A General Analytical Framework for Fast Solving Nonlinear MPC Problems in the Linear Koopman Space

A General Analytical Framework for Fast Solving Nonlinear MPC Problems in the Linear Koopman Space

Lorenzo Calogero, Mattia Boggio, Carlo Novara, and Alessandro Rizzo

Abstract— **The Koopman operator stands as a powerful framework to transform nonlinear dynamical systems into equivalent linear ones within a lifted state space. Its application can be extended to nonlinear optimal control problems, enabling their efficient solution in the linear Koopman space. However, a systematic methodology to analytically derive a suitable basis of Koopman observables and handle the operator infinite-dimensionality is still lacking. In this paper, we propose a comprehensive analytical framework to efficiently solve Nonlinear Model Predictive Control (NMPC) problems in the linear Koopman space. We present a general procedure to derive a basis of observables that lifts both the nonlinear prediction model and nonlinear state constraints of NMPC, obtaining a quadratic program in the Koopman lifted space (denoted as Koopman NMPC, in short K-NMPC) that closely approximates the original NMPC solution. Additionally, we propose a general method to arbitrarily reduce the dimensionality of the Koopman lifted space, lowering the K-NMPC complexity and handling the infinite-dimensional case. We validate our K-NMPC approach in simulation, showcasing its solid performance and execution times, which are over ten times lower than classic NMPC.**

I. Introduction

The Koopman operator has gained traction in the realm of system modeling and control [1], due to its ability to encapsulate the dynamics of nonlinear systems into a "lifted" state space with linear dynamics. Original and lifted spaces are mapped through a basis of observable functions, which span an invariant subspace of the Koopman operator [2]–[4].

The Koopman operator can be used to tackle nonlinear optimal control problems, including Nonlinear Model Predictive Control (NMPC) [5]. Specifically, the nonlinear prediction model and nonlinear state constraints of NMPC can be lifted into the Koopman space and transformed into their equivalent linear counterparts, yielding nonlinear programs (NLP) to be replaced by quadratic programs (QP), which can be solved with superior efficiency and performance [6].

The Koopman operator unleashes its full potential when the basis of observables can be derived analytically. While analytical approaches have been explored in the literature, they face inherent challenges that make their application quite restricted. Eigenfunction-based methods [4], [7] find closed-form observables only for few classes of systems and, in general, Koopman eigenfunctions cannot be inverted to recover the original system state (which is crucial to enable control) [3]. Methods like Carleman linearization [8] derive observables by expanding in series the system dynamics, limiting to the polynomial space and not effectively exploiting the original system structure. Moreover, analytical methods often struggle to systematically address the case of infinite-dimensional bases of observables [3], [7]. As a result, data-driven techniques have emerged to obtain an approximated finite-dimensional Koopman lifting; yet, they also come with their own inherent shortcomings. Extended Dynamic Mode Decomposition (EDMD) [9] – the leading data-driven Koopman approach – relies on system measurements and a handpicked dictionary of observables [3]. However, to achieve a satisfactory accuracy, a very large dictionary is typically required [5], and, in general, the resulting lifted system cannot generalize beyond the initial data set. On the other hand, automated techniques to learn the dictionary [10], [11] lack interpretability and cannot incorporate human expertise in selecting potentially useful observables. Notably, data-driven methods directly identify an approximated finite-dimensional lifted system, rather than performing a tailored reduction on the infinite-dimensional one [7].

In this work, we address the limitations of the existing methods discussed above, namely, the systematic derivation of suitable Koopman observables and the handling of infinite-dimensional Koopman lifted systems. We propose a systematic procedure to analytically lift generic input-affine nonlinear systems, starting from an initial set of handpicked observables and iteratively generating the whole basis. We also present a general approach to reduce infinite-dimensional Koopman lifted systems to a finite dimension of choice. To determine the lowest dimension achieving the best accuracy for the lifted system, we evaluate its multi-step prediction error for increasing dimensions.

The proposed method is employed to lift a general NMPC problem, including both the nonlinear prediction model and nonlinear state constraints, obtaining an equivalent QP-MPC in the Koopman space – referred to as Koopman NMPC (in short, K-NMPC) – which closely approximates the original NMPC solution. In this context, reducing the dimensionality of the Koopman lifted space directly reduces also the computational complexity of the K-NMPC problem. Some works have applied the Koopman operator to NMPC [12], [13]; yet, they all rely on data-driven methods to lift the prediction model only. By converse, our method features

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a fully analytical and general framework, lifting also the NMPC constraints, and incorporating dimension reduction.

Our approach is validated on the real-world case study of mobile robot navigation in a cluttered environment. K-NMPC manages to proficiently attain the control task, while always respecting the original NMPC constraints. We demonstrate that K-NMPC closed-loop trajectories closely approximate those given by NMPC, with K-NMPC outperforming NMPC in terms of computation time by over an order of magnitude.

Outline: Section II briefly recalls the Koopman operator theory and presents our analytical method to derive the basis of observables. Section III formulates the NMPC problem. Section IV outlines the process of lifting the NMPC problem into the Koopman space, yielding the quadratic K-NMPC problem. Section V describes the general method to reduce the dimensionality of the Koopman lifted space. Section VI validates the K-NMPC approach in simulation. Our conclusions are drawn in Section VII.

