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ORIGINAL ARTICLE

Calderón–Zygmund theory on some Lie groups of exponential growth

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Abstract

Let $G = N \rtimes A$, where N is a stratified Lie group and $A = \mathbb{R}_+$ acts on N via automorphic dilations. We prove that the group G has the Calderón–Zygmund property, in the sense of Hebisch and Steger, with respect to a family of flow measures and metrics. This generalizes in various directions previous works by Hebisch and Steger and Martini et al., and provides a new approach in the development of the Calderón–Zygmund theory in Lie groups of exponential growth. We also prove a weak-type (1,1) estimate for the Hardy–Littlewood maximal operator naturally arising in this setting.

KEYWORDS

Calderón–Zygmund theory, exponential growth groups, Hardy–Littlewood maximal function, nondoubling spaces

1 | INTRODUCTION

In the past century, the classical Calderón–Zygmund theory has been developed in the Euclidean setting and, more generally, on spaces of homogeneous type, see among others [6, 16, 36]. In the following years, many efforts have been made to generalize such theory in various nondoubling settings, both of polynomial and exponential growth (see, e.g., [3, 7, 8, 11, 17, 28, 30, 32, 39, 40, 42, 45]).

In this paper, we are specifically interested in the approach of one of the contributions listed above. Namely, the seminal paper [17] of some 20 years ago by Hebisch and Steger, in which they introduced an abstract Calderón–Zygmund theory based on the following definition.

Definition 1.1. A metric measure space (X, d, μ), with $\mu(X) = \infty$, has the Calderón–Zygmund property (CZP) if there exists $C_0 \geq 1$ such that, for every $f \in L^1(\mu)$ and $\alpha > 0$, there exist a countable family of sets $\mathcal{E}(f, \alpha) = \{E_j\}$, positive numbers r_j, and points $x_j \in X$ for which $f = g + \sum_j b_j$, in such a way that, for every $j \in \mathbb{N}$,

(a) $|g| \leq C_0\alpha$ μ-almost everywhere;
(b) $b_j = 0$ on $X \setminus E_j$;
(c) $\sum_j \|b_j\|_1 \leq C_0\|f\|_1$ and $\int_{E_j} b_j \, d\mu = 0$;
(d) $E_j \subset B(x_j, C_0 r_j)$;
(e) $\sum_j \mu(E_j^*) \leq \frac{C_0}{\alpha}\|f\|_1$, where $E_j^* = \{x : d(x, E_j) < r_j\}$.

In such case, we let $\mathcal{E} = \{E \in \mathcal{E}(f, \alpha) : f \in L^1(\mu), \alpha > 0\}$, and we say that (X, d, μ) has the CZP with respect to the family $\mathcal{E}$, and that $\mathcal{E}$ is a CZ family for (X, d, μ).

Observe that properties (a), (b), and (c) in Definition 1.1 only concern (X, μ) as a measure space, and do not depend in any way from the choice of a metric d on X. We will say that $L^1(\mu)$ admits a CZ decomposition with respect to the family $\mathcal{E}$ if (a), (b), and (c) hold true.

In [17], the authors provided evidence that spaces enjoying the CZP constitute a fertile environment to develop harmonic analysis (in particular singular integrals theory) which goes beyond the comfort zone of the spaces of homogeneous type. Indeed, while all spaces of homogeneous type have the CZP, the class of spaces with the CZP is strictly larger, and it even includes some natural and well-studied spaces of exponential growth. In the discrete setting, a first example of such a class is provided by homogeneous trees with the natural distance and the canonical flow measure [17]. It was later shown in [24] that the CZP actually extends to any tree (non-necessarily homogeneous) with any locally doubling flow measure (not necessarily the canonical one). Also in the continuous setting, on which we focus in this paper, there exist nontrivial examples of spaces of exponential growth enjoying the CZP. Consider the group $G = N \rtimes A$, where N is a stratified Lie group and $A = \mathbb{R}_+$ acts on N via automorphic dilations. Let d_G be a suitably chosen Carnot–Carathéodory metric on G and ρ a right Haar measure on G. Then, (G, d_G, ρ) has the CZP. This result was first proved in [17, Lemma 5.1] for the case $N = \mathbb{R}^n$ (i.e., when G is a so called $ax + b$ group), and then extended to the general case of arbitrary stratified Lie group N in [28, Theorem 3.20].

