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Solving 2D Exterior Soft Scattering Elastodynamic Problems by BEM and by FEM-BEM Coupling Using Potentials

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Abstract. We consider the decomposition into scalar potentials for the simulation of transient 2D soft scattering elastic wave propagation problems in unbounded isotropic homogeneous media. The vector elastodynamic equation is reformulated in terms of two scalar wave equations, that are coupled by the Dirichlet boundary conditions. These are successively solved by using their associated space-time Boundary Integral Equation (BIE) representations. The corresponding Boundary Element Method (BEM) is obtained by combining a time convolution quadrature formula with a classical space collocation method. Then, the same boundary integral representation and its discretization are used to define a non-reflecting condition to be imposed on an artificial boundary delimiting the exterior computational domain of interest. In this latter a Finite Element Method (FEM) is applied.

THE MODEL PROBLEM

Let $\Omega^i \subset \mathbb{R}^2$ be an open, bounded and rigid domain, whose boundary Γ is assumed to be closed and smooth. We aim at studying the propagation of elastic waves in the homogeneous isotropic elastic medium $\Omega^e := \mathbb{R}^2 \setminus \overline{\Omega^i}$, with a Dirichlet datum $\mathbf{g}$ prescribed on its boundary and, for brevity, with null body force and trivial initial conditions. Assuming small variations of the vector field $\mathbf{u}^e(\mathbf{x}, t) = (\mathrm{u}_1^e(\mathbf{x}, t), \mathrm{u}_2^e(\mathbf{x}, t))$, $\mathbf{x} = (x_1, x_2)$, this latter is uniquely defined by the following system:

$$\rho \frac{\partial^2 \mathbf{u}^e}{\partial t^2}(\mathbf{x}, t) - (\lambda + \mu)\nabla(div\mathbf{u}^e)(\mathbf{x}, t) - \mu \nabla^2 \mathbf{u}^e(\mathbf{x}, t) = \mathbf{0} \qquad \mathbf{x} \in \Omega^e, t \in (0, T] \tag{1}$$

$$\mathbf{u}^e(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma, t \in (0, T] \tag{2}$$

$$\mathbf{u}^e(\mathbf{x}, 0) = \mathbf{0} \qquad \mathbf{x} \in \Omega^e \tag{3}$$

$$\mathbf{u}_t^e(\mathbf{x}, 0) = \mathbf{0} \qquad \mathbf{x} \in \Omega^e, \tag{4}$$

where $\rho > 0$ is the constant material density, $\lambda > 0$ and $\mu > 0$ are the Lamé constants.

In nearly incompressible media such as soft tissues, the simulation of elastic wave propagation based on displacement formulations are penalized by the fact that the shear (S) waves propagate much more slowly than the pressure (P) waves. In the case of homogeneous media, by applying a classical Helmholtz decomposition, the splitting of the displacement field as the sum of the gradient and of the rotational of two scalar potentials allows for decoupling the two dynamics and for constructing discretization spaces adapted, in principle, to each type of wave. We consider here this approach and propose two numerical strategies which allow to describe the propagation of P-waves and S-waves: the first one is based on the reformulation of the two scalar equations in terms of their space-time BIE formulation; the second one consists in a coupling of a BEM with a FEM.

In particular, we decompose the unknown displacement by two unknown scalar potentials $\mathbf{u}^e = \nabla\varphi_P^e + \mathbf{curl}\ \varphi_S^e$ where, for a generic scalar function $w = w(x_1, x_2)$, its vectorial curl is defined as $\mathbf{curl}\ w = (\partial_{x_2} w, -\partial_{x_1} w)$ (see [1]). The unknowns φ_P^e and φ_S^e are called Primary (or longitudinal) and Secondary (or transverse) waves. Referring to [2] for details, we recall the main relations that allow us to rewrite the elastodynamic equation in terms of a couple of wave