Notation: In the following, $x = [x_i]_{i=1}^n \in \mathbb{R}^n$ is the vector with components $x_i \in \mathbb{R}$; $(x)_I$ is the vector collecting the components of $x \in \mathbb{R}^n$ indexed by the set $I \subset \{1, \dots, n\}$; $\|x\|_W = \sqrt{x^\top W x}$ is the weighted 2-norm of x; given $x, y \in \mathbb{R}^n$, any relation $x \lesseqqgtr y$ is considered component-wise, i.e., $(x)_i \lesseqqgtr (y)_i$, $\forall\, i \in \{1, \dots, n\}$.

II. Koopman Operator Theoretical Framework

Consider a continuous-time (CT) input-affine dynamical system, evolving on the finite-dimensional manifold $\mathcal{X} \subseteq \mathbb{R}^{n_x}$,

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) = f_0(x(t)) + \sum_{j=1}^{n_u} g_j(x(t)) u_j(t), \\
x(0) &= x_0, \quad t \in \mathbb{R}_{\geq 0},
\end{aligned} \tag{1}
$$

where $x \in \mathcal{X}$ and $u = [u_j]_{j=1}^{n_u} \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ are the state and input vectors, respectively. For system (1), the following regularity assumptions are considered:

Assumption 1: a) $f(x, \cdot)$ is piecewise continuous; b) $f(\cdot, u)$ is locally Lipschitz continuous; c) $f_0(\cdot)$ and $g_j(\cdot)$, $j = 1, \dots, n_u$, are locally Lipschitz continuous.

Assumption 2: The input signal $u(t) \in \mathscr{U}$, where $\mathscr{U}$ is the space of piecewise continuous signals.

With Assumptions 1a-1c, the Picard-Lindelöf theorem guarantees the existence of a unique solution

$$
x(t) = x_0 + \int_{t_0}^{t} f(x(\tau), u(\tau)) d\tau \equiv \varphi_t(x_0, u(t)) \tag{2}
$$

of system (1), in the time interval $[t_0, t_0 + \epsilon]$ ($t_0 = 0$, $\epsilon > 0$), starting from x_0 [14]. The family of one-parameter maps $\varphi_t : \mathcal{X} \times \mathscr{U} \to \mathcal{X}$ represents the flow of system (1), which generates the system trajectories $x(t)$.

From system (1), evolving in the state space $\mathcal{X}$, the Koopman operator is used to obtain an equivalent bilinear system, evolving in a lifted state space $\mathcal{Z}$. These two systems are put in relation through a set of functions $\phi : \mathcal{X} \to \mathbb{R}$, called observable functions (or, more briefly, observables).

Definition 1 (Koopman operator [2]): Let $\mathscr{F} \subseteq \mathcal{C}^1$ be a Banach space of continuously differentiable observables $\phi : \mathcal{X} \to \mathbb{R}$. The family of Koopman operators $\mathcal{K}^{t,u} : \mathscr{F} \to \mathscr{F}$ associated with the family of maps φ_t is defined as follows:

$$
\mathcal{K}^{t,u} \phi(\cdot) = \phi \circ \varphi_t(\cdot, u(t)), \quad \forall\, \phi \in \mathscr{F}, \tag{3}
$$

where $\circ$ denotes the function composition operator.

Remark 1: In Eq. (3)*, the family of Koopman operators is parameterized by t and the input signal $u(t) \in \mathscr{U}$.*

From Definition 1, we observe that the Koopman operator (3) provides the evolution of the observables along the trajectories of system (1), i.e., $\mathcal{K}^{t,u}\phi(x_0) = \phi \circ \varphi_t(x_0, u(t)) = \phi \circ x(t) = \phi(x(t))$, $\forall\, \phi \in \mathscr{F}$; these are denoted as lifted trajectories. To obtain the dynamics of the lifted trajectories (i.e., the lifted dynamics), we can compute, $\forall\, \phi \in \mathscr{F}$, the time derivative of ϕ along the system trajectory $x(t)$, i.e.,

$$
\begin{aligned}
\dot{\phi}(x(t)) &= \frac{\partial}{\partial x}\phi(x)\, \dot{x} = \nabla_x \phi(x)\, f(x, u) \equiv L_f \phi(x) \\
&= L_{f_0}\phi(x) + \sum_{j=1}^{n_u} L_{g_j}\phi(x) u_j,
\end{aligned} \tag{4}
$$

where $L_\star \equiv \star \cdot \nabla$ denotes the Lie derivative.

Now, consider a countable (possibly infinite) basis of observables $\Phi = \{\phi_i(x)\}_{i=1}^{N_o} \subset \mathscr{F}$, $N_o \in \mathbb{R} \cup \{+\infty\}$, spanning the N_o-dimensional linear subspace $\mathscr{F}_{N_o} \subseteq \mathscr{F}$. We denote with $\phi(x) = [\phi_i(x)]_{i=1}^{N_o}$ the basis Φ arranged as a vector of observables.

Assumption 3: For system (1) *and basis Φ, it holds that*

$$
L_{f_0}\phi(x) \in \operatorname{span}(\Phi), \quad L_{g_j}\phi(x) \in \operatorname{span}(\Phi) \tag{5}
$$

with $j = 1, \dots, n_u$ and $\forall\, \phi \in \Phi$.

