetimes of coherent systems have been widely studied in reliability and survival theories. Starting from the paper by Kochar et al. [1], in which for the first time conditions are given for two coherent systems with iid components to be comparable in the st (usual stochastic), hr (hazard rate), and lr (likelihood ratio) orders, a wide range of results have been given for other orders and under less restrictive assumptions (e.g., for components with dependent lifetimes). See for example Navarro et al. [2] and Arriaza and Sordo [3] and references therein for a comprehensive overview.

Many stochastic orders have been considered in this framework but, to the best of the authors' knowledge, no result describes conditions for the lifetimes of two systems to be comparable in icx (increasing convex order), although the icx order is a stochastic comparison of extreme interest in the reliability field. The aim of this note is to fill this gap; a first general result describing conditions for the ordering in icx of two lifetimes defined through distortions of the same distribution is given, and its applications in comparing coherent systems and order statistics are then described. Regarding the icx order, some results in a different direction are given in Li and Li [4], in which the preservation of the icx order is studied in relation to the formation of series and parallel systems with dependent components.

For the purpose of presenting the results, useful notions are briefly recalled here, starting from the definition of the stochastic orderings mentioned along the paper. For it, let X and Y be two absolutely continuous random variables with support $S \subseteq (0, +\infty)$, finite means and respective distribution functions F and G, and let $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively, be the corresponding survival functions. Then:

- X is said to be smaller than Y in the st (*usual stochastic*) order, denoted $X \le_{st} Y$, if $E[\psi(X)] \le E[\psi(Y)]$ for all increasing functions $\psi : \mathbb{R} \to \mathbb{R}$ such that the expectations exist, or, alternatively, if $\overline{F}(t) \le \overline{G}(t)$ for all $t \in S$;
- X is said to be smaller than Y in the icx (*increasing convex*) order, denoted $X \le_{icx} Y$, if $E[\psi(X)] \le E[\psi(Y)]$ for all increasing and convex functions $\psi : \mathbb{R} \to \mathbb{R}$ such that the expectations exist, or, alternatively, if

$$\int_x^\infty \overline{F}(t)dt \le \int_x^\infty \overline{G}(t)dt \quad \text{for all} \quad x \in S;$$

- X is said to be smaller than Y in the icv (*increasing concave*) order, denoted $X \le_{icv} Y$, if $E[\psi(X)] \le E[\psi(Y)]$ for all increasing and concave functions $\psi : \mathbb{R} \to \mathbb{R}$ such that the expectations exist, or, alternatively, if

$$\int_0^x F(t)dt \ge \int_0^x G(t)dt \quad \text{for all} \quad x \in S.$$

We refer the reader to Shaked and Shanthikumar [5] and Belzunce et al. [6] for properties and details about these stochastic orders, and to Barlow and Proschan [7] and Navarro [8] for their applications in reliability theory. We just point out here that, clearly, the st order implies both the icx and icv orders. We remark that all these stochastic orders imply that the order is preserved by the respective means. Moreover, from $X \le_{icx} Y$ it follows $E[(X-t)^+] \le E[(Y-t)^+]$, which means that by the icx order one can compare the expected survival times after a fixed threshold t, which may represent the end of a warranty, contract or burn-in period.

Closures of stochastic orders with respect to various reliability operations have been widely studied in reliability literature. In particular, a number of papers provide conditions for which the lifetimes of two different coherent systems, whose definition is recalled here, are ordered according to some specific stochastic order. Given a set components having lifetimes $X_1, \dots, X_n$, a *coherent system* is a system having lifetime $T = \phi(X_1, \dots, X_n)$ where $\phi : [\mathbb{R}^+]^n \to \mathbb{R}^+$ is increasing and non-constant in each component (see Barlow and Proschan, [7], for formal definitions and basic properties of coherent systems). The lifetimes $X_1, \dots, X_n$ of the components can be identically distributed or not, and can be independent or dependent. In the latter case, the dependence is commonly described by means of the copula or the survival copula of the vector $(X_1, \dots, X_n)$ (see, e.g., page 59 in Navarro, [8], for details and a brief introduction on copulas and survival copulas). In the specific case where the components' lifetimes have a common distribution F, then the distribution function F_T of T can be expressed as

$$F_T(t) = h(F(t)), \quad \text{for all} \quad t \in \mathbb{R}^+,$$

where the function $h : [0,1] \to [0,1]$ only depends on ϕ and on the copula of $(X_1, \dots, X_n)$ (see Theorem 2.11 in Navarro [8]). The functions h one obtains in this manner are special cases of *distortion functions*, that is, left continuous and increasing functions such that $h(0) = 0$ and $h(1) = 1$. Note that a similar representation holds also for the survival function of the lifetime T: $\overline{F}_T(t) = 1 - F_T(t) = 1 - h(1 - \overline{F}(t)) = \overline{h}(\overline{F}(t))$, where $\overline{h}(u) = 1 - h(1-u)$ is again a "distortion function", commonly called *dual distortion function* in reliability literature. Details on distortion representation of lifetimes of systems can be found in Chapter 2 in Navarro [8], while specific examples of distortion (or dual distortion) functions for coherent systems (with independent or dependent components) are found in the references mentioned earlier.

In the particular case that the lifetimes $X_1, \dots, X_n$ are independent and identically distributed (iid), the distortion function associated to a coherent system having lifetime $T = \phi(X_1, \dots, X_n)$ is entirely described by the so called *Samaniego's signature* of the system, introduced in Samaniego [9] (see also Samaniego, [10], for further details). The idea behind the signature of a coherent system is that the lifetime T always coincides with that of one of the order statistics corresponding to $X_1, \dots, X_n$. Indeed, if $X_{i:n}$ represents the i-th smallest component lifetime, with $i \in \{1, \dots, n\}$, then we have $T \in \{X_{1:n}, \dots, X_{n:n}\}$ with probability one. Thus, we may identify a probability vector $\alpha = (\alpha_1, \dots, \alpha_n)$ where

$$\alpha_i = P[T = X_{i:n}], \quad i = \{1, \dots, n\}.$$

The lifetime distribution of any coherent system with iid components is entirely described by the vector α, referred as the system's signature. An explicit expression of the distortion function can be then provided in terms of the signature of the system and Bernstein polynomials as follows. For it, recall that the survival function of the i-th order statistic of a random vector of size n with iid components having survival function $\overline{F}$ is given by

$$\overline{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} (1 - \overline{F}(t))^j (\overline{F}(t))^{n-j}.$$

Hence, one can write $\overline{F}_{i:n}(t) = \overline{h}_{i:n}(\overline{F}(t))$, being $\overline{h}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} (1-t)^j t^{n-j}$. Moreover, by recalling that the $n+1$ Bernstein basis polynomials of degree n are defined as

$$g_j(t) = \binom{n}{j} (1-t)^j t^{n-j},$$

for $j \in \{0, 1, \dots, n\}$, the function $\overline{h}_{i:n}(t)$ can be expressed as $\overline{h}_{i:n}(t) = \sum_{j=0}^{i-1} g_j(t)$. Given the signature $\alpha = (\alpha_1, \dots, \alpha_n)$ of a coherent system, one can finally define as $\overline{H}_\alpha(t)$ the distortion function given by $\overline{H}_\alpha(t) = \sum_{i=1}^n \alpha_i \overline{h}_{i:n}(t)$. Then, the distortion function $\overline{H}_\alpha(t)$ can be rewritten as

$$\overline{H}_\alpha(t) = \sum_{i=1}^n \alpha_i \overline{h}_{i:n}(t) = \sum_{i=1}^n \alpha_i \sum_{j=0}^{i-1} g_j(t) = \sum_{j=0}^{n-1} g_j(t) \sum_{i=j+1}^n \alpha_i = \sum_{j=0}^{n-1} g_j(t) \overline{P}_{\alpha,j} \tag{1.1}$$

where $\overline{P}_{\alpha,j} = \sum_{i=j+1}^n \alpha_i$.