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equations. In particular, by using the decomposition of the Dirichlet datum on Γ, $\mathbf{g} = \nabla\varphi_P^e + \mathbf{curl}\,\varphi_S^e$, and introducing the anti-clockwise oriented unit tangent vector $\tau_\Gamma = (n_{\Gamma,2}, -n_{\Gamma,1})$, $n_\Gamma = (n_{\Gamma,1}, n_{\Gamma,2})$ being the ingoing unit normal vector on Γ, the following relations hold:

$$\frac{\partial\varphi_P^e}{\partial n_\Gamma} - \frac{\partial\varphi_S^e}{\partial\tau_\Gamma} = \mathbf{g}\cdot n_\Gamma, \qquad \frac{\partial\varphi_S^e}{\partial n_\Gamma} + \frac{\partial\varphi_P^e}{\partial\tau_\Gamma} = \mathbf{g}\cdot\tau_\Gamma.$$

Hence, introducing the P and S-wave speeds defined by $v_P = \sqrt{(\lambda+2\mu)/\rho}$ and $v_S = \sqrt{\mu/\rho}$, we obtain that the exterior elastodynamics problem is equivalent (see [1]) to the exterior potentials problem

$$\frac{\partial^2\varphi_P^e}{\partial t^2} - v_P^2\nabla^2\varphi_P^e = 0 \qquad (\mathbf{x}, t) \in \Omega^e \times (0, T] \tag{5}$$

$$\frac{\partial^2\varphi_S^e}{\partial t^2} - v_S^2\nabla^2\varphi_S^e = 0 \qquad (\mathbf{x}, t) \in \Omega^e \times (0, T] \tag{6}$$

$$\frac{\partial\varphi_P^e}{\partial n_\Gamma} = \frac{\partial\varphi_S^e}{\partial\tau_\Gamma} + \mathbf{g}\cdot n_\Gamma =: \frac{\partial\varphi_S^e}{\partial\tau_\Gamma} + g_{n_\Gamma} \qquad (\mathbf{x}, t) \in \Gamma \times (0, T] \tag{7}$$

$$\frac{\partial\varphi_S^e}{\partial n_\Gamma} = -\frac{\partial\varphi_P^e}{\partial\tau_\Gamma} + \mathbf{g}\cdot\tau_\Gamma =: -\frac{\partial\varphi_P^e}{\partial\tau_\Gamma} + g_{\tau_\Gamma} \qquad (\mathbf{x}, t) \in \Gamma \times (0, T], \tag{8}$$

endowed with null initial conditions.

THE BOUNDARY ELEMENT METHOD. By applying the Time Dependent Boundary Integral Equation (TDBIE) representation to both equations (5) and (6), and by considering the coupling relations (7) and (8), we can analogously reformulate (5)–(8) as follows:

$$\left(\frac{1}{2}\mathcal{I} + \mathcal{K}_P\right)\varphi_P^e(\mathbf{x}, t) - (\mathcal{V}_P(\partial_{\tau_\Gamma}\varphi_S^e))(\mathbf{x}, t) = (\mathcal{V}_P\, g_{n_\Gamma})(\mathbf{x}, t), \quad \left(\frac{1}{2}\mathcal{I} + \mathcal{K}_S\right)\varphi_S^e(\mathbf{x}, t) + (\mathcal{V}_S(\partial_{\tau_\Gamma}\varphi_P^e))(\mathbf{x}, t) = (\mathcal{V}_S\, g_{\tau_\Gamma})(\mathbf{x}, t), \tag{9}$$

for $(\mathbf{x}, t) \in \Gamma \times (0, T]$ and where, for $\star := P, S$,

$$(\mathcal{V}_\star\psi)(\mathbf{x}, t) := \int_0^t \int_\Gamma G_\star(\mathbf{x}-\mathbf{y}, t-s)\psi(\mathbf{y}, s)\, \mathrm{d}\Gamma_\mathbf{y}\mathrm{d}s, \qquad (\mathcal{K}_\star\lambda)(\mathbf{x}, t) := \int_0^t \int_\Gamma G_{n,\star}(\mathbf{x}-\mathbf{y}, t-s)\lambda(\mathbf{y}, s)\, \mathrm{d}\Gamma_\mathbf{y}\mathrm{d}s \tag{10}$$