By Assumption 3, $\mathscr{F}_{N_o}$ is an invariant subspace of the Koopman operator [4].

Theorem 1: Let the CT system (1) *and the basis of observables $\Phi = \{\phi_i(x)\}_{i=1}^{N_o}$ be given. Let $x(t)$ be the solution of system* (1) *and $z(t)$ the solution of the bilinear system*

$$
\dot{z} = Az + \sum_{j=1}^{n_u} B_j z u_j, \quad z(0) = \phi(x_0). \tag{6}
$$

If Assumption 3 is satisfied, then the two solutions are equivalent through the following map:

$$
z(t) = \phi(x(t)), \quad x(t) = \phi^{-1}(z(t)), \quad \forall\, t \geq 0. \tag{7}
$$

We denote system (6) as Koopman lifted system and $z \in \mathbb{R}^{n_z}$, $n_z = N_o$, as lifted state. The Koopman lifted system evolves in the lifted state space $\mathcal{Z}$.

Remark 2: A sufficient condition for the existence of the inverse map ϕ^{-1} in Eq. (7) *is that the states x are included as the first n_x observables, i.e., $(\phi(x))_{\{1,\dots,n_x\}} = x$. In this way, $x = \phi^{-1}(z) = (z)_{\{1,\dots,n_x\}}$.*

Remark 3: For Assumption 3 to hold, an infinite-dimensional basis Φ may be needed, yielding an infinite-dimensional Koopman lifted system (6)*.*

A. Finding the Basis of Koopman Observables

In Algorithm 1, we describe a general procedure to iteratively generate a basis of observables $\Phi = \{\phi_i(x)\}_{i=1}^{N_o}$ for system (1), satisfying Assumption 3, and construct the related Koopman lifted system (6).

Algorithm 1 Koopman observables basis and lifted system

Input: System (1); $\Phi_{\rm in} = \{x_i\}_{i=1}^{n_x} \cup \{\psi_{{\rm in},i}(x)\}_{i=1}^{N_{\rm in}}$
Output: Φ; lifted system (6)

$\Phi \leftarrow \Phi_{\rm in}$, $\phi(x) \leftarrow \phi_{\rm in}(x)$, $N_o \leftarrow n_x + N_{\rm in}$, $i \leftarrow 1$
while $i \le N_o$ **do**
  Compute $\alpha(x) = L_{f_0}\phi_i(x)$
  Expand $\alpha(x) = a_i'^\top \phi(x) + \alpha_{\rm res}(x)$, $a_i' \in \mathbb{R}^{N_o}$
  Expand $\alpha_{\rm res}$ as a linear combination of nonlinear terms: $\alpha_{\rm res}(x) = a_i''^\top \psi_\alpha(x)$, a_i'' real vector
  Store each component of $\psi_\alpha(x)$ in the set Ψ_α
  for $j = 1, \dots, n_u$ **do**
    Compute $\beta_j(x) = L_{g_j}\phi_i(x)$
    Expand $\beta_j(x) = b_{i,j}'^\top \phi(x) + \beta_{j,{\rm res}}(x)$, $b_{i,j}' \in \mathbb{R}^{N_o}$
    Expand $\beta_{j,{\rm res}}$ as a linear combination of nonlinear terms: $\beta_{j,{\rm res}}(x) = b_{i,j}''^\top \psi_{\beta,j}(x)$, $b_{i,j}''$ real vector
    Store each component of $\psi_{\beta,j}(x)$ in the set $\Psi_{\beta,j}$
  end for
  Add the new observables:
  $\Phi \leftarrow \Phi \cup \Psi_\alpha \cup \left(\bigcup_{j=1}^{n_u} \Psi_{\beta,j}\right)$,
  $\phi(x) \leftarrow [\phi(x)^\top, \psi_\alpha(x)^\top, \psi_{\beta,1}(x)^\top, \dots, \psi_{\beta,n_u}(x)^\top]^\top$,
  $N_o \leftarrow |\Phi|$, $i \leftarrow i+1$
end while
for $i = 1, \dots, N_o$ **do**
  $a_i = [a_i'^\top,\ a_i''^\top,\ 0^\top]^\top \in \mathbb{R}^{N_o}$,
  $b_{i,j} = [b_{i,j}'^\top,\ 0_{|\Psi_\alpha|}^\top,\ b_{i,j}''^\top,\ 0^\top]^\top \in \mathbb{R}^{N_o}$, $j = 1, \dots, n_u$
end for
$A = [a_1, \dots, a_{N_o}]^\top$,
$B_j = [b_{1,j}, \dots, b_{N_o,j}]^\top$, $j = 1, \dots, n_u$
return Φ, A, $\{B_j\}_{j=1}^{n_u}$

The basis Φ is incrementally built. Φ is initialized with a set of observables $\Phi_{\rm in}$, containing the system states $\{x_i\}_{i=1}^{n_x}$ (so to ensure the existence of the inverse map ϕ^{-1}, see Remark 2) and, possibly, some other functions of choice $\{\psi_{{\rm in},i}(x)\}_{i=1}^{N_{\rm in}}$ (arising from, e.g., nonlinear state constraints, see Section IV). For each observable $\phi_i(x) \in \Phi$, $i \le |\Phi|$, the related lifted dynamics (4) is computed, separating the autonomous ($L_{f_0}\phi_i(x)$) and input-dependent parts ($L_{g_j}\phi_i(x)$, $j = 1, \dots, n_u$). Each of these parts is expanded as a linear combination of the current observables in Φ and a nonlinear residual ($\alpha_{\rm res}$, $\beta_{j,{\rm res}}$). The residuals are in turn expanded as a linear combination of nonlinear terms (ψ_α, $\psi_{\beta,j}$); such nonlinear terms are appended to Φ as new observables of the basis. The above procedure is iterated until a finite-dimensional basis Φ is found, yielding no further residual terms, or after a maximum number of observables has been generated. The latter case is addressed in Section V through the dimensionality reduction of the Koopman lifted space.