The notions of coherent systems and their signatures have been further generalized in Boland and Samaniego [11] with the concept of *mixed systems*. In practice, a mixed system is any system with lifetime T whose survival function can be represented as in (1.1), but without constraints on the signature α, that is, without necessarily representing a real coherent system. However, it can be proved that mixed systems can be realized in practice by using a randomization device which chooses a coherent system according to a fixed probability distribution (see Section 1.3.3 in Navarro, [8], for details).

2 | Main Results

It is a well-known fact and easy to prove that, given a lifetime X having survival function $\overline{F}$, and denoted by $X_{\overline{h}}$ the random lifetime with survival function defined as $\overline{F}_{\overline{h}}(\cdot) = \overline{h}(\overline{F}(\cdot))$, where $\overline{h}$ is any dual distortion function, then one has $X_{\overline{h}_1} \leq_{st} X_{\overline{h}_2}$ if, and only if, $\overline{h}_1(t) \leq \overline{h}_2(t)$ for all $t \in [0,1]$ (see, e.g., Navarro et al. [2]). However, just in few cases one has that distortion functions associated to coherent systems do not cross each other. This, in particular, occurs when the systems contain the same number of components. An interesting motivating case is provided for example in Kochar et al. [1] (see the systems in Figure 4). The following result describes conditions such that, in these cases, the lifetimes of the two systems are comparable in the weaker icx order and, consequently, also in terms of the expected values (although they are not comparable in the usual stochastic order).

Theorem 2.1. *Let $\overline{h}_1$ and $\overline{h}_2$ be dual distortion functions, $t_0 \in (0,1)$ such that $\overline{h}_1(t) \leq \overline{h}_2(t)$ for $t \in [0,t_0]$ and $\overline{h}_1(t) \geq \overline{h}_2(t)$ for $t \in [t_0,1]$ and $\int_0^1 \left[\overline{h}_1(t) - \overline{h}_2(t)\right] dt \leq 0$. Let X be an absolutely continuous random variable with support $S \subseteq (0,+\infty)$, survival function $\overline{F}$ and probability density function (pdf) f. If*

$$f(t) \geq f(\overline{F}^{-1}(t_0)) \text{ for } t \in (\inf S, \overline{F}^{-1}(t_0)) \quad \text{and}$$
$$f(t) \leq f(\overline{F}^{-1}(t_0)) \text{ for } t \in (\overline{F}^{-1}(t_0), \sup S), \tag{2.1}$$

then $X_{\overline{h}_1} \leq_{icx} X_{\overline{h}_2}$.

Proof. From the definition of the increasing convex order, $X_{\overline{h}_1} \leq_{icx} X_{\overline{h}_2}$ is equivalent to have

$$\int_x^{+\infty} \overline{F}_{X_{\overline{h}_1}}(t)dt \leq \int_x^{+\infty} \overline{F}_{X_{\overline{h}_2}}(t)dt, \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.2}$$

Since the survival functions of $X_{\overline{h}_1}$ and $X_{\overline{h}_2}$ are given by $\overline{h}_1(\overline{F}(\cdot))$ and $\overline{h}_2(\overline{F}(\cdot))$, respectively, and X is a random variable with support S, the relation in (2.2) can be equivalently reformulated as

$$\int_x^{\sup S} \left[\overline{h}_1(\overline{F}(t)) - \overline{h}_2(\overline{F}(t))\right] dt \leq 0, \quad \text{for all } x \in S. \tag{2.3}$$

With the change of variable $z = \overline{F}(t)$, and defining $\delta(t) = \overline{h}_1(t) - \overline{h}_2(t)$, (2.3) is reformulated as

$$\int_0^{\overline{F}(x)} \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz \leq 0, \quad \text{for all } x \in S. \tag{2.4}$$

The sign of the integrand function in (2.4) is determined by the numerator $\delta(t)$, being the denominator positive. Hence, from the hypothesis, the integrand is non-positive for $z \in [0,t_0]$ and non-negative for $z \in [t_0,1]$. This means that if $\overline{F}(x) \leq t_0$, then the integrand is non-positive over the entire integration domain and then the integral is non-positive, and the inequality in (2.4) is satisfied. If $\overline{F}(x) > t_0$, the integrand is non-negative for $z \in [t_0, \overline{F}(x)]$ and then we need to establish if the integral from 0 to $\overline{F}(x)$ takes a positive or negative sign. For $\overline{F}(x) > t_0$, we have

$$\frac{d}{dx}\int_{t_0}^{\overline{F}(x)} \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz = -f(x)\frac{\delta(\overline{F}(x))}{f(\overline{F}^{-1}(\overline{F}(x)))} = -\delta(\overline{F}(x)) \leq 0,$$

hence it is decreasing with respect to x and we just need to show the result for $x = \inf S$, which means $\overline{F}(\inf S) = 1$. Thus, we want to show that

$$\int_0^1 \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz \leq 0.$$

By using the properties of the integral, this is equivalent to

$$\int_0^{t_0} \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz + \int_{t_0}^1 \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz \leq 0,$$

and then

$$\int_{t_0}^1 \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz \leq \int_0^{t_0} \frac{-\delta(z)}{f(\overline{F}^{-1}(z))} dz. \tag{2.5}$$

Note that the integrand functions in LHS and RHS of (2.5) are non-negative. Moreover, by the assumptions on f, for $z \in (t_0, 1)$, that is, $\overline{F}^{-1}(z) \in (\inf S, \overline{F}^{-1}(t_0))$, we have

$$\frac{\delta(z)}{f(\overline{F}^{-1}(z))} \leq \frac{\delta(z)}{f(\overline{F}^{-1}(t_0))},$$

and for $z \in (0, t_0)$, which means $\overline{F}^{-1}(z) \in (\overline{F}^{-1}(t_0), \sup S)$,

$$\frac{-\delta(z)}{f(\overline{F}^{-1}(z))} \geq \frac{-\delta(z)}{f(\overline{F}^{-1}(t_0))}.$$

Then, about the integrals in LHS and RHS in (2.5), we have

$$\int_{t_0}^1 \frac{\delta(z)}{f(\overline{F}^{-1}(z))} dz \leq \int_{t_0}^1 \frac{\delta(z)}{f(\overline{F}^{-1}(t_0))} dz,$$
$$\int_0^{t_0} \frac{-\delta(z)}{f(\overline{F}^{-1}(z))} dz \geq \int_0^{t_0} \frac{-\delta(z)}{f(\overline{F}^{-1}(t_0))} dz.$$

