A Game-Theoretic Approach to Robust NMPC via Pontryagin's Minimum Principle and Penalty Functions

Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 59-16 (2025) 265–270

A Game-Theoretic Approach to Robust NMPC via Pontryagin's Minimum Principle and Penalty Functions ⋆

Michele Pagone * **Lorenzo Calogero** * **Alessandro Rizzo** * **Carlo Novara** *

* *Department of Electronics and Telecommunications, Politecnico di Torino, 10129 Turin, Italy (e-mail: {michele.pagone, lorenzo.calogero, alessandro.rizzo, carlo.novara}@polito.it).*

Abstract:
In this paper, we propose a novel robust Nonlinear Model Predictive Control (RNMPC) strategy that combines the classic min-max formulation for robust optimization with game theory. The RNMPC control problem is defined as a zero-sum differential game, in which control input and disturbance act as opposing players. Such a problem is solved leveraging the Pontryagin's Minimum Principle (PMP), which recasts it as a two-point boundary value problem (TPBVP), which can be efficiently solved with a low computational burden. State constraints are incorporated within the RNMPC problem by including suitable penalty functions within the min-max stage cost. The optimal solution is computed as the Nash equilibrium (NE) of the differential game, of which we prove the existence and employ its structure to obtain a more numerically efficient version of the TPBVP. The effectiveness of the proposed RNMPC strategy is validated in simulation on the real-world case study of an unmanned ground vehicle (UGV), demonstrating its superiority over the non-robust case in both attaining the control task and delivering a more energy-efficient control action.

Keywords:
Robust Control, Nonlinear Model Predictive Control, Game Theory, Unmanned Ground Vehicles.

1. INTRODUCTION

Nonlinear Model Predictive Control (NMPC) stands as the most prominent technique for achieving optimal control of multivariate nonlinear systems. Its success stems from its ability to handle nonlinear dynamics while satisfying constraints – possibly nonlinear and non-convex – on the system inputs, outputs, and states (Camacho and Bordons, 2013; Calogero et al., 2024).

NMPC operates by predicting the system evolution over a finite time horizon. In general, such a prediction relies on a simplified or approximate model of the plant, especially when the actual system is highly complex, partially unknown, or influenced by external disturbances. As a result, inaccuracies in the prediction occur, potentially leading to a significant degradation of the NMPC performance (Calogero et al., 2025). This aspect necessitates the development of robust extensions to traditional NMPC (henceforth called Robust NMPC, in short RNMPC). In this context, several research efforts have been made (see, e.g., Ping et al. (2022) and the references therein).

A popular approach, introduced by Mayne et al. (2011), is the Tube-Based RNMPC, which relies on an ancillary controller that enforces closed-loop trajectories to stay within a tube-shaped region surrounding the reference. However, apart from the linear case, determining such a tube is a challenging problem.
Alternative approaches include learning-based (Kimaev and Ricardez-Sandoval, 2019; Lucia and Karg, 2018), stochastic (Thangavel et al., 2018; Sanchez et al., 2020), and H_∞-based (Magni et al., 2001; Chen et al., 1997; Goulart et al., 2009) RNMPC. Still, all these methods suffer from high computational complexity, rendering them impractical for online applications. Moreover, learning-based methods are heavily reliant on high-quality training datasets, which may be unavailable for several applications; stochastic methods require accurate modeling of the uncertainty to not compromise the controller performance; lastly, H_∞-based methods rely on an offline-computed precompensation H_∞ policy, that is later combined with an auxiliary control term arising from the min-max optimal control problem.

To overcome the above limitations, Pagone et al. (2023) proposed a min-max RNMPC based on the Pontryagin's Minimum Principle (PMP) (Pontryagin et al., 1986) –

⋆ L. Calogero is supported by the European Union NextGenerationEU (NGEU)–Piano Nazionale di Ripresa e Resilienza (PNRR) Project and the Italian Ministry of University and Research (MUR) under Grant 352/2022. A. Rizzo is supported by the Sustainable Mobility National Research Center (MOST) and the European Union NGEU–PNRR Project (Mission 4, Component 2, Investment 1.4–D.D. 1033 17/06/2022) under Grant CN00000023.

10.1016/j.ifacol.2025.10.114

henceforth called PMP-RNMPC. This methodology formulates the min-max optimal control problem as a zero-sum differential game, where the control input and the uncertainty/disturbance act as opposing players (Deka et al., 2021; Jordana et al., 2022; Sarkar et al., 2021); consequently, the optimal solution coincides with the Nash equilibrium (NE) of the differential game (Bernhard, 2021; Joshi et al., 2022; Pagone et al., 2023). The PMP is leveraged to recast the RNMPC differential game as a two-point boundary value problem (TPBVP), which can be effectively solved with a low computational burden.
Nonetheless, at its current state, the PMP-RNMPC approach presents some shortcomings, among which the lack of a state tracking term in the cost function and the inability to handle state constraints.

In this paper, we pursue the PMP-RNMPC line of research, aiming to fill the gaps that arise from its current limitations. Specifically, our contribution is three-fold:

(1) We propose a PMP-RNMPC scheme for state tracking towards a reference trajectory. Differently from the previous work by the same authors (Pagone et al., 2023), we incorporate a time-varying state tracking term in the stage cost of the min-max optimal control problem.
(2) Unlike Pagone et al. (2023), we augment the PMP-RNMPC strategy with state constraints, which are handled by including suitable penalty functions within the min-max problem cost function (Pagone et al., 2024).
(3) We prove the existence of a NE for the differential game, that is also the solution of the min-max RNMPC problem for the state constrained framework. Additionally, we show how this latter NE allows to define a joint Hamiltonian, exhibiting convexity in the input and concavity in the uncertainty; this result greatly simplifies the min-max problem, enabling the use of a single set of covectors in the TPBVP.