III. Nonlinear MPC Formulation

Let us now consider system (1) as a CT nonlinear plant to control, using Nonlinear MPC (NMPC). By considering a control signal $u(t)$ for plant (1) that is constant over each time interval $[kT_s, (k+1)T_s]$, $k \in \mathbb{Z}_{\ge 0}$ (thus satisfying Assumption 2), where T_s is the discrete time step, we can compute a discretization of plant (1) as follows:

$$x_{k+1} = x_k + \int_{kT_s}^{(k+1)T_s} f(x(\tau), u_k)d\tau \approx f_d(x_k, u_k), \quad (8)$$

where $x_k = x(kT_s)$, $u_k = u(kT_s)$, and f_d is obtained by approximating the integral in Eq. (8) with some discretization method of choice [15].

We formulate the NMPC optimal control problem as follows, for each $k \ge 0$:

$$\begin{aligned} \min_{\hat{u},\hat{x}} \quad & \sum_{i=0}^{N_p-1} \|\hat{x}_i - x_r\|_Q^2 + \|\hat{u}_i\|_R^2 && (9a)\\ \text{s.t.} \quad & \forall\, i = 0, \dots, N_p - 1, \\ & \hat{x}_0 = x_k, \quad \hat{x}_{i+1} = f_d(\hat{x}_i, \hat{u}_i), && (9b)\\ & \hat{u}_i \in \mathcal{U}, \quad \hat{x}_i \in \mathcal{X}, \quad c_x(\hat{x}_i) \le 0. && (9c) \end{aligned}$$

In Eq. (9), $\hat{u} = \{\hat{u}_i\}_{i=0}^{N_p-1}$ and $\hat{x} = \{\hat{x}_i\}_{i=0}^{N_p}$ are the inputs and states predicted i steps ahead at time k, respectively, which act as decision variables; Eq. (9a) reports the cost function; Eq. (9b) reports the nonlinear prediction model constraints (with prediction horizon N_p), given by Eq. (8); Eq. (9c) reports input and state constraints, where $\mathcal{X}$, $\mathcal{U}$ are convex and compact polytopes, i.e., $\mathcal{X} = \{x \in \mathbb{R}^{n_x} : H_x x \le h_x\}$, $\mathcal{U} = \{u \in \mathbb{R}^{n_u} : H_u u \le h_u\}$, and $c_x : \mathbb{R}^{n_x} \to \mathbb{R}^{N_{c_x}}$ is a generic nonlinear function; x_r is the reference state; $Q \succeq 0$, $R \succ 0$ are weighting matrices. The optimal solution of the NMPC problem (9) is denoted by $(\hat{u}^*, \hat{x}^*)$.

The plant (1) is controlled via the one-step receding horizon policy, i.e., at each k, the first optimal predicted input $\hat{u}_0^*$ is applied to plant (1) over the time interval $[kT_s, (k+1)T_s]$.

IV. Koopman Nonlinear MPC

The NMPC problem (9) can be solved in the Koopman space by constructing a lifted QP-MPC problem in the lifted state z, henceforth denoted as K-NMPC. Such problem is significantly more tractable and faster to solve compared to the NLP in Eq. (9). The NMPC solution can be retrieved from the K-NMPC solution by means of the map (7).

To lift the NMPC problem (9) in the Koopman space, three main steps have to be performed:

1) generate with Algorithm 1 a basis of observables for the nonlinear system (1) and the nonlinear state constraints (9c), obtaining the Koopman lifted system (6);
2) convert the NLP (9) in the original state x into a QP in the lifted state z;
3) reduce the dimension of the lifted state z, in order to either reduce the QP complexity or render tractable the case of an infinite-dimensional Koopman lifted system.

Step 1 is dealt with in Sections IV-A and IV-B, step 2 in Section IV-C, and step 3 in Section V.

A. Observables Generation and Lifted System

The initial set of Koopman observables $\Phi_{\rm in}$ is chosen as

$$\begin{aligned} &\Phi_{\rm in} = \{x_i\}_{i=1}^{n_x} \cup \{c_{x,i}(x)\}_{i=1}^{N_{c_x}}, \\ &\phi_{\rm in}(x) = [x^\top,\ c_x(x)^\top]^\top. \end{aligned} \quad (10)$$

Thus, we choose the initial observables to comprise both the system states x and the nonlinear state constraints function $c_x(x) = [c_{x,i}(x)]_{i=1}^{N_{c_x}}$ in Eq. (9c). If $c_{x,i}(x)$, $i = 1, \ldots, N_{c_x}$, is a linear combination of nonlinear terms, each term can be individually set as an initial observable.