are the well known single and double layer operators associated with the scalar wave equation, whose fundamental solution is $G_\star(r, t) = \frac{1}{2\pi}H\left(t - \frac{r}{v_\star}\right) / \sqrt{t^2 - \frac{r^2}{v_\star^2}}$ and $G_{n_\Gamma,\star} := \partial_{n_\Gamma}G_\star$.

We discretize the space-time integral equations in (9) defined on Γ by combining a Lubich convolution quadrature in time and a collocation method in space. To this aim, we start by introducing the time integral discretization by means of the convolution quadrature. We split the interval $[0, T]$ into N steps of equal length $\Delta_t = T/N$ and collocate equations (9) at the times $t_n = n\Delta_t$, $n = 0, \ldots, N$. After having exchanged the order of integration, the time integrals appearing in the definition of the operators $\mathcal{V}_\star$ and $\mathcal{K}_\star$ in (10) are discretized by means of the following Lubich convolution quadrature formula associated with the BDF2 method:

$$(\mathcal{V}_\star\psi)(\mathbf{x}, t_n) \approx \sum_{j=0}^{n} \int_\Gamma \omega_{n-j}\left(\Delta_t; \widehat{G}_\star(r)\right)\psi^j(\mathbf{y})\, \mathrm{d}\Gamma_\mathbf{y}, \qquad (\mathcal{K}_\star\lambda)(\mathbf{x}, t_n) \approx \sum_{j=0}^{n} \int_\Gamma \omega_{n-j}\left(\Delta_t; \widehat{G}_{n_\Gamma,\star}(r)\right)\lambda^j(\mathbf{y})\, \mathrm{d}\Gamma_\mathbf{y} \tag{11}$$

for $n = 0, \ldots, N$, where we have set $\psi^j(\mathbf{y}) := \psi(\mathbf{y}, t_j)$ and $\lambda^j(\mathbf{y}) := \lambda(\mathbf{y}, t_j)$. In (11) the coefficients $\omega_n(\Delta_t; J_\star(r))$ denote the quadrature weights associated with the Laplace transforms $J_\star = \widehat{G}_\star, \widehat{G}_{n,\star}$, and they are defined by

$$\omega_j(\Delta_t; J_\star) = \frac{1}{2\pi\imath}\int_{|z|=\varrho} J_\star\left(\frac{\gamma(z)}{\Delta_t}\right) z^{-(j+1)}\mathrm{d}z, \tag{12}$$

$\gamma(z) = 3/2 - 2z + 1/2z^2$ being the characteristic quotient of the BDF method of order 2 and ϱ is such that for $|z| \le \varrho$ the corresponding $\gamma(z)$ lies in the domain of analyticity of $J_\star$. By introducing the polar coordinate $z = \varrho\mathrm{e}^{\imath\vartheta}$, the convolution coefficients are approximated by the following formula

$$\omega_j(\Delta_t; J_\star) \approx \frac{\varrho^{-j}}{L} \sum_{\ell=0}^{L-1} J_\star\left(\frac{\gamma(\varrho \mathrm{e}^{\imath \ell \frac{2\pi}{L}})}{\Delta_t}\right) \mathrm{e}^{-\imath j \ell \frac{2\pi}{L}}, \qquad j = 0, \ldots, N, \tag{13}$$

where the interval $[0, 2\pi]$ has been partitioned into L subintervals of equal length. All the $\omega_j(\Delta_t; J_\star)$ are then computed simultaneously by the FFT with $O(N \log N)$ flops. The Laplace transforms $J_\star$, in (12) and (13) are

$$\widehat{G}_\star(r, s) = \frac{1}{2\pi} K_0\left(\frac{rs}{v_\star}\right), \qquad \widehat{G}_{n_\Gamma,\star}(r, s) = -\frac{s}{2\pi} K_1\left(\frac{rs}{v_\star}\right) \frac{\partial r}{\partial n_\Gamma},$$

where $K_0(z)$ and $K_1(z)$ are the second kind modified Bessel functions of order 0 and 1, respectively.