The proposed RNMPC strategy is validated on the real-world case study of an unmanned ground vehicle (UGV), tasked to track a reference state trajectory in a cluttered environment. Simulations demonstrate the effectiveness of RNMPC in proficiently attaining the control task in presence of disturbance, with superior performance over the nominal NMPC scheme. Moreover, it is observed that RNMPC is capable to provide a more energy-efficient control action compared to the nominal case.

1.1 Outline

The remainder of the paper is structured as follows. In Section 2, we introduce the problem statement. In Section 3, we present the PMP-based solution of the RNMPC differential game. In Section 4, our RNMPC strategy is validated in simulation on the real-world UGV case study. Our conclusions are drawn in Section 5, along with perspectives for future research avenues.

1.2 Notation

In the following, $z = [z_i]_{i=1}^{n} \in \mathbb{R}^n$ is a column vector with components z_i. $A = [a_{i,j}]_{i=1,\ldots,m}^{j=1,\ldots,n} \in \mathbb{R}^{m\times n}$ is a matrix with entries $a_{i,j}$. $(z)_I$ is the column vector collecting the components of $z \in \mathbb{R}^n$ indexed by $I \subset \{1,\ldots,n\}$. $\|z\|_W^2 \equiv z^\top W z$ is the (squared) weighed norm of z. $\frac{\partial(\cdot)}{\partial z} = \left[\frac{\partial(\cdot)_i}{\partial z_j}\right]$ is the Jacobian operator with respect to z. Given $x, y \in \mathbb{R}^n$, any relation $x \lesseqqgtr y$ is considered component-wise, i.e., $(x)_i \lesseqqgtr (y)_i$, $\forall\, i \in \{1,\ldots,n\}$.

2. PROBLEM STATEMENT

Consider a continuous-time (CT) input-affine nonlinear dynamical system, affected by an affine disturbance, i.e.,

$$\begin{aligned}\dot{x}(t) &= f(x(t),u(t),w(t)) = \\ &= f_0(x(t)) + g(x(t))u(t) + h(x(t))w(t), \qquad (1)\end{aligned}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ and $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ are the state and input vectors at time $t \in \mathbb{R}_{\geq 0}$, respectively; $w(t) \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is a vector representing an unmeasured disturbance or uncertainty acting on the system at $t \geq 0$. In the following, $w(t)$ shall be treated as an exogenous disturbance/uncertainty.

Assumption 1. *f_0, g, and h are C^1-smooth on $\mathcal{X}$.*

Assumption 2. *$\mathcal{X}$ is a convex polytope, i.e., $\mathcal{X} = \{x \in \mathbb{R}^{n_x} : H_x x \leq h_x\}$; $\mathcal{U}$ and $\mathcal{W}$ are hyperrectangles, i.e., $\mathcal{U} = \{u \in \mathbb{R}^{n_u} : \underline{u} \leq u \leq \overline{u}\}$, $\mathcal{W} = \{w \in \mathbb{R}^{n_w} : \underline{w} \leq w \leq \overline{w}\}$.*

Assumption 3. *System* (1) *is controllable and the system state is measured at each time instant $t = kT_s \equiv t_k$, $k \in \mathbb{Z}_{\geq 0}$, $T_s > 0$.*

Remark 1. *The quantity $w(t)$ can represent, alternatively, a system uncertainty (coming from an incomplete knowledge of the system dynamics), a parametric uncertainty, or an additive disturbance. In the following, we shall refer to it as adversary input.*

2.1 Robust NMPC controller

The purpose of this work is to design a Robust Nonlinear MPC (RNMPC) strategy for system (1); the main control task to be attained is state tracking towards a reference trajectory $x_r(t)$. Indeed, the key purpose of RNMPC is to ensure reliable control performance and stability in presence of uncertainties, disturbances, and model inaccuracies. This becomes crucial when traditional NMPC fails to deliver an effective and stabilizing control action due to its inability to explicitly account for uncertainties.

The RNMPC strategy is formulated as a min-max optimal control problem (OCP), where, at each time instant t_k, the cost function is to be minimized with respect to the control input $u(\tau)$ over the worst-case adversary input $w(\tau)$, $\tau \in [t_k, t_k + T_p]$, i.e.,

$$(u^*, w^*) = \arg\min_u \max_w J_w(x,u,w) \qquad (2a)$$

$$\text{s.t.} \quad \forall\, \tau \in [t_k, t_k + T_p],$$

$$\dot{\hat{x}}(\tau) = f(\hat{x}(\tau), u(\tau), w(\tau)), \quad \hat{x}(t_k) = x(t_k), \qquad (2b)$$

$$u(\tau) \in \mathcal{U}, \quad w(\tau) \in \mathcal{W}, \quad \hat{x}(\tau) \in \mathcal{X}, \qquad (2c)$$

$$c_{\mathrm{nl}}(\hat{x}(\tau)) \leq 0. \qquad (2d)$$

Here, together with the linear input and state constraints in Eq. (2c) (see Assumption 2), additional nonlinear state constraints are defined in Eq. (2d). Thus, state constraints can be combined into a single function $c : \mathbb{R}^{n_x} \to \mathbb{R}^{N_c}$, being N_c the total number of state constraints, as follows:

$$c(\hat{x}(\tau)) = \begin{bmatrix} H_x\hat{x}(\tau) - h_x \\ c_{\text{nl}}(\hat{x}(\tau)) \end{bmatrix} = [c_i(\hat{x}(\tau))]_{i=1}^{N_c} \leq 0. \quad (3)$$

The cost function J_w in Eq. (2a) is constructed to fulfill three requisites:

(1) attain the main control task (i.e., state trajectory tracking);
(2) include the effect of the adversary input w;
(3) enforce state constraints (3) by means of penalty functions (Pagone et al., 2024), allowing their removal from problem (2).

