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Variational evolution of discrete one-dimensional second-order functionals

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Abstract

A variational scheme of evolution (minimizing movements) is applied to a sequence of functionals converging to the prototypical second-order functional with free-discontinuities. The method provides a function which matches the expected evolution of the free-discontinuity limit functional.

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1 Introduction

In this paper we focus on the problem of defining an energy-driven evolution related to the prototypical second-order-functional with free discontinuities

$$\frac{1}{2}\int_0^1 |u''(x)|^2 \mathrm{d}x + 2\#S(u) + \#S(u'),$$

by an approximation scheme which involves energies F_ε defined on finite-dimensional spaces. Here, u is piecewise smooth and $\#S(u)$ and $\#S(u')$ denote the number of discontinuity points of u and u', respectively.

The goal is the extension of an analogous first-order result obtained in [1], where first-order functionals with discontinuities only in u have been dealt with, and the evolution along the approximating functionals F_ε is compared with the evolution according to the limit functional. We note that such functionals are related to Griffith fracture energies, and have been extensively analyzed in the context of the evolution of brittle fracture ([2], [3]), while the second-order energies above can be interpreted as depending on the curvature of a rod with possibility of bending and eventual fracture as extreme bending (see, for instance, Barchiesi et al. [4] for an approach to second-gradient theories).

The so-called method of *minimizing movements* ([5], [6], [7], [8]) provides a general procedure to define curves of maximal slope (in an extended meaning) for functionals which are not necessarily smooth. Given the "state space" X of a system (e.g. an Hilbert space) and an initial datum u^0, the evolution driven by an "energy" functional $\mathscr{F}\colon X \to \mathbb{R}$ arises as a limit function of an iterative-minimization process. Given a (small) time step τ, a sequence $(u^k)_k$ of states is defined from u^0 by a recursive minimization:

$$u^k \text{ minimizes } \quad v \mapsto \mathscr{F}(v) + \frac{1}{2\tau}\|v - u^{k-1}\|_X^2.$$

This means that the state u^k aims to decrease the energy while keeping a distance of order τ from u^{k-1}. A piecewise-constant function $u_\tau\colon [0,+\infty) \to X$ is thus defined by setting $u_\tau(t) = u^{\lfloor t/\tau \rfloor}$. A *minimizing movement* for $\mathscr{F}$ from u^0 is any pointwise limit u of a (sub)sequence $\left(u_{\tau_n}\right)_n$ with $\tau_n \to 0$. For instance, in the classical case of a convex functional $\mathscr{F}$, the function u turns out to be the absolutely continuous solutions of the differential inclusion $u'(t) \in -\partial\mathscr{F}\big(u(t)\big)$. Actually, by a careful choice of the minimization functional, the method has proven to be widely applicable in a variety of fields (Calculus of Variations, PDE, Geometric Measure Theory, ...).

A natural question emerges when we apply the scheme to a converging sequence F_ε of functionals (think of the parameter ε as varying along an infinitesimal positive sequence). Assume that $F_\varepsilon \xrightarrow{\Gamma} F$ as $\varepsilon \to 0$ (Γ convergence with respect to a suitable topology), and define, for every ε and τ, the piecewise-constant evolution $u_{\varepsilon,\tau}$ as above. In Braides [7] (see also [9]), the notion of *minimizing movement along a sequence* is introduced, as the uniform limit, on compact sets of $[0,+\infty)$, of a sequence u_{ε_n,τ_n}. In general, the result depends on the mutual rate of convergence to zero of ε and τ. An assumption which guarantees the independence from the ε-τ regime is the convexity of F_ε (see [7] §11.1; see also [10] and [11]); unfortunately, physical models (and the one considered in this paper, too) often do not meet this requirement. A weaker condition has been introduced by Colombo and Gobbino [12] for the analysis of curves of maximal slope, and extended in [13] to the study of minimizing movements (see also [9] Section 2.3.3). Again, this condition will not be met by our energies.

Here, we are concerned with free-discontinuity functionals. Consider the basic model of the one-dimensional Mumford-Shah functional

$$F(u) = \frac{1}{2}\int_0^1 |u'(x)|^2 \,\mathrm{d}x + \#S(u),$$

where u is a piecewise-H^1 function on $(0,1)$, and $S(u)$ denotes its set of discontinuities (jump points). A well-known discrete approximation (see [14], [7] and [15], [16] in the context of fracture mechanics) is given by

$$F_\varepsilon(u) = \sum_i \psi\left(\frac{u_{i+1} - u_i}{\sqrt{\varepsilon}}\right),$$

where u is defined on $[0,1]\cap\varepsilon\mathbb{Z}$ and u_i denotes its value on the i-th node; here, ψ can be chosen to be an even function which is quadratic in a neighborhood of the origin (with $\psi''(0)=1$) and takes value 1 at infinity (see Figure 1). A study of the minimizing movements along F_ε is contained in [1] (actually in the mechanical frame of the Lennard-Jones-type potentials). If the minimizing movement is computed with the constraint $\tau \ll \varepsilon^2$, it can be proved that the singularities do not move at least in a right neighborhood of the initial time $t=0$, and in each of the subintervals of $(0,1)$ determined by the singularities, the initial function evolves according to the heat equation with Neumann boundary conditions. Thus, in this case, we recover the minimizing movement for the Mumford-Shah functional F, as computed in [7], Chapter 7. Note that functionals F_ε above may not satisfy the Colombo-Gobbino condition in [13]. In particular, we may have a sequence of local minimizers for F_ε, which, taken as initial data, give constant evolutions, converging to a function which is not a critical point for F and from which the evolution is not constant. To overcome this possibilities we have to assume that the data are "well-prepared", so that we have no energy gap in the limit. This is a usual requirement in non-convex evolution problems, as for example in the approach by Sandier and Serfaty [17].

The present paper contains a partial extension of the above result to the second-order case. The rôle of the Mumford-Shah functional is now played by

$$F(u) = \frac{1}{2}\psi''(0)\int_0^1 |u''(x)|^2 \mathrm{d}x + 2\#S(u) + \#S(u'),$$

where u is piecewise-H^2 on $(0,1)$ and $S(u')$ stands for the set of continuity points where u' is discontinuous (crease points). The starting result is the discrete approximation (Braides [18]) of F which adjusts to the second order the functionals F_ε previously introduced: now

$$F_\varepsilon(u) = \sum_i \psi\Big(\frac{u_{i+1}-2u_i+u_{i-1}}{\varepsilon\sqrt{\varepsilon}}\Big),$$

where ψ is as above (see Figure 1). We show again that a suitable mutual rate ε-τ of convergence guarantees a unique minimizing movement which presents stability of the singularities with respect to the initial datum; in addition, on each interval where the evolution is smooth, it satisfies the fourth-order equation which is the expected result for the minimizing movement of the limit functional F. Let us briefly outline the structure of the paper in more detail.

In §3 we introduce the discrete approximating functionals F_ε; we specify the notation about discrete functions and define the quadratic interpolation already introduced in [18]. A significant notion is that of singular point, i.e. a node i where the second-difference quotient $(\Delta_\varepsilon u)_i = (u_{i+1}-2u_i+u_{i-1})/\varepsilon^2$ exceeds a given threshold.

In §4 we detail the scheme of minimizing movements along the sequence F_ε; the main result of this section is the compactness result in Theorem 4.7 and Corollary 4.9, which relies on a basic compactness theorem for piecewise-H^2 functions (see the result, from [19] and [20], recalled in Theorem 2.3). So far we have a minimizing movement $u \in C^{1/2}([0,+\infty); L^2(0,1))$ with $u(\cdot,t)$ piecewise-$H^2(0,1)$.

In §5 we first prove a regularity result for u: the pointwise second-order derivative of $u(\cdot,t)$ is H^2 globally on $(0,1)$ (see Theorem 5.1); moreover, the second and third derivatives vanish on the boundary of $(0,1)$. Furthermore, (Theorem 5.5 and Corollary 5.6), we show that $u(x,\cdot)$ is $H^1(0,T)$ for every $T>0$ and $u_t = -\psi''(0)(u_{xx})_{xx}$.

In §6 we deal with the issue of the evolution of singular points of the minimizing movement u; we limit ourselves to provide a sufficient condition which entails the stability of the singular points of the initial datum in a small interval after the initial time $t=0$.

In §7, under the assumption of the stability of the singular points, we prove the vanishing of the second derivative on jump and crease points, and also of the third derivatives on jump points. These result will be used in the final section to prove the uniqueness of the minimizing movement; this comes from the uniqueness of the solution of the equation $u_t = -\psi''(0)(u_{xx})_{xx}$ with the specified boundary conditions, considered in a domain $(a,b)\times(0,T)$ where (a,b) is any of the intervals determined by the singular (fixed) points of u. Though we do not address the study of the minimizing movements for the functional F, it turns out (see §8) that this fourth-order equation corresponds to the evolution of F if we assume that the singular points remain fixed.

Actually, some remarks are now in order.

In a similar way as in the first order case [1], we require the condition $\tau \ll \varepsilon^4$ on the mutual rate ε-τ of convergence (and a bound on the L^2 norm of the initial datum), to obtain a sufficient condition for the stability of singular points from $t=0$. In this second-order case, too, the hypothesis is used to give an estimate on the growth (with respect to the time index) of the second-difference quotients. This appears to be a technical issue, since the final result of the commutativity of the procedure (minimizing movement along a Γ-converging sequence and minimizing movement of the limit) should hold in general in the opposite case when $\varepsilon \ll \tau$ (see [9], Theorem 2.1).

The vanishing of the third derivatives on jump points is obtained by adding, with respect to the initial natural structural requirements, a suitable (mild) growth assumption on ψ.

Moreover, we point out that the complete result is limited to the case of one singular point (jump or crease), without tackling the problem of localizing the method.

Finally, a possible extension to more than one space variable appears to be a difficult task; this is supported by the fact that only partial results dealing with the approximation of second-order free-discontinuity functionals are available ([21], [22]).

2 Preliminaries

Function spaces Let (a,b) be a bounded open interval. We denote by $\mathscr{H}^k(a,b)$ (with $k=1,2$) the space of piecewise-H^k functions on (a,b), i.e. the space of functions $u\colon (a,b)\to\mathbb{R}$ which admit a partition $a=x_0<x_1<\ldots<x_m=b$ with the property that $u\in H^k(x_{j-1},x_j)$ for each $j=1,\ldots,m$.

If $u\in\mathscr{H}^1(a,b)$ and $x_0,\ldots,x_m$ is a partition of (a,b) as above, then $u\in C^0([x_{j-1},x_j])$ and the traces

$$u^+(a),\quad u^\pm(x_j),\quad u^-(b)$$

are well defined for each $j = 1, \dots, m-1$. Moreover, u is absolutely continuous on each interval (x_{j-1}, x_j): we denote by u' the classical derivative of u, which exists a.e. on (a, b). We denote by $S(u)$ the set of discontinuity points *(jump points)* of u.

If $u \in \mathscr{H}^2(a, b)$ then $u' \in \mathscr{H}^1(a, b)$; in particular, the traces

$$(u')^+(a), \quad (u')^\pm(x_j), \quad (u')^-(b)$$

are well defined for each $j = 1, \dots, m-1$, and on each interval (x_{j-1}, x_j) the classical derivative u'' of u' exists a.e. We denote by $S(u')$ the set of points where u is continuous but u' is discontinuous *(crease points)*.

In the sequel we will consider functions u of the "space-time" variable $(x, t) \in (a, b) \times [0, +\infty)$; in such a case, if $u^t := u(\cdot, t) \in \mathscr{H}^2(a, b)$, we denote by $u_x(\cdot, t)$ and $u_{xx}(\cdot, t)$ the derivatives $(u^t)'$ and $(u^t)''$ as defined above.

If $(X, \|\cdot\|)$ is a Banach space and $T > 0$, we shall denote by $L^p(0, T; X)$ the space of (strongly) measurable functions $u\colon [0, T] \to X$ such that

$$\int_0^T \|u(t)\|^p \, \mathrm{d}t < +\infty.$$

(See, e.g., [23], § 2.19, [24], Chapters II and III). For the following result see, e.g., [24] Corollary III.13 and Theorem IV.1).

Theorem 2.1. *Let X be a reflexive Banach space. Let $1 \leq p < +\infty$. Then the dual space of $L^p(0, T; X)$ can be isometrically identified with $L^q(0, T; X')$, where $1/p + 1/q = 1$.*

In particular, we deduce that $L^2(0, T; X)$ is reflexive if X is reflexive. We will use this result with $X = H_0^2(0, 1)$.

Compactness results We state, in the one-dimensional case, the classical compactness and closure theorem for SBV functions (see [25], Theorems 4.8 and 4.7, where the general n-dimensional setting is considered, and [26], Theorem 7.3, for the one-dimensional case). Recall that, if $v \in \mathscr{H}^1(a, b)$, then the jump part of the derivative of v is the measure $D^j v$ defined by $D^j v = \sum_{x \in S(v)} (v^+(x) - v^-(x)) \delta_x$ (here δ_x denotes the usual Dirac measure on x).

Theorem 2.2. *Let (v_n) be a sequence of functions in $\mathscr{H}^1(a, b)$ such that*

$$\sup_n \left(\int_a^b |v_n'(x)|^2 \, \mathrm{d}x + \#S(v_n) + \|v_n\|_\infty \right) < +\infty.$$

Then there exists a subsequence $(v_{n_h})_h$ and a function $v \in \mathscr{H}^1(a, b)$ such that

$$v_{n_h} \to v \quad \textit{strongly in } L^2(a, b); \quad v_{n_h}' \rightharpoonup v' \quad \textit{weakly in } L^2(a, b).$$

Moreover, $D^j v_{n_h} \rightharpoonup D^j v$ weakly in the sense of measures (i.e., for every $\varphi \in C^0([a, b])$ vanishing on a and b it turns out that $\int_a^b \varphi D^j v_{n_h} \to \int_a^b \varphi D^j v$).*

As to the second order, we will need the following compactness result from [19] (see also [20], Theorem 7).

Theorem 2.3. *Let (a,b) be a bounded open interval, and (z_n) a sequence in $\mathscr{H}^2(a,b)$. Assume that*

$$\sup_n \Big[\int_a^b ((z_n'')^2 + (z_n)^2)\,\mathrm{d}x + \#S(z_n) + \#S(z_n')\Big] < +\infty. \tag{2.1}$$

Then there exist a subsequence $(z_{n_h})_h$ and a function $z_0 \in \mathscr{H}^2(a,b)$ such that

$$\begin{aligned} &z_{n_h} \to z_0 \quad \text{a.e. and strongly in } L^1(a,b); \quad z_{n_h} \rightharpoonup z_0 \quad \text{weakly in } L^2(a,b);\\ &z_{n_h}' \to z_0' \quad \text{a.e. in } (a,b), \text{ and } z_{n_h}'' \rightharpoonup z_0'' \quad \text{weakly in } L^2(a,b). \end{aligned}$$

Remark 2.4. Since (2.1) implies the L^2-boundedness of (z_n), the subsequence $(z_{n_h})_h$ given by the theorem also satisfies:

$$z_{n_h} \to z_0 \quad \text{strongly in } L^q(a,b) \text{ for every } 1 \le q < 2.$$

3 The functional F_ε

In this section we introduce the functional F_ε. Though its natural domain is a space of discrete functions, we will need to consider suitable affine and quadratic interpolations. We collect some results which will be useful in the sequel.

Notation for discrete functions Let $\varepsilon > 0$ be given. If u is a function $[0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$, we denote the value $u(i\varepsilon)$ simply by u_i; therefore, we also write u as an indexed family $(u_i)_{i=0,1,\dots,N_\varepsilon}$ where $N_\varepsilon = \lfloor 1/\varepsilon \rfloor$ ($\lfloor a \rfloor$ denotes the integer part of a). It will be useful to define u_i for $i = N_\varepsilon + 1$, too: we thus set the value $u_{N_\varepsilon+1}$ in such a way that

$$\frac{1}{2}(u_{N_\varepsilon+1} + u_{N_\varepsilon-1}) = u_{N_\varepsilon} \tag{3.1}$$

(this choice makes the second-order difference-quotient $\varepsilon^{-2}(u_{N_\varepsilon+1} - 2u_{N_\varepsilon} + u_{N_\varepsilon-1})$ null).

By u we will also denote the piecewise-constant extension to $[0, \varepsilon(N_\varepsilon + 1))$ defined by

$$u(x) = u_i \quad \text{with } i = \lfloor x/\varepsilon \rfloor, \qquad \text{if } 0 \le x < \varepsilon(N_\varepsilon + 1). \tag{3.2}$$

The L^p norms of u are defined by taking this piecewise-constant extension into account; we set

$$\|u\|_{L^p}^p = \|u\|_{L^p((0,\varepsilon(N_\varepsilon+1))}^p = \varepsilon \sum_{i=0}^{N_\varepsilon} |u_i|^p\,.$$

The functional F_ε Let $\psi\colon \mathbb{R} \to [0,+\infty)$ be a function satisfying the following conditions:

$$\begin{aligned} &\psi(z) = \psi(-z), \qquad \psi(0) = 0;\\ &\psi \text{ is convex on the interval } [0, z_0] \text{ and concave on } [z_0, +\infty);\\ &\lim_{z\to+\infty} \psi(z) = \gamma, \quad \text{with } \gamma > 0;\\ &\psi(z) \ge \nu_0 z^2 \text{ if } |z| \le z_0 \quad \text{for a suitable } \nu_0 > 0. \end{aligned}$$

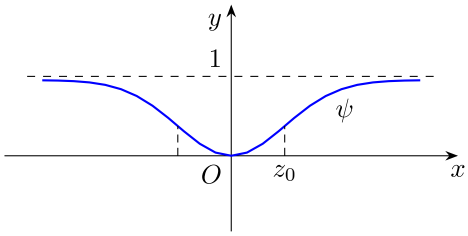

Figure 1 - Function ψ.