From Φ_{in}, we apply Algorithm 1 to obtain the basis of observables Φ and the Koopman lifted system (6), i.e.,

$$\begin{aligned} \Phi &= \Phi_{\text{in}} \cup \{\psi_i(x)\}_{i=1}^{N_{\text{new}}}, \\ \phi(x) &= [\phi_{\text{in}}(x)^\top,\ \psi(x)^\top]^\top, \end{aligned} \tag{11}$$

where $\psi(x) = [\psi_i(x)]_{i=1}^{N_{\text{new}}}$ are the newly generated observables; $|\Phi| = N_o = n_x + N_{c_x} + N_{\text{new}} = n_z$.

B. Lifted System Bilinear Terms Relaxation

To obtain a linear prediction model for the K-NMPC, we relax the bilinear terms of the Koopman lifted system (6) by linearizing them around a point $(\overline{z}, \overline{u})$. Without loss of generality, we set $\overline{u} = 0$. Each bilinear term is linearized as

$$\begin{aligned} B_j z u_j &\approx B_j \overline{z}\, \overline{u}_j + \frac{\partial(B_j z u_j)}{\partial z}(\overline{z}, \overline{u}_j)(z - \overline{z}) + \\ &\frac{\partial(B_j z u_j)}{\partial u_j}(\overline{z}, \overline{u}_j)(u_j - \overline{u}_j) = B_j \overline{z}\, \overline{u}_j + B_j \overline{u}_j (z - \overline{z}) + \\ &B_j \overline{z}(u_j - \overline{u}_j) = B_j \overline{z} u_j, \quad j = 1, \ldots, n_u. \end{aligned} \tag{12}$$

The relaxed lifted system is then a linear parameter-varying (LPV) system, with parameter $\overline{z}$, i.e.,

$$\dot{z} = Az + \sum_{j=1}^{n_u} B_j \overline{z} u_j \equiv Az + \overline{B}(\overline{z}) u, \tag{13}$$

where $\overline{B}(\overline{z}) = [B_1 \overline{z},\ \ldots,\ B_{n_u} \overline{z}] \equiv B(\overline{z})$. The relaxed lifted system (13) is discretized as in Eq. (8). Being the system linear, the integral can be solved in closed-form, obtaining the following linear DT system:

$$\begin{aligned} z_{k+1} &= e^{AT_s} z_k + (e^{AT_s} - I) A^{-1} B(\overline{z}) u_k \\ &\equiv A_d z_k + B_d(\overline{z}) u_k. \end{aligned} \tag{14}$$

C. K-NMPC Formulation

After having derived the discrete-time (DT) relaxed lifted system (14), we can formulate the K-NMPC problem. Let us rewrite the lifted state as $z = [z_x^\top,\ z_{c_x}^\top,\ z_\psi^\top]^\top$, where z_x, z_{c_x}, and z_ψ are the states related to the observables x, $c_x(x)$, and $\psi(x)$, respectively. Then, the K-NMPC optimal control problem is the following, for each $k \geq 0$:

$$\min_{\hat{u}, \hat{z}} \quad \sum_{i=0}^{N_p - 1} \|\hat{z}_{x,i} - x_r\|_Q^2 + \|\hat{u}_i\|_R^2 \tag{15a}$$

$$\text{s.t.} \quad \forall\, i = 0, \ldots, N_p - 1,$$

$$\hat{z}_0 = \phi(x_k), \quad \hat{z}_{i+1} = A_d \hat{z}_i + B_{d,k} \hat{u}_i, \tag{15b}$$

$$\hat{u}_i \in \mathcal{U}, \quad \hat{z}_{x,i} \in \mathcal{X}, \quad \hat{z}_{c_x,i} \leq 0. \tag{15c}$$

In Eq. (15), $\hat{u} = \{\hat{u}_i\}_{i=0}^{N_p - 1}$ and $\hat{z} = \{\hat{z}_i\}_{i=0}^{N_p}$ are the inputs and lifted states predicted i steps ahead at time k, respectively; Eq. (15b) reports the linear prediction model constraints, given by the relaxed lifted system (14), where $B_{d,k} = B_d(\phi(x_k))$; the constraints in the original state x (9c) are expressed in the lifted state z as in Eq. (15c), using the maps $z_x = x$ (for the linear ones) and $z_{c_x} = c_x(x)$ (for the nonlinear ones); finally, in the cost function (15a), the original state x is replaced by z_x. The optimal solution of the K-NMPC problem (15) is denoted by $(\hat{u}^*, \hat{z}^*)$.

The optimal solution of the NMPC problem (9) can be obtained by computing the solution of the K-NMPC problem (15) and applying to it the map (7). It is worth noticing, however, that, having relaxed the bilinearity of system (6), solving the K-NMPC problem leads to an approximation of the NMPC solution, i.e.,

$$\hat{x}^*_{\text{NMPC}} \approx \phi^{-1}(\hat{z}^*_{\text{K-NMPC}}), \quad \hat{u}^*_{\text{NMPC}} \approx \hat{u}^*_{\text{K-NMPC}}. \tag{16}$$

V. Dimensionality Reduction of the Koopman Lifted Space

As noted in Remark 3, an infinite-dimensional basis of Koopman observables may be needed to exactly lift system (1). In this section, we provide a general approach to reduce the basis of observables and the Koopman lifted system (6) to a finite dimension of choice. The proposed method also reduces the dimension (and, thus, the computational complexity) of the underlying K-NMPC problem (15).