Hence, to satisfy the inequality in (2.5), it is enough to show that

$$\int_{t_0}^1 \frac{\delta(z)}{f(\overline{F}^{-1}(t_0))} dz \leq \int_0^{t_0} \frac{-\delta(z)}{f(\overline{F}^{-1}(t_0))} dz$$

holds. By simplifying the common positive factor $f(\overline{F}^{-1}(t_0))$, the relation becomes

$$\int_{t_0}^1 \delta(z)dz \leq \int_0^{t_0} -\delta(z)dz,$$

which can be rewritten as

$$\int_0^1 \delta(z)dz \leq 0$$

and is satisfied by assumption. □

Note that an alternative proof of Theorem 2.1 can be also given by making use of Theorem 4.A.22 in Shaked and Shanthikumar [5] or Theorem 2.3.7 in Belzunce et al. [6]; the hypotheses on the pdf in (2.1) and of non-negativity of $\int_0^1 \left[\overline{h}_1(t) - \overline{h}_2(t)\right] dt$ serve to satisfy the condition $E[X_{\overline{h}_1}] \leq E[X_{\overline{h}_2}]$ required in those theorems. In this regard, we remark that the focus of the proof of

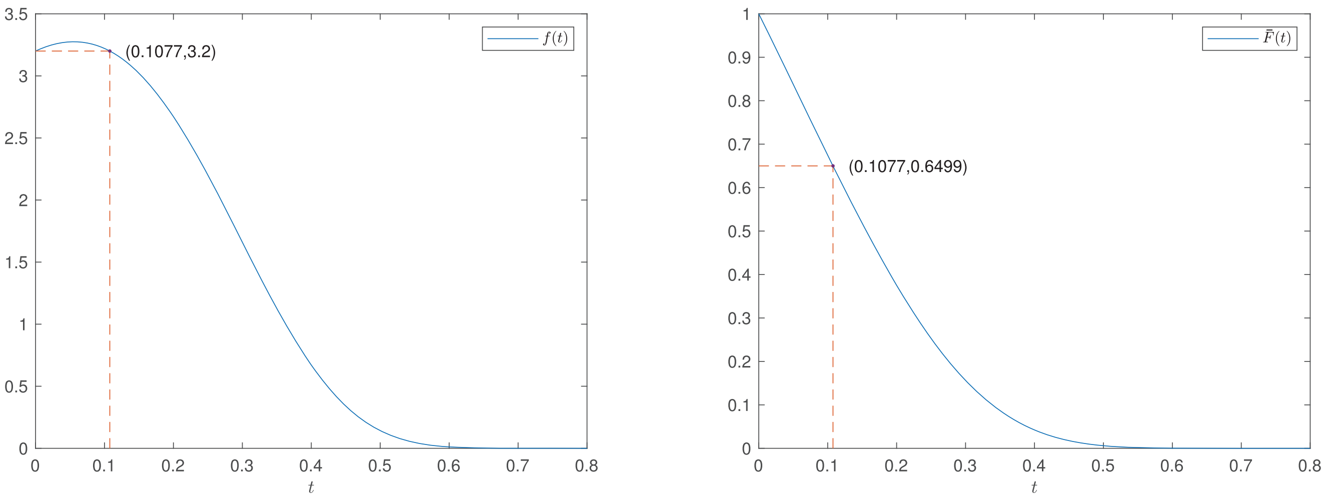

FIGURE 1 | Plot of the pdf f in (2.6) (left) and of the corresponding survival function $\overline{F}$ (right).

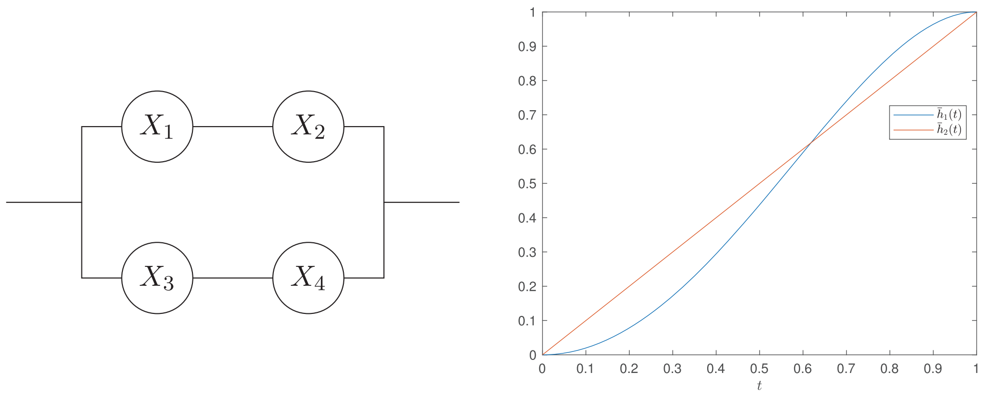

FIGURE 2 | The structure function of the system with lifetime T (left) and plot of the functions $\overline{h}_1(t)$ and $\overline{h}_2(t)$ (right) in Example 2.1.

Theorem 2.1 is about showing that the expectations are ordered in that sense and we prefer not using it as hypothesis since this is one of the main consequences of the icx order.

An example of distribution which can satisfy the assumptions in (2.1) is the Gompertz distribution with shape parameter $\eta = 0.8$ and scale parameter $b = 4$, with pdf given by

$$f(x) = b\eta \exp(\eta + bx - \eta \exp(bx)), \;\; x > 0. \tag{2.6}$$

In Figure 1, we plot the pdf f and the corresponding survival function $\overline{F}$. To satisfy the assumptions, the point of intersection of the dual distortion functions t_0 has to be such that $\overline{F}^{-1}(t_0) \geq 0.1077$ which means $t_0 \leq \overline{F}(0.1077) \approx 0.6499$. Note that this is satisfied, for instance, by the dual distortions considered in Example 3.2. Furthermore, in the following example, we show how Theorem 2.1 can be used to provide a bound on the expected lifetime of a coherent system.

Example 2.1. Consider a coherent system with four independent and identically distributed components X_1, X_2, X_3, X_4 and lifetime $T = \max(\min(X_1, X_2), \min(X_3, X_4))$, whose structure is represented in Figure 2 (left). Then, the dual distortion of this coherent system is given as $\overline{h}_1(t) = 1 - (1 - t^2)^2$. Considering $\overline{h}_2(t) = t$, we plot $\overline{h}_1$ and $\overline{h}_2$ in Figure 2 (right) and observe that they satisfy the assumptions in Theorem 2.1 with $t_0 \approx 0.6180$. Thus, if the components are distributed according to the Gompertz distribution with parameters $\eta = 0.8$ and $b = 4$, since $t_0 \leq 0.6499$, all the assumptions in Theorem 2.1 are met and we have $T \leq_{icx} X_1$. In particular, we derive $E(T) \leq E(X_1) \approx 0.1728$ obtaining a simple upper bound for the expected lifetime of the system.