It follows that ψ is non-decreasing on $[0, +\infty)$. In the sequel we will assume $\gamma = 1$ (see Figure 1).

We require that ψ is C^1 on $\mathbb{R}$ and C^2 in a neighbourhood of 0. In particular,

$$\psi'(0) = 0, \qquad |\psi'(z)| \leq C|z| \text{ for every } z \in \mathbb{R},$$

for a suitable constant $C > 0$.

On the space of the discrete functions $(u_i)_i$ on $[0, 1]$ we consider the functional

$$\begin{aligned} F_\varepsilon(u) &= \sum_{i=1}^{N_\varepsilon} \psi\big(\frac{u_{i+1} - 2u_i + u_{i-1}}{\varepsilon\sqrt{\varepsilon}}\big) \\ &= \sum_{i=1}^{N_\varepsilon} \varepsilon\varphi_\varepsilon\big(\frac{u_{i+1} - 2u_i + u_{i-1}}{\varepsilon^2}\big) = \sum_{i=1}^{N_\varepsilon} \varepsilon\varphi_\varepsilon\big((\Delta_\varepsilon u)_i\big), \end{aligned}$$

where

$$\varphi_\varepsilon(z) = \frac{1}{\varepsilon}\psi(\sqrt{\varepsilon}z), \qquad (\Delta_\varepsilon u)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\varepsilon^2} \qquad (1 \leq i \leq N_\varepsilon).$$

Notice that in the definition of F_ε the sum over i actually extends up to $N_\varepsilon - 1$, since $(\Delta_\varepsilon u)_{N_\varepsilon} = 0$ in view of the definition (3.1) of $u_{N_\varepsilon+1}$.

Let us point out some properties of φ_ε:

- For every $\zeta \geq z_0$ there exists $\nu(\zeta) > 0$ such that

  $$\varphi_\varepsilon(z) \geq \nu(\zeta)z^2 \qquad \text{if } |z| \leq \zeta/\sqrt{\varepsilon}. \tag{3.3}$$

  The constant $\nu(\zeta)$ can be taken independent of ζ if this latter varies in a bounded set. Clearly, we can assume $\nu(z_0) = \nu_0$.

- $\varepsilon\varphi_\varepsilon(\frac{z}{\varepsilon}) \to 1$ as $\varepsilon \to 0$ for every $z \neq 0$;

- $\varphi'_\varepsilon(z) \to \psi''(0)z$ as $\varepsilon \to 0$ uniformly with respect to z in bounded sets.

Γ-convergence Let us first consider the "pointwise" convergence of F_ε on piecewise-C^2 functions.

Remark 3.1. Since F_ε acts on discrete functions, for any given piecewise-C^2 function $u\colon [0, 1] \to \mathbb{R}$ let us define a suitable discretization on which we compute F_ε. Let $0 = x_0 < x_1 < \ldots < x_m = 1$ be such that $u_{|(x_j,x_{j+1})} \in C^2([x_j, x_{j+1}])$ (i.e.

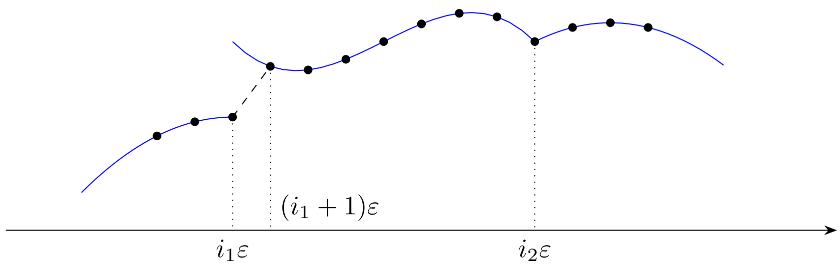

Figure 2 - Piecewise-C^2 left-continuous function (solid blue line) u_ε. The second-order difference quotients centered in $i_1\varepsilon$ (discontinuity point) and in $(i_1+1)\varepsilon$ are both unbounded with respect to ε. As to the discontinuity points of the derivative, $i_2\varepsilon$ has unbounded second-order difference quotient.

$u_{|(x_j,x_{j+1})}$ can be extended to a C^2 function on the closed interval). By possibly changing the values on $\{x_0, x_1, \dots, x_m\}$, we can assume that u is continuous from the left and that it is continuous in $x=0$ (define $u(0)=u(0^+)$ and $u(x_j)=u(x_j^-)$ if $j>0$). For every ε we can find a piecewise-affine change of variable (tending uniformly to the identity as $\varepsilon\to 0$) which "moves" the singularities of u onto $\varepsilon\mathbb{Z}$. More precisely, if $i_j\varepsilon \le x_j < (i_j+1)\varepsilon$ (x_j are the singular points as above), consider $l_\varepsilon\colon [0,1]\to[0,1]$ defined as the piecewise-affine function whose graph interpolates the points $(0,0), (i_j\varepsilon, x_j)$ (with $j=1,\dots,m-1$) and $(1,1)$; then set $u_\varepsilon = u\circ l_\varepsilon$.

Let us compute the limit of $F_\varepsilon(u_\varepsilon)$ as $\varepsilon\to 0$. Let i_j ($j=1,\dots,m-1$) be as above, and $i_0=0, i_m=N_\varepsilon$. Then u_ε is C^2 (with equibounded second derivative) on each interval $[i_j\varepsilon, i_{j+1}\varepsilon]$. Let us split the sum which expresses $F_\varepsilon(u_\varepsilon)$ into various terms. For each $j=1,\dots,m-1$ let I_j, be the set of indices i defined as follows (see also Figure 2):

- if εi_j is a discontinuity point of $u_\varepsilon (= u\circ l_\varepsilon)$, then I_j is the set of indices $i\in\mathbb{Z}\cap[0,1]$ such that $i_j+1<i<i_{j+1}$;
- if, at the point εi_j, the function u_ε is continuous, but u'_ε is discontinuous, then I_j denotes the set of indices $i\in\mathbb{Z}\cap[0,1]$ such that $i_j<i<i_{j+1}$;

Thus, in both cases, if $i\in I_j$ then the second-difference quotient $(\Delta_\varepsilon u)_i$ is bounded uniformly with respect to ε (recall that u_ε is left continuous). Then, for every j

$$(3.4)\qquad \sum_{i\in I_j}\left|\psi(\sqrt{\varepsilon}(\Delta_\varepsilon u_\varepsilon)_i) - \frac{1}{2}\psi''(0)\varepsilon(\Delta_\varepsilon u_\varepsilon)_i^2\right| \to 0 \quad \text{as } n\to+\infty.$$

Consider now the two types of singular points:

- if $\xi=\varepsilon i_j$ is a discontinuity point of u_ε, then $u_\varepsilon(\xi)=u_\varepsilon(\xi^-)$ and $|(\Delta_\varepsilon u_\varepsilon)_{i_j}| \sim |u_\varepsilon(\xi^+)-u_\varepsilon(\xi^-)|/\varepsilon^2 \sim \delta/\varepsilon^2$, with $\delta>0$. An analogous estimate holds for $|(\Delta_\varepsilon u_\varepsilon)_{i_j+1}|$ (see Figure 2). The corresponding terms $\psi\big(\sqrt{\varepsilon}(\Delta_\varepsilon u_\varepsilon)_i\big)$ tend to 1 as $\varepsilon\to 0$.

– if εi_j is a discontinuity point of u'_ε, then $|(\Delta_\varepsilon u_\varepsilon)_{i_j}| \sim \delta/\varepsilon$, for some $\delta > 0$; again, the term $\psi\big(\sqrt{\varepsilon}(\Delta_\varepsilon u_\varepsilon)_{i_j}\big)$ tends to 1 as $\varepsilon \to 0$.

So, in the limit as $\varepsilon \to 0$, each discontinuity point counts double, while each discontinuity point of the derivative weighs one. Together with (3.4) this implies that

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(u), \tag{3.5}$$
$$\text{where} \qquad F(u) = \frac{1}{2}\psi''(0)\int_0^1 (u'')^2(x)\mathrm{d}x + 2\#S(u) + \#S(u').$$

The functional F can be defined on $\mathscr{H}^2(0,1)$. In [18] the following result is proved (for an introduction to Γ-convergence see [26]).

Theorem 3.2. *Let F be defined as in (3.5) on $\mathscr{H}^2(0,1)$. The functionals F_ε Γ-converge to F with respect to the $L^1(0,1)$ convergence on bounded sets of $L^2(0,1)$.*

Singular set. Interpolations Here we define suitable interpolations of the discrete functions introduced above.

For a function $(u_i)_i$ on $[0,1]\cap\varepsilon\mathbb{Z}$ a key rôle will be played by the "singular set" of the points i where the second-order difference quotient exceeds a fixed threshold. Let $\zeta \ge z_0$ (where $z_0 > 0$ is the inflection point of ψ); we define

$$I_\varepsilon^\zeta(u) = \{i \in \mathbb{Z} : 1 \le i \le N_\varepsilon - 1,\ |(\Delta_\varepsilon u)_i| \ge \zeta/\sqrt{\varepsilon}\} \tag{3.6}$$

(for $i = N_\varepsilon$ the difference quotient is null by the definition of $u_{N_\varepsilon+1}$). For future reference we point out a simple estimate for $\#I_\varepsilon^\zeta(u)$:

$$\#I_\varepsilon^\zeta(u) \le \frac{F_\varepsilon(u)}{\psi(\zeta)} \le \frac{F_\varepsilon(u)}{\psi(z_0)}. \tag{3.7}$$

Indeed, by the monotonicity of φ_ε on $[0,+\infty)$, we have

$$F_\varepsilon(u) \ge \sum_{i\in I_\varepsilon^\zeta(u)} \varepsilon\varphi_\varepsilon\big((\Delta_\varepsilon u)_i\big) \ge \big(\#I_\varepsilon^\zeta(u)\big)\varepsilon\varphi_\varepsilon(\zeta/\sqrt{\varepsilon}) = \big(\#I_\varepsilon^\zeta(u)\big)\psi(\zeta). \tag{3.8}$$

We will denote by $\overline{u}$ the piecewise-affine function on $[0,\varepsilon(N_\varepsilon+1)]$ which interpolates the values (u_i) for $0 \le i \le N_\varepsilon+1$. Following [18], to every $u\colon [0,1]\cap\varepsilon\mathbb{Z} \to \mathbb{R}$ we also associate a function $\tilde{u}$ which is a quadratic smoothing of $\overline{u}$ for the indices $i \notin I_\varepsilon^\zeta$, and which globally keeps the value of the curvature (see Figure 3). For the exact definition we need to note that, for every $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, and given four real values u_-, u_+, m_-, m_+, there exists a unique quadratic (or linear) function $\tilde{u}$ satisfying

$$\begin{cases} \tilde{u}(x_0-\varepsilon/2) = u_-,\ \tilde{u}(x_0+\varepsilon/2) = u_+ \\ \tilde{u}'(x_0-\varepsilon/2) = m_-,\ \tilde{u}'(x_0+\varepsilon/2) = m_+ \end{cases} \tag{3.9}$$

if and only if $(m_+ + m_-)/2 = (u_+ - u_-)/\varepsilon$. In particular, this condition is satisfied if

$$\begin{aligned} & x_0 = \varepsilon i, \\ & u_- = \overline{u}(x_0-\varepsilon/2) = \frac{u_{i-1}+u_i}{2}, \quad u_+ = \overline{u}(x_0+\varepsilon/2) = \frac{u_i+u_{i+1}}{2}, \\ & m_- = \frac{u_i-u_{i-1}}{\varepsilon}, \quad m_+ = \frac{u_{i+1}-u_i}{\varepsilon}. \end{aligned} \tag{3.10}$$

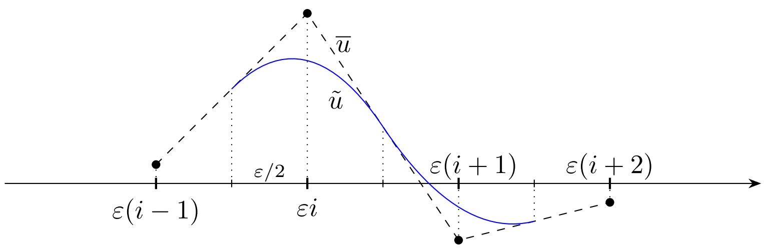

Figure 3 - Piecewise interpolation $\overline{u}$ (dashed line) and quadratic smoothing $\tilde{u}$ (solid line) near points εi and $\varepsilon(i+1)$ assuming that $i, i+1$ do not belong to the singular set $I_\varepsilon^\zeta(u)$.

Moreover, it turns out that the (constant) $\tilde{u}''$ takes the value

$$\tilde{u}'' = \frac{m_+ - m_-}{\varepsilon} = (\Delta_\varepsilon u)_i\,.$$

Definition 3.3 (Quadratic smoothing)**.** *Let $u\colon [0,1]\cap\varepsilon\mathbb{Z} \to \mathbb{R}$ and let $\overline{u}\colon [0,\varepsilon(N_\varepsilon+1)] \to \mathbb{R}$ be the piecewise-affine extension of u introduced above. Let $\zeta \geq z_0$ be fixed; with respect to this threshold, define $\tilde{u}\colon [0,\varepsilon(N_\varepsilon+1)] \to \mathbb{R}$ as the function which coincides with the quadratic function characterized by* (3.9) *and* (3.10) *on each interval $[\varepsilon(i-1/2), \varepsilon(i+1/2)]$ with $1 \leq i \leq N_\varepsilon$ and $i \notin I_\varepsilon^\zeta(u)$, and which equals $\overline{u}$ otherwise.*

Proposition 3.4. *Let u be as above. Then:*

i) $\tilde{u} \in \mathscr{H}^2(0,1) \cap C([0,1])$, *and* $S(\tilde{u}') = I_\varepsilon^\zeta(u)$. *Moreover,*

$$F_\varepsilon(u) \geq \nu(\zeta)\int_0^1 |\tilde{u}''|^2\,\mathrm{d}x + \psi(\zeta)\#S(\tilde{u}'),$$

where $\nu(\zeta)$ is as in (3.3).

ii) $|\tilde{u} - \overline{u}| \leq \dfrac{1}{8}\varepsilon\sqrt{\varepsilon}\,\zeta.$

iii) For every $x \in [\varepsilon i, \varepsilon(i+1)]$, with $0 \leq i \leq N_\varepsilon$,

$$|\overline{u}(x) - u(x)| \leq |u_{i+1} - u_i|.$$

Moreover, for every $p \geq 1$ there exist positive constants c_p and C_p (independent of u) such that

$$c_p\|u\|_{L^p} \leq \|\overline{u}\|_{L^p(0,1)} \leq \|\overline{u}\|_{L^p((0,\varepsilon(N_\varepsilon+1))} \leq C_p\|u\|_{L^p}.$$

Proof. *i)* The inequality in (*i*) follows from the estimate in (3.8) for the singular part, and from (3.3) and the following estimate for the other part:

$$\sum_{i\notin I_\varepsilon^\zeta(u)} \varepsilon\varphi_\varepsilon\big((\Delta_\varepsilon u)_i\big) = \int_0^{\varepsilon(N_\varepsilon+1)} \varphi_\varepsilon(\tilde{u}'')\,\mathrm{d}x \geq \nu(\zeta)\int_0^1 |\tilde{u}''|^2\,\mathrm{d}x.$$

ii) Let $1 \le i \le N_\varepsilon$, with $i \notin I^\zeta_\varepsilon(u)$. Let us take the interval $[\varepsilon(i-\frac{1}{2}), \varepsilon i]$ into account (an analogous argument holds for $[\varepsilon i, \varepsilon(i+\frac{1}{2})]$). The function $\overline{u}$ coincides with the linear part of the Taylor expansion of the quadratic polynomial $\tilde{u}$ with respect to the point $\varepsilon(i-\frac{1}{2})$; hence, on the interval $(\varepsilon(i-\frac{1}{2}), \varepsilon i)$ we have

$$|\tilde{u}(x) - \overline{u}(x)| = \frac{1}{2}|(\Delta_\varepsilon u)_i|(\varepsilon/2)^2 \le \frac{1}{8}\varepsilon\sqrt{\varepsilon}\,\zeta. \tag{3.11}$$

iii) Recall that on each interval $[\varepsilon i, \varepsilon(i+1)]$, with $0 \le i \le N_\varepsilon$, the function $\overline{u}$ is a convex combination of the values u_i and u_{i+1}:

$$\overline{u}(x) = u_i + \lambda(x)(u_{i+1} - u_i), \quad \text{with } \lambda(x) = (x - \varepsilon i)/\varepsilon. \tag{3.12}$$

Then the first inequality immediately follows; moreover, on this interval

$$|\overline{u}|^p \le \max(|u_i|^p, |u_{i+1}|^p) \le |u_i|^p + |u_{i+1}|^p$$

Therefore $\int_0^{\varepsilon(N_\varepsilon+1)} |\overline{u}|^p \,\mathrm{d}x \le \varepsilon\Big(|u_0|^p + 2\sum_{i=1}^{N_\varepsilon}|u_i|^p + |u_{N_\varepsilon+1}|^p\Big)$; since,

$$|u_{N_\varepsilon+1}| = |2u_{N_\varepsilon} - u_{N_\varepsilon-1}| \le 2|u_{N_\varepsilon}| + |u_{N_\varepsilon-1}|,$$

we have $|u_{N_\varepsilon+1}|^p \le 2^{p-1}((2|u_{N_\varepsilon}|)^p + |u_{N_\varepsilon-1}|^p)$; hence $\|\overline{u}\|_{L^p((0,\varepsilon(N_\varepsilon+1))} \le C_p\|u\|_{L^p}$, for a suitable C_p.