Specifically, J_w is defined as follows:

$$J_w(x, u, w) = \int_{t_k}^{t_k+T_p} \Big(\ell(\hat{x}(\tau), u(\tau)) - \ell_w(w(\tau)) + K(\hat{x}(\tau))\Big) d\tau + \varphi(\hat{x}(t_k + T_p)), \quad (4)$$

where

$$\ell(\hat{x}(\tau), u(\tau)) = \|\hat{x}(\tau) - x_r(\tau)\|_Q^2 + \|u(\tau)\|_R^2, \quad (5a)$$
$$\ell_w(w(\tau)) = \|w(\tau)\|_S^2, \quad (5b)$$
$$\varphi(\hat{x}(t_k + T_p)) = \|\hat{x}(t_k + T_p) - x_r(t_k + T_p)\|_P^2, \quad (5c)$$
$$K(\hat{x}(\tau)) = \sum_{i=1}^{N_c} \kappa(c_i(\hat{x}(\tau))). \quad (5d)$$

Here, $Q \succeq 0$, $R \succ 0$, $S \succ 0$, and $P \succeq 0$ are weighting matrices of suitable dimensions; ℓ is the tracking stage cost; ℓ_w is the adversary input stage cost; φ is the terminal cost; $\kappa : \mathbb{R} \to \mathbb{R}$ is a penalty function such that $\kappa(c) \gg 0$ if $c = 0$ and $\kappa(c) \to 0$ if $c \to -\infty$ ($c < 0$).

Assumption 4. κ is C^1-smooth on $\mathbb{R}$.

We see that the term K (5d) becomes significantly larger than the stage costs ℓ and ℓ_w when the state $\hat{x}(\tau)$ approaches the boundary of the constraint set defined in Eq. (3), i.e., $K(\hat{x}(\tau)) \gg \max\{\ell(\hat{x}(\tau), u(\tau)), \ell_w(w(\tau))\}$ for $c(\hat{x}(\tau)) \to 0$; instead, K is (almost) null when far from the boundary. Therefore, the inclusion of the term K (5d) in the cost function J_w (4) incorporates soft state constraints (3) within problem (2) (Pagone et al., 2024); this allows to remove the hard state constraints in Eqs. (2c) and (2d), at the price of obtaining a slight approximation of the optimum (u^*, w^*).

By the cost function J_w in Eq. (4), the OCP (2) can be seen as a zero-sum differential game, whose players are the control input u and the adversary input w, with payoff functions equal to J_w and $-J_w$, respectively. In other words, the goal of the control input u is to minimize the payoff function, while the goal of the adversary input w is to maximize it (i.e., minimize its opposite). In this setting, the optimum (u^*, w^*) corresponds to the saddle point of the game, i.e., the Nash equilibrium (NE); then, the optimal cost J_w^* corresponds to the value of the payoff for the control input at this NE. It is also worth remarking that, in this symmetric scenario, each one of the two players has to make their own choice without any a-priori information on the strategy taken by the opponent (Pagone et al., 2023).

The optimal control input u^* is fed to system (1) according to a receding horizon policy: only the first sample $u^*(t_k)$ is applied to the system, keeping it constant over the time interval $[t_k, t_{k+1}]$, i.e., $u(t) = u^*(t_k)$, $\forall t \in [t_k, t_{k+1}]$. The remainder of the solution u^* is discarded.

2.2 Nominal NMPC controller

In a non-robust setting, a nominal NMPC controller can be defined. Such a controller neglects the adversary input contribution, both in the prediction model (2b) and in the cost function (4). The nominal NMPC OCP is formulated as follows, at each t_k:

$$u^* = \arg\min_u J(x, u) \quad (6a)$$
$$\text{s.t.} \quad \forall\, \tau \in [t_k, t_k + T_p],$$
$$\dot{\hat{x}}(\tau) = \tilde{f}(\hat{x}(\tau), u(\tau)), \quad \hat{x}(t_k) = x(t_k), \quad (6b)$$
$$u(\tau) \in \mathcal{U}, \quad (6c)$$

where, recalling Eqs. (5) and (1),

$$J(x, u) = \int_{t_k}^{t_k+T_p} \Big(\ell(\hat{x}(\tau), u(\tau)) + K(\hat{x}(\tau))\Big) d\tau + \varphi(\hat{x}(t_k + T_p)), \quad (7)$$
$$\tilde{f}(\hat{x}(\tau)) = f_0(\hat{x}(\tau)) + g(\hat{x}(\tau))u(\tau). \quad (8)$$

3. ROBUST NMPC DIFFERENTIAL GAME SOLUTION VIA PONTRYAGIN'S MINIMUM PRINCIPLE

In order to solve the min-max RNMPC OCP (2), we leverage the Pontryagin's Minimum Principle (PMP) (Pontryagin et al., 1986).