Consider another coherent system with lifetime $T^{(d)}$, with the same structure and marginal distribution of the components as before. This time assume that X_1 and X_2 are dependent with survival copula $K(\cdot, \cdot)$, and the same copula describes the dependency between X_3 and X_4, while (X_1, X_2) and (X_3, X_4) are independent. Then, the dual distortion of this coherent system is given as $\overline{h}_1^{(d)}(t) = 2K(t, t) - K^2(t, t)$. Now, assume that K is a Frank copula given as

$$K(u, v) = -\frac{1}{\alpha} \log\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right).$$

Considering the case $\alpha = -0.5$, we plot $\overline{h}_1^{(d)}$ and $\overline{h}_2$ in Figure 3 and observe that they satisfy the assumptions in Theorem 2.1 with

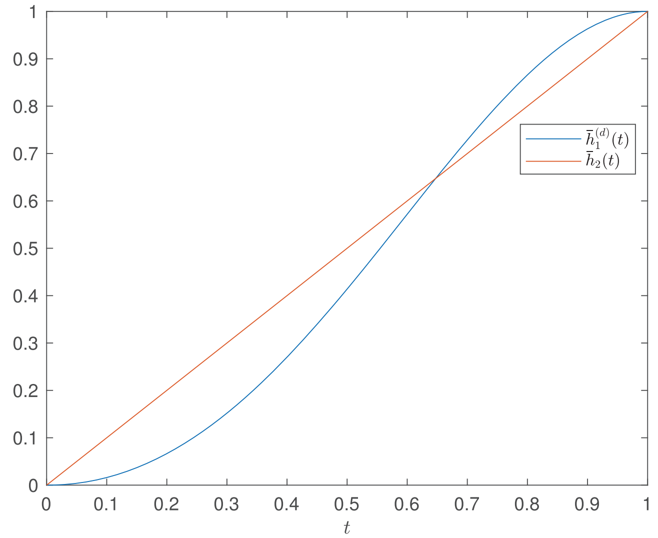

FIGURE 3 | Plot of the functions $\overline{h}_1^{(d)}(t)$ and $\overline{h}_2(t)$ in Example 2.1.

$t_0 \approx 0.6473 < 0.6499$. Thus, we have $T^{(d)} \leq_{icx} X_1$ and, in particular, $E(T^{(d)}) \leq E(X_1) \approx 0.1728$.

A particular case of interest in which the assumptions in (2.1) are easily verified without taking into account the value of t_0 is the one in which the survival function is convex over the support of the random variable, which implies that the pdf is decreasing. Hence, we state the following corollary.

Corollary 2.1. *Let X be an absolutely continuous random variable with support $S \subseteq (0, +\infty)$ and convex survival function $\overline{F}$ on S. Let $\overline{h}_1$ and $\overline{h}_2$ be dual distortion functions and $t_0 \in (0, 1)$ such that $\overline{h}_1(t) \leq \overline{h}_2(t)$ for $t \in [0, t_0]$ and $\overline{h}_1(t) \geq \overline{h}_2(t)$ for $t \in [t_0, 1]$. If $\int_0^1 \left[\overline{h}_1(t) - \overline{h}_2(t)\right] dt \leq 0$, then $X_{\overline{h}_1} \leq_{icx} X_{\overline{h}_2}$.*

In addition, it can be pointed out that the convexity of the survival function of components' lifetimes required in the statement above is a quite common assumption in reliability analysis; for example if $\overline{F}$ is the underlying survival function of a renewal processes, then the delay survival function (that is, the survival function of the "time till next renewal" of the process if it had started in the infinite past) is convex (see Sengupta and Nanda, [12], on this aim and for other examples of convex survival functions in reliability studies). Furthermore, other examples of distributions whose survival function is convex over the support are, among the others, the exponential, Lomax, Pareto type I, and the Gompertz (for suitable choices of the parameters) distributions.

A simple immediate example of application of Corollary 2.1, dealing with the lifetime of a coherent system with and without independence among components' lifetimes, is provided in the next section (see Example 3.1).

A similar statement can be proved for the icv order, but under different assumptions on the pdf f. The assumptions imply that the result can be applied only to variables with bounded support, thus that its interest is mainly restricted to the analysis of past lifetimes or inactivity times of systems (see, e.g., Navarro et al. [13]). The proof of the statement is not given here since it follows in analogy with the one of Theorem 2.1. Here, X_h denotes the lifetime having cumulative distribution function F_h defined as $F_h(\cdot) = h(F(\cdot))$ for a distortion function h.

Theorem 2.2. *Let h_1 and h_2 be distortion functions, $t_0 \in (0, 1)$ such that $h_1(t) \geq h_2(t)$ for $t \in [0, t_0]$ and $h_1(t) \leq h_2(t)$ for $t \in [t_0, 1]$ and $\int_0^1 \left[h_1(t) - h_2(t)\right] dt \geq 0$. Let X be an absolutely continuous random variable with support $S \subset (0, +\infty)$, distribution function F and pdf f. If f satisfies*

$$f(t) \leq f(F^{-1}(t_0)) \text{ for } t \in (\inf S, F^{-1}(t_0)) \quad \text{and}$$
$$f(t) \geq f(F^{-1}(t_0)) \text{ for } t \in (F^{-1}(t_0), \sup S) \tag{2.7}$$

then $X_{h_1} \leq_{icv} X_{h_2}$.

Note that to satisfy the assumptions in (2.7), the support of X cannot be unbounded because in that case the pdf would have been converging to zero at infinity violating $f(t) \geq f(F^{-1}(t_0))$ from a certain point onwards. In analogy with Corollary 2.1, we can provide an easier condition for the pdf by assuming the convexity of the cumulative distribution function which in turn implies that the pdf is increasing.

Corollary 2.2. *Let X be an absolutely continuous random variable with support $S \subset (0, +\infty)$ and convex distribution function F on S. Let h_1 and h_2 be distortion functions and $t_0 \in (0, 1)$ such that $h_1(t) \geq h_2(t)$ for $t \in [0, t_0]$ and $h_1(t) \leq h_2(t)$ for $t \in [t_0, 1]$. If $\int_0^1 \left[h_1(t) - h_2(t)\right] dt \geq 0$, then $X_{h_1} \leq_{icv} X_{h_2}$.*

The next statement provides simple algebraic conditions on the signatures α and β of two coherent systems having the same number n of iid components such that the corresponding lifetimes are comparable in icx order. For it, recall the notation introduced before Equation (1.1), and note that by the properties of the signature, we have $\overline{P}_{\alpha,0} = 1$. Denote in a similar manner the distortion $\overline{H}_\beta(t)$ that refers to the system having signature $\beta = (\beta_1, \ldots, \beta_n)$.

Note that, by using (1.1) the difference between $\overline{H}_\alpha(t)$ and $\overline{H}_\beta(t)$ can be expressed as

$$\begin{aligned}\overline{H}_\alpha(t) - \overline{H}_\beta(t) &= \sum_{j=0}^{n-1} g_j(t)\left(\overline{P}_{\alpha,j} - \overline{P}_{\beta,j}\right) \\ &= \sum_{j=1}^{n-1} g_j(t)\left(\overline{P}_{\alpha,j} - \overline{P}_{\beta,j}\right),\end{aligned} \tag{2.8}$$

being both $\overline{P}_{\alpha,0}$ and $\overline{P}_{\beta,0}$ equal to 1. By introducing the notation $c_j = \overline{P}_{\alpha,j} - \overline{P}_{\beta,j}$ the difference in (2.8) is expressed by

$$\overline{H}_\alpha(t) - \overline{H}_\beta(t) = \sum_{j=1}^{n-1} g_j(t) c_j. \tag{2.9}$$

To provide a result which does not rely on the particular value of the intersection point between the dual distortion functions, we formulate the statement by assuming the convexity of the survival function.

