Summary

This thesis is divided into five main chapters, preceded by a short introductory chapter that briefly recalls some preliminary concepts. Each chapter is self-consistent, although the last three share some underlying questions and concepts.

The first chapter is, for the most part, taken from [57], and its argument lies in descriptive set theory. A particularly effective tool in descriptive set theory is games, specifically infinite two-player, perfect information games. Unlike game theorists, who typically seek to find equilibrium strategies, set theorists are interested in the existence of winning strategies for either of the two players and in the implications that this has on the mathematical entities (e.g., topological spaces, functions, measure spaces) involved in the game. This chapter introduces a game that characterizes Baire class 1 functions between separable metrizable spaces. We show that the determinacy of our game (i.e. the statement "one of the two players has a winning strategy") is equivalent to a generalization of Baire's characterization theorem for Baire class 1 functions, and that both these statements hold under AD (the axiom of determinacy) and in Solovay's model.

The second chapter is largely derived from [6], co-authored with Alessandro Andretta. This chapter is about the relationship between two weak variants of the axiom of choice, specifically the axiom of countable choice (AC_ω) and the axiom of dependent choice (DC). These two axioms have a local version: given a set X, $\mathsf{AC}_\omega(X)$ asserts that every countable collection of nonempty subsets of X admits a choice function, while $\mathsf{DC}(X)$ asserts that every binary and total relation on X has an infinite chain. It is well-known that DC implies AC_ω. We show that it is consistent with ZF that there exists a set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds but $\mathsf{AC}_\omega(A)$ fails.

The third chapter lies at the intersection of lattice theory and combinatorial set theory. It addresses an open question posed by S. Z. Ditor in 1984 [22]: Given a positive integer n, is there a lattice of cardinality $\aleph_n$ whose principal ideals are finite and whose elements have at most $n+1$ lower covers? We show that such lattices exist in the constructible universe and, therefore, that their existence is consistent with ZFC.

The fourth chapter is for the most part taken from [7], co-authored with Alessandro Andretta. In this chapter, we generalize a result of A. Törnquist and W. Weiss [70] by studying the connection between the existence of Σ^1_2 Sierpiński's coverings of $\mathbb{R}^n$, and a numeric invariant of the join-semilattice of constructibility real degrees known as breadth. Additionally, we investigate the relationship between the breadth of the constructibility real degrees and the size of the continuum.

In the last chapter, we analyze the structure of the constructibility real degrees in the side-by-side Sacks model. The key feature of Sacks forcing is adding a particularly tame generic real of minimal degree of constructibility. Hence, it is natural to seek an understanding of the structure of the constructibility real degrees in models of ZFC obtained by forcing (over the constructible universe) with products or iterations of Sacks forcing. Much is already known regarding iterations and finite products. In this chapter, we address the case that has been less explored: the structure of the constructibility degrees of the reals in the model of ZFC obtained by forcing over L with a countable-support product of infinitely many Sacks forcings. Among other results, we show that, in such a model, the join-semilattice of the constructibility real degrees is rigid, meaning that it has no non-trivial automorphisms.