The aim of this paper is to enrich further the fauna of noncompact Lie groups of exponential growth treatable in the context of the abstract Calderón–Zygmund theory described above. Our setting is the following. We consider the same groups G as in [28] and a class $\mathcal{Z}$ of left-invariant vector fields having nonvanishing vertical component. Given a vector field $Z \in \mathcal{Z}$, we say that a measure μ is a Z-flow if it is absolutely continuous with respect to ρ and its Radon–Nykodim derivative φ is right-invariant with respect to the multiplication by $\exp(tZ)$, $t \in \mathbb{R}$. We introduce the class $\mathcal{F}_Z$ of measures μ on G that are Z-flows and such that (N, d_N, μ_N) is doubling, where $d\mu_N(n) := \varphi(n, 1)dn$ (see Section 2) and d_N is a Carnot–Carathéodory metric on N. We then construct an associated family $\mathcal{D}^Z$ of subsets of G, and we define a flow metric d_Z (see Sections 3, and 5, respectively, for their precise definitions). It is important to point out that the space (G, d_Z, μ) has exponential growth, and hence it is nondoubling, at least when φ is bounded away from zero on G. Our main result is the following.

Theorem 1.2. *For every vector field $Z \in \mathcal{Z}$ and any measure $\mu \in \mathcal{F}_Z$ the metric measure space (G, d_Z, μ) has the CZP with respect to the family $\mathcal{D}^Z$.*

Theorem 1.2 is strictly more general than the result of [28] previously cited. Indeed, right Haar measures belong to $\mathcal{F}_Z$ for any $Z \in \mathcal{Z}$, so that in Theorem 1.2 one can always choose, in particular, $\mu = \rho$, as in [28]. In general, $\mathcal{F}_Z$ is a large class of measures, we refer to Remark 2.3 for a specific example. Moreover, (see Section 6) when Z is the vertical vector field X_0 (see Section 2), then $d_Z = d_G$. That said, [28, Theorem 3.20] can be rephrased saying that (G, d_Z, μ) has the CZP with respect to the family $\mathcal{D}^Z$ if $Z = X_0$ and $\mu = \rho$.

We point out that our proof of Theorem 1.2 is new (even in the known case in which $d_Z = d_G$ and $\mu = \rho$), since we cannot exploit the left-invariance of the metric nor that of the measure.

As mentioned before, having the CZP is a key ingredient to develop a theory of singular integrals on a metric measure space. In particular, Theorem 1.2 implies boundedness properties for a class of linear integral operators on (G, d_Z, μ) whose kernels satisfy a Hörmander-type condition, see Theorem 5.4. A first natural project in this direction would be the study of the boundedness properties of the Riesz transform associated with a flow Laplacian on G (i.e., a Laplacian operator self-adjoint on $L^2(\mu)$). Such boundedness has already been studied on the $ax + b$ group (in [14, 17, 27, 34]), on homogeneous trees (in [17, 22]), and on nonhomogeneous trees in [29]. We are not addressing this or other applications here, leaving it for a possible follow-up work or for other interested mathematicians.

We now briefly describe the structure of the paper. In Section 2, we recall the basic notions on stratified Lie groups and their rank one extensions, and we introduce the family of vector fields $\mathcal{Z}$ and the class of measures $\mathcal{F}_Z$, where $Z \in \mathcal{Z}$.

In Section 3, we introduce the class of *admissible cylinders* in G, which resembles the class of admissible sets first appeared in the $ax + b$ group in [15], which in turn inspired [17, 28]. Note that part of the difficulty here is to find a natural geometric shape for such sets, while in different nondoubling settings (see, e.g., [32, 39]) the underlying manifold is Euclidean space equipped with a nondoubling measure, so that cubes are the standard cubes with sides parallel to the axes and dilations are used to define the subclass of cubes that is suitable for the CZ theory. We then prove that if $\mu \in \mathcal{F}_Z$, then (G, μ) admits a family $\mathcal{D}^Z$ of (admissible) dyadic sets (Theorem 3.9). As a consequence, we deduce that $L^1(\mu)$ admits a CZ decomposition with respect to $\mathcal{D}^Z$ (Theorem 3.10).

In Section 4, we consider the problem of the boundedness of the Hardy–Littlewood maximal function associated with the admissible cylinders introduced in Section 3. By means of a covering lemma for admissible sets (Proposition 4.4), in Theorem 4.2 we are able to show that the maximal function is of weak type (1,1).

In Section 5, we introduce a flow metric d_Z and, by means of some geometric lemmas, we are able to show that (d) and (e) in Definition 1.1 are satisfied on (G, d_Z, μ) by the sets in $\mathcal{D}^Z$. This, together with Theorem 3.10, completes the proof of our main result, Theorem 1.2.