Finally, we have to estimate $\|\overline{u}\|$ from below by means of $\|u\|_{L^p}$. Let us take (3.12) into account and consider two cases. First assume that u_i and u_{i+1} have the same sign; consider $\max(|u_i|, |u_{i+1}|)$ and, e.g., assume it is given by $|u_i|$. Then

$$\begin{aligned}\int_{\varepsilon i}^{\varepsilon(i+1)} |\overline{u}(x)|^p \,\mathrm{d}x &\ge |u_i|^p \int_{\varepsilon i}^{\varepsilon(i+1)} (1-\lambda(x))^p \,\mathrm{d}x \\ &= \frac{1}{p+1}\varepsilon|u_i|^p = \frac{1}{p+1}\varepsilon\max(|u_i|^p, |u_{i+1}|^p).\end{aligned}$$

Otherwise, let $u_i\,u_{i+1} < 0$, and assume, e.g., that $\max(|u_i|, |u_{i+1}|) = |u_i|$. Let $\overline{x} \in (\varepsilon i, \varepsilon(i+1))$ be such that $\overline{u}(\overline{x}) = 0$; it can be easily checked (and it is geometrically clear) that $\overline{x} - \varepsilon i \ge \varepsilon/2$ and that in the interval $(\varepsilon i, \overline{x})$ the function $\overline{u}$ can be estimated by the affine interpolation of the values u_i and 0 in εi and $\varepsilon(i+\frac{1}{2})$, respectively, i.e.

$$|\overline{u}(x)| \ge |u_i(1 - \frac{x - \varepsilon i}{\varepsilon/2})|.$$

Then

$$\begin{aligned}\int_{\varepsilon i}^{\varepsilon(i+1)} |\overline{u}(x)|^p \,\mathrm{d}x \ge \int_{\varepsilon i}^{\varepsilon(i+\frac{1}{2})} |\overline{u}(x)|^p \,\mathrm{d}x &\ge \int_{\varepsilon i}^{\varepsilon(i+\frac{1}{2})} |u_i(1 - \frac{x-\varepsilon i}{\varepsilon/2})|^p \,\mathrm{d}x \\ &= \frac{1}{2(p+1)}\varepsilon|u_i|^p = \frac{1}{2(p+1)}\varepsilon\max(|u_i|^p, |u_{i+1}|^p)).\end{aligned}$$

In either case

$$\int_{\varepsilon i}^{\varepsilon(i+1)} |\overline{u}(x)|^p \,\mathrm{d}x \ge \frac{1}{2(p+1)}\varepsilon\max(|u_i|^p, |u_{i+1}|^p)) \ge \frac{1}{2(p+1)}\frac{\varepsilon}{2}(|u_i|^p + |u_{i+1}|^p).$$

Let us now sum up for $i = 0, \ldots, N_\varepsilon - 1$:

$$\|\overline{u}\|_{L^p(0,1)} \geq \|\overline{u}\|_{L^p(0,\varepsilon N_\varepsilon)} \geq c_p\Big(\varepsilon \sum_{i=0}^{N_\varepsilon} |u_i|^p\Big)^{1/p}$$

for a suitable $c_p > 0$. $\square$

Remark 3.5. Let $u\colon [0,1] \cap \mathbb{Z} \to \mathbb{R}$ and let v be the continuous piecewise-quadratic function given by (3.9) and (3.10) on *each* interval $[\varepsilon(i-1/2), \varepsilon(i+1/2)]$ with $1 \leq i \leq N_\varepsilon$ (and equal to $\overline{u}$ outside $[\varepsilon/2, \varepsilon(N_\varepsilon + \frac{1}{2})]$); this means that the quadratic smoothing is considered for *every* i, not only in the complement of $I_\varepsilon^\zeta(u)$. It turns out that

$$\|v\|_{L^2(0,\varepsilon(N_\varepsilon+1))} \leq C\|u\|_{L^2},$$

where C is a constant independent of u and ε. Indeed, for every $1 \leq i \leq N_\varepsilon$, on $[\varepsilon(i - \frac{1}{2}), \varepsilon i]$ we have $|\overline{u}| \leq |u_{i-1}| + |u_i|$, so that, as in (3.11),

$$|v(x)| \leq |u_{i-1}| + |u_i| + \frac{1}{8}|u_{i+1} - 2u_i + u_{i-1}|$$

We conclude by computing the L^2 norm on each interval (recall that $u_{N_\varepsilon+1}$ is defined in terms of u_{N_ε} and $u_{N_\varepsilon-1}$).

4 Minimizing movements along F_ε. Compactness.

As mentioned in the Introduction, we apply the so-called method of the *minimizing movements* to the functionals F_ε, but we allow the spatial-discretization parameter ε to vary as the time-discretization step goes to zero (*"minimizing movements along a sequence"*, according to [7]; see also [9]). In Theorem 4.7 we give an existence result; then, in the next sections, we will show some regularity properties of the limit function, and prove that it satisfies a fourth-order equation of evolution type.

Discrete evolution For each $\varepsilon > 0$ let $u_\varepsilon^0\colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$ be a given function and let $\tau > 0$ be fixed. We now recursively define a sequence $u_{\varepsilon,\tau}^k$ ($k \in \mathbb{N}$) of real-valued functions on $[0,1] \cap \varepsilon\mathbb{Z}$; to keep the notation more readable, we denote $u_{\varepsilon,\tau}^k$ simply by u^k (if the context is clear), so that u_i^k stands for the i-th value of u^k.

We define the function u^0 just as the initial datum u_ε^0 fixed above, while for any $k \geq 1$, the function u^k is required to be a minimizer of

$$G_{\varepsilon,\tau}^k(v) := F_\varepsilon(v) + \frac{1}{2\tau}\sum_{i=0}^{N_\varepsilon} \varepsilon|v_i - u_i^{k-1}|^2\,, \tag{4.1}$$

among all possible $v\colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$. Thus, the index k acts as the "time variable" of the evolution.

For the initial datum u_ε^0 we require that:

Hyp 1) There exists $M_0 > 0$ such that $\|u_\varepsilon^0\|_{L^2} \leq M_0$ for any $\varepsilon > 0$.

Hyp 2) There exists $M > 0$ such that $F_\varepsilon(u^0_\varepsilon) \le M$ for any $\varepsilon > 0$.

Let us now point out two simple results we will use later on.

Proposition 4.1. For every $k \in \mathbb{N}$ we have

$$F_\varepsilon(u^k) \le F_\varepsilon(u^{k-1}) \quad \text{and} \quad \sum_{i=0}^{N_\varepsilon} \varepsilon |u^k_i - u^{k-1}_i|^2 \le 2\tau\big[F_\varepsilon(u^{k-1}) - F_\varepsilon(u^k)\big]. \tag{4.2}$$

Indeed, the minimality of u^k with respect to the test function $v = u^{k-1}$ implies that:

$$F_\varepsilon(u^k) + \frac{1}{2\tau}\sum_{i=0}^{N_\varepsilon} \varepsilon |u^k_i - u^{k-1}_i|^2 \le F_\varepsilon(u^{k-1}),$$

from which both inequalities follow.

Since u^k is the solution of a minimum problem in finite dimension, we get the following classical optimality conditions.

Proposition 4.2 (Optimality conditions)**.** *Let $(u^k)_k$ be defined recursively as above. Then, for $i = 2, \dots, N_\varepsilon - 2$, the following equation holds:*

$$\begin{aligned}
&\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_1\big) + \frac{\varepsilon^2}{\tau}(u^k_0 - u^{k-1}_0) = 0,\\
&\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_2\big) - 2\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_1\big) + \frac{\varepsilon^2}{\tau}(u^k_1 - u^{k-1}_1) = 0,\\
&\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_{i+1}\big) - 2\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_i\big) + \varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_{i-1}\big) + \frac{\varepsilon^2}{\tau}(u^k_i - u^{k-1}_i) = 0,\\
&\hspace{20em}(i = 2, \dots, N_\varepsilon - 2),\\
&-2\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_{N_\varepsilon-1}\big) + \varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_{N_\varepsilon-2}\big) + \frac{\varepsilon^2}{\tau}(u^k_{N_\varepsilon-1} - u^{k-1}_{N_\varepsilon-1}) = 0,\\
&\varphi'_\varepsilon\big((\Delta_\varepsilon\, u^k)_{N_\varepsilon-1}\big) + \frac{\varepsilon^2}{\tau}(u^k_{N_\varepsilon} - u^{k-1}_{N_\varepsilon}) = 0.
\end{aligned}$$

Remark 4.3. Recall the rôle of k as the time parameter, and that we have the pointwise convergence of φ'_ε to $z \mapsto \psi''(0)z$, uniformly on bounded sets. Therefore, these optimality conditions are expected to give rise to the equation $\psi''(0)(u_{xx})_{xx} + u_t = 0$ in the limit as $\varepsilon, \tau \to 0$.

Remark 4.4. Let us collect here two more estimates which we will use in the arguments of the following sections.
i) By Proposition 4.1 and *(Hyp 2)* we have

$$\varepsilon |u^k_i - u^{k-1}_i|^2 \le 2M\tau \qquad \text{for every } 0 \le i \le N_\varepsilon \text{ and } k \in \mathbb{N}.$$

ii) Since

$$|(\Delta_\varepsilon u^k)_i - (\Delta_\varepsilon u^{k-1})_i| \le \varepsilon^{-2}\big(|u^k_{i+1} - u^{k-1}_{i+1}| + 2|u^k_i - u^{k-1}_i| + |u^k_{i-1} - u^{k-1}_{i-1}|\big),$$

we deduce that

$$|(\Delta_\varepsilon u^k)_i - (\Delta_\varepsilon u^{k-1})_i| \le 6\sqrt{M}\left(\frac{\tau}{\varepsilon^5}\right)^{1/2}. \tag{4.3}$$

Compactness The compactness result we are going to state (Theorem 4.7) is rather standard in the theory of minimizing movement.

Let (ε_n) and (τ_n) be positive infinitesimal sequences; when no confusion may arise we will still use ε and τ in place of ε_n and τ_n. Moreover, we fix a threshold ζ for the definition of the singular set I_ε^ζ (see (3.6)). For the sake of simplicity, we use the following notation:

- F_n denotes the functional F_{ε_n};
- u_n^k denotes the function $u^k_{\varepsilon_n,\tau_n}$ defined on $[0,1]\cap\varepsilon_n\mathbb{Z}$ by recursive minimization of the functional in (4.1); as a function on $[0,\varepsilon(N_\varepsilon+1)]$ it is the piecewise-constant extension given by (3.2);
- for any $t\geq 0$ let $u_n(\cdot,t)=u_n^k$, with $k=\lfloor t/\tau_n\rfloor$ (in particular, $u_n(\cdot,0)=u_n^0$). Therefore, $\tilde{u}_n(\cdot,t)$ denotes the quadratic smoothing of $u_n(\cdot,t)$ according to Definition 3.3.

As in [7], Proposition 7.1, the following estimate holds.

Proposition 4.5. *For any $s,t\geq 0$, with $s<t$, we have*

$$\|u_n(\cdot,t)-u_n(\cdot,s)\|_{L^2}\leq \big(2F_n(u^0_{\varepsilon_n})\big)^{1/2}\sqrt{t-s+\tau_n}.$$

Proof. Let $\lfloor t/\tau_n\rfloor=k$ and $\lfloor s/\tau_n\rfloor=h$; assume $k>h$. Then

$$\|u_n(\cdot,t)-u_n(\cdot,s)\|_{L^2}\leq\sum_{l=h+1}^{k}\|u_n^l-u_n^{l-1}\|_{L^2}\,.$$

By Proposition 4.1 we estimate the right-hand side by

$$\sum_l\sqrt{2\tau_n}\Big(F_n(u_n^{l-1})-F_n(u_n^l)\Big)^{1/2}$$
$$\leq\sqrt{2\tau_n}\sqrt{k-h}\left(\sum_l\Big(F_n(u_n^{l-1})-F_n(u_n^l)\Big)\right)^{1/2}\leq\sqrt{2\tau_n}\sqrt{k-h}\Big(F_n(u_n^0)\Big)^{1/2}.$$

We conclude by the inequality $\tau_n(k-h)\leq t-s+\tau_n$. $\square$

In particular, since $u_n(\cdot,0)=u^0_{\varepsilon_n}$, by assumptions *(Hyp 1)* and *(Hyp 2)*, for every $T>0$

$$u_n(\cdot,t)\quad\text{are equibounded in } L^2(0,1),\ \text{uniformly with respect to } t\in[0,T]. \tag{4.4}$$

Let us deduce other useful estimates. Apply Proposition 3.4. From (i) we get, for every $t\geq 0$

$$F_n\big(u_n(\cdot,t)\big)\geq\nu(\zeta)\int_0^1|(\tilde{u}_n)_{xx}(x,t)|^2\,\mathrm{d}x+\psi(\zeta)\#S((\tilde{u}_n)_x(\cdot,t)).$$

From Proposition 4.1 and assumption *(Hyp 2)*, we have

$$F_n\big(u_n(\cdot,t)\big)\leq F_n(u^0_{\varepsilon_n})\leq M. \tag{4.5}$$

Moreover, for any $T > 0$, by (4.4) and by Proposition 3.4 ((*ii*) and (*iii*)), we get the L^2-boundedness of $\big(\tilde{u}_n(\cdot,t)\big)$, uniformly with respect to $t \in [0,T]$. It is easily seen that these bounds are independent of the threshold ζ, if this latter varies in a bounded set.

We gather these facts in the following lemma.

Lemma 4.6. *For every $T \geq 0$*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \Big[\int_0^1 \big(|(\tilde{u}_n)_{xx}(x,t)|^2 + |\tilde{u}_n(x,t)|^2\big)\,\mathrm{d}x + \#\big(S((\tilde{u}_n)_x(\cdot,t))\big)\Big] < +\infty.$$

This supremum is uniform with respect to the threshold ζ defining $\tilde{u}$, if ζ varies in a bounded set.

Theorem 4.7. *Let u^0_ε satisfy* (Hyp 1) *and* (Hyp 2). *Let $\tilde{u}_n(\cdot,t)$ be the function associated to $u_n(\cdot,t)$ according to Definition 3.3. There exists a function $u\colon [0,1]\times[0,+\infty) \to \mathbb{R}$ such that*

$$u(\cdot,t) \in \mathscr{H}^2(0,1) \qquad \textit{for every } t \geq 0,$$

and, up to a subsequence, for every $t \geq 0$

$$\begin{aligned} &\tilde{u}_n(\cdot,t) \to u(\cdot,t) \quad \textit{strongly in } L^q(0,1) \textit{ for every } 1 \leq q < 2,\\ &\tilde{u}_n(\cdot,t) \rightharpoonup u(\cdot,t) \qquad \textit{weakly in } L^2(0,1),\\ &(\tilde{u}_n)_{xx}(\cdot,t) \rightharpoonup u_{xx}(\cdot,t) \textit{ weakly in } L^2(0,1). \end{aligned}$$

Remark 4.8. The uniform estimate given by Lemma 4.6 implies that, for every $T > 0$, the (sub)sequences $\tilde{u}_n$ and $(\tilde{u}_n)_{xx}$ in the above theorem weakly converge in $L^2(Q_T)$, where $Q_T = (0,1)\times(0,T)$.

Proof. By Lemma 4.6 we can apply Theorem 2.3 to the sequence $\big(\tilde{u}_n(\cdot,t)\big)_n$ for every $t \geq 0$. By a diagonal argument we can find a subsequence $(\tilde{u}_{n_h})_h$ such that $\big(\tilde{u}_{n_h}(\cdot,q)\big)_h$ converges strongly in $L^1(0,1)$ to a function $u(\cdot,q)$ for every $q \in \mathbb{Q}$, $q \geq 0$.

Let now $t \geq 0$ be fixed. We are going to show that $\big(\tilde{u}_{n_h}(\cdot,t)\big)_h$ is a Cauchy sequence in $L^1(0,1)$. To simplify the notation, let us drop the subscript h.