As stated in Section 2.1, the optimum (u^*, w^*) is a NE of the differential game arising from Eqs. (2)-(5). According to the definition of NE (Bressan, 2011), (u^*, w^*) is a solution of the min-max OCP (2) if and only if the following conditions hold simultaneously:

$$u^* = \arg\min_u J_w(x, u, w), \quad (9a)$$
$$w^* = \arg\max_w J_w(x, u, w) = \arg\min_w -J_w(x, u, w), \quad (9b)$$

over the choice of all control inputs $u(\tau) \in \mathcal{U}$ and adversary inputs $w(\tau) \in \mathcal{W}$, $\tau \in [t_k, t_k + T_p]$, and subject to the prediction model constraints (2b).

For each condition (9a) and (9b), we define the corresponding Hamiltonian as

$$H^{(u)}(\hat{x}, u, w, \lambda^{(u)}) = \|\hat{x} - x_r\|_Q^2 + \|u\|_R^2 - \|w\|_S^2 + K(\hat{x}) + \lambda^{(u)\top}(f_0(\hat{x}) + g(\hat{x})u + h(\hat{x})w), \quad (10a)$$
$$H^{(w)}(\hat{x}, u, w, \lambda^{(w)}) = -\|\hat{x} - x_r\|_Q^2 - \|u\|_R^2 + \|w\|_S^2 - K(\hat{x}) + \lambda^{(w)\top}(f_0(\hat{x}) + g(\hat{x})u + h(\hat{x})w), \quad (10b)$$

where $\lambda^{(u)}$ and $\lambda^{(w)}$ are the covectors of the minimization and maximization problem, respectively.

Then, by the PMP (Pontryagin et al., 1986), the following system of differential equations holds:

$$\frac{\partial H^{(u)}}{\partial \hat{x}} + \dot{\lambda}^{(u)\top} = 0, \quad \frac{\partial H^{(w)}}{\partial \hat{x}} + \dot{\lambda}^{(w)\top} = 0, \quad (11a)$$
$$\frac{\partial H^{(u)}}{\partial u} = 0, \quad \frac{\partial H^{(w)}}{\partial w} = 0, \quad u \in \mathcal{U}, \quad w \in \mathcal{W}, \quad (11b)$$

$$\dot{\hat{x}} = f_0(\hat{x}) + g(\hat{x})u + h(\hat{x})w, \tag{11c}$$

subject to boundary conditions

$$\hat{x}(t_k) = x(t_k), \tag{11d}$$

$$\frac{\partial \varphi}{\partial \hat{x}}(\hat{x}(t_k+T_p)) - \lambda^{(u)\top}(t_k+T_p) = 0, \tag{11e}$$

$$\frac{\partial \varphi}{\partial \hat{x}}(\hat{x}(t_k+T_p)) + \lambda^{(w)\top}(t_k+T_p) = 0. \tag{11f}$$

In Eq. (11), the unknowns are the predicted state $\hat{x}(\tau)$, the control input $u(\tau)$, the adversary input $w(\tau)$, and the covectors $\lambda^{(u)}(\tau)$, $\lambda^{(w)}(\tau)$, over the time interval $\tau \in [t_k, t_k + T_p]$.

Importantly, by the PMP, the solutions of Eq. (11) coincide with those of the min-max OCP (2).

By Eq. (10), Eq. (11b) is equal to

$$2u^\top R^\top + \lambda^{(u)\top} g(\hat{x}) = 0 \;\Rightarrow\; u = -\frac{1}{2}R^{-1}g^\top(\hat{x})\lambda^{(u)}, \tag{12a}$$

$$2w^\top S^\top + \lambda^{(w)\top} h(\hat{x}) = 0 \;\Rightarrow\; w = -\frac{1}{2}S^{-1}h^\top(\hat{x})\lambda^{(w)}. \tag{12b}$$

The control input and adversary input constraints in Eq. (11b) can be seamlessly included in Eq. (12) as follows (Pagone et al., 2024):

$$u^* = \operatorname*{sat}_{\mathcal{U}}\Big(-\frac{1}{2}R^{-1}g^\top(\hat{x})\lambda^{(u)}\Big), \tag{13a}$$

$$w^* = \operatorname*{sat}_{\mathcal{W}}\Big(-\frac{1}{2}S^{-1}h^\top(\hat{x})\lambda^{(w)}\Big), \tag{13b}$$

where sat(·) is the component-wise saturation operator,

$$\Big(\operatorname*{sat}_{\mathcal{Z}}(z)\Big)_i = \begin{cases} \underline{z}_i & \text{if } z_i < \underline{z}_i, \\ z_i & \text{if } \underline{z}_i \le z_i \le \overline{z}_i, \\ \overline{z}_i & \text{if } z_i > \overline{z}_i, \end{cases}$$
$$= \min\{\max\{z_i, \underline{z}_i\}, \overline{z}_i\}, \quad i = 1, \ldots, n_z, \tag{14}$$

being $\mathcal{Z} = \{z \in \mathbb{R}^{n_z} : \underline{z} \le z \le \overline{z}\}$ a hyperrectangle (by Assumption 2, $\mathcal{U}$ and $\mathcal{W}$ are hyperrectangles).

Specifically, Eq. (13a) provides the optimal control input $u^*(\tau)$, $\tau \in [t_k, t_k + T_p]$, as function of the predicted state $\hat{x}(\tau)$ and covector $\lambda^{(u)}(\tau)$.