For the space discretization, we assume that Γ is described by a smooth global parametrization $\mathbf{x} = \eta(\vartheta) = (\eta_1(\vartheta), \eta_2(\vartheta))$, with $\vartheta \in [0, 1]$ and we recall that $\partial_{\tau_\Gamma} \varphi_\star^{e,j}(\mathbf{x}) = \partial_{\xi_\Gamma} \varphi_\star^{e,j}(\eta(\vartheta))$, where ξ_Γ denotes the curvilinear abscissa on Γ. Then, after having reduced the integration on Γ into the equivalent one defined on $[0, 1]$, we apply a nodal collocation boundary element method with piecewise linear basis functions $\{N_k\}_{k=1}^{M+1}$ associated to a uniform partition $\{\vartheta_k\}_{k=1}^{M+1}$ of $[0, 1]$. We approximate the unknowns $\varphi_\star^{e,j}(\mathbf{x})$ and $\partial_{\xi_\Gamma} \varphi_\star^{e,j}(\mathbf{x})$ for $\mathbf{x} \in \Gamma$ by

$$\varphi_\star^{e,j}(\eta(\vartheta)) \approx \sum_{k=1}^{M+1} \varphi_{\star,k}^{e,j} N_k(\vartheta), \quad \partial_{\xi_\Gamma} \varphi_\star^{e,j}(\eta(\vartheta)) \approx \sum_{k=1}^{M+1} \varphi_{\star,k}^{e,j} \partial_{\xi_\Gamma} N_k(\vartheta).$$

By introducing the matrices

$$(\mathbf{V}_\star^n)_{m,k} := \frac{1}{2\pi} \frac{\varrho^{-n}}{L} \sum_{\ell=0}^{L-1} \left(\int_0^1 K_0\left(\frac{r_m z}{v_\star}\right) N_k(\vartheta) \, \|\eta'(\vartheta)\| \, \mathrm{d}\vartheta \right) e^{-\frac{\imath n \ell 2\pi}{L}}$$

$$(\widetilde{\mathbf{V}}_\star^n)_{m,k} := \frac{1}{2\pi} \frac{\varrho^{-n}}{L} \sum_{\ell=0}^{L-1} \left(\int_0^1 K_0\left(\frac{r_m z}{v_\star}\right) \partial_{\xi_\Gamma} N_k(\vartheta) \, \|\eta'(\vartheta)\| \, \mathrm{d}\vartheta \right) e^{-\frac{\imath n \ell 2\pi}{L}}$$

$$(\mathbf{K}_\star^n)_{m,k} := -\frac{1}{2\pi} \frac{\varrho^{-n}}{L} \sum_{\ell=0}^{L-1} \left(\int_0^1 s K_1\left(\frac{r_m z}{v_\star}\right) \frac{\partial r}{\partial n_\Gamma} N_k(\vartheta) \, \|\eta'(\vartheta)\| \, \mathrm{d}\vartheta \right) e^{-\frac{\imath n \ell 2\pi}{L}}$$

where $z = \gamma(\varrho e^{\imath \ell 2\pi/L})/\Delta_t$ and $r_m = \|\eta(\vartheta_m) - \eta(\vartheta)\|$, and setting $\mathbf{g}_{n_\Gamma}^j = \left[g_{n_\Gamma}^j(\eta(\vartheta_1)), \ldots, g_{n_\Gamma}^j(\eta(\vartheta_{M+1}))\right]^T$ and $\mathbf{g}_{\tau_\Gamma}^j = \left[g_{\tau_\Gamma}^j(\eta(\vartheta_1)), \ldots, g_{\tau_\Gamma}^j(\eta(\vartheta_{M+1}))\right]^T$, we get the following system