Let $\sigma > 0$ be fixed; we can find $\delta_\sigma > 0$ and $n_\sigma \in \mathbb{N}$ such that

$$\sqrt{\delta_\sigma + \tau_n} < \sigma \qquad \text{for every } n \geq n_\sigma.$$

Let $q \in \mathbb{Q}$ be such that $q \geq 0$ and $|q-t| < \delta_\sigma$. Since $(\tilde{u}_n(\cdot,q))$ converges in $L^1(0,1)$, we can also assume that

$$\|\tilde{u}_n(\cdot,q) - \tilde{u}_m(\cdot,q)\|_{L^1(0,1)} < \sigma \qquad \text{for every } n,m \geq n_\sigma.$$

For any $n,m \geq n_\sigma$ we have:

$$\begin{aligned} \|\tilde{u}_n(\cdot,t) - \tilde{u}_m(\cdot,t)\|_{L^1(0,1)} &\leq \|\tilde{u}_n(\cdot,t) - \tilde{u}_n(\cdot,q)\|_{L^1(0,1)}\\ &\quad + \|\tilde{u}_n(\cdot,q) - \tilde{u}_m(\cdot,q)\|_{L^1(0,1)} + \|\tilde{u}_m(\cdot,q) - \tilde{u}_m(\cdot,t)\|_{L^1(0,1)}. \end{aligned}$$

The second term on the right-hand side is bounded by σ; as to the first one (the third term is analogous), we have:

$$\begin{aligned}\|\tilde{u}_n(\cdot,t)-\tilde{u}_n(\cdot,q)\|_{L^1(0,1)} &\leq \|\tilde{u}_n(\cdot,t)-\tilde{u}_n(\cdot,q)\|_{L^2(0,1)}\\ &\leq \|\tilde{u}_n(\cdot,t)-\overline{u}_n(\cdot,t)\|_{L^2(0,1)}+\|\overline{u}_n(\cdot,t)-\overline{u}_n(\cdot,q)\|_{L^2(0,1)}\\ &\qquad\qquad + \|\overline{u}_n(\cdot,q)-\tilde{u}_n(\cdot,q)\|_{L^2(0,1)}\,.\end{aligned}$$

By Proposition 3.4 the first and third term on the right-hand side are bounded by $\zeta\varepsilon_n^{3/2}/8$: we can choose n_σ in such a way that this quantity is less than σ (recall that $n \geq n_\sigma$). Moreover, note that $\overline{u}_n(\cdot,t)-\overline{u}_n(\cdot,q)$ coincides with the piecewise-affine interpolation of $u_n(\cdot,t)-u_n(\cdot,q)$; thus, Proposition 3.4 *(iii)*, Proposition 4.5 and assumption *(Hyp 2)* yield

$$\begin{aligned}\|\overline{u}_n(\cdot,t)-\overline{u}_n(\cdot,q)\|_{L^2(0,\varepsilon(N_\varepsilon+1))} &\leq C_p\|u_n(\cdot,t)-u_n(\cdot,q)\|_{L^2}\\ &\leq C_p(2M)^{1/2}\sqrt{|t-q|+\tau_n} \leq \sigma C_p(2M)^{1/2}.\end{aligned}$$

By collecting all the estimates, we get that, if $n, m \geq n_\sigma$

$$\|\tilde{u}_n(\cdot,t)-\tilde{u}_m(\cdot,t)\|_{L^1(0,1)} \leq C\sigma$$

for a suitable constant C. We have proved that $\big(\tilde{u}_n(\cdot,t)\big)$ is a Cauchy sequence in $L^1(0,1)$. We denote the limit by $u(\cdot,t)$, too.

So far we have proved that there exists a subsequence $(\tilde{u}_{n_h})_h$ such that

$$\tilde{u}_{n_h}(\cdot,t) \to u(\cdot,t) \qquad \text{in } L^1(0,1)$$

for every $t \geq 0$. An application of Theorem 2.3 and the subsequent remark to the sequence $\big(\tilde{u}_{n_h}(\cdot,t)\big)$ now yields that $u(\cdot,t) \in \mathscr{H}^2(0,1)$ and that the stated convergences hold. $\square$

Corollary 4.9. *For every $t \geq 0$ the piecewise-constant functions $u_n(\cdot,t)$ converge to $u(\cdot,t)$ in $L^q(0,1)$ for every $1 \leq q < 2$. Moreover, $u \in C^{1/2}\big([0,+\infty);L^2(0,1)\big)$, and $(u_n)_n$ weakly converges to u in $L^2(Q_T)$, where $Q_T = (0,1)\times(0,T)$, for every $T > 0$.*

Proof. Let $t \geq 0$ be fixed; since $u(\cdot,t) \in \mathscr{H}^2(0,1)$, there exist $0 = x_0 < x_1 < \ldots < x_m = 1$ such that $u(\cdot,t) \in H^2(x_{j-1},x_j)$ for every $j = 1,\ldots,m$; in particular, $u(\cdot,t) \in C^0([x_{j-1},x_j])$. For every n define the piecewise-constant function v_n as

$$(v_n)_i = u(\varepsilon_n i,t), \qquad i = 0,\ldots,N_\varepsilon$$

(choose $(v_n)_i = \lim_{x\to\varepsilon i-} u(x,t)$ if εi is discontinuity point). It is easy to check that the (piecewise-constant) functions v_n and the (piecewise-affine) functions $\overline{v}_n$ converge to $u(\cdot,t)$ in $L^2(0,1)$.

Let now $q \in [1,2)$, and let us prove that $u_n(\cdot,t) \to u(\cdot,t)$ in $L^q(0,1)$. We have

$$\|u_n(\cdot,t)-u(\cdot,t)\|_{L^q(0,1)} \leq \|u_n(\cdot,t)-v_n\|_{L^q(0,1)}+\|v_n-u(\cdot,t)\|_{L^q(0,1)}\,.$$

The second term on the right-hand side tends to zero; as to the first one, by Proposition 3.4 we have

$$
\begin{aligned}
c_q\|u_n(\cdot,t)-v_n\|_{L^q(0,1)} \le \|\overline{u}_n(\cdot,t)-\overline{v}_n\|_{L^q(0,1)} &\le \|\overline{u}_n(\cdot,t)-\tilde{u}_n(\cdot,t)\|_{L^q(0,1)} \\
&+ \|\tilde{u}_n(\cdot,t)-u(\cdot,t)\|_{L^q(0,1)} + \|u(\cdot,t)-\overline{v}_n\|_{L^q(0,1)}\,.
\end{aligned}
$$

As mentioned above, the third term on the right-hand side tends to zero. By Theorem 4.7 the second term tends to zero, too, and so is the first one taking the estimate of Proposition 3.4 (*ii*) into account.

The L^q convergence just proved, together with the uniform estimate in (4.4), implies the weak-$L^2(0,1)$ convergence of the functions $u_n(\cdot,t)$ to $u(\cdot,t)$ for every $t\ge 0$, and therefore it yields that (u_n) weakly converges to u in $L^2(Q_T)$.

Finally, the L^q convergence entails a.e. convergence (up to a subsequence); then, by applying Fatou's Lemma to the estimate in Proposition 4.5, we conclude that $u\in C^{1/2}\big([0,+\infty);L^2(0,1)\big)$. $\square$

Remark 4.10. Since the (sub)sequence $u_n(\cdot,t)$ does not depend on the threshold ζ which enter the definition of $\tilde{u}_n(\cdot,t)$, from the preceding corollary we deduce that the limit u in Theorem 4.7 does not depend on ζ.

5 Limit evolution equation

In this section we establish a regularity result (Theorem (5.1)) for the limit evolution u of Theorem 4.7, and we prove that u satisfies a fourth-order equation (see Theorem 5.5).

Regularity of u_{xx} Let u be a limit function as in Theorem 4.7. On the line of Remark 4.3, we use the L^2-bound of the terms $|u_i^k-u_i^{k-1}|$ (from Proposition 4.1) to obtain an estimate for the second-order difference quotient of u_{xx}. More precisely, we get the following result.

Theorem 5.1. *Let u be as in Theorem 4.7. Then*

$$u_{xx}\in L^2(0,T;H_0^2(0,1)) \qquad \textit{for every } T>0.$$

In particular,

$$u_{xx}(0,t)=u_{xx}(1,t)=0 \quad \textit{and} \quad (u_{xx})_x(0,t)=(u_{xx})_x(1,t)=0$$

for a.e. $t>0$.

Since u is independent of the choice of the threshold ζ (see Remark 4.10), we assume $\zeta=z_0$.

To simplify the notation, let us drop the subscript n in ε_n. Let u_n be a (sub)sequence as in Theorem 4.7; for every $k\ge 0$ let $(w_n^k)_i$ be defined, for every $i\in\mathbb{Z}$, by $(\varepsilon=\varepsilon_n)$

$$
(5.1) \qquad (w_n^k)_i = \begin{cases} \varphi_\varepsilon'\big((\Delta_\varepsilon u_n^k)_i\big), & \text{if } 1\le i\le N_\varepsilon; \\ 0 & \text{otherwise in } \mathbb{Z}. \end{cases}
$$

We denote by w_n the following piecewise-constant extension of $(w_n^k)_i$: if $(x,t) \in \mathbb{R} \times [0,+\infty)$ let

$$
(5.1') \qquad w_n(x,t) = (w_n^k)_i \qquad i = \lfloor x/\varepsilon \rfloor,\ k = \lfloor t/\tau \rfloor.
$$

For future reference we extract the following result from the proof of Theorem 5.1.

Lemma 5.2. *Let w_n be as above, with (u_n) given by Theorem 4.7. Then, for every $t \geq 0$*

$$
w_n(\cdot,t) \rightharpoonup \psi''(0) u_{xx}(\cdot,t) \qquad \textit{weakly in } L^2(0,1),
$$

where u is the limit of $(\tilde{u}_n)$ according to Theorem 4.7. Moreover, for every $T > 0$ the sequence $(w_n(\cdot,t))$ is bounded in $L^2(0,1)$ uniformly with respect to $t \in [0,T]$.

Remark 5.3. The uniform boundedness of $\|w_n(\cdot,t)\|_{L^2(0,1)}$ with respect to $t \in [0,T]$ implies that

$$
\sup_{t \in [0,T]} \|u_{xx}(\cdot,t)\|_{L^2(0,1)} < +\infty,
$$

and

$$
w_n \rightharpoonup \psi''(0) u_{xx} \qquad \text{weakly in } L^2(Q_T).
$$

Proof (of Lemma 5.2) As in Lemma 3.3 in [1], we exploit the uniform convergence of φ_ε' to $z \mapsto \psi''(0)z$ on bounded intervals, in particular on $[0, z_0]$.

Let $t \geq 0$ be fixed. Let χ_n be the characteristic function of the union of all the intervals $\varepsilon[i - \frac{1}{2}, i + \frac{1}{2}]$ with $i \in I_n^+ := I_\varepsilon^{z_0}\big(u_n(\cdot,t)\big)$ (in the rest of the proof we frequently drop the variable t). Let us split w_n as the sum

$$
w_n = \chi_n w_n + (1 - \chi_n) w_n
$$

and deal with each term separately.

The sequence $(\chi_n w_n)$ is bounded in $L^2(0,1)$; indeed, since ψ' is decreasing on $[z_0, +\infty)$:

$$
(5.2) \qquad \int_0^1 |\chi_n w_n|^2 \,\mathrm{d}x \leq \varepsilon (\# I_n^+) \varphi_\varepsilon'(z_0/\sqrt{\varepsilon})^2 = (\# I_n^+) \psi'(z_0)^2 \leq \frac{M}{\psi(z_0)} \psi'(z_0)^2
$$

by (3.7) and (4.5). An analogous estimate gives the strong convergence to zero in $L^1(0,1)$:

$$
\int_0^1 |\chi_n w_n| \,\mathrm{d}x \leq \varepsilon (\# I_n^+) \varphi_\varepsilon'(z_0/\sqrt{\varepsilon}) = \sqrt{\varepsilon} (\# I_n^+) \psi'(z_0) \leq \sqrt{\varepsilon} \frac{M}{\psi(z_0)} \psi'(z_0) \to 0
$$

as $n \to \infty$. Therefore

$$
\chi_n w_n \rightharpoonup 0 \qquad \text{weakly in } L^2(0,1).
$$

Let us now consider the term $(1 - \chi_n) w_n$. Let us notice that $(\tilde{u}_n)_{xx} = 0$ on each interval $\varepsilon(i - \frac{1}{2}, i + \frac{1}{2})$, with $i \in I_n^+$, and on $[0, \varepsilon/2] \cup [\varepsilon(N_\varepsilon - \frac{1}{2}), \varepsilon(N_\varepsilon + 1)])$, while $(\tilde{u}_n)_{xx} = \Delta_\varepsilon u_n (= \Delta_\varepsilon u_n(\cdot,t))$ otherwise; thus

$$
(1 - \chi_n) w_n = \varphi_\varepsilon'\big((\tilde{u}_n)_{xx}\big).
$$

Let $C > 0$ be such that $|\psi'(z)| \le C|z|$ for every $z \in \mathbb{R}$. Then

$$(5.3) \qquad |\varphi'_\varepsilon\big((\tilde{u}_n)_{xx}\big)| = \Big|\frac{1}{\sqrt{\varepsilon}}\psi'\big(\sqrt{\varepsilon}(\tilde{u}_n)_{xx}\big)\Big| \le C|(\tilde{u}_n)_{xx}|.$$

Hence $\varphi'_\varepsilon\big((\tilde{u}_n)_{xx}(\cdot,t)\big)$, i.e. $(1-\chi_n)w_n(\cdot,t)$, is bounded in $L^2(0,1)$ (recall Lemma 4.6), so that it weakly converges in $L^2(0,1)$ *up to a subsequence*. Let us show that the limit is $\psi''(0)u_{xx}(\cdot,t)$: since it is independent of the subsequence, we conclude that the whole sequence converges.

Let

$$\begin{aligned}\gamma_\varepsilon &= \varphi'_\varepsilon\big((\tilde{u}_n)_{xx}(\cdot,t)\big) - \psi''(0)(\tilde{u}_n)_{xx}(\cdot,t)\\ &= \frac{1}{\sqrt{\varepsilon}}\psi'\big(\sqrt{\varepsilon}(\tilde{u}_n)_{xx}(\cdot,t)\big) - \psi''(0)(\tilde{u}_n)_{xx}(\cdot,t).\end{aligned}$$

Let us prove that $\gamma_\varepsilon \to 0$ in $L^1(0,1)$. Since $\gamma_\varepsilon = 0$ where $(\tilde{u}_n)_{xx}(\cdot,t) = 0$, we have

$$\begin{aligned}&\int_0^1 |\gamma_\varepsilon(x)|\,\mathrm{d}x\\ &\quad\le \left(\int_{\{(\tilde{u}_n)_{xx}\neq 0\}} \left|\frac{\psi'\big(\sqrt{\varepsilon}(\tilde{u}_n)_{xx}(x,t)\big)}{\sqrt{\varepsilon}(\tilde{u}_n)_{xx}(x,t)} - \psi''(0)\right|^2 \mathrm{d}x\right)^{1/2} \|(\tilde{u}_n)_{xx}(\cdot,t)\|_{L^2(0,1)}.\end{aligned}$$

Since $(\tilde{u}_n)_{xx}(\cdot,t)$ is bounded in $L^2(0,1)$, we have $\sqrt{\varepsilon}(\tilde{u}_n)_{xx}(\cdot,t) \to 0$ a.e. (up to a subsequence); hence, the integrand function on the right-hand side of the previous inequality tends pointwise to zero as $n \to +\infty$. Moreover, it is bounded by $\big(C + \psi''(0)\big)^2$ (the constant C is as above). We conclude by the Dominated Convergence Theorem.

Therefore, $\varphi'_\varepsilon\big((\tilde{u}_n)_{xx}(\cdot,t)\big) = \psi''(0)(\tilde{u}_n)_{xx}(\cdot,t) + \gamma_\varepsilon$, where $\gamma_\varepsilon \to 0$ in L^1. Now, the L^2-weak convergence of $(\tilde{u}_n)_{xx}(\cdot,t)$ to $u_{xx}(\cdot,t)$ implies that $\varphi'_\varepsilon\big((\tilde{u}_n)_{xx}(\cdot,t)\big)$ tends to $\psi''(0)u_{xx}(\cdot,t)$ with respect to the L^1-weak topology. This limit must coincide with the L^2-weak limit.

Finally, the uniform $L^2(0,1)$-boundedness of $(w_n(\cdot,t))_n$ for $t \in [0,T]$ follows from the decomposition $w_n = \chi_n w_n + (1-\chi_n)w_n$ and the $L^2(0,1)$-boundedness of $(\chi_n w_n)_n$ and of $((1-\chi_n)w_n)_n$, both uniform with respect to t (see (5.2) and (5.3), and recall Lemma 4.6). □

Before addressing the proof of Theorem 5.1 we need to introduce the quadratic smoothing of $w_n(\cdot,t)$ in (5.1) and (5.1′); more precisely

(5.4) let $\omega_n(\cdot,t)$ be the quadratic smoothing of $w_n(\cdot,t)$, obtained by applying, to the function $i \mapsto (w_n^k)_i$ with $k = \lfloor t/\tau \rfloor$, and *to each node* in $\varepsilon\mathbb{Z}$, the quadratic smoothing used in Definition 3.3.

Proof (of Theorem 5.1). *Step 1.* Let us provide suitable estimates to get a weak compactness for the sequence (ω_n).

By Proposition 4.1 we have

$$\sum_{i=0}^{N_\varepsilon} \varepsilon|(u_n^k)_i - (u_n^{k-1})_i|^2 \le 2\tau[F_n(u_n^{k-1}) - F_n(u_n^k)].$$

Fix $T > 0$ and let $K_\tau = \lfloor T/\tau \rfloor + 1$; let us sum with respect to $k = 1, \dots, K_\tau$ and take assumption *(Hyp 2)* into account:

$$\sum_{k=1}^{K_\tau} \sum_{i=0}^{N_\varepsilon} \varepsilon |(u_n^k)_i - (u_n^{k-1})_i|^2 \le 2\tau F_n(u_n^0) \le 2\tau M.$$

Apply now the optimality conditions of Proposition 4.2 and remind the definition (5.1) of w_n^k; then

$$(u_n^k)_i - (u_n^{k-1})_i = -\tau \frac{(w_n^k)_{i+1} - 2(w_n^k)_i + (w_n^k)_{i-1}}{\varepsilon^2}$$

for $i = 2, \dots, N_\varepsilon - 2$ and also for $i = 0, 1$, and $i = N_\varepsilon - 1, N_\varepsilon$ (recall that $(\Delta_\varepsilon u_n^k)_{N_\varepsilon}$ vanishes by the definition (3.1) of the value in $N_\varepsilon + 1$). Therefore, from the inequality above we get

$$\sum_{k=1}^{K_\tau} \sum_{i=0}^{N_\varepsilon} \varepsilon\tau \left| \frac{(w_n^k)_{i+1} - 2(w_n^k)_i + (w_n^k)_{i-1}}{\varepsilon^2} \right|^2 \le 2M.$$

The i-th second-order difference quotient in this formula equals the second derivative of the function $\omega_n(\cdot, t)$, with $k = \lfloor t/\tau \rfloor$, on the interval $\varepsilon[i - \frac{1}{2}, i + \frac{1}{2}]$ for every $i \in \mathbb{Z}$. Notice that $\omega_n(x, t) = 0$ for every $x \le -\varepsilon/2$ and $x \ge \varepsilon(N_\varepsilon + \frac{1}{2})$. Therefore, if J is an open interval containing $[0, 1]$, for n sufficiently large we have

$$\omega_n \in L^2(0, T; H_0^2(J))$$

($t \mapsto \omega_n(\cdot, t)$ is measurable since it is piecewise constant). Moreover,

$$\int_\tau^T \int_J |(\omega_n)_{xx}|^2(x, t)\, \mathrm{d}x\mathrm{d}t \le 2M \tag{5.5}$$

(here $\tau = \tau_n \to 0$).