Theorem 2.3. *Let X be an absolutely continuous random variable with support $S \subseteq (0, +\infty)$ and convex survival function $\overline{F}(\cdot)$ on S. Consider two signatures α and β such that*

(i) *there exists an index* $k \in \{1, \ldots, n-1\}$ *such that* $c_i \geq 0$ *for* $i \leq k$ *and* $c_i \leq 0$ *for* $i > k$, *with at least a strict inequality in both cases,*

(ii) $\sum_{j=1}^{n-1} \overline{P}_{\alpha,j} \leq \sum_{j=1}^{n-1} \overline{P}_{\beta,j}$.

Then $X_\alpha \leq_{icx} X_\beta$, *where* X_α *and* X_β *denote the lifetimes of the systems having signatures* α *and* β, *respectively.*

Proof. First, we observe that by (2.9), the difference $\overline{H}_\alpha(t) - \overline{H}_\beta(t)$ is expressed as a Bernstein polynomial of degree n with Bernstein coefficients c_j, $j \in \{1, \ldots, n-1\}$. Then, from Bertot et al. [14], it is known that if all the coefficients have the same sign, then the polynomial is guaranteed to have no roots inside the interval $(0, 1)$ and if the coefficients taken in order exhibit exactly one sign change, then the polynomial is guaranteed to have exactly one root inside the interval $(0, 1)$. Under the assumptions, the latter is the case and then there exists a unique t_0 for which the equality $\overline{H}_\alpha(t_0) = \overline{H}_\beta(t_0)$ occurs. Furthermore, considering the limit

$$\lim_{t\to 0^+} \frac{1}{\sum_{j=1}^{n-1} g_j(t) c_j} = \lim_{t\to 0^+} \frac{1}{t \sum_{j=1}^{n-1} \binom{n}{j} c_j (1-t)^j t^{n-j-1}} = -\infty$$

where the sign is determined by the maximum non-null coefficient c_j, we conclude that $\overline{H}_\alpha(t) < \overline{H}_\beta(t)$ for $t \in (0, t_0)$ and $\overline{H}_\alpha(t) > \overline{H}_\beta(t)$ for $t \in (t_0, 1)$.

Considering the integral of the difference $\overline{H}_\alpha(t) - \overline{H}_\beta(t)$, we have

$$\begin{aligned}\int_0^1 \left[\overline{H}_\alpha(t) - \overline{H}_\beta(t)\right] dt &= \int_0^1 \sum_{j=1}^{n-1} g_j(t) c_j \, dt \\ &= \sum_{j=1}^{n-1} c_j \int_0^1 g_j(t) dt = \frac{1}{n+1} \sum_{j=1}^{n-1} c_j \leq 0\end{aligned}$$

from the assumption $\sum_{j=1}^{n-1} \overline{P}_{\alpha,j} \leq \sum_{j=1}^{n-1} \overline{P}_{\beta,j}$ and by using the properties of the Bernstein polynomials, being

$$\begin{aligned}\int_0^1 g_j(t) dt &= \int_0^1 \binom{n}{j} (1-t)^j t^{n-j} dt \\ &= \binom{n}{j} \int_0^1 (1-t)^{(j+1)-1} t^{(n-j+1)-1} dt \\ &= \binom{n}{j} \frac{j!(n-j)!}{(n+1)!} = \binom{n}{j} \frac{1}{n+1} \binom{n}{j}^{-1} = \frac{1}{n+1}.\end{aligned}$$

Then, all the assumptions of Corollary 2.1 are met and we conclude that $X_\alpha \leq_{icx} X_\beta$. □

Remark 2.1. Assuming $\alpha \leq_{icx} \beta$ guarantees that (ii) in Theorem 2.3 is satisfied. In fact, $\alpha \leq_{icx} \beta$ holds if, and only if, $\sum_{j=k}^{n-1} \overline{P}_{\alpha,j} \leq \sum_{j=k}^{n-1} \overline{P}_{\beta,j}$ for $k \in \{1, \ldots, n-1\}$, which is exactly the assumption (ii) for $k = 1$.

The result in Theorem 2.1 can be generalized to the case in which the distortion functions $\overline{h}_1$ and $\overline{h}_2$ have more than one intersection in the interval $(0, 1)$ as stated in the following theorem, whose proof is not given here since it can be derived in analogy with the one of Theorem 2.1. For the sake of simplicity, we provide the statement assuming that the survival function is convex, but it is straightforward to adapt the assumptions in (2.1) to the case of more than one intersection between the dual distortion functions.

Theorem 2.4. *Let* X *be an absolutely continuous random variable with support* $S \subseteq (0, +\infty)$ *and convex survival function* $\overline{F}(\cdot)$ *on* S. *Let* $\overline{h}_1$ *and* $\overline{h}_2$ *be dual distortion functions and* $0 = t_0 < t_1 < \cdots < t_k = 1$, $k \geq 2$, *such that* $\overline{h}_1(t_j) = \overline{h}_2(t_j)$ *for* $j \in \{1, \ldots, k-1\}$ *and the difference* $\overline{h}_1(t) - \overline{h}_2(t)$ *taking the sign of* $(-1)^j$ *for* $t \in (t_{j-1}, t_j)$, $j \in \{1, \ldots, k\}$. *If* $\int_{t_{2s}}^{t_{2s+2}} \left[\overline{h}_1(t) - \overline{h}_2(t)\right] dt \leq 0$ *for* $s \in \{0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1\}$, *then* $X_{\overline{h}_1} \leq_{icx} X_{\overline{h}_2}$.

Note that if one wants to compare two systems with a different number of components, then it is possible to make such a comparison by using the extended signatures which are recalled here (see Samaniego, [10], for further details). Considering $s = (s_1, \ldots, s_n)$ as the signature of a coherent or mixed systems with n independent and identically distributed components with a common survival function $\overline{F}$, then the coherent or mixed system with $n+1$ independent and identically distributed components with common survival function $\overline{F}$ and signature

$$s^* = \left(\frac{n}{n+1} s_1, \frac{1}{n+1} s_1 + \frac{n-1}{n+1} s_2, \ldots, \frac{n}{n+1} s_n\right)$$

has the same lifetime distribution as the n-component system with signature s.

Let X be a random variable with support $S \subseteq (0, +\infty)$. Consider a random vector of size n with independent and identically distributed components having the same distribution of X. It is well-known that the relations $X_{1:n} \leq_{st} X \leq_{st} X_{n:n}$ hold, which imply $X_{1:n} \leq_{icx} X \leq_{icx} X_{n:n}$. However, X and $X_{i:n}$, $i \in \{2, \ldots, n-1\}$, are not comparable in the usual stochastic order. Hence, we want to study a possible relation between them in the increasing convex order. To apply Theorem 2.3, we consider α as the signature of the i-th order statistic, with $i \in \{2, \ldots, n-1\}$, that is, $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$ and $\beta = (\frac{1}{n}, \ldots, \frac{1}{n})$ as the signature of the system given by X itself, obtained as extended signature as described in the previous paragraph.