A. Lifted System with a Reduced Number of Observables

Let us set a-priori the maximum number of observables $\overline{N}_o < N_o$ (where N_o may be infinite) to be generated by Algorithm 1. By setting $\overline{N}_o$, the lifted equations associated with the lastly generated observables will contain a residual nonlinear term. In detail, let us denote the observables as

$$\phi(x) = \begin{bmatrix} x \\ c_x(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \phi'(x) \\ \phi''(x) \end{bmatrix}, \tag{17}$$

where $\phi'(x) : \mathbb{R}^{n_x} \to \mathbb{R}^{n'_z}$, $\phi''(x) : \mathbb{R}^{n_x} \to \mathbb{R}^{n''_z}$, $n_z = \overline{N}_o = n'_z + n''_z$. The reduced Koopman lifted system has then the following form:

$$\begin{aligned} \dot{z}'(t) &= A' z(t) + \sum_{i=1}^{n_u} B'_i z(t) u_i(t), \\ \dot{z}''(t) &= A'' z(t) + \sum_{i=1}^{n_u} B''_i z(t) u_i(t) + f_{\text{res}}(z, u), \\ z(0) &= \phi(x_0), \quad z = [z'^\top,\ z''^\top]^\top. \end{aligned} \tag{18}$$

In Eq. (18), f_{res} is a residual nonlinear term containing the dynamics of the unlifted part of the system. This means that the solutions of the exactly-lifted system (6) and the reduced system (18) are still equal.

B. Reduced Lifted System Relaxation

As for the exactly-lifted case in Section IV-B, we relax both the bilinear terms and the nonlinear residual of the reduced system (18) by linearizing them around a point $(\overline{z}, \overline{u})$. Without loss of generality, we set $\overline{u} = 0$. Each j-th bilinear term, $j = 1, \ldots, n_u$, is linearized as in Eq. (12),

$$B'_j z u_j \approx B'_j \overline{z} u_j, \quad B''_j z u_j \approx B''_j \overline{z} u_j, \tag{19}$$

while the nonlinear residual $f_{\text{res}}(z, u)$ is linearized as

$$f_{\rm res}(z,u) \approx f_{\rm res}(\overline{z},0) + \frac{\partial f_{\rm res}}{\partial z}(\overline{z},0)(z-\overline{z}) + \frac{\partial f_{\rm res}}{\partial u}(\overline{z},0)u \equiv J_{{\rm res},z}(\overline{z})z + J_{{\rm res},u}(\overline{z})u + b''(\overline{z}). \quad (20)$$

The relaxed reduced lifted system is then an affine parameter-varying (APV) system, with parameter $\overline{z}$, i.e.,

$$\begin{aligned} \dot{z}' &= A'z + \sum_{i=1}^{n_u} B_i'\overline{z}u_i = A'z + \overline{B}'(\overline{z})u, \\ \dot{z}'' &= A''z + \sum_{i=1}^{n_u} B_i''\overline{z}u_i + J_{{\rm res},z}(\overline{z})z + J_{{\rm res},u}(\overline{z})u + b''(\overline{z}) \\ &= \left(A'' + J_{{\rm res},z}(\overline{z})\right)z + \left(\overline{B}''(\overline{z}) + J_{{\rm res},u}(\overline{z})\right)u + b''(\overline{z}) \\ \Rightarrow \dot{z} &= \begin{bmatrix} A' \\ A''+J_{{\rm res},z}(\overline{z}) \end{bmatrix} z + \begin{bmatrix} \overline{B}'(\overline{z}) \\ \overline{B}''(\overline{z})+J_{{\rm res},u}(\overline{z}) \end{bmatrix} u + \begin{bmatrix} 0 \\ b''(\overline{z}) \end{bmatrix} \\ &\equiv A(\overline{z})z + B(\overline{z})u + b(\overline{z}), \end{aligned} \quad (21)$$

where $\overline{B}'(\overline{z}) = [B_1'\overline{z}, \ \ldots, \ B_{n_u}'\overline{z}]$ and $\overline{B}''(\overline{z}) = [B_1''\overline{z}, \ \ldots, \ B_{n_u}''\overline{z}]$. The relaxed reduced lifted system (21) is discretized as in Eq. (14), obtaining

$$z_{k+1} = A_d(\overline{z})z_k + B_d(\overline{z})u_k + b_d(\overline{z}). \quad (22)$$

C. Reduced K-NMPC Formulation

As done in Section IV-C, from the DT relaxed reduced lifted system (22), we formulate the K-NMPC optimal control problem as follows, for each $k \geq 0$:

$$\min_{\hat{u},\hat{z}} \sum_{i=0}^{N_p-1} \|\hat{z}_{x,i} - x_r\|_Q^2 + \|\hat{u}_i\|_R^2 \quad (23a)$$

$$\text{s.t.} \quad \forall\, i = 0,1,\ldots,N_p-1,$$

$$\hat{z}_0 = \phi(x_k), \quad \hat{z}_{i+1} = A_{d,k}\hat{z}_i + B_{d,k}\hat{u}_i + b_{d,k}, \quad (23b)$$

$$\hat{u}_i \in \mathcal{U}, \quad \hat{z}_{x,i} \in \mathcal{X}, \quad \hat{z}_{c_x,i} \leq 0, \quad (23c)$$

where $A_{d,k}{=}A_d(\phi(x_k))$, $B_{d,k}{=}B_d(\phi(x_k))$, $b_{d,k}{=}b_d(\phi(x_k))$. Also in the reduced case, an approximation of the solution of the NMPC problem (9) can be obtained from the solution of the K-NMPC problem (23), as in Eq. (16).