The sequence $(\omega_n(\cdot, t))$ is bounded in $L^2(J)$, too, uniformly with respect to $t \in [0, T]$: this follows from the L^2-boundedness of the piecewise-constant function $w_n(\cdot, t)$ (see Lemma 5.2), and the argument in Remark 3.5. The interpolation results for Sobolev spaces (see, e.g., [27], Theorem 5.2) imply the boundedness of (ω_n) in the reflexive Banach space $L^2(0, T; H_0^2(J))$. We conclude that, up to a subsequence, (ω_n) has a weak limit ω in $L^2(0, T; H_0^2(J))$ (see Section 2). By the arbitrariness of σ we conclude that $\omega \in L^2(0, T; H_0^2(0, 1))$.

Let us show that $\omega = \psi''(0) u_{xx}$.

Step 2. Let $t \in [0, T]$ be fixed, and let $\overline{w}_n(\cdot, t)$ be the piecewise-affine interpolation of $w_n(\cdot, t)$. Here we show that

$$\overline{w}_n(\cdot, t) \rightharpoonup \psi''(0) u_{xx}(\cdot, t) \qquad \text{weakly in } L^2(0, 1). \tag{5.6}$$

Since (w_n) is bounded in $L^2(0, 1)$ by Lemma 5.2 (for simplicity's sake, here we drop the dependence on t), $(\overline{w}_n)$ is bounded in $L^2(0, 1)$ by Proposition 3.4; therefore, it is enough to check the convergence on piecewise-constant functions, hence it is enough to check that

$$\int_a^b \overline{w}_n \to \int_a^b w, \qquad \text{where } w = \psi''(0) u_{xx}(\cdot, t),$$

for every interval $(a,b) \subseteq (0,1)$. Moreover, by the weak convergence of (w_n) to w, we can just verify that $\int_a^b (\overline{w}_n - w_n)\mathrm{d}x \to 0$.

Since the L^2-boundedness implies the uniform integrability, we can approximate the interval (a,b) by an interval $(l\varepsilon, m\varepsilon)$ with $l,m \in \mathbb{Z}$, and prove that $\int_{l\varepsilon}^{m\varepsilon} (\overline{w}_n - w_n)\mathrm{d}x \to 0$. We have

$$\begin{aligned}\int_{l\varepsilon}^{m\varepsilon} [\overline{w}_n(x) - w_n(x)]\,\mathrm{d}x &= \sum_{i=l}^{m-1} \int_{i\varepsilon}^{(i+1)\varepsilon} \frac{x-i\varepsilon}{\varepsilon}\big((w_n)_{i+1} - (w_n)_i\big)\,\mathrm{d}x \\ &= \frac{1}{2}\varepsilon \sum_{i=l}^{m-1} \big((w_n)_{i+1} - (w_n)_i\big) = \frac{1}{2}\varepsilon\big((w_n)_m - (w_n)_l\big).\end{aligned}$$

The last term tends to 0 as $\varepsilon \to 0$ since

$$|\varepsilon\big((w_n)_m - (w_n)_l\big)| \leq \varepsilon\sqrt{2}\big(|(w_n)_m|^2 + |(w_n)_l|^2\big)^{1/2} \leq \sqrt{2\varepsilon}\|w_n\|_{L^2} \to 0\,.$$

Therefore $\int_{l\varepsilon}^{m\varepsilon} [\overline{w}_n(x) - w_n(x)]\,\mathrm{d}x \to 0$, and (5.6) is proved.

Step 3. Let $\overline{w}_n$ be as in the previous step. Then, for every $\sigma > 0$:

$$\overline{w}_n - \omega_n \to 0 \qquad \text{in } L^2((\sigma,T)\times(0,1)).$$

Indeed, in each interval $[\varepsilon(i-\frac{1}{2}), \varepsilon(i+\frac{1}{2})]$, with $i = 1,\dots,N_\varepsilon$ we have:

$$|\overline{w}_n(\cdot,t) - \omega_n(\cdot,t)| \leq \frac{1}{8}\varepsilon^2|(\Delta_\varepsilon w_n)_i| = \frac{1}{8}\varepsilon^2|(\omega_n)_{xx}(\cdot,t)|$$

(recall the equality in (3.11)). Then, for every $\sigma > 0$ and n sufficiently large, by (5.5)

$$\|\overline{w}_n - \omega_n\|_{L^2((\sigma,T)\times(0,1))} \leq \frac{1}{8}\varepsilon^2\|(\omega_n)_{xx}\|_{L^2((\sigma,T)\times(0,1))} \leq \frac{1}{8}\varepsilon^2 2M\,.$$

The last term tends to zero as $\varepsilon = \varepsilon_n \to 0$.

Step 4. Since $(\overline{w}_n(\cdot,t))$ is bounded in $L^2(0,1)$ *uniformly* with respect to $t \in [0,T]$, the convergence (5.6) of Step 2 implies that $\overline{w}_n \rightharpoonup \psi''(0)u_{xx}$ weakly in $L^2(Q_T)$. This, together with the convergence of Step 3, implies that $\omega_n \rightharpoonup \psi''(0)u_{xx}$ weakly in $L^2((\sigma,T)\times(0,1))$ for every $\sigma > 0$. Since the weak convergence $L^2(0,T;H_0^2(J))$ implies the weak convergence in $L^2(Q_T)$, we conclude that $\omega = \psi''(0)u_{xx} \in L^2(0,T;H_0^2(0,1))$. $\square$

Remark 5.4. For future reference, we notice that the previous proof yields that the sections $\omega_n(\cdot,t)$ have the following properties:

a) $\omega_n(\cdot,t)$ is bounded in $L^2(0,1)$, uniformly with respect to $t \in [0,T]$.

b) for every $\sigma > 0$
$$\int_\sigma^T \int_J |(\omega_n)_{xx}|^2(x,t)\,\mathrm{d}x\mathrm{d}t \leq 2M.$$

c) for a.e. $t \in [0,T]$
$$\omega_n(\cdot,t) \rightharpoonup \psi''(0)u_{xx}(\cdot,t) \qquad \text{weakly in } L^2(0,1).$$

Indeed, (a) follows from the L^2-boundedness of the piecewise-constant function $w_n(\cdot,t)$ (see Lemma 5.2), and the argument in Remark 3.5. As to (b), it is a consequence of (5.5). Finally, Step 3 of the previous proof entails that, up to a subsequence, $\overline{w}_n(\cdot,t) - \omega_n(\cdot,t) \to 0$ in $L^2(0,1)$. Now, Step 2 allows to get (c) (the uniqueness of the limit guarantees that the whole sequence converges).

The evolution equation We now address the fourth-order equation (5.9). Let us first recall some formulas. Consider the real numbers

$$a_1, a_2, \dots, a_l, a_{l+1} \qquad b_1, b_2, \dots, b_l \qquad c_0, c_1, c_2, \dots, c_l.$$

It turns out that (the term a_{l+1} does not enter this formula, but will be used in (5.8)):

$$\sum_{j=1}^{l-1} a_j(b_{j+1} - b_j) = a_l b_l - a_1 b_1 - \sum_{j=1}^{l-1} (a_{j+1} - a_j) b_{j+1}. \tag{5.7}$$

If we apply this equality with $b_j = c_j - c_{j-1}$ $(j = 1, \dots, l)$, then we have

$$\sum_{j=1}^{l-1} a_j(c_{j+1} - 2c_j + c_{j-1}) = a_l(c_l - c_{l-1}) - a_1(c_1 - c_0) - \sum_{j=1}^{l-1} (a_{j+1} - a_j)(c_{j+1} - c_j).$$

Finally, by applying again (5.7) to the last term we get:

$$\sum_{j=1}^{l-1} a_j(c_{j+1} - 2c_j + c_{j-1}) = \gamma_{1,l} + \sum_{j=1}^{l-1} (a_{j+2} - 2a_{j+1} + a_j) c_{j+1}\,, \tag{5.8}$$

where

$$\gamma_{1,l} = [a_j(c_j - c_{j-1}) - (a_{j+1} - a_j)c_j]_{j=1}^{j=l}\ .$$

Theorem 5.5. *Let u be as in Theorem 4.7. Then*

$$u_t = -\psi''(0)(u_{xx})_{xx}\,, \tag{5.9}$$

in the sense that for every $T > 0$ and $\phi \in C_c^\infty(Q_T)$, with $Q_T = (0,1) \times (0,T)$, we have

$$\int_{Q_T} u(x,t)\phi_t(x,t)\,\mathrm{d}x\mathrm{d}t = \psi''(0) \int_{Q_T} u_{xx}(x,t)\phi_{xx}(x,t)\,\mathrm{d}x\mathrm{d}t. \tag{5.9$'$}$$

Proof. Let $T > 0$ be fixed and $\phi \in C_c^\infty(Q_T)$. Define

$$\phi_i^k = \phi(i\varepsilon, k\tau), \qquad \text{with } i, k \in \mathbb{Z}$$

$(\varepsilon = \varepsilon_n,\ \tau = \tau_n)$. Let $K_\tau = \lfloor T/\tau \rfloor$. By (5.7):

$$\begin{aligned} A_n :&= \sum_{i=0}^{N_\varepsilon} \sum_{k=0}^{K_\tau - 1} \varepsilon\tau (u_n^k)_i \frac{\phi_i^{k+1} - \phi_i^k}{\tau} \\ &= \varepsilon\tau \sum_{i=0}^{N_\varepsilon} \left[(u_n^{K_\tau})_i \phi_i^{K_\tau} - (u_n^0)_i \phi_i^0 - \sum_{k=0}^{K_\tau - 1} \frac{(u_n^{k+1})_i - (u_n^k)_i}{\tau}\ \phi_i^{k+1} \right] \end{aligned}$$

Since ϕ has compact support in Q_T, we have $\phi_i^0 = 0$ for every i and we can assume that $\phi_i^{K_\tau} = 0$ for every i if τ is sufficiently small. Furthermore, for any $m \in \mathbb{N}$ fixed, if ε and τ are sufficiently small, we can also suppose that, for every k,

$$\phi_i^k = 0 \quad \text{for every } i \text{ such that } 0 \le i \le m \text{ or } N_\varepsilon - m \le i \le N_\varepsilon.$$

Therefore, we can apply the optimality conditions stated in Proposition 4.2 for $i = 2, \dots, N_\varepsilon - 2$:

$$\begin{aligned} A_n = \frac{\tau}{\varepsilon} \sum_{i=2}^{N_\varepsilon-2} \sum_{k=0}^{K_\tau-1} \Big[\big(\varphi_\varepsilon'\big((\Delta_\varepsilon\, u_n^{k+1})_{i+1}\big) \\ -2\varphi_\varepsilon'\big((\Delta_\varepsilon\, u_n^{k+1})_i\big) + \varphi_\varepsilon'\big((\Delta_\varepsilon\, u_n^{k+1})_{i-1}\big)\big)\Big]\, \phi_i^{k+1}\,. \end{aligned} \tag{5.10}$$

By (5.8), taking the vanishing of ϕ on the boundary into account:

$$\begin{aligned} A_n &= \frac{\tau}{\varepsilon} \sum_{i=2}^{N_\varepsilon-2} \sum_{k=0}^{K_\tau-1} \varphi_\varepsilon'\big((\Delta_\varepsilon\, u_n^{k+1})_{i+1}\big)\big(\phi_{i+2}^{k+1} - 2\phi_{i+1}^{k+1} + \phi_i^{k+1}\big) \\ &= \varepsilon\tau \sum_{i=2}^{N_\varepsilon-2} \sum_{k=1}^{K_\tau} \varphi_\varepsilon'\big((\Delta_\varepsilon\, u_n^{k})_{i+1}\big) \frac{\phi_{i+2}^k - 2\phi_{i+1}^k + \phi_i^k}{\varepsilon^2}\,. \end{aligned}$$

If we introduce $\eta_n^{(2)}$ as the piecewise-constant function on Q_T defined by:

$$\eta_n^{(2)}(x,t) = \frac{\phi_{i+2}^k - 2\phi_{i+1}^k + \phi_i^k}{\varepsilon^2}, \qquad \text{for } x \in \varepsilon[i+1, i+2), \text{ and } k = \lfloor t/\tau \rfloor,$$

(recall that ϕ has compact support on Q_T), then we can write

$$A_n = \int_{Q_T} w_n(x,t)\eta_n^{(2)}(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

where w_n was defined in (5.1$'$). Since $(\eta_n^{(2)})$ converges uniformly to ϕ_{xx} on Q_T, the convergence result of Lemma 5.2 and Remark 5.3 allow to pass to the limit on the right-hand side, and get

$$\psi''(0) \int_{Q_T} u_{xx}(x,t)\phi_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, from the definition of A_n we have

$$A_n = \int_{Q_T} u_n(x,t)\eta_n^{(1)}(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

where:

$$\eta_n^{(1)}(x,t) = \frac{\phi_i^{k+1} - \phi_i^k}{\tau}, \qquad \text{for } x \in \varepsilon[i, i+1), \text{ and } k = \lfloor t/\tau \rfloor.$$

It turns out that $(\eta_n^{(1)})_n$ converges to ϕ_t uniformly in Q_T. Therefore, by the weak-L^2 convergence stated in Corollary 4.9, we get

$$\lim_{n\to\infty} A_n = \int_{Q_T} u(x,t)\phi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

thus proving the stated equality. □

Recall now that (Theorem 5.1) $(u_{xx})_{xx} \in L^2(Q_T)$. Then we have the following result.

Corollary 5.6. *Let u be as in Theorem 4.7. Then $u(x,\cdot) \in H^1(0,T)$ for a.e. $x \in (0,1)$, and the weak derivative is given by*

$$u_t = -\psi''(0)(u_{xx})_{xx}.$$

Proof. Let $\varphi \in C_c^\infty(0,1)$ and $\eta \in C_c^\infty(0,T)$. Apply (5.9′) with $\phi(x,t) = \varphi(x)\eta(t)$. Since $u_{xx}(\cdot,t)$ is in $H^2(0,1)$ for a.e. $t \geq 0$, we have

$$\begin{aligned}\int_0^1 \left(\int_0^T u(x,t)\eta'(t)\mathrm{d}t\right)\varphi(x)\mathrm{d}x &= \psi''(0)\int_0^T \left(\int_0^1 u_{xx}(x,t)\varphi''(x)\mathrm{d}x\right)\eta(t)\mathrm{d}t \\ &= \psi''(0)\int_0^T \left(\int_0^1 (u_{xx})_{xx}(x,t)\varphi(x)\mathrm{d}x\right)\eta(t)\mathrm{d}t \\ &= \psi''(0)\int_0^1 \left(\int_0^T (u_{xx})_{xx}(x,t)\eta(t)\mathrm{d}t\right)\varphi(x)\mathrm{d}x.\end{aligned}$$

By the arbitrariness of φ

$$\int_0^T u(x,t)\eta'(t)\mathrm{d}t = \psi''(0)\int_0^T (u_{xx})_{xx}(x,t)\eta(t)\mathrm{d}t \qquad \text{for a.e. } x \in (0,1).$$

The set of measure zero up to which this equality holds can be chosen independently of η since $C_c^\infty(0,T)$ contains a countable subset which is dense with respect to C^1-norm. We conclude that $u(x,\cdot) \in H^1(0,T)$ for a.e. $x \in (0,1)$.

In the following proposition we

□

6 Singular points: stability of the initial datum

The precise analysis of the evolution of a singular point of the initial datum appears to be a difficult task. In this section we give a simple sufficient condition on the initial datum which guarantees that a singular point (jump or crease) does not move, at least in a (small) right neighborhood of the initial time.

In this section u denotes a limit function as in Theorem 4.7. Let us specify the conditions on the initial datum; for reference convenience, we collect the settings in the following assumption, to be considered in addition to (*Hyp 1)* and (*Hyp 2)* introduced in Section 4:

> *Hyp 3)* Let $u^0\colon [0,1] \to \mathbb{R}$ be a piecewise-C^2 function, and let u^0_ε be the function defined in Remark 3.1 (hence the singularities of u^0_ε are on $\varepsilon\mathbb{Z}$). The initial datum $(u^0_\varepsilon)_i$ is the discretization of the piecewise-C^2 function u^0_ε (i.e. $(u^0_\varepsilon)_i = u^0_\varepsilon(i\varepsilon)$). According to the general notation, u^0_ε will also denote the piecewise-constant extension of this function.

Again, ε will denote ε_n and we will write u_n^0 instead of $u_{\varepsilon_n}^0$. It turns out that

$$u_n^0 \to u^0 \quad \text{in } L^2(0,1); \qquad F_n(u_n^0) \to F(u^0). \tag{6.1}$$

Indeed, the first one is a simple check, while the convergence of $F_\varepsilon(u_n^0)$ follows from (3.5). Thus, we can see (*Hyp 3)* as an assumption of "well-preparedness" of the initial datum: the discretization u_n^0 is the right one to recover the functional F in the limit.