Theorem 2.5. *Let* X *be an absolutely continuous random variable with support* $S \subseteq (0, +\infty)$ *and convex survival function* $\overline{F}(\cdot)$ *on* S. *Consider a random vector of size* n *with independent and identically distributed components having the same distribution of* X. *Then, we have*

$$X_{i:n} \leq_{icx} X \quad \text{for} \quad \begin{cases} i \in \{1, \ldots, \frac{n+1}{2}\}, & \text{if } n \text{ is odd}, \\ i \in \{1, \ldots, \frac{n}{2}\}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Consider the signatures α and β, with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$, and $\beta_1 = \cdots = \beta_n = \frac{1}{n}$. Then, for $j \in \{1, \ldots, n-1\}$, we have

$$\overline{P}_{\alpha,j} = \sum_{k=j+1}^{n} \alpha_k = \begin{cases} 1, & \text{for } j \in \{1, \ldots, i-1\}, \\ 0, & \text{for } j \in \{i, \ldots, n-1\}, \end{cases}$$

and

$$\overline{P}_{\beta,j} = \sum_{k=j+1}^{n} \beta_k = \sum_{k=j+1}^{n} \frac{1}{n} = \frac{n-j}{n}.$$

Then, we obtain

$$c_j = \overline{P}_{\alpha,j} - \overline{P}_{\beta,j} = \begin{cases} 0, & \text{for } j = 1, \\ 1 - \frac{n-j}{n} \ (> 0), & \text{for } j \in \{2, \dots, i-1\}, \\ -\frac{n-j}{n} \ (< 0), & \text{for } j \in \{i, \dots, n-1\}, \end{cases}$$

that is, the assumption (i) in Theorem 2.3 is satisfied. Regarding the assumption (ii), we consider the sums

$$\sum_{j=1}^{n-1} \overline{P}_{\alpha,j} = \sum_{j=1}^{i-1} 1 = i - 1$$

and

$$\sum_{j=1}^{n-1} \overline{P}_{\beta,j} = \sum_{j=1}^{n-1} \frac{n-j}{n} = \frac{1}{n} \sum_{j=1}^{n-1} (n-j)$$
$$= \frac{1}{n} \sum_{j=1}^{n-1} j = \frac{1}{n} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2}.$$

Then, the inequality $\sum_{j=1}^{n-1} \overline{P}_{\alpha,j} \leq \sum_{j=1}^{n-1} \overline{P}_{\beta,j}$ is satisfied if, and only if,

$$i - 1 \leq \frac{n-1}{2} \Leftrightarrow i \leq \frac{n+1}{2}$$

that is, for $i \in \{1, \dots, \frac{n+1}{2}\}$ if n is odd and for $i \in \{1, \dots, \frac{n}{2}\}$ if n is even. Thus, the proof is completed by Theorem 2.3. □

3 | Examples

The first example describes an immediate application of Corollary 2.1.

Example 3.1. Consider the coherent system with lifetime $T = \min(X_1, \max(X_2, X_3))$ having dual distortion function

$$\overline{h}(u) = K(u, u, 1) + K(u, 1, u) - K(u, u, u), \quad u \in [0, 1].$$

where K denotes the survival copula of the vector (X_1, X_2, X_3) (see Section 2.4 in Navarro [8], or Navarro et al. [2], for details). In case the components have iid lifetimes, that is, in the case $K(u_1, u_2, u_3) = u_1 u_2 u_3$, then one gets for the lifetime T_1 of the corresponding system the dual distortion function

$$\overline{h}_1(u) = 2u^2 - u^3.$$

In the case (X_1, X_2, X_3) has a Clayton survival copula with parameter $\alpha > 0$, that is, if $K(u_1, u_2, u_3) = \left(\sum_{i=1}^{3} u_i^{-\alpha} - 2\right)^{-1/\alpha}$, then the corresponding lifetime T_2 of the system is described by the dual distortion function

$$\overline{h}_2(u) = 2(2u^{-\alpha} - 1)^{-1/\alpha} - (3u^{-\alpha} - 2)^{-1/\alpha}.$$

Figure 4 shows the graphs of the two distortion functions, with $\alpha = 5$ for $\overline{h}_2$.

As one can see, $\overline{h}_1$ and $\overline{h}_2$ cross each other, thus the two corresponding lifetimes cannot be ordered in the strong usual stochastic order. However, assuming convexity for the survival functions of the X_i (for example, assuming they are exponentially

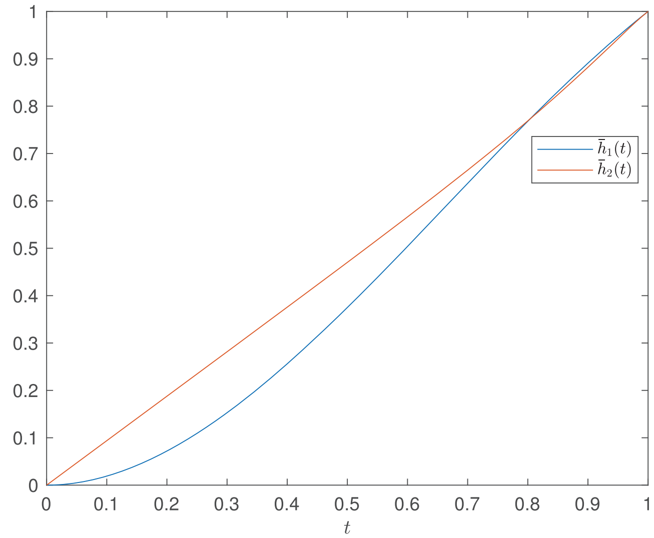

FIGURE 4 | Plot of the functions $\overline{h}_1(t)$ and $\overline{h}_2(t)$ in Example 3.1.

distributed), one can graphically observe that $\overline{h}_1$ and $\overline{h}_2$ satisfy the assumptions of Corollary 2.1 (a long and straightforward verification can also be done analytically), thus that $T_1 \leq_{icx} T_2$ holds. In particular, we have $E(T_1) \leq E(T_2)$, while T_1 and T_2 are not comparable in the usual stochastic order.

Remark 3.1. Note that if the assumptions in (2.1) are not satisfied then the conclusion of Theorem 2.1 may not hold. Consider, for instance, a Weibull distribution with survival function $\overline{F}(t) = e^{-t^7}$, $t > 0$ and whose pdf f is first increasing and then decreasing. The pdf satisfies $f(0) = 0$ and then regardless of the value of t_0 it would not satisfy $f(t) \geq f(\overline{F}^{-1}(t_0))$ for $t \in (0, \overline{F}^{-1}(t_0))$. Consider for instance the dual distortion functions $\overline{h}_1(t) = t^4 + 4(1-t)t^3$ (2nd order statistic in a sample of size 4) and $\overline{h}_2(t) = 2t^2 - t^3$ (the system with lifetime $\min(X_1, \max(X_2, X_3))$, as considered in Example 3.1). Then $\overline{h}_1$ and $\overline{h}_2$ satisfy the assumptions in Theorem 2.1 with $t_0 = \frac{2}{3}$ and $\int_0^1 [\overline{h}_1(t) - \overline{h}_2(t)]dt = -\frac{1}{60}$ but the conclusion does not hold as, for instance, $\int_0^{+\infty} [\overline{h}_1(\overline{F}(t)) - \overline{h}_2(\overline{F}(t))]dt \approx 0.0012 > 0$. In Figure 5, we plot the graphs of the functions $\overline{h}_1(\cdot)$, $\overline{h}_2(\cdot)$, $\overline{h}_1(\overline{F}(\cdot))$ and $\overline{h}_2(\overline{F}(\cdot))$.

The second example refers to Theorem 2.3, that is, to the case of two systems having the same number of iid components, but different signatures.