VI. Simulations and Results

To validate the K-NMPC strategy, we carry out simulations on the real-world case study of mobile robot navigation in a cluttered environment.

A. Implementation Details

Simulations are performed with MATLAB® (ver. 2023b) on a 13th Gen Intel Core™ i7 CPU at 1.7 GHz. The K-NMPC problem (23) is modeled with YALMIP [16] and solved with MOSEK (interior-point). The classic NMPC problem (9) is modeled with CasADi [17] and solved with Ipopt.

B. Mobile Robot Navigation in a Cluttered Environment

The mobile robot nonlinear model, employed for both NMPC prediction and as plant to control, is the kinematic unicycle model, i.e.,

$$\dot{\xi} = v\cos\psi, \qquad \dot{\eta} = v\sin\psi, \qquad \dot{v} = a, \qquad \dot{\psi} = \omega, \quad (24)$$

where (ξ,η) is the planar position, ψ the heading angle, v the longitudinal velocity, a the longitudinal acceleration, and ω the angular velocity; the system states and inputs are $x = [\xi,\eta,v,\psi]^\top$ and $u = [a,\omega]^\top$, respectively ($n_x = 4$, $n_u = 2$). System (24) is input-affine, matching Eq. (1).

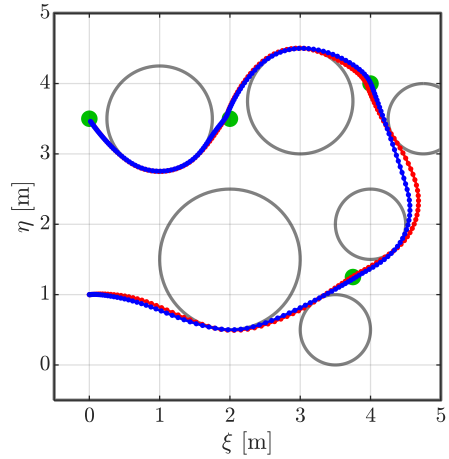

Fig. 1. Simulation of mobile robot navigation in a cluttered environment, reporting: robot closed-loop trajectories (NMPC, K-NMPC); reference states x_r; obstacles safety ellipsoids.

The control task is a regulation control problem, with 4 reference states $\{x_r^{(n)}\}_{n=1}^4$ (i.e., the robot target positions); the transition between successive references, i.e., $x_r^{(n)} \to x_r^{(n+1)}$, is triggered when $\|x_k - x_r^{(n)}\|_\infty \leq \epsilon_r$, being x_k the current robot state and ϵ_r a fixed tolerance.

1) Nonlinear State Constraints and Initial Set of Observables: Nonlinear state constraints are given by the obstacles the robot has to avoid. We assume that each obstacle can be fully enclosed in a safety ellipsoid. Then, the constraints are

$$c_{x,i}(x) = \frac{(\xi - c_{\xi,i})^2}{(l_{\xi,i}/2)^2} + \frac{(\eta - c_{\eta,i})^2}{(l_{\eta,i}/2)^2} \geq 1, \quad (25)$$

where $i = 1,\ldots,N_{\rm obst}$, being $N_{\rm obst}$ the number of obstacles; $[c_{\xi,i}, c_{\eta,i}]^\top$ and $(l_{\xi,i}, l_{\eta,i})$ are the center and sizes of the i-th safety ellipsoid, respectively. From Eq. (25), we can define the initial set of observables $\Phi_{\rm in}$ for system (24) as described in Section IV-A, i.e.,

$$\Phi_{\rm in} = \{\xi,\eta,v,\psi,\xi^2,\eta^2\}, \qquad |\Phi_{\rm in}| = 6, \quad (26)$$

which includes the states of system (24) and the nonlinear terms (i.e., ξ^2, η^2) introduced by constraints (25).

2) Observables Generation and Lifted System: The basis of observables Φ for system (24) and the related Koopman lifted system are generated through Algorithm 1, obtaining a finite-dimensional basis of 27 observables, i.e.,

$$\begin{aligned} \Phi = \Phi_{\rm in} \cup \{&v{\rm c}_\psi, v{\rm s}_\psi, \xi v{\rm c}_\psi, \eta v{\rm s}_\psi, {\rm c}_\psi, {\rm s}_\psi, v^2{\rm c}_\psi^2, \xi v{\rm s}_\psi, \\ &\xi{\rm c}_\psi, v^2{\rm s}_\psi^2, \eta v{\rm c}_\psi, \eta{\rm s}_\psi, v{\rm c}_\psi^2, v^2{\rm c}_\psi{\rm s}_\psi, \xi{\rm s}_\psi, v{\rm s}_\psi^2, \\ &\eta{\rm c}_\psi, {\rm c}_\psi^2, v{\rm c}_\psi{\rm s}_\psi, {\rm s}_\psi^2, {\rm c}_\psi{\rm s}_\psi\}, \quad |\Phi| = 27, \end{aligned} \quad (27)$$

where ${\rm c}_\psi \equiv \cos\psi$ and ${\rm s}_\psi \equiv \sin\psi$. Therefore, system (24) admits an exact finite-dimensional Koopman lifted system, with dimension $n_z = |\Phi| = 27$.