Simple consequences of this are gathered in the following lemma.

Lemma 6.1. *a)* $u(\cdot,0) = u^0$.

b) For every $t \geq 0$ *we have* $F\big(u(\cdot,t)\big) \leq F(u^0)$*, where* F *is defined in* (3.5).

Proof. (a) follows from Corollary 4.9 and the convergence of (u_n^0) to u^0. As to (b), by Proposition 4.1, for every $t \geq 0$ and for every n

$$F_n(u_n(\cdot,t)) \leq F_n(u_n^0).$$

Thus, by (4.4), Theorem 3.2, Corollary 4.9 and the convergence of $F_n(u_n^0)$ we have

$$F(u(\cdot,t)) \leq \liminf_{n\to\infty} F_n(u_n^0) = F(u^0), \qquad \text{for every } t \geq 0. \quad \square$$

Remark 6.2. For every $T > 0$ the norms $\|u(\cdot,t)\|_{L^2(0,1)}$ are uniformly bounded for $t \in [0,T]$ (recall that $u \in C^{1/2}(0,T;L^2(0,1))$); by Remark 5.3 the norms $\|u_{xx}(\cdot,t)\|_{L^2(0,1)}$ are uniformly bounded, too. Therefore, the following standard interpolation result (the proof of which can be obtained, e.g., from that of Lemma 5.4 in [27]), implies a uniform bound for the H^2-norms $\|u(\cdot,t)\|_{H^2(I)}$ on any interval I where $u(\cdot,t)$ has the H^2-regularity.

Lemma 6.3 (Interpolation)**.** *Let* I *be an open interval with length* $\lambda > 0$*. Let* $1 \leq p < +\infty$*. Then for every* $v \in H^2(I)$ *and for every* $s \in I$

$$|v'(s)|^2 \leq C\left(\lambda \int_I |v''(r)|^2\,\mathrm{d}r + \lambda^{-3}\int_I |v(r)|^2\,\mathrm{d}r\right),$$

where $C = 144$.

Jump points The next result gives a sufficient condition to exclude that a jump point in the initial datum is immediately regularized. The argument takes advantage of inequality (4.3), which estimates the distance between the difference quotients at two consecutive time steps: if τ is sufficiently small with respect to ε, then the difference quotient at a jump point in the initial datum needs a positive time to fall below the threshold.

Theorem 6.4 (Jump point)**.** *Let* $\overline{x} \in (0,1)$*,* $J_1 = (0,\overline{x})$*, and* $J_2 = (\overline{x},1)$*. Let* $u^0\colon [0,1] \to \mathbb{R}$ *satisfy*

$$u^0\big|_{J_i} \in C^2(\overline{J}_i) \qquad (i = 1,2), \quad \overline{x} \in S(u^0).$$

Moreover, assume that

$$\frac{1}{2}\psi''(0)\int_0^1 |(u^0)''|^2\,\mathrm{d}x < 1, \qquad \tau_n = o(\varepsilon_n^4).$$

Then there exists $\sigma > 0$ *such that* $S\big(u(\cdot,t)\big) = \{\overline{x}\}$ *and* $S(u_x(\cdot,t)) = \emptyset$ *for every* $t \in [0,\sigma]$.

Proof. By assumption $F(u^0) < 3$. Hence, Lemma 6.1 (b) gives that

$$2\#S(u(\cdot,t)) + \#S(u_x(\cdot,t)) \leq 2.$$

Thus, the theorem is proved if we show that $\overline{x} \in S(u(\cdot,t))$ for t in a suitable right neighborhood of zero.

Let β be such that $F(u^0) < \beta < 3$; since $\lim_{z\to+\infty}\psi(z) = 1$, we can find $z_\beta \geq z_0$ such that $\beta/\psi(z_\beta) < 3$.

As in the proof of Lemma 6.1, for every $t \geq 0$ we have

$$\limsup_{n\to+\infty} F_n(u_n(\cdot,t)) \leq \lim_{n\to+\infty} F_n(u_n^0) = F(u^0) < \beta.$$

Therefore, we can assume that

$$F_n(u_n(\cdot,t)) \leq \beta \qquad \text{for every } n \in \mathbb{N} \text{ and } t \geq 0.$$

Define $I_{n,k}$ as the set of the singular points of $u_n^k = u_n(\cdot,t)$ with threshold $\zeta = z_\beta$ (here $\lfloor t/\tau \rfloor = k$; *for the sake of simplicity we often drop the subscript* n *from* ε_n *and* τ_n*);* i.e.

$$I_{n,k} = I_\varepsilon^{z_\beta}(u_n^k) = \{i \in \mathbb{Z} :\ 1 \leq i \leq N_\varepsilon - 1,\ |(\Delta_\varepsilon u_n^k)_i| \geq \frac{z_\beta}{\sqrt{\varepsilon}}\}. \tag{6.2}$$

By (3.7)

$$\#I_{n,k} \leq \frac{\beta}{\psi(z_\beta)} < 3.$$

Hence $I_{n,k}$ has at most two elements.

Let us now fix $n \in \mathbb{N}$ and let $i_0 = \lfloor \overline{x}/\varepsilon \rfloor$ (clearly $i_0 = i_0(n)$). Since we agreed that u^0 was left-continuous (recall *(Hyp 3)* and Remark 3.1), by the definition of u_ε^0 we deduce that $i_0, i_0+1 \in I_{n,0}$ for n sufficiently large. Define

$$k_n = \sup\{h \geq 0 :\ i_0, i_0+1 \in I_{n,k} \text{ for every } k \leq h\}.$$

a) We now prove that

$$\liminf_{n\to+\infty} k_n\tau_n = l \in (0,+\infty] \tag{6.3}$$

(if $k_n = +\infty$ then we agree that $k_n\tau_n = +\infty$). Let us argue by contradiction and assume that $\liminf_{n\to+\infty} k_n\tau_n = 0$. Therefore, there exists a subsequence of $(k_n\tau_n)_n$ which converges to 0: for the sake of simplicity we assume that $k_n\tau_n \to 0$ (in particular, $k_n \in \mathbb{N}$).

From the definition of k_n, we get that one of the indices i_0 or i_0+1 is not in I_{n,k_n+1}. For instance, let $i_0+1 \notin I_{n,k_n+1}$, i.e.

$$|(\Delta_\varepsilon u_n^{k_n+1})_{i_0+1}| < \frac{z_\beta}{\sqrt{\varepsilon}}.$$

Then, by (4.3) and the assumption $\tau = o(\varepsilon^4)$, we have

$$|(\Delta_\varepsilon u_n^{k_n+1})_{i_0+1} - (\Delta_\varepsilon u_n^{k_n})_{i_0+1}| \leq \frac{z_\beta}{\sqrt{\varepsilon}}$$

for every n sufficiently large. Thus

$$|(\Delta_\varepsilon u_n^{k_n})_{i_0+1}| < \frac{2z_\beta}{\sqrt{\varepsilon}}.$$

Let $v_n = u_n^{k_n} : [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$, and let $\tilde{v}_n$ be the usual quadratic smoothing of v_n according to Definition 3.3 and with respect to the threshold $\zeta = 2z_\beta$. Then $\tilde{v}_n$ has, at most, $i_0\varepsilon$ as a singular point: indeed,

$$\{i_0, i_0+1\} = I_{n,k_n} = I_\varepsilon^{z_\beta}(u_n^{k_n}) \supseteq I_\varepsilon^{2z_\beta}(u_n^{k_n}), \quad \text{and} \quad i_0+1 \notin I_\varepsilon^{2z_\beta}(u_n^{k_n}).$$

Thus

$$\tilde{v}_n \in H^1(0, \varepsilon(N_\varepsilon+1)) \cap H^2\big((0, \varepsilon(N_\varepsilon+1)) \setminus \{i_0\varepsilon\}\big).$$

Apply now the interpolation Lemma 6.3 to each of the intervals $(0, i_0\varepsilon)$ and $(i_0\varepsilon, 1)$, and note that $v_n = u_n(\cdot, t)$ with $t \geq 0$ such that $\lfloor t/\tau_n \rfloor = k_n$, for instance $t = t_n := k_n\tau_n$. Then, by Lemma 4.6 (applied with $\zeta = 2z_\beta$ as a threshold for the quadratic smoothing) the norms $\|\tilde{v}_n\|_{H^1}$ on $(0, \varepsilon(N_\varepsilon+1))$, are equibounded. In particular, $(\tilde{v}_n)_n$ is a bounded sequence in $H^1(0,1)$. Let us prove that the weak-H^1 limit v coincides with u^0, which contradicts that u^0 has a jump point.

Let $\eta > 0$ be fixed; let $n_\eta \in \mathbb{N}$ and $t_\eta > 0$ be such that (recall Proposition 4.5)

$$\|u_n(\cdot,t) - u_n(\cdot,s)\|_{L^2} \leq \eta \qquad \text{for every } n \geq n_\eta \text{ and } t, s \in [0, t_\eta].$$

Clearly, we can also assume that $\|u_n(\cdot,0) - u^0\|_{L^2} \leq \eta$ if $n \geq n_\eta$. Then $\|u_n(\cdot,t) - u^0\|_{L^2} \leq 2\eta$ for every $n \geq n_\eta$ and $t \in [0, t_\eta]$. By the assumption that $t_n := k_n\tau_n \to 0$ we can suppose that $t_n \leq t_\eta$, so that

$$\|u_n(\cdot, t_n) - u^0\|_{L^2} \leq 2\eta.$$

Therefore $v_n = u_n(\cdot, t_n) \to u^0$ in $L^2(0,1)$. Finally, we prove that $(v_n)_n$ and $(\tilde{v}_n)_n$ share the same L^2 limit, which yields that $v = u^0$.

We have (recall that $\overline{v}_n$ is the usual piecewise-affine interpolation of $(v_n)_i$)

$$\|\tilde{v}_n - v_n\|_{L^2(0,1)} \leq \|\tilde{v}_n - \overline{v}_n\|_{L^2(0,1)} + \|\overline{v}_n - v_n\|_{L^2(0,1)} \ ;$$

the first term on the right-hand side goes to zero with $\varepsilon = \varepsilon_n$ by Proposition 3.4 (*ii*). As to the second one, for every $i = 0, \ldots, N_\varepsilon$, on the interval $[\varepsilon i, \varepsilon(i+1)]$ it turns out that

$$|\overline{v}_n - v_n|^2 \leq |(v_n)_{i+1} - (v_n)_i|^2 = \varepsilon^2 |\overline{v}_n'|^2 = \varepsilon \int_{\varepsilon i}^{\varepsilon(i+1)} |\overline{v}_n'|^2 \, \mathrm{d}x,$$

from which

$$\|\overline{v}_n - v_n\|_{L^2(0,1)} \leq \varepsilon \|\overline{v}_n'\|_{L^2(0,\varepsilon(N_\varepsilon+1))}.$$

In order to conclude, it is enough to prove that $(\overline{v}_n')_n$ is bounded in L^2. We have (the norms are considered with respect to $(0, \varepsilon(N_\varepsilon+1))$):

$$\|\overline{v}_n'\|_{L^2} \leq \|\overline{v}_n' - \tilde{v}_n'\|_{L^2} + \|\tilde{v}_n'\|_{L^2} \ ;$$

the second term on the right-hand side is bounded since $(\tilde{v}_n)_n$ is bounded in H^1, as pointed out above. As to the first one, we have to consider only the intervals $[(i-\frac{1}{2})\varepsilon, (i+\frac{1}{2})\varepsilon]$ where $\tilde{v}_n$ is quadratic (otherwise $\overline{v}_n' - \tilde{v}_n' = 0$). Recall that on each half-interval $[\varepsilon(i-\frac{1}{2}), \varepsilon i]$ (and analogously on $[\varepsilon i, \varepsilon(i+\frac{1}{2})]$), $\overline{v}_n$ is the linear part of the Taylor expansion of $\tilde{v}_n$ with respect to the point $\varepsilon(i-\frac{1}{2})$; therefore, on this interval

$$|\tilde{v}_n' - \overline{v}_n'| = |(\tilde{v}_n - \overline{v}_n)'| \le \frac{\varepsilon}{2}|\tilde{v}_n''|.$$

Thus

$$\|\tilde{v}_n' - \overline{v}_n'\|_{L^2} \le \frac{1}{2}\varepsilon\|\tilde{v}_n''\|_{L^2}.$$

We conclude by applying Lemma 4.6, as already argued above.

b) Now we make use of (6.3) to get the existence of $\sigma > 0$ such that $\overline{x} \in S\big(u(\cdot,t)\big)$ for every $t \in [0,\sigma]$.

By (6.3) we can assume that there exists $\sigma > 0$ such that

$$k_n\tau_n \ge \sigma \qquad \text{for every } n \in \mathbb{N}.$$

It follows that for any fixed $t \in [0,\sigma]$ we have $k = k(t) := \lfloor t/\tau_n \rfloor \le k_n$. Hence,

$$I_{n,k_n} = I_\varepsilon^{z_\beta}\big(u_n(\cdot,t)\big) = \{i_0, i_0+1\}.$$

Let $\tilde{u}_n(\cdot,t)$ be the quadratic smoothing of u_n^k according to Definition 3.3, with respect to the singular set given by the threshold $\zeta = z_\beta$, i.e., the set $I_{n,k}$. It turns out that

$$\tilde{u}_n(\cdot,t) \in H^2(A_n), \qquad \text{with } A_n = (0, i_0\varepsilon) \cup ((i_0+1)\varepsilon, 1).$$

By Lemma 4.6 there exists $C > 0$ such that for every $n \in \mathbb{N}$

$$\|\tilde{u}_n(\cdot,t)\|_{L^2(0,1)} \le C, \qquad \|(\tilde{u}_n)_{xx}(\cdot,t)\|_{L^2(0,1)} \le C$$

Fix $\delta > 0$ and let n_δ be such that for $n \ge n_\delta$ we have $\varepsilon = \varepsilon_n < \delta$; then $i_0\varepsilon, (i_0+1)\varepsilon \in (\overline{x}-\delta, \overline{x}+\delta)$. We can now apply the interpolation Lemma 6.3 to each interval $(0, \overline{x}-\delta)$, $(\overline{x}+\delta, 1)$ and get the existence of $C' > 0$ such that

$$\|\tilde{u}_n(\cdot,t)\|_{H^1(A_\delta)} \le C', \qquad \text{with } A_\delta = (0, \overline{x}-\delta) \cup (\overline{x}+\delta, 1)$$

for every $n \ge n_\delta$. Note that C' can be chosen independently of δ and $t \in (0,\sigma]$; then, the weak-H^1 limit of $\tilde{u}_n(\cdot,t)$ on A_δ, which must coincide with $u(\cdot,t)$ (by Theorem 4.7) satisfies

$$\|u(\cdot,t)\|_{H^1(A_\delta)} \le C', \qquad \text{for every } \delta \text{ and } t \in (0,\sigma].$$

It follows that $u(\cdot,t) \in H^1\big((0,1)\setminus\{\overline{x}\}\big)$ and $\|u(\cdot,t)\|_{H^1((0,1)\setminus\{\overline{x}\})} \le C'$. Now, it is easy to see that for t in a right neighborhood of 0 the function $u(\cdot,t)$ has a jump in $\overline{x}$; otherwise we could find a sequence $t_j \to 0$ such that $u(\cdot,t_j) \in H^1(0,1)$ and $\|u(\cdot,t_j)\|_{H^1(0,1)}$ is equibounded: this implies that $u^0 = u(\cdot,0)$, which is the L^2 limit of $u(\cdot,t_j)$ ($u \in C^{1/2}\big([0,+\infty); L^2(0,1)\big)$), is also the weak-$H^1$ limit. Hence $u^0 \in H^1(0,1)$, which contradicts the fact that $\overline{x}$ is a jump point for u^0. $\square$

Crease points An analogous result holds for the discontinuity points of $(u^0)'$.

Theorem 6.5 (Crease point). *Let $\hat{x} \in (0,1)$, $J_1 = (0,\hat{x})$, and $J_2 = (\hat{x},1)$. Let $u^0 \colon [0,1] \to \mathbb{R}$ satisfy*

$$u^0\big|_{J_i} \in C^2(\overline{J}_i) \qquad (i=1,2), \quad u^0 \in H^1(0,1), \qquad \hat{x} \in S\big((u^0)'\big).$$

Assume that

$$\frac{1}{2}\psi''(0)\int_0^1 |(u^0)''|^2\,\mathrm{d}x < 1, \qquad \tau_n = o(\varepsilon_n^4).$$

Then there exists $\sigma > 0$ such that $S\big(u(\cdot,t)\big) = \emptyset$ and $S(u_x(\cdot,t)) = \{\hat{x}\}$ for every $t \in [0,\sigma]$.

Proof. Let us argue along the line of Theorem 6.4. By assumption $F(u^0) < 2$. Hence, Lemma 6.1 (b) gives that

$$2\#S(u(\cdot,t)) + \#S(u_x(\cdot,t)) \le 1.$$

Thus, the theorem is proved if we show that $\hat{x} \in S(u_x(\cdot,t))$ for t in a suitable right neighborhood of zero.

By assumption, $F(u^0) < 2$; then, as in the first part of the proof of Theorem 6.4, we fix $F(u^0) < \beta < 2$ and let $z_\beta \ge z_0$ be such that $\beta/\psi(z_\beta) < 2$, and assume that $F_n(u_n(\cdot,t)) \le \beta$ for every $n \in \mathbb{N}$ and $t \ge 0$. Moreover, let $I_{n,k}$ be as in (6.2). Since, by (3.7), $\#I_{n,k} \le \frac{\beta}{\psi(z_\beta)} < 2$, the set $I_{n,k}$ has at most one element.