$$\frac{1}{2} \mathbf{\Phi}_P^{e,n} + \sum_{j=0}^{n} \mathbf{K}_P^{n-j} \mathbf{\Phi}_P^{e,j} - \sum_{j=0}^{n} \widetilde{\mathbf{V}}_P^{n-j} \mathbf{\Phi}_S^{e,j} = \sum_{j=0}^{n} \mathbf{V}_P^{n-j} \, \mathbf{g}_{n_\Gamma}^j, \qquad \frac{1}{2} \mathbf{\Phi}_S^{e,n} + \sum_{j=0}^{n} \mathbf{K}_S^{n-j} \mathbf{\Phi}_S^{e,j} + \sum_{j=0}^{n} \widetilde{\mathbf{V}}_S^{n-j} \mathbf{\Phi}_P^{e,j} = \sum_{j=0}^{n} \mathbf{V}_S^{n-j} \, \mathbf{g}_{\tau_\Gamma}^j \tag{14}$$

in the unknowns $\mathbf{\Phi}_\star^{e,n} = [\varphi_{\star,1}^{e,n}, \ldots, \varphi_{\star,M+1}^{e,n}]^T$, $\star = P, S$.

For the computation of the solution $\mathbf{u}$ of Problem (1)-(4), a post processing evaluation of integrals is required, for which the Laplace transforms of the derivatives of $G_\star$ and $G_{n_\Gamma,\star}$ with respect to the variables $x_i, i = 1, 2$ are needed. We refer to [2] for details on such issue. This procedure turns out to be useful if one is interested in knowing the vector field $\mathbf{u}$ at some points far away form the boundary Γ, while it may result not efficient if $\mathbf{u}$ is needed at many points close to Γ. In this latter case, the coupling of a FEM with a BEM turns out to be an alternative approach.

THE FEM-BEM COUPLING. To describe the FEM-BEM coupling, we start by introducing an artificial smooth boundary $\mathcal{B}$ that surrounds the physical domain Ω^i and we reduce the infinite domain Ω^e to a finite computational one Ω, which is bounded internally by Γ and externally by $\mathcal{B}$. Then, we use the integral representations (9) to define on $\mathcal{B} \times [0, T]$ a couple of scalar Time Dependent Non Reflectiong Boundary Conditions (TD-NRBCs). In this case, the integration on Γ appearing in (10) and (11) is replaced by that on $\mathcal{B}$. To restrict the original problem in the finite computational domain Ω, we impose the continuity transmission conditions of the P and S-waves as well as of their normal derivatives on the artificial boundary $\mathcal{B}$. Hence, denoting by φ_P and φ_S the restriction of the solutions φ_P^e

and φ_S^e to Ω, we couple the variational formulation of (5) and (6) with the strong ones of the TD-NRBCs. Thus, introducing the bilinear form $a(u,w) = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla w(\mathbf{x}) \mathrm{d}\mathbf{x}$, the $L^2(\Omega)$ scalar product $(u,w)_\Omega = \int_\Omega u(\mathbf{x}) w(\mathbf{x}) \mathrm{d}\mathbf{x}$ and the bilinear forms associated with the duality product $b_D(u,w) = \langle u, w \rangle_D$, for $D = \Gamma, \mathcal{B}$, we reformulate (5)-(8) in Ω as follows: for any $t \in (0,T]$, find $\varphi_P(t), \varphi_S(t) \in H^1(\Omega)$, $\lambda_P(t) = (\partial_{n_\mathcal{B}} \varphi_P)(t)$, $\lambda_S(t) = (\partial_{n_\mathcal{B}} \varphi_S)(t) \in H^{-1/2}(\mathcal{B})$ ($n_\mathcal{B}$ denoting the unit normal vector defined on $\mathcal{B}$ and pointing outside Ω) such that