Example 3.2. Consider the signatures $\alpha = (0, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0)$ and $\beta = (0, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ of two coherent systems with five components given in Navarro and Rubio [15]. It can be checked that α and β are not ordered in the usual stochastic order being $\alpha_5 < \beta_5$ and $\alpha_3 + \alpha_4 + \alpha_5 > \beta_3 + \beta_4 + \beta_5$. This means that we have at least one intersection between the functions $\overline{H}_\alpha(t)$ and $\overline{H}_\beta(t)$ in $(0, 1)$. Then, we evaluate $\overline{P}_{\alpha,j}$ and $\overline{P}_{\beta,j}$ and the coefficients c_j, for $j \in \{1, 2, 3, 4\}$ as reported in Table 1.

Then, we observe that the coefficients c_j satisfy the assumption (i) in Theorem 2.3 with $k = 2$, being $c_1 = 0$, $c_2 > 0$, $c_3 = 0$ and $c_4 < 0$. In Figure 6, we plot the functions $\overline{H}_\alpha(t)$ and $\overline{H}_\beta(t)$ from which we can also observe that they have exactly one intersection point

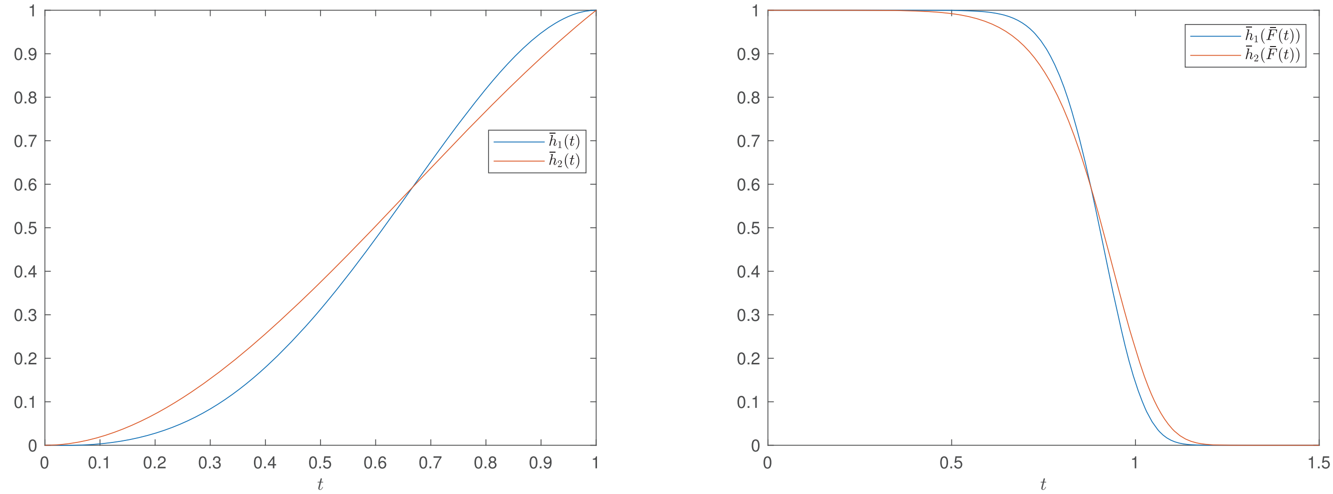

FIGURE 5 | Plot of the functions $\overline{h}_1(t)$ and $\overline{h}_2(t)$ (left) and of the functions $\overline{h}_1(\overline{F}(t))$ and $\overline{h}_2(\overline{F}(t))$ (right) in Remark 3.1.

TABLE 1 | $\overline{P}_{\alpha,j}$, $\overline{P}_{\beta,j}$ and c_j for α and β in Example 3.2.

	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	
$\overline{P}_{\cdot,1}$	1	1	$c_1 = 0$
$\overline{P}_{\cdot,2}$	$\frac{4}{5}$	$\frac{3}{5}$	$c_2 = \frac{1}{5}$
$\overline{P}_{\cdot,3}$	$\frac{2}{5}$	$\frac{2}{5}$	$c_3 = 0$
$\overline{P}_{\cdot,4}$	0	$\frac{1}{5}$	$c_4 = -\frac{1}{5}$

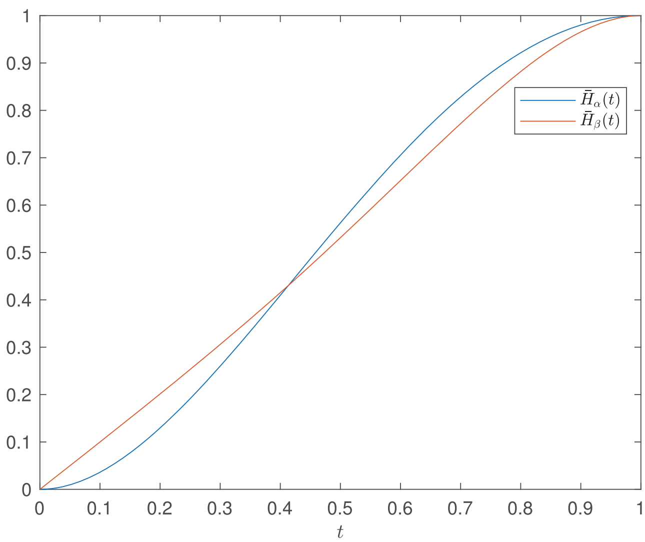

FIGURE 6 | Plot of the functions $\overline{H}_\alpha(t)$ and $\overline{H}_\beta(t)$ in Example 3.2.

in $(0, 1)$ as expected. It can be checked that the relation $\alpha \leq_{icx} \beta$ holds, but we can also just observe that $\sum_{j=1}^{4} \overline{P}_{\alpha,j} = \sum_{j=1}^{4} \overline{P}_{\beta,j} = \frac{11}{5}$ so that the assumption (ii) in Theorem 2.3 is satisfied. Then, we conclude $X_\alpha \leq_{icx} X_\beta$ for all the random variables with support $S \subseteq (0, +\infty)$ and convex survival function on S.

In the following example, we give an application of Theorem 2.4 in which we consider a case of distortions having two intersections in the interval $(0, 1)$.

Example 3.3. Consider a mixed system with four independent and identically distributed components with a convex survival function $\overline{F}(\cdot)$ and signature $(0.3, 0.05, 0.65, 0)$ which means dual distortion function $\overline{h}_1(t)$ given as $\overline{h}_1(t) = t^4 + 2.8t^3(1-t) + 3.9t^2(1-t)^2$. We want to compare this system in the icx order with the one-component system having the same survival function $\overline{F}(\cdot)$, that is, $\overline{h}_2(t) = t$. In Figure 7 we plot the functions $\overline{h}_1$ and $\overline{h}_2$ over the interval $(0, 1)$ and observe that they intersect in $t_1 = 0.6667$ and $t_2 = 0.7143$.

With the notation introduced in the statement of Theorem 2.4, we have $k = 3$ and then we have to consider only the integral of the difference $h_1(t) - h_2(t)$ over $(0, t_2)$ which is clearly negative from the plot of the functions h_1 and h_2, and then we can conclude $X_{h_1} \leq_{icx} X_{h_2}$, that is, $X_{h_1} \leq_{icx} X$.