By replacing Eq. (13) into Eq. (11) and recalling Eqs. (10) and (5c), we obtain the following two-point boundary value problem (TPBVP):

$$\dot{\lambda}^{(u)} = -2Q(\hat{x} - x_r) - \frac{\partial K}{\partial \hat{x}}^\top - \Big(\frac{\partial f_0}{\partial \hat{x}} + u^{*\top}(\hat{x}, \lambda^{(u)})\frac{\partial g}{\partial \hat{x}} + w^{*\top}(\hat{x}, \lambda^{(w)})\frac{\partial h}{\partial \hat{x}}\Big)^\top \lambda^{(u)}, \tag{15a}$$

$$\dot{\lambda}^{(w)} = 2Q(\hat{x} - x_r) + \frac{\partial K}{\partial \hat{x}}^\top - \Big(\frac{\partial f_0}{\partial \hat{x}} + u^{*\top}(\hat{x}, \lambda^{(u)})\frac{\partial g}{\partial \hat{x}} + w^{*\top}(\hat{x}, \lambda^{(w)})\frac{\partial h}{\partial \hat{x}}\Big)^\top \lambda^{(w)}, \tag{15b}$$

$$\dot{\hat{x}} = f_0(\hat{x}) + g(\hat{x})u^*(\hat{x}, \lambda^{(u)}) + h(\hat{x})w^*(\hat{x}, \lambda^{(w)}), \tag{15c}$$

subject to boundary conditions

$$\hat{x}(t_k) = x(t_k), \tag{15d}$$

$$\lambda^{(u)}(t_k+T_p) = 2P(\hat{x}(t_k+T_p) - x_r(t_k+T_p)), \tag{15e}$$

$$\lambda^{(w)}(t_k+T_p) = -2P(\hat{x}(t_k+T_p) - x_r(t_k+T_p)). \tag{15f}$$

Solving Eq. (15) for $\hat{x}(\tau)$ and $\lambda^{(u)}(\tau)$ allows to compute the optimal control input $u(\tau)$ from Eq. (13a).

The TPBVP (15) exhibits a tight relation between the covectors $\lambda^{(u)}$ and $\lambda^{(w)}$. In the following, we expand the results from Pagone et al. (2023), where such a relation was found only for a special case of the OCP (2).

Theorem 1. Let the TPBVP in Eq. (15) *be given. Then, it holds that*

$$\lambda^{(w)}(\tau) = -\lambda^{(u)}(\tau), \quad \tau \in [t_k, t_k + T_p], \tag{16}$$

which is an admissible NE of the differential game arising from the min-max OCP (2).

Proof. From the boundary conditions in Eqs. (15e) and (15f), it holds that $\lambda^{(w)}(t_k + T_p) = -\lambda^{(u)}(t_k + T_p)$.

Now, adding together Eqs. (15a) and (15b) yields

$$\dot{\lambda}^{(u)} + \dot{\lambda}^{(w)} = \frac{d}{dt}(\lambda^{(u)} + \lambda^{(w)}) = -\Big(\frac{\partial f_0}{\partial \hat{x}} + u^\top \frac{\partial g}{\partial \hat{x}} + w^\top \frac{\partial h}{\partial \hat{x}}\Big)^\top (\lambda^{(u)} + \lambda^{(w)}). \tag{17}$$

Necessarily, if a couple $(\lambda^{(u)}(\tau), \lambda^{(v)}(\tau))$ solves the TP-BVP (15), then it satisfies Eq. (17), $\forall\, \tau \in [t_k, t_k + T_p]$.

The most immediate way to fulfill such a necessary condition (consistently with the boundary conditions in Eqs. (15e) and (15f)) is by setting $\lambda^{(w)}(\tau) = -\lambda^{(u)}(\tau)$. □

Remark 2. From Theorem 1, we observe that:

- *A joint Hamiltonian H and a common covector λ of the min-max OCP* (2) *can be defined. H can be picked as $H^{(u)}$ and $\lambda = \lambda^{(u)} = -\lambda^{(w)}$, or vice-versa*[1]*. Hence,*

$$u^* = \arg\min_u H(x, u, w, \lambda), \tag{18a}$$

$$w^* = \arg\min_w H(x, u, w, -\lambda). \tag{18b}$$

Consequently, the TPBVP (15) *reduces to*

$$\dot{\lambda} = -2Q(\hat{x} - x_r) - \frac{\partial K}{\partial \hat{x}}^\top - \Big(\frac{\partial f_0}{\partial \hat{x}} + u^{*\top}(\hat{x}, \lambda)\frac{\partial g}{\partial \hat{x}} + w^{*\top}(\hat{x}, -\lambda)\frac{\partial h}{\partial \hat{x}}\Big)^\top \lambda, \tag{19a}$$

$$\dot{\hat{x}} = f_0(\hat{x}) + g(\hat{x})u^*(\hat{x}, \lambda) + h(\hat{x})w^*(\hat{x}, -\lambda), \tag{19b}$$

subject to boundary conditions

$$\hat{x}(t_k) = x(t_k), \tag{19c}$$

$$\lambda(t_k+T_p) = 2P(\hat{x}(t_k+T_p) - x_r(t_k+T_p)). \tag{19d}$$

where the unknowns are $\hat{x}(\tau)$ and $\lambda(\tau)$, $\tau \in [t_k, t_k + T_p]$.

- *It exists a NE of the min-max OCP* (2) *coinciding with the joint Hamiltonian saddle point. In this situation, a pair (u^*, w^*) exists such that $H(u^*, w) \le H(u^*, w^*) \le H(u, w^*)$.*
- *Being the joint Hamiltonian convex in u and concave in w (i.e., separable), it is sufficient to find only the solution of one TPBVP, as reported in Eq.* (19).

In the end, by Theorem 1 and Remark 2, we have shown that the optimal control input $u^*(\tau)$ solving the min-max RNMPC OCP (2) can be found by solving the TPBVP in Eq. (19) for $\hat{x}(\tau)$ and $\lambda(\tau)$, and inserting these latter quantities into Eq. (13a).