For every fixed $n \in \mathbb{N}$, let $i_0 = \lfloor \hat{x}/\varepsilon \rfloor$ ($\varepsilon = \varepsilon_n$ and $i_0 = i_0(n)$); by the definition of u^0_ε we have $i_0 \in I_{n,0}$. Set $k_n = \sup\{h \ge 0 : \ i_0 \in I_{n,k} \text{ for every } k \le h\}$.

$a)$ Let us prove that

$$\liminf_{n\to+\infty} k_n\tau_n = l \in (0,+\infty]. \tag{6.4}$$

As in the corresponding part of the previous proof, we assume that $k_n\tau_n \to 0$ (in particular, $k_n \in \mathbb{N}$). By the definition of k_n we have $i_0 \notin I_{n,k_n+1}$, i.e. $|(\Delta_\varepsilon u_n^{k_n+1})_{i_0}| < \frac{z_\beta}{\sqrt{\varepsilon}}$. As above, by (4.3) and the assumption $t = o(\varepsilon^4)$, we have $|(\Delta_\varepsilon u_n^{k_n+1})_{i_0} - (\Delta_\varepsilon u_n^{k_n})_{i_0}| \le \frac{z_\beta}{\sqrt{\varepsilon}}$ for every n sufficiently large. Thus $|(\Delta_\varepsilon u_n^{k_n})_{i_0}| < \frac{2z_\beta}{\sqrt{\varepsilon}}$. Let $v_n = u_n^{k_n} \colon [0,1] \cap \varepsilon\mathbb{Z} \to \mathbb{R}$, and let $\tilde{v}_n$ be the usual quadratic smoothing of v_n according to Definition 3.3 and with respect to the threshold $\zeta = 2z_\beta$. Then $\tilde{v}_n$ has no singular point, i.e. it is in $H^2(0,1)$. Notice now that $v_n = u_n(\cdot,t)$, with $t > 0$ such that $\lfloor t/\tau_n \rfloor = k_n$ (for instance $t = t_n := k_n\tau_n$); by Lemma 4.6 (applied with $\zeta = 2z_\beta$ as a threshold for the quadratic smoothing) $\tilde{v}_n$ has bounded second derivatives. The interpolation Lemma 6.3 yields now that $(\tilde{v}_n)_n$ is a bounded sequence in $H^2(0,1)$: up to a subsequence it has a weak limit v. As in the proof of Theorem 6.4 we can get $v = u^0$, which contradicts that $\hat{x}$ is a jump point for $(u^0)'$.

$b)$ Parallel to step (b) of the previous proof, we can now assume the existence of $\sigma > 0$ such that $k_n\tau_n \ge \sigma$ for every $n \in \mathbb{N}$. Then the quadratic smoothing $\tilde{u}_n(\cdot,t)$ of $u_n(\cdot,t)$ for $t \in [0,\sigma]$ has only $\hat{x}_\varepsilon = i_0\varepsilon$ as a singular point, since $k = k(t) := \lfloor t/\tau_n \rfloor \le k_n$. Therefore, by Lemma 4.6 and by Lemma 6.3 applied to each of the intervals $(0,\hat{x}_\varepsilon)$ and $(\hat{x}_\varepsilon,1)$, there exists $C>0$ such that for every $\delta>0$ and n sufficiently large, and for every $t\in[0,\sigma]$

$$\|\tilde{u}_n(\cdot,t)\|_{H^1(0,1)} \le C, \quad \|\tilde{u}_n(\cdot,t)\|_{H^2(A_\delta)} \le C,$$

where $A_\delta=(0,\hat{x}-\delta)\cup(\hat{x}+\delta,1)$. It follows that $u(\cdot,t)\in H^1(0,1)\cap H^2(A_\delta)$ and, by the arbitrariness of δ, we have $u(\cdot,t)\in H^2\big((0,1)\setminus\{\hat{x}\}\big)$, too, with a uniform bound for the H^2 norm on $(0,1)\setminus\{\hat{x}\}$. This allows to conclude that $\hat{x}$ has to be a jump point for $u_x(\cdot,t)$ for every t sufficiently small: indeed, otherwise we could find a sequence $t_j\to 0$ such that $u(\cdot,t_j)\in H^2(0,1)$, and the uniform bound on the H^2 norm would imply that the L^2 limit $u(\cdot,0)=u^0$ (hence the weak-H^2 limit) is in $H^2(0,1)$, which contradicts the assumption about $\hat{x}$. □

7 Differential conditions on singular points

In this section we show that the second and third (spatial) derivatives of the evolution function $u(\cdot,t)$ vanish on the jump points and on the endpoints of the interval; on crease points we can prove that the second derivative vanishes. This results will be used in the next section to prove the uniqueness of the evolution.

We expect that a crease point for $u(\cdot,t)$ arises as a limit of a sequence $x_h=i_h\varepsilon_{n_h}$ of points where the difference between the right and left derivatives of the discrete approximations is bounded away from zero. Lemma 7.3 make this precise.

As to a jump point in the limit function $u(\cdot,t)$, we expect (see also Figure 2) that it emerges from the convergence of two points where the second-difference quotient is above the threshold. While we cannot guarantee that these quotients exceed $1/\varepsilon^2$ (as in the example of Figure 2), we prove here that we can approximate a jump point $\overline{x}$ by two distinct points whose difference quotients exceed $1/\varepsilon$, i.e., they belong to the singular set introduced in the following definition.

Definition 7.1. *Let $u=(u_i)_i$ be a real function on $[0,1]\cap\varepsilon\mathbb{Z}$. Let*

$$I_\varepsilon^+(u)=\{i\in\mathbb{Z}: 1\le i\le N_\varepsilon-1,\ |(\Delta_\varepsilon u)_i|\ge 1/\varepsilon\}.$$

Before addressing the problem of the boundary conditions on singular points, we need some preliminary results. Notice that if $i\notin I_\varepsilon^+(u)$ then

$$\Big|\frac{u_{i+1}-u_i}{\varepsilon}-\frac{u_i-u_{i-1}}{\varepsilon}\Big| = |\overline{u}'_+(i\varepsilon)-\overline{u}'_-(i\varepsilon)|\le 1.$$

Consider an interval $[\alpha,\beta]$, and assume that it contains no points of $I_\varepsilon^+(u_n(\cdot,t))$. Since the number of singular points of $\tilde{u}_n(\cdot,t)$ (i.e. in $I_\varepsilon^{z_0}(u_n(\cdot,t))$ is uniformly bounded with respect to n (and t), we can estimate $\tilde{u}'_n(x,t)-\tilde{u}'_n(y,t)$, for any $x,y\in[\alpha,\beta]$, by $\int_\alpha^\beta|\tilde{u}''_n(\xi,t)|\,\mathrm{d}\xi$ up to an additive constant which counts the singular points in the interval. More precisely, we have the following result.

Lemma 7.2. *Let $t\ge 0$ and let $(\alpha,\beta)\subseteq[0,1]$ be an interval which contains no points of $I_\varepsilon^+\big(u_n(\cdot,t)\big)$. Then, there exists a constant C, which depends only on the constant M in* (Hyp 2)*, such that:*

a) for every $x, y \in [\alpha, \beta]$

$$|\tilde{u}_n'(x,t) - \tilde{u}_n'(y,t)| \le C + \int_\alpha^\beta |\tilde{u}_n''(\xi,t)|\,\mathrm{d}\xi$$

(if x or y are singular points and $x < y$ we consider the right derivative in x and the left one in y);

b) for every $x \in [\alpha, \beta]$

$$|\tilde{u}_n'(x,t)| \le C\big(1 + \|\tilde{u}_n''(\cdot,t)\|_{L^2(\alpha,\beta)} + (\beta-\alpha)^{-3/2}\|\tilde{u}_n(\cdot,t)\|_{L^2(\alpha,\beta)}\big).$$

Proof. a) Let $x, y \in [\alpha, \beta]$, with $x < y$; denote by $x_1 < x_2 < \ldots < x_N$ the elements of $I_\varepsilon^{z_0}\big(u_n(\cdot,t)\big)$ in the interval (x, y). By assumption, $x_j \notin I_\varepsilon^+\big(u_n(\cdot,t)\big)$; thus, if we set $v = \tilde{u}_n(\cdot,t)$,

$$|v_+'(x_j) - v_-'(x_j)| \le 1.$$

Consider the case $N = 1$ and let $x < x_1 < y$; since $v \in H^2\big((\alpha, x_1) \cup (x_1, \beta)\big)$, we have

$$\begin{aligned}|v'(x) - v'(y)| &\le |v'(x) - v_-'(x_1)| + |v_-'(x_1) - v_+'(x_1)| + |v_+'(x_1) - v'(y)| \\ &\le \int_x^{x_1} |v''(\xi)|\,\mathrm{d}\xi + 1 + \int_{x_1}^y |v''(\xi)|\,\mathrm{d}\xi \le 1 + \int_\alpha^\beta |v''(\xi)|\,\mathrm{d}\xi.\end{aligned}$$

This inequality clearly holds if $\alpha \le x < y < x_1$, too, or $x_1 < x < y \le \beta$; moreover, if x (or y) is a singular point of v, by $v'(x)$ ($v'(y)$) we intend the appropriate unilateral derivative. In the general case ($N \ge 1$) it turns out that

$$|v'(x) - v'(y)| \le N + \int_\alpha^\beta |v''(\xi)|\,\mathrm{d}\xi. \tag{7.1}$$

By (3.7) and (4.5), the value N can be majorized through the constant M in *(Hyp 2)* only.

b) The interval $[\alpha, \beta]$ splits into the $(N+1)$ intervals $[\alpha, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_N, \beta]$; therefore, the length of one of them, say J, is at least $(\beta-\alpha)/(N+1)$. In (7.1) let $y \in J$ and $x \in [\alpha, \beta]$; then ($\beta - \alpha \le 1$)

$$|v'(x)| \le |v'(y)| + N + \int_\alpha^\beta |v''(\xi)|\,\mathrm{d}\xi \le |v'(y)| + N + \|v''\|_{L^2(\alpha,\beta)}$$

By Lemma 6.3 applied to the interval J

$$|v'(y)| \le 12\big(\lambda^{1/2}\|v''\|_{L^2(J)} + \lambda^{-3/2}\|v\|_{L^2(J)}\big),$$

where $\lambda = |J|$. Since $(\beta-\alpha)/(N+1) \le \lambda \le 1$ we get the inequality in (b). $\square$

Let us now address the problem of the boundary conditions on singular points. Let w_n and ω_n be as in (5.1), (5.1′) and (5.4). Apply Fatou's Lemma to the estimate of Remark 5.4 (b); then

$$\int_\sigma^T \liminf_{n\to+\infty}\Big(\int_0^1 |(\omega_n)_{xx}|^2(x,t)\,\mathrm{d}x\Big)\mathrm{d}t \le 2M.$$

Thus, for a.e. $t \in [0,T]$

$$\liminf_{n\to+\infty} \int_0^1 |(\omega_n)_{xx}|^2(x,t)\,\mathrm{d}x < +\infty$$

(notice that the exceptional set of t may vary if we consider a subsequence).

Fix such a t: then there exists a subsequence $(\omega_{n_h})_h$ (possibly depending on t) with

$$\int_0^1 |(\omega_{n_h})_{xx}|^2(x,t)\,\mathrm{d}x \le C$$

for a suitable constant C. By Remark 5.4 (a) and the interpolation results for Sobolev spaces (see, e.g., [27], Theorem 5.2), we get that $(\omega_{n_h})_{xx}$ is bounded in $H^2(0,1)$. Therefore, we can assume that $(\omega_{n_h})_{xx}(\cdot,t)$ weakly converges in $H^2(0,1)$. From Remark 5.4 (c) we deduce that the limit is $\psi''(0)u_{xx}(\cdot,t)$:

$$(\omega_{n_h})_{xx}(\cdot,t) \rightharpoonup \psi''(0)u_{xx}(\cdot,t) \quad \text{weakly in } H^2(0,1). \tag{7.2}$$

Recall that on each interval $\varepsilon[i-\frac{1}{2}, i+\frac{1}{2}]$

$$(\omega_n)_{xx}(\cdot,t) = (\Delta_\varepsilon w_n^k)_i, \qquad k = \lfloor t/\tau \rfloor.$$

Then, the previous L^2 estimate can be written as

$$\sum_i \varepsilon|(\Delta_\varepsilon w_{n_k}^k)_i|^2 \le C.$$

It follows that $|(\Delta_\varepsilon w_{n_h}^k)_i| \le \sqrt{C/\varepsilon}$, and, by (3.11),

$$|\omega_{n_h}(\cdot,t) - w_{n_h}^k| \le c\varepsilon\sqrt{\varepsilon} \qquad \text{on } [0,1], \tag{7.3}$$

for a suitable constant c.

Now we deal with crease and jump points separately.

Crease points Here, we assume that $\hat{x}$ is a singular point of $u_x(\cdot,t)$ for every t in a right neighborhood of zero. A sufficient condition for this setting is given in Theorem 6.5.

Lemma 7.3. *Let $\hat{x} \in S\big(u_x(\cdot,t)\big)$ for every $t \in [0,\sigma]$. Let $\omega_{n_h}(\cdot,t)$ be as above. We can extract a further subsequence (not relabelled) in such a way that there exists $x_h = i_h\varepsilon$ (with $\varepsilon = \varepsilon_{n_h}$) with*

$$x_h \to \hat{x}, \qquad \liminf_{h\to+\infty} \varepsilon\big(\Delta_\varepsilon u_{n_h}(\cdot,t)\big)_{i_h} > 0.$$

Proof. Let us consider two cases.

$a)$ First assume that for every $\delta > 0$ and $\overline{n} \in \mathbb{N}$ there exists $n \ge \overline{n}$ and $i \in I_\varepsilon^+\big(u_n(\cdot,t)\big)$ such that $|i\varepsilon - \hat{x}| \le \delta$ (here $\varepsilon = \varepsilon_n$). By choosing $\delta = 1/k$ (with $k \in \mathbb{N}$), we get a sequence $x_k = i_k\varepsilon_{n_k}$ of singular points which converge to $\hat{x}$ and satisfy

$$\varepsilon\big(\Delta_\varepsilon u_{n_k}(\cdot,t)\big)_{i_k} \ge 1 \qquad (\varepsilon = \varepsilon_{n_k})$$

$b)$ If the assumption in (a) does not hold, then there exists a neighbourhood U of $\hat{x}$ such that for every n sufficiently large $U \cap I_\varepsilon^+\big(u_n(\cdot,t)\big) = \emptyset$. Then, by

Lemma 7.2 (and the estimates of Lemma 4.6), the sequence $v_n := \tilde{u}_n'(\cdot, t)$ is an equibounded sequence of piecewise-H^1 functions on U. We can apply Theorem 2.2: there exists a subsequence (possibly depending on t) $(v_{n_k})_k$ and a piecewise-H^1 function v such that

$$\begin{aligned} &v_{n_k} \to v \quad \text{strongly in } L^2(0,1), \\ &D^j v_{n_k} \rightharpoonup D^j v \quad \text{weakly}^* \text{ in the sense of measures.} \end{aligned}$$

Since the number of jump points of v_n is equibounded, possibly passing to a further subsequence we can assume that for every k they stay in an arbitrarily small compact neighborhood K of a finite set. On each interval of $U \setminus K$, the $L^1(0,1)$-convergence of $\tilde{u}_n(\cdot, t)$ to $u(\cdot, t)$ (see Theorem 4.7) and of $v_{n_k} = \tilde{u}_{n_k}'(\cdot, t)$ to v (see above) yields that $v = u_x(\cdot, t)$. We conclude that the equality holds on U up to a finite number of points.

The lower semicontinuity of the total variation (see [25], Theorem 1.59) gives

$$0 < \alpha := |(u_x)_+(\hat{x}, t) - (u_x)_-(\hat{x}, t)| = |D^j v|(U) \le \liminf_{k \to +\infty} |D^j v_{n_k}|(U).$$

If $\hat{x}$ was not in S, we could choose U in such a way that for k sufficiently large v_{n_k} has no jump point in U: the right-hand side of the previous inequality would vanish, giving a contradiction. Select now U satisfying $U \cap S = \{\hat{x}\}$; for every k we can choose a jump point $x_k = i_k \varepsilon$ of v_{n_k} converging to $\hat{x}$ and such that

$$\varepsilon\big(\Delta_\varepsilon u_{n_k}(\cdot, t)\big)_{i_k} = |(v_{n_k}')_+(x_k) - (v_{n_k}')_-(x_k)| \ge \frac{1}{2}\alpha/N,$$

where N is an upper bound for the number of singular points. $\square$

We are now in a position to prove the following result.