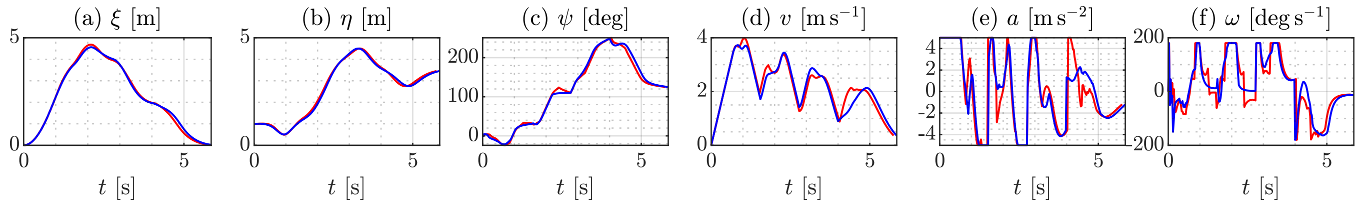

Fig. 2. Closed-loop state and input trajectories of the mobile robot navigation in a cluttered environment: NMPC (——); K-NMPC (——).

3) Dimension Reduction: Even if system (24) admits an exact finite-dimensional lifted system, we would like to further reduce its dimension as described in Section V, so to have a less complex K-NMPC problem.

To determine the "best" dimension reduction, we empirically evaluate the prediction accuracy of the reduced Koopman lifted system, with respect to a trajectory of choice $x(t)$ of system (24), for increasing number of observables $\phi(x)=[\phi_i(x)]_{i=1}^{\overline{N}_o}$, with $\overline{N}_o=6,\dots,27$.

Through this procedure, we observe that, after 12 observables, a steep reduction in the prediction error occurs, after which it does not further reduce and its value is approximately 0 (i.e., it is close to the integration tolerances).

Therefore, we can confidently reduce the dimension of the Koopman lifted system to $n_z = 12$.

4) Simulation Results: The plant (24) is controlled with K-NMPC (23) and, for comparison, also with classic NMPC (9). Controllers data is the following: $T_s = 0.025\,\mathrm{s}$; $N_p = 20$; $Q = \mathrm{diag}([20, 20, 0.1, 10^{-3}]^\top)$; $R = \mathrm{diag}([10^{-3}, 10^{-3}]^\top)$; $\mathcal{U} = \{u \in \mathbb{R}^{n_u} : u_{\mathrm{lb}} \leq u \leq u_{\mathrm{ub}}\}$, $u_{\mathrm{lb}} = [-5\,\mathrm{m\,s^{-2}},\ -\pi\,\mathrm{rad\,s^{-1}}]^\top$, $u_{\mathrm{ub}} = [5\,\mathrm{m\,s^{-2}},\ \pi\,\mathrm{rad\,s^{-1}}]^\top$; $\mathcal{X} = \{x \in \mathbb{R}^{n_x} : x_{\mathrm{lb}} \leq x \leq x_{\mathrm{ub}}\}$, $x_{\mathrm{lb}} = [-0.5\,\mathrm{m},\ -0.5\,\mathrm{m},\ -10\,\mathrm{m\,s^{-1}},\ -\infty]^\top$, $x_{\mathrm{ub}} = [5\,\mathrm{m},\ 5\,\mathrm{m},\ 10\,\mathrm{m\,s^{-1}},\ +\infty]^\top$.

Figure 1 reports the cluttered environment and the robot closed-loop trajectories. We see that K-NMPC manages to successfully attain the control task and effectively avoids all the obstacles. Moreover, K-NMPC and NMPC trajectories are remarkably similar, as expected by Eq. (16). This is further confirmed by Figure 2, reporting the detailed comparison of the closed-loop trajectories for each state and input.

5) Execution Time: Execution time data has been collected over 20 simulations: K-NMPC execution time is within $[2.79, 9.93]\,\mathrm{ms}$, with average value $4.66\,\mathrm{ms}$, outperforming classic NMPC by more than one order of magnitude, being its execution time within $[27.14, 126.60]\,\mathrm{ms}$, with average value $57.68\,\mathrm{ms}$.

VII. Conclusions

In this paper, we presented a comprehensive analytical framework to efficiently solve NMPC problems in the linear Koopman space. We proposed a systematic procedure to iteratively derive a basis of observables for generic input-affine nonlinear systems. A general approach to reduce the dimension of the resulting lifted system was also presented, to handle the case of infinite dimensionality. This approach was employed to lift a general NMPC problem, including its constraints, obtaining an equivalent QP-MPC problem in the Koopman space (K-NMPC), which closely approximates the original NMPC solution. Simulation results showcased the proficiency of our K-NMPC approach in attaining the given control task, providing closed-loop trajectories that are very close to the NMPC ones, and outperforming NMPC execution time by over an order of magnitude.

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