[1] One can pick $H^{(w)}$ as joint Hamiltonian. In this case, the signs of the covectors are inverted.

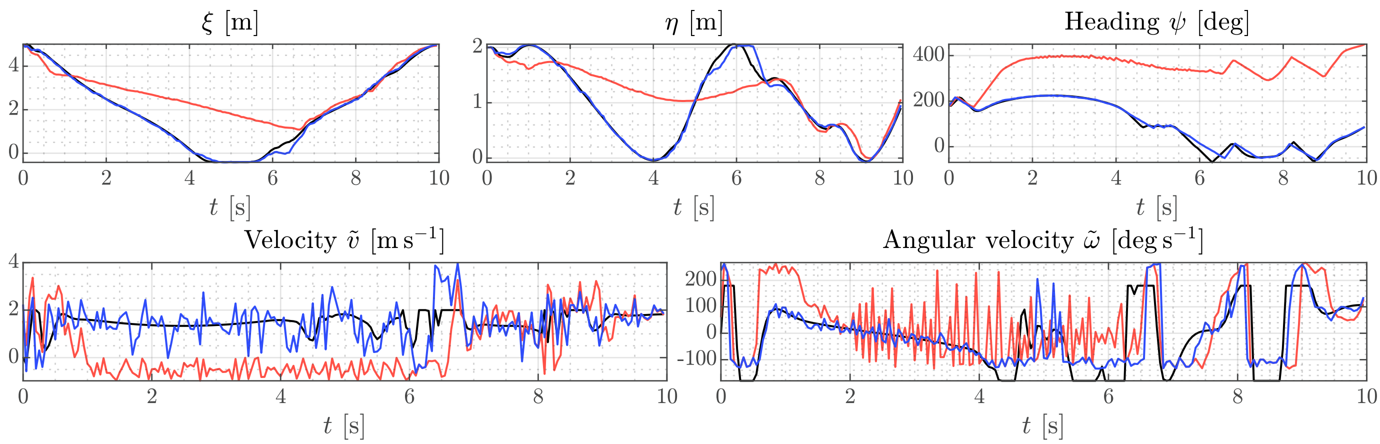

Fig. 1. Closed-loop states $x(t)$ and perturbed inputs $\tilde{u}(t) = [\tilde{v}(t), \tilde{\omega}(t)]^\top = [v(t) + w_1(t), \omega(t) + w_2(t)]^\top$: unperturbed NMPC (—); nominal NMPC (—); RNMPC (—).

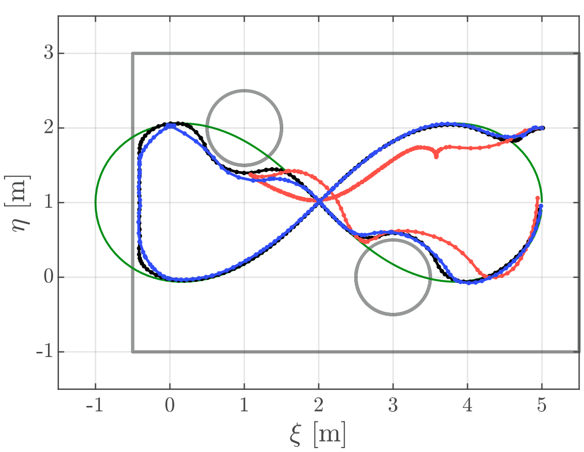

Fig. 2. UGV planar motion: unperturbed NMPC (—•—); nominal NMPC (—•—); RNMPC (—•—); reference x_r (—); obstacles (—).

4. SIMULATED EXAMPLE

The proposed game-theoretic robust NMPC strategy is validated in simulation on a real-world application scenario. Specifically, we consider the case of an Unmanned Ground Vehicle (UGV), whose planar motion is governed by the kinematic unicycle equations, i.e.,

$$\dot{\xi} = v\cos\psi, \qquad \dot{\eta} = v\sin\psi, \qquad \dot{\psi} = \omega, \tag{20}$$

where (ξ, η) is the planar position, ψ the heading angle, v the longitudinal velocity, and ω the angular velocity; the system states and inputs are $x = [\xi, \eta, \psi]^\top$ and $u = [v, \omega]^\top$, respectively ($n_x = 3$, $n_u = 2$); the system is input-affine. The system input is constrained by $\mathcal{U} = \{u \in \mathbb{R}^2 : \underline{u} \le u \le \overline{u}\}$, with $\underline{u} = [-2 \text{ m s}^{-1}, -180 \text{ deg s}^{-1}]^\top$, $\overline{u} = [2 \text{ m s}^{-1}, 180 \text{ deg s}^{-1}]^\top$.

Now, we introduce in system (20) an exogenous affine disturbance $w = [w_1, w_2]^\top$ ($n_w = 2$), altering the system inputs as follows:

$$\dot{\xi} = (v + w_1)\cos\psi, \quad \dot{\eta} = (v + w_1)\sin\psi, \quad \dot{\psi} = \omega + w_2. \tag{21}$$

The disturbance w is bounded by the set $\mathcal{W} = \{w \in \mathbb{R}^2 : \underline{w} \le w \le \overline{w}\}$, $0 \le \underline{w}_i \le \overline{w}_i$, $i = 1, 2$. Hence, w models an erroneous excess of velocity and an oversteering to the left of the UGV. Over time, we consider $w(t)$ as a piecewise constant signal, i.e., $w(t) = w_k$, $\forall\, t \in [t_k, t_{k+1}]$, $k \ge 0$, and $w_k \sim U_{\mathcal{W}}$, being $U_{\mathcal{W}}$ the uniform probability distribution over the set $\mathcal{W}$. Finally, data for w is as follows: $\underline{w} = [1 \text{ m s}^{-1}, 45 \text{ deg s}^{-1}]^\top$, $\overline{w} = [2 \text{ m s}^{-1}, 90 \text{ deg s}^{-1}]^\top$.