$$\frac{d^2}{dt^2}(\varphi_P(t), \psi_P)_\Omega + v_P^2 a(\varphi_P(t), \psi_P) - v_P^2 b_\Gamma(\partial_{\tau_\Gamma} \varphi_S(t), \psi_P) - v_P^2 b_\mathcal{B}(\lambda_P(t), \psi_P) = v_P^2 (g_{n_\Gamma}(t), \psi_P)_\Gamma \quad \forall\, \psi_P \in H^1(\Omega) \tag{15}$$

$$\frac{d^2}{dt^2}(\varphi_S(t), \psi_S)_\Omega + v_S^2 a(\varphi_S(t), \psi_S) + v_S^2 b_\Gamma(\partial_{\tau_\Gamma} \varphi_P(t), \psi_S) - v_S^2 b_\mathcal{B}(\lambda_S(t), \psi_S) = v_S^2 (g_{\tau_\Gamma}(t), \psi_S)_\Gamma \quad \forall\, \psi_S \in H^1(\Omega) \tag{16}$$

$$\frac{1}{2}\varphi_P(t)(\mathbf{x}) - (\mathcal{K}_P \varphi_P)(t)(\mathbf{x}) + (\mathcal{V}_P(\lambda_P))(t)(\mathbf{x}) = 0 \qquad \mathbf{x} \in \mathcal{B} \tag{17}$$

$$\frac{1}{2}\varphi_S(t)(\mathbf{x}) - (\mathcal{K}_S \varphi_S)(t)(\mathbf{x}) + (\mathcal{V}_S(\lambda_S))(t)(\mathbf{x}) = 0 \qquad \mathbf{x} \in \mathcal{B}, \tag{18}$$

together with the associated initial conditions. For the discretization of (15)-(16) we apply, for the space approximation, a finite element method based on piecewise linear approximation of the unknowns φ_P and φ_S associated with a uniform triangulation of the computational domain Ω and, for the approximation in time, a Crank-Nicolson marching scheme. For the approximation of (17)-(18), we apply the same technique previously described to solve the couple of TD-BIEs. To define the global numerical scheme, we introduce $\mathcal{S}$, the set of the indices of the nodes $\{\mathbf{x}_i\}_{i \in \mathcal{S}}$ of the triangular mesh, including those lying on Γ, and $\{N_i^\Omega\}_{i \in \mathcal{S}}$ the standard piecewise linear finite element basis functions defined on the triangulation. We split the total set of indices $\mathcal{S} = \mathcal{S}^I \cup \mathcal{S}^\mathcal{B}$ into the set $\mathcal{S}^I$ of interior mesh nodes (including those on Γ) and $\mathcal{S}^\mathcal{B}$ of the mesh nodes lying on the artificial boundary $\mathcal{B}$. Then, denoting by $\{N_i^\mathcal{B}\}_{i \in \mathcal{S}^\mathcal{B}}$, with $N^\mathcal{B}(\mathbf{x}) = N^\Omega_{|_\mathcal{B}}(\mathbf{x})$, the non-vanishing piecewise linear continuous functions defined on the boundary $\mathcal{B}$, we write the matrix form of the discrete Galerkin scheme associated with (15) and (16) as follows (see [3] for details)

$$(\mathbf{M} + \alpha v_P^2 \mathbf{A})\mathbf{\Phi}_P^{n+1} - \alpha v_P^2 \mathbf{B}\mathbf{\Phi}_S^{n+1} - \alpha v_P^2 \mathbf{Q}\Lambda_P^{n+1} = (\mathbf{M} - \alpha v_P^2 \mathbf{A})\mathbf{\Phi}_P^n + \alpha v_P^2 \mathbf{B}\mathbf{\Phi}_S^n + \alpha v_P^2 \mathbf{Q}\Lambda_P^n + \Delta_t \mathbf{M}\mathbf{z}_P^n + \alpha v_P^2 \mathbf{g}_{n_\Gamma}^n \tag{19}$$