4 | Additional Results for Mixtures of Order Statistics

In this final section we describe some results dealing with comparisons in icx order between order statistics and their mixtures, which follow as corollaries of those described in Section 2. Before proceeding, we fix the notation used to describe such mixtures. Consider a random variable X with support $S \subseteq (0, +\infty)$ and survival function $\overline{F}$, and a random vector of size n with iid components having the same distribution of X. Then, we denote by $Y_{i,j:n}$ the random variable with survival function $\overline{G}_{i,j:n}(\cdot) = \frac{1}{2}\left(\overline{F}_{i:n}(\cdot) + \overline{F}_{j:n}(\cdot)\right)$, that is, the uniform mixture of $X_{i:n}$ and $X_{j:n}$.

The first statement immediately follows from Theorem 2.3 just observing that $\overline{G}_{i,j:n}$ is the distribution of the mixed system having signature α where all the α_i are equal zero except those with indices i and j, that assume value $1/2$. This result will be further applied in the next statements.

Lemma 4.1. *Let X be an absolutely continuous random variable with support $S \subseteq (0, +\infty)$ and convex survival function $\overline{F}$ on S. Consider a random vector of size n with iid components*

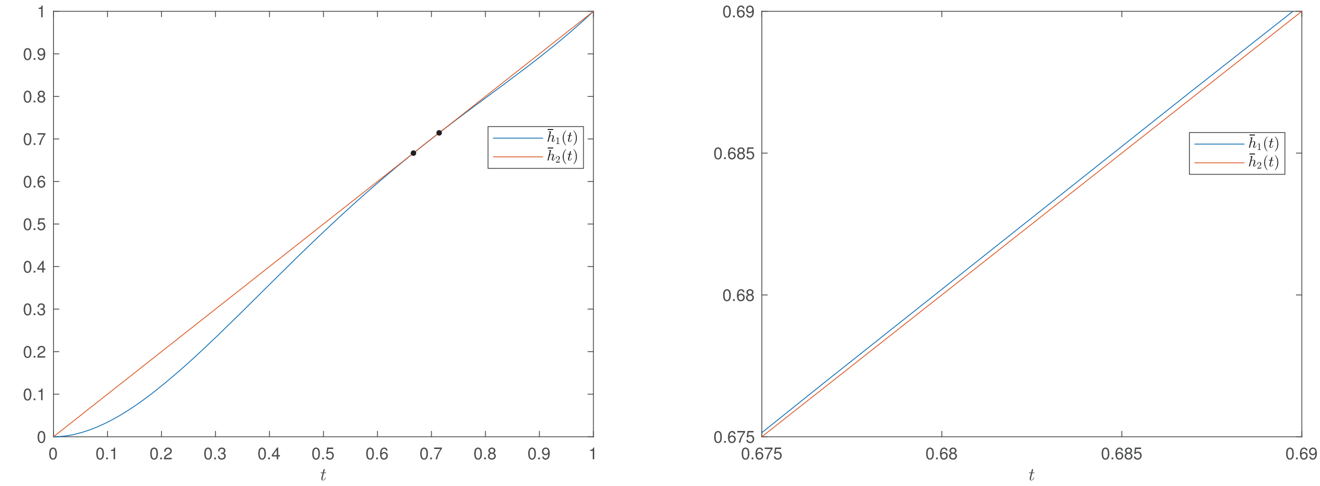

FIGURE 7 | Plot of the functions $\overline{h}_1(t)$ and $\overline{h}_2(t)$ with marked intersection points in Example 3.3 (left) and restriction to a subinterval of (t_1, t_2) to better visualize the inequality $\overline{h}_2(t) \leq \overline{h}_1(t)$ in such interval (right).

having the same distribution of X. Then, we have $Y_{i-s+1,i+s-1:n} \leq_{icx} Y_{i-s,i+s:n}$, for $s = 1, 2, \ldots, \min\{n-i, i-1\}$.

Note that for $s = 1$ the result in Lemma 4.1 gives $X_{i:n} \leq_{icx} Y_{i-1,i+1:n}$, that is, that, whenever $\overline{F}$ is convex, the order statistic of any order i is always smaller in icx order than the uniform mixture of pair of orders statistics or orders symmetric with respect to i. In fact, for $s = 1$, we have $Y_{i-1+1,i+1-1:n} = Y_{i,i:n}$ with survival function $\overline{G}_{i,i:n}(\cdot) = \frac{1}{2}\left(\overline{F}_{i:n}(\cdot) + \overline{F}_{i:n}(\cdot)\right) = \overline{F}_{i:n}(\cdot)$. By combining the result of Lemma 4.1 with this fact, one can immediately derive the following result.

Lemma 4.2. *Let X be an absolutely continuous random variable with support $S \subseteq (0, +\infty)$ and convex survival function $\overline{F}$ on S. Consider a random vector of size n with independent and identically distributed components having the same distribution of X. Let $s^* = \min\{n-i, i-1\}$. Then, we have the following chain of stochastic inequalities*

$$X_{i:n} \leq_{icx} Y_{i-1,i+1:n} \leq_{icx} Y_{i-2,i+2:n} \leq_{icx} \cdots \leq_{icx} Y_{i-s^*,i+s^*:n}.$$

In particular, we deduce $X_{i:n} \leq_{icx} Y_{i-s,i+s:n}$, for $s = 1, 2, \ldots, s^*$.

Before proceeding with the last result, we recall that $X_{l:n} \leq_{st} X_{m:n}$ if $l \leq m \leq n$ and then also $X_{l:n} \leq_{icx} X_{m:n}$.

Theorem 4.1. *Let X be an absolutely continuous random variable with support $S \subseteq (0, +\infty)$ and convex survival function $\overline{F}(\cdot)$ on S. Consider a random vector of size n with independent and identically distributed components having the same distribution of X. If k, i and r are three indices such that $1 \leq k < i < r \leq n$ and $i \leq \frac{k+r}{2}$, then $X_{i:n} \leq_{icx} Y_{k,r:n}$.*

Proof. Let $s = \min\{i-k, r-i\}$. Since $2i \leq k + r$, we have $i - k \leq r - i$ and then $s = i - k$. In addition, with the notation introduced in Lemma 4.2, we have $s \leq s^*$, in fact, $s = i - k \leq r - i \leq n - i$ and $s = i - k \leq i - 1$. Then, by Lemma 4.2, we have

$$X_{i:n} \leq_{icx} Y_{i-s,i+s:n} = Y_{k,2i-k:n}.$$

From the assumptions, $2i - k \leq r$ and then $X_{2i-k:n} \leq_{st} X_{r:n}$ which implies $X_{2i-k:n} \leq_{icx} X_{r:n}$. Then, we also have $Y_{k,2i-k:n} \leq_{icx} Y_{k,r:n}$ and we can conclude $X_{i:n} \leq_{icx} Y_{k,r:n}$. □

5 | Conclusions

One of the main interests in the context of reliability theory is to stochastically compare coherent systems. The comparisons are usually made in terms of stochastic orders with well-known results about, for instance, the usual stochastic, the hazard rate and the likelihood ratio orders. In this paper, we have introduced a criteria to compare systems in terms of the weaker increasing convex order (and therefore also in terms of the means) even when the previous stochastic comparisons do not hold. The results are formulated also in terms of the signatures of the coherent system and with a special regard to order statistics and mixtures of them. Several illustrative examples show how to apply the results given in the paper.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

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