Combining Eqs. (20) and (21), we obtain the complete system model, matching Eq. (1), i.e.,

$$\dot{x} = f_0(x) + g(x)u + h(x)w, \tag{22a}$$

$$f_0(x) = 0, \quad g(x) = h(x) = \begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}. \tag{22b}$$

The UGV is tasked to track a planar lemniscate reference trajectory, given by $x_r(t) = [\xi_r(t), \eta_r(t), \psi_r(t)]^\top = [a\frac{\cos(2\pi t/T_{\text{sim}})}{1+\sin^2(2\pi t/T_{\text{sim}})} + c_\xi, b\frac{\sin(2\pi t/T_{\text{sim}})\cos(2\pi t/T_{\text{sim}})}{1+\sin^2(2\pi t/T_{\text{sim}})} + c_\eta, 0]^\top$, being T_{sim} the total simulation time, $[c_\xi, c_\eta]^\top$ and (a, b) the center and sizes of the curve, respectively. The reference trajectory partially crosses some infeasible regions of space, which are defined by linear state constraints $\mathcal{X}$ ($\mathcal{X} = \{x \in \mathbb{R}^3 : \underline{x} \le x \le \overline{x}\}$, with $\underline{x} = [-0.5 \text{ m}, -1 \text{ m}, -\infty \text{ deg}]^\top$, $\overline{x} = [5.5 \text{ m}, 3 \text{ m}, +\infty \text{ deg}]^\top$) and nonlinear state constraints $c_{\text{nl}}(x)$, as in Eq. (3). Nonlinear state constraints are circular obstacles, defined as

$$(c_{\text{nl}}(x))_i = r_i^2 - (\xi - c_{\xi,i})^2 - (\eta - c_{\eta,i})^2, \tag{23}$$

where $i = 1, \ldots, N_{\text{obst}}$, being N_{obst} the number of obstacles, $[c_{\xi,i}, c_{\eta,i}]^\top$ and r_i the center and radius of the i-th obstacle, respectively.
Data for $x_r(t)$ and obstacles can be inferred from Figure 2.

For state constraints, a Gaussian-like penalty function (5d) is employed, i.e., $\kappa(c) = \alpha e^{-\beta c^2}$ (Pagone et al., 2024), with $\alpha = 10^3$, $\beta = 9 \times 10^2$.

System (22) is controlled by the RNMPC in Eq. (2), whose optimal control input is computed by solving the TPBVP in Eq. (19). For comparison, system (22) is also controlled by the nominal NMPC in Eq. (6), both with and without the disturbance w (in the latter case, we shall refer to it as unperturbed NMPC).
All simulations last $T_{\text{sim}} = 10$ s. Both controllers (2) and (6) are set up with the same parameters, namely, $T_s = 50$ ms, $T_p = 10\,T_s$, $Q = \text{diag}(10^2, 10^2, 10^{-3})$, $R = \text{diag}(1, 10^{-1})$, $P = Q$, $S = 10^{-1} I_2$.

Both the nominal NMPC (6) and the TPBVP (19) for the RNMPC (2) are formulated with CasADi and solved with Ipopt. Simulations are performed in MATLAB® 2023b on a 13th Gen Intel® Core™ i7 CPU at 1.7 GHz.

Simulation results are reported in Figures 1 and 2. It can be noticed how the nominal NMPC is unable to attain the tracking task, providing a closed-loop trajectory that is drastically different from that given by the unperturbed NMPC. By converse, the RNMPC manages to effectively track the reference, with a closed-loop trajectory that, even in presence of the disturbance, closely resembles the unperturbed NMPC one.
These latter observations can be quantified by computing the L_2-norm of the signals $\Delta u_{\mathrm{rob}}(t) = \tilde{u}_{\mathrm{rob}}(t) - u_{\mathrm{unp}}(t)$ and $\Delta u_{\mathrm{nom}}(t) = \tilde{u}_{\mathrm{nom}}(t) - u_{\mathrm{unp}}(t)$, i.e., $\|\tilde{u}_\star - u_{\mathrm{unp}}\|_2 = (\int_0^{T_{\mathrm{sim}}} \|\tilde{u}_\star(\tau) - u_{\mathrm{unp}}(\tau)\|^2 d\tau)^{1/2}$, $\star = \mathrm{rob}, \mathrm{nom}$, obtaining $\Delta u_{\mathrm{rob}} = 5.6136$ and $\Delta u_{\mathrm{nom}} = 8.8545$. This means that not only the RNMPC is able to achieve better tracking performance, but also provides a more energy-efficient control authority with respect the nominal case.

5. CONCLUSIONS

In this paper, we proposed a novel robust NMPC (RNMPC) strategy, that combines the classic min-max formulation for robust optimization with game theory. Specifically, we formulated the RNMPC control problem as a zero-sum differential game, where control input and disturbance act as opposing players. Such a problem is solved leveraging the Pontryagin's Minimum Principle (PMP), yielding a two-point boundary value problem (TPBVP). The optimal solution is then computed as the Nash equilibrum (NE) of the differential game, of which we proved the existence. State constraints have been incorporated within the min-max cost function by means of suitable penalty functions. This entails the removal of hard state constraints at the price of obtaining a slight approximation of the optimal control input.