Proposition 7.4. *Assume that $\hat{x}$ is a singular point of $u_x(\cdot, t)$ for every t in $[0, \sigma]$. Then $u_{xx}(\hat{x}, t) = 0$.*

Proof. Let $x_h = i_h \varepsilon$ be as in the previous lemma. For h sufficiently large it turns out that

$$|(\Delta_\varepsilon u_{n_h})_{i_h}| \ge \frac{\alpha}{\varepsilon}, \qquad [\varepsilon = \varepsilon_{n_h}],$$

for a suitable constant $\alpha > 0$. We can assume that $\alpha/\varepsilon \ge z_0/\sqrt{\varepsilon}$; hence

$$|(w_{n_h}^k)_{i_h}| = |\varphi_\varepsilon'\big((\Delta_\varepsilon u_{n_h})_{i_h}\big)| \le \frac{z_0}{\sqrt{\varepsilon}} \psi'(\frac{z_0}{\sqrt{\varepsilon}}).$$

This, together with (7.3), entails that

$$|\omega_{n_h}(x_h, t)| \le |\omega_{n_h}(x_h, t) - (w_{n_h}^k)_{i_h}| + |(w_{n_h}^k)_{i_h}| \le c\big[\varepsilon\sqrt{\varepsilon} + \frac{z_0}{\sqrt{\varepsilon}} \psi'(\frac{z_0}{\sqrt{\varepsilon}})\big].$$

The weak convergence of ω_{n_h} in $H^2(0,1)$ (see (7.2)) implies, in particular, the uniform convergence on $(0,1)$; since $z\psi'(z) \to 0$ as $z \to +\infty$, we conclude that $u_{xx}(\hat{x}, t) = 0$. $\square$

Jump points Let us now address the problem of the approximation of a jump point through a pair of singular points in I_ε^+ (see Definition 7.1).

Let u^0 be as in *(Hyp 3)* of Section 6; assume that $\overline{x} \in S\big(u(\cdot,t)\big)$ for every $t \in [0,\sigma]$ (for instance this is the case of Theorem 6.4).

Lemma 7.5. *There exists a subsequence (u_{n_h}) of (u_n) satisfying the following property: for every $t \in [0,\sigma]$ and for every $h \in \mathbb{N}$ there exist two distinct indices $i_1, i_2 \in I_\varepsilon^+\big(u_{n_h}(\cdot,t)\big)$ such that $|i_1\varepsilon_{n_h} - \overline{x}| \le 1/h$ and $|i_2\varepsilon_{n_h} - \overline{x}| \le 1/h$.*

Proof. The stated property can be proved by showing that the following assumption gives a contradiction: there exists $\delta > 0$ and $\overline{n} \in \mathbb{N}$ such that for every $n \ge \overline{n}$ there exists $t_n \in [0,\sigma]$ with the property that the set $[\overline{x}-\delta, \overline{x}+\delta]$ has at most one point of the form $i\varepsilon_n$ with $i \in I_\varepsilon^+\big(u_n(\cdot,t_n)\big)$.

Assume this is true; then, for every $n \ge \overline{n}$ we can find at most one point $\xi_n = i_n\varepsilon_n \in (\varepsilon_n\mathbb{Z}) \cap (\overline{x}-\delta, \overline{x}+\delta)$ with $i_n \in I_\varepsilon^+\big(u_n(\cdot,t_n)\big)$. Let us prove that

(7.4) a subsequence of $\big(\tilde{u}_n(\cdot,t_n)\big)$ is bounded in the H^1-norm on a suitable neighbourhood U of $\overline{x}$.

Up to subsequence we can assume that (ξ_n) has a limit $\overline{\xi} \in [\overline{x}-\delta, \overline{x}+\delta]$. Let us first consider the case $\overline{\xi} > \overline{x}$ (the case $\overline{\xi} < \overline{x}$ follows by the same argument). Fix $\gamma \in (\overline{x}, \overline{\xi})$: we can suppose that $\xi_n > \gamma$ for every n. Apply now Lemma 7.2 to the interval $[\overline{x}-\delta, \gamma]$ with $t = t_n$: we get the uniform boundedness of $\tilde{u}_n'(\cdot,t_n)$ (the bound depends on δ).

Consider now the case $\overline{\xi} = \overline{x}$. We can apply Lemma 7.2 both to the interval $[\overline{x}-\delta, \xi_n]$ and to the interval $[\xi_n, \overline{x}+\delta]$ with $t = t_n$; again, since the length of the intervals are bounded from below, we get a uniform estimate for $\tilde{u}_n'(\cdot,t_n)$ on both intervals.

Thus, we have proved (7.4). From this we deduce that, up to a subsequence, $\tilde{u}_n(\cdot,t_n)$ has a weak limit v in $H^1(U)$. Let us show that $v = u(\cdot,\overline{t})$ for some $\overline{t} \in [0,\sigma]$.

We can assume that $t_n \to \overline{t}$, for some $\overline{t} \in [0,\sigma]$. We have:

$$\|u_n(\cdot,t_n) - u(\cdot,\overline{t})\|_{L^1(U)} \le \|u_n(\cdot,t_n) - u_n(\cdot,\overline{t})\|_{L^1(U)} + \|u_n(\cdot,\overline{t}) - u(\cdot,\overline{t})\|_{L^1(U)}\,.$$

The first term on the right-hand side tends to zero by Proposition 4.5, while for the second one we apply Corollary 4.9. Therefore

$$u_n(\cdot,t_n) \to u(\cdot,\overline{t}) \qquad \text{in } L^1(U).$$

Thus, it is enough to show that $u_n(\cdot,t_n) - \tilde{u}_n(\cdot,t_n) \to 0$ in $L^1(U)$. Taking Proposition 3.4 (ii) into account, this is implied by $u_n(\cdot,t_n) - \overline{u}_n(\cdot,t_n) \to 0$ in $L^1(U)$. Notice now that on any interval $[i\varepsilon, (i+1)\varepsilon)$ we have

$$\begin{aligned}|u_n(x,t_n) - \overline{u}_n(x,t_n)| &\le |u_n((i+1)\varepsilon,t_n) - u_n(i\varepsilon,t_n)| \\ &= \varepsilon\ |\tilde{u}_n'\big((i+\frac{1}{2})\varepsilon,t_n\big)|\end{aligned}$$

We have proved above that $\tilde{u}_n'(.,t_n)$ is bounded on U independently of n; therefore, $u_n(x,t_n) - \overline{u}_n(x,t_n) \to 0$ uniformly on U. Thus, $v = u(\cdot,\overline{t})$ on U.

We conclude that $u(\cdot,\overline{t})$ has no jump point in $\overline{x}$, against the assumption. $\square$

In a perfectly analogous way to Proposition 7.4, we can now prove the vanishing of the second derivative on jump points. As to the third derivative, we need a stronger assumption on the function ψ.

Proposition 7.6. *Assume that $\overline{x} \in S\big(u(\cdot,t)\big)$ for every $t \in [0,\sigma]$. Then*

a) $u_{xx}(\overline{x},t) = 0$.

b) If

$$\limsup_{z\to+\infty} z^4\psi'(z) < +\infty. \tag{7.5}$$

then $(u_{xx})_x(\overline{x},t) = 0$ *for every* $t \in [0,\sigma]$.

Proof. Let u_{n_h} and the indices i_1, i_2 be as in the previous proposition. Set $x^1_{n_h} = i_1\varepsilon$ and $x^2_{n_h} = i_2\varepsilon$. We can follow the proof of Proposition 7.4 with $x^1_{n_h}$ and $x^2_{n_h}$ in place of x_{n_h} (and $\alpha = 1$). Then ($j = 1,2$):

$$|\omega_{n_h}(x^j_{n_h},t)| \le c\big[\varepsilon\sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\psi'(\frac{1}{\sqrt{\varepsilon}})\big].$$

The uniform convergence on $(0,1)$ implies now that $u_{xx}(\overline{x},t) = 0$.

Assume now that ψ satisfies (7.5). Then

$$|\omega_{n_h}(x^j_{n_h},t)| \le c\varepsilon\sqrt{\varepsilon} \qquad (j = 1,2).$$

By Lagrange's Theorem, there exists $\xi_h \in (x^1_{n_h}, x^2_{n_h})$ with

$$|\omega'_{n_h}(\xi_h,t)| \le c\sqrt{\varepsilon}$$

(notice that $|i_1 - i_2| \ge 1$, hence $|x^1_{n_h} - x^2_{n_h}| \ge \varepsilon$). Clearly, $\xi_h \to \overline{x}$. By (7.2), the sequence $\omega'_{n_h}(\cdot,t)$ weakly converges in $H^1(0,1)$ to $u_{xxx}(\cdot,t)$; hence we have the uniform convergence, and we conclude that $(u_{xx})_x(\overline{x},t) = 0$. $\square$

8 Uniqueness of the minimizing movement

For the sake of simplicity, here we assume that the initial datum presents *only one singular point* (jump or crease). We also assume that this point remains singular in a time interval $[0,\sigma]$. Recall that sufficient conditions for such a behaviour are given in Theorem 6.4 and Theorem 6.5.

Let u be a limit function as in Theorem 4.7. Assume that the initial datum u^0_ε satisfies *(Hyp 1)* and *(Hyp 2)* of Section 4, and *(Hyp 3)* of Section 6, i.e.

Hyp 1) There exists $M_0 > 0$ such that $\|u^0_\varepsilon\|_{L^2} \le M_0$ for any $\varepsilon > 0$.

Hyp 2) There exists $M > 0$ such that $F_\varepsilon(u^0_\varepsilon) \le M$ for any $\varepsilon > 0$.

Hyp 3) Let $u^0 \colon [0,1] \to \mathbb{R}$ be a piecewise-C^2 function, and let u^0_ε be the function defined in Remark 3.1 (hence the singularities of u^0_ε are on $\varepsilon\mathbb{Z}$). The initial datum $(u^0_\varepsilon)_i$ is the discretization of the piecewise-C^2 function u^0_ε (i.e. $(u^0_\varepsilon)_i = u^0_\varepsilon(i\varepsilon)$).

We recall that

- $u(\cdot,t) \in \mathscr{H}^2(0,1)$ for every $t \ge 0$ (Theorem 4.7);

- $u_{xx} \in L^2(0,T;H_0^2(0,1))$ for every $T>0$ (Theorem 5.1);
- $u(x,\cdot) \in H^1(0,T)$ for a.e. $x \in (0,1)$, and the weak derivative is given by $u_t = -\psi''(0)(u_{xx})_{xx}$ (Corollary 5.6).

Moreover, $u(\cdot,t)$ satisfies suitable differential conditions on jump and crease points (see Propositions 7.6 and 7.4).

Within this frame, the theorem below guarantees the uniqueness of the minimizing movement u.

Theorem 8.1. *Let u be a limit function as in Theorem 4.7. Assume that there exists $\sigma > 0$ such that either of the following assumptions holds:*

a) $S\big(u(\cdot,t)\big) = \{\overline{x}\}$ and $S\big(u_x(\cdot,t)\big) = \emptyset$ for every $t \in [0,\sigma]$; moreover, ψ satisfies (7.5), i.e.

$$\limsup_{z\to+\infty} z^4\psi'(z) < +\infty.$$

b) $S\big(u(\cdot,t)\big) = \emptyset$ and $S\big(u_x(\cdot,t)\big) = \{\hat{x}\}$ for every $t \in [0,\sigma]$.

Then the function u is uniquely determined.

Proof. *a*) Consider the interval $(0,\overline{x})$ (the same proof holds for $(\overline{x},1)$). Notice that $(u_{xx})_{xx}$, hence u_t, is a square integrable function on Q_T for every $T>0$; in particular, $u\,u_t \in L^1(Q_T)$.

Since $\overline{x}$ is the only singular point, $u(\cdot,t) \in H^2(0,\overline{x})$ for every $t \in (0,\sigma)$ (actually, it belongs to H^4). As a consequence, we can integrate by parts in the integral below, for a.e. $t \in (0,\sigma)$:

$$\begin{aligned}\frac{1}{\psi''(0)}\int_0^{\overline{x}} u(x,t)u_t(x,t)\,\mathrm{d}t &= -\int_0^{\overline{x}} u(x,t)(u_{xx})_{xx}(x,t)\,\mathrm{d}x\\ &= \int_0^{\overline{x}} u_x(x,t)(u_{xx})_x(x,t)\,\mathrm{d}x - u(u_{xx})_x\Big|_0^{\overline{x}}\\ &= -\int_0^{\overline{x}} u_{xx}(x,t)u_{xx}(x,t)\,\mathrm{d}x + u_xu_{xx}\Big|_0^{\overline{x}}\\ &= -\int_0^{\overline{x}} \big(u_{xx}(x,t)\big)^2\,\mathrm{d}x \le 0.\end{aligned}$$

Here, we have employed the vanishing of the second and third derivatives on the boundary of $(0,1)$ (since $u_{xx}(\cdot,t) \in H_0^2(0,1)$) and on the jump point $\overline{x}$ (Proposition 7.6).

Now, fix $s \in (0,\sigma)$ and integrate the left-hand side with respect to $t \in (0,s)$; we can apply Fubini's Theorem since $u\,u_t \in L^1(Q_T)$; hence

$$\begin{aligned}0 \ge \int_0^s \mathrm{d}t \int_0^{\overline{x}} u(x,t)u_t(x,t)\,\mathrm{d}x &= \int_0^{\overline{x}} \mathrm{d}x \int_0^s u(x,t)u_t(x,t)\,\mathrm{d}t\\ = \frac{1}{2}\int_0^{\overline{x}} \mathrm{d}x \int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\big(u(x,t)\big)^2\,\mathrm{d}t &= \frac{1}{2}\int_0^{\overline{x}} \big[u^2(x,s) - u^2(x,0)\big]\,\mathrm{d}x\end{aligned}$$

(indeed, $u^2(x,\cdot) \in H^1(0,T)$ and its derivative is $2uu_t$). We have proved that

$$\|u(\cdot,s)\|_{L^2(0,\overline{x})} \le \|u(\cdot,0)\|_{L^2(0,\overline{x})} \quad \text{for every } s \in [0,\sigma].$$

This implies the uniqueness of u in a standard way: if u_1 and u_2 were two functions satisfying the same assumptions displayed above, then their difference $u = u_1 - u_2$ would have the same properties (equation $u_t = -u_{xxxx}$ on $(0, \overline{x})$ is linear), but $u(x, 0) \equiv 0$; therefore, $\|u(\cdot, s)\|_{L^2(0,\overline{x})} = 0$, too, i.e. $u \equiv 0$.

b) We argue on the line of the proof in (*a*); however, since $\hat{x}$ is the only singularity, it turns out that $u(\cdot, t) \in H^1(0, 1)$ and we can integrate by parts on the whole $(0, 1)$:

$$\begin{aligned}\frac{1}{\psi''(0)}\int_0^1 u(x,t)u_t(x,t)\,\mathrm{d}t &= -\int_0^1 u(x,t)(u_{xx})_{xx}(x,t)\,\mathrm{d}x \\ &= \int_0^1 u_x(x,t)(u_{xx})_x(x,t)\,\mathrm{d}x - u(u_{xx})_x\big|_0^1 \\ &= \int_0^1 u_x(x,t)(u_{xx})_x(x,t)\,\mathrm{d}x\end{aligned}$$

(the third derivative vanishes on 0 and 1). Now, we split the integral on $(0, \hat{x})$ and $(\hat{x}, 1)$; on each of them $u(\cdot, t)$ is H^1 and we can integrate by parts. For instance, on $(0, \hat{x})$:

$$\begin{aligned}\int_0^{\hat{x}} u_x(x,t)(u_{xx})_x(x,t)\,\mathrm{d}x &= -\int_0^{\hat{x}} \big(u_{xx}(x,t)\big)^2\,\mathrm{d}x + u_x u_{xx}\big|_0^{\hat{x}} \\ &= -\int_0^{\hat{x}} \big(u_{xx}(x,t)\big)^2\,\mathrm{d}x \le 0.\end{aligned}$$

Here we have applied the null boundary conditions of the second derivative on 0 and $\hat{x}$ (Proposition 7.4). We now conclude in the same way as in the case of a jump point. □

Though we do not tackle the characterization of the minimizing movements for the functional F, if we assume that the singular points remain fixed during the evolution, the problem reduces to a convex one. Indeed, consider, for instance, an interval (a, b) between two jump points, and define

$$\mathscr{F}(v) = \begin{cases} \frac{1}{2}\psi''(0)\int_a^b (v''(x))^2\,\mathrm{d}x & \text{if } v \in H^2(a,b), \\ +\infty & \text{otherwise in } L^2(a,b). \end{cases}$$

As showed in [6], the minimizing movement from u^0 is given by the absolutely continuous function $w\colon [0, +\infty) \to L^2(a, b)$, with square integrable derivative, which solves the problem

$$\begin{cases} w'(t) \in -\partial\mathscr{F}(w(t)) \\ w(0) = u^0. \end{cases}$$

Let u be the evolution function considered above (Theorem 8.1); it solves

$$\begin{cases} u_t(x,t) = -\psi''(0)u_{xxxx}(x,t) & \text{in } (a,b)\times[0,\sigma], \\ u(\cdot,0) = u^0; \quad u_{xx}(x,t) = u_{xxx}(x,t) = 0 & \text{if } x = a \text{ and } x = b. \end{cases} \tag{8.1}$$

Notice that for every $v \in H^2(a,b)$ and $t \in [0,\sigma]$

$$\begin{aligned}\mathscr{F}(v) &\geq \mathscr{F}\big(u(\cdot,t)\big) + \psi''(0)\int_a^b u_{xx}(x,t)\big(v''(x) - u_{xx}(x,t)\big)\,\mathrm{d}x \\ &= \mathscr{F}\big(u(\cdot,t)\big) + L\big(v - u(\cdot,t)\big).\end{aligned}$$

Because of the boundary conditions satisfied by $u(\cdot,t)$, we can integrate by parts and express the linear functional L as follows

$$L(v) = \psi''(0)\int_a^b u_{xxxx}(x,t)(v(x) - u(x,t))\,\mathrm{d}t$$

(in case of a crease point we have to proceed by splitting the integral as in the proof of Theorem 8.1). Hence $\psi''(0)u_{xxxx}(\cdot,t)$ is in $\partial\mathscr{F}\big(u(\cdot,t)\big)$. We conclude that, in this particular case, the solution of (8.1) gives the evolution of F.

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