$$(\mathbf{M} + \alpha v_S^2 \mathbf{A})\mathbf{\Phi}_S^{n+1} + \alpha v_S^2 \mathbf{B}\mathbf{\Phi}_P^{n+1} - \alpha v_S^2 \mathbf{Q}\Lambda_S^{n+1} = (\mathbf{M} - \alpha v_S^2 \mathbf{A})\mathbf{\Phi}_S^n - \alpha v_S^2 \mathbf{B}\mathbf{\Phi}_P^n + \alpha v_S^2 \mathbf{Q}\Lambda_S^n + \Delta_t \mathbf{M}\mathbf{z}_S^n + \alpha v_S^2 \mathbf{g}_{\tau_\Gamma}^n, \tag{20}$$

where $\alpha = \Delta_t^2/4$ and the mass, stiffness and boundary matrices are defined by

$$\mathbf{M}_{ij} = \left(N_i^\Omega, N_j^\Omega\right)_\Omega, \quad \mathbf{A}_{ij} = a(N_i^\Omega, N_j^\Omega), \quad \mathbf{B}_{ij} = \int_\Gamma N_i^\Omega(\mathbf{x})(\partial_{\tau_\Gamma} N_j^\Omega)(\mathbf{x}) \mathrm{d}\Gamma, \quad i,j \in \mathcal{S}$$

$$\mathbf{Q}_{ij} = \int_\mathcal{B} N_i^\Omega(\mathbf{x}) N_j^\mathcal{B}(\mathbf{x}) \mathrm{d}\mathcal{B}, \quad i \in \mathcal{S}, j \in \mathcal{S}^\mathcal{B}.$$

The unknowns $\mathbf{\Phi}_\star^n = \{\varphi_\star^{j,n}\}_{j \in \mathcal{S}}$ and $\Lambda_\star^n = \{\lambda_\star^{j,n}\}_{j \in \mathcal{S}^\mathcal{B}}$, $\star = P, S$ are the nodal values of the functions $\varphi_\star^n(\mathbf{x})$ and $\lambda_\star^n(\mathbf{x})$ associated with the nodes of the triangular mesh. The terms $\mathbf{g}_\square^n$ in (19) and (20) are the column vectors whose j-th component, with $j \in \mathcal{S}$, are defined by $\mathbf{g}_{\square,j}^n = (g_\square^{n+1} + g_\square^n, N_j^\Omega)_\Gamma$, $\square = \tau_\Gamma, n_\Gamma$. The unknown vector $\mathbf{z}_\star^n$ represents the approximation of the partial derivative $z_\star := \partial\varphi_\star/\partial t$, and it is updated at each time step by the formula $\mathbf{z}_\star^{n+1} = \Delta_t/2(\mathbf{\Phi}_\star^{n+1} - \mathbf{\Phi}_\star^n) - \mathbf{z}_\star^n$. Finally, we combine (19) and (20) with the analogous of (14) defined on $\mathcal{B}$, which now read

$$\frac{1}{2}\mathbf{\Phi}_P^n + \sum_{j=0}^n \mathbf{K}_P^{n-j}\mathbf{\Phi}_P^n + \sum_{j=0}^n \mathbf{V}_P^{n-j}\Lambda_P^j = 0, \qquad \frac{1}{2}\mathbf{\Phi}_S^n + \sum_{j=0}^n \mathbf{K}_S^{n-j}\mathbf{\Phi}_S^j + \sum_{j=0}^n \mathbf{V}_S^{n-j}\Lambda_S^j = 0. \tag{21}$$

Examples and applications of the above mentioned methods, including numerical tests which confirm the stability and convergence properties of the proposed approaches, can be found in [2] and [3].

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