The proposed RNMPC strategy was tested on the real-world case study of an unmanned ground vehicle (UGV), demonstrating its effectiveness in proficiently attaining the tracking control task in presence of disturbance, with superior performance over the nominal NMPC scheme, also in terms of energy efficiency of the control action.

The promising results presented in this paper pave the way for several avenues of future research. First, effort should be placed in incorporating state constraints analytically within the Pontryagin-based optimal control problem, eliminating the need for penalty functions, that could lead to poor problem conditioning. Second, a further theoretical study on the closed-loop stability of the proposed RNMPC will be conducted.

ACKNOWLEDGEMENTS

This manuscript reflects only the authors' views and opinions, neither the European Union nor the European Commission can be considered responsible for them.

REFERENCES

Bernhard, P. (2021). Pursuit-Evasion Games and Zero-Sum Two-Person Differential Games. In J. Baillieul and T. Samad (eds.), *Encyclopedia of Systems and Control*, 1780–1785. Springer.

Bressan, A. (2011). Noncooperative Differential Games. *Milan J. Math.*, 79, 357–427.

Calogero, L., Pagone, M., Cianflone, F., Gandino, E., Karam, C., and Rizzo, A. (2025). Neural Adaptive MPC with Online Metaheuristic Tuning for Power Management in Fuel Cell Hybrid Electric Vehicles. *IEEE Trans. Autom. Sci. Eng.*, 22, 11540–11553.

Calogero, L., Pagone, M., and Rizzo, A. (2024). Enhanced Quadratic Programming via Pseudo-Transient Continuation: An Application to Model Predictive Control. *IEEE Control Syst. Lett.*, 8, 1661–1666.

Camacho, E.F. and Bordons, C. (2013). *Model Predictive Control*. Advanced Textbooks in Control and Signal Processing. Springer, London, U.K.

Chen, H., Scherer, C.W., and Allgöwer, F. (1997). A Game Theoretic Approach to Nonlinear Robust Receding Horizon Control of Constrained Systems. In *Proc. Am. Control Conf.*

Deka, S.A., Lee, D., and Tomlin, C.J. (2021). Towards Cyber-Physical Systems Robust to Communication Delays: A Differential Game Approach. *IEEE Control Syst. Lett.*, 6, 2042–2047.

Goulart, P.J., Kerrigan, E.C., and Alamo, T. (2009). Control of Constrained Discrete-Time Systems With Bounded ℓ_2 gain. *IEEE Trans. Autom. Control*, 54(5), 1105–1111.

Jordana, A., Hammoud, B., Carpentier, J., and Righetti, L. (2022). Stagewise Newton Method for Dynamic Game Control With Imperfect State Observation. *IEEE Control Syst. Lett.*, 6, 3241–3246.

Joshi, A.A., Chatterjee, D., and Banavar, R.N. (2022). Robust Discrete-Time Pontryagin Maximum Principle on Matrix Lie Groups. *IEEE Trans. Autom. Control*, 67(7), 3545–3552.

Kimaev, G. and Ricardez-Sandoval, L.A. (2019). Nonlinear model predictive control of a multiscale thin film deposition process using artificial neural networks. *Chem. Eng. Sci.*, 207, 1230–1245.

Lucia, S. and Karg, B. (2018). A deep learning-based approach to robust nonlinear model predictive control. *IFAC-PapersOnLine*, 51(20), 511–516.

Magni, L., Nijmeijer, H., and van der Schaft, A.J. (2001). A receding-horizon approach to the nonlinear H_∞ control problem. *Automatica*, 37(3), 429–435.

Mayne, D.Q., Kerrigan, E.C., van Wyk, E.J., and Falugi, P. (2011). Tube-based robust nonlinear model predictive control. *Int. J. Robust Nonlinear Control*, 21(11), 1341–1353.

Pagone, M., Boggio, M., Novara, C., Proskurnikov, A., and Calafiore, G.C. (2024). Continuous-time nonlinear model predictive control based on Pontryagin Minimum Principle and penalty functions. *Int. J. Control.*

Pagone, M., Zino, L., and Novara, C. (2023). A Pontryagin-Based Game-Theoretic Approach for Robust Nonlinear Model Predictive Control. In *Proc. IEEE Conf. Decis. Control*, 5532–5537.

Ping, X., Hu, J., Lin, T., Ding, B., Wang, P., and Li, Z. (2022). A Survey of Output Feedback Robust MPC for Linear Parameter Varying Systems. *IEEE/CAA J. Autom. Sinica*, 9(10), 1717–1751.

Pontryagin, L.S., Boltyanski, V.G., Gamkrelidze, R.V., and Mishchenko, E.F. (1986). *Mathematical Theory of Optimal Processes*. Gordon and Breach Science Publishers, Montreux, Switzerland.

Sanchez, J.C., Gavillan, F., and Vazquez, R. (2020). Chance-constrained Model Predictive Control for Near Rectilinear Halo Orbit spacecraft rendezvous. *Aerosp. Sci. Technol.*, 100.

Sarkar, R., Patil, D., Mulla, A.K., and Kar, I.N. (2021). Finite-Time Consensus Tracking of Multi-Agent Systems Using Time-Fuel Optimal Pursuit Evasion. *IEEE Control Syst. Lett.*, 6, 962–967.

Thangavel, S., Lucia, S., Paulen, R., and Engell, S. (2018). Dual robust nonlinear model predictive control: A multi-stage approach. *J. Process Control*, 72, 39–51.