In Section 6, we compare our result with those previously available in the literature. First, we observe, as mentioned before, that when the vector field Z is vertical then $d_Z = d_G$, and therefore the result of [28] (and, a fortiori, that of [17]) can be improved to: (G, d_G, μ) has the CZP with respect to the family $\mathcal{D}^Z$ for *any* $\mu \in \mathcal{F}_Z$ (and not only $\mu = \rho$) if Z is the vertical vector field. Next, we prove that when $N = \mathbb{R}^n$ we can even say more. Indeed, we show that in this case d_G is equivalent to d_Z for *any* $Z \in \mathcal{Z}$. Hence, we can improve the result of [17] further to: for $N = \mathbb{R}^n$, (G, d_G, μ) has the CZP with respect to the family $\mathcal{D}^Z$ for any $Z \in \mathcal{Z}$, and any $\mu \in \mathcal{F}_Z$. We dedicate the last part of the section, and of the paper, to investigating whether, always with d_G as underlying metric, the same level of generality in the choice of the family of sets and of the measure can be attained also when N is nonabelian. We give a negative answer to this question by providing a counterexample in the extended Heisenberg group $\mathbb{H}^1_e$, also known as the shearlet group [10]. In particular, in Theorem 6.9 we consider a particular vector field Z and we show that $(\mathbb{H}^1_e, d_{\mathbb{H}^1_e}, \rho)$, which is known to have the CZP with respect to the family $\mathcal{D}^{X_0}$ already from [28], does not have the CZP with respect to $\mathcal{D}^Z$.

Throughout the work, we write $f(x) \lesssim g(x)$ if there exists a uniform constant $C > 0$, such that $f(x) \leq Cg(x)$, for every x, and we write $f(x) \approx g(x)$ if it is both $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$. Constants carrying a numerical subscript, such as $C_1, C_2, \dots$ are meant to maintain their value across the whole paper, while C will be used (typically in proofs) for a generic constant whose value may change from line to line.

2 | PRELIMINARIES AND NOTATION

A Lie algebra $\mathfrak{n}$ is said to be stratified of step $S \in \mathbb{N}$, $S \geq 1$, if it admits a vector space decomposition

$$\mathfrak{n} = \bigoplus_{j=1}^{S} \mathfrak{n}_j, \quad \text{with} \quad [\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}.$$

Every $\mathfrak{n}_j$ is a layer and $M := \sum_{j=1}^{S} j \dim(\mathfrak{n}_j)$ is called the homogeneous dimension of $\mathfrak{n}$. A stratified Lie algebra $\mathfrak{n}$ can be equipped with a derivation ∂ such that $\mathfrak{n}_j$ is the eigenspace of ∂ corresponding to the eigenvalue j. A stratified Lie group N is a simply connected Lie group whose Lie algebra $\mathfrak{n}$ is stratified.

Any stratified Lie algebra, and then group, is nilpotent, hence unimodular. The push-forward of the Lebesgue measure on $\mathfrak{n}$ via $\exp_N : \mathfrak{n} \to N$ is a left and right Haar measure on N, which we fix and denote by dn. The formula $D_a = \exp_N((\log a)\partial)$ defines a family of automorphic dilations $(D_a)_{a\in A}$ on N. Hence, the Lie group $A = (\mathbb{R}_+, \cdot)$ acts on N via $D_a : N \to N$ and we can consider the corresponding semidirect product $G = N \rtimes A$, namely the product $N \times A$ endowed with the multiplication

$$(n, a)(n', a') := (nD_a(n'), aa'), \qquad n, n' \in N, \ a, a' \in A.$$

The neutral element of G is $1_G = (1_N, 1)$ and the inverse of $(n, a) \in G$ is $(n, a)^{-1} = (D_{1/a}(n^{-1}), 1/a)$. The group G is a solvable Lie group, and the Lie algebra $\mathfrak{g}$ of G is naturally identified with the semidirect product of Lie algebras $\mathfrak{n} \rtimes \mathfrak{a}$ (see Sections 3.14– 3.15 in [44]), namely $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, with

$$[(X, Y), (X', Y')]_{\mathfrak{g}} := ([X, X']_{\mathfrak{n}} + \partial_Y X' - \partial_{Y'} X, 0), \qquad X, X' \in \mathfrak{n}, \ Y, Y' \in \mathfrak{a},$$

where ∂_Y denotes the differential at $Y \in \mathfrak{a}$ of the map $a \mapsto D_a$, hence a derivation of $\mathfrak{a}$. A left and a right Haar measures λ and ρ on G are given by

$$d\lambda(n, a) = a^{-M-1} dn da \qquad d\rho(n, a) = a^{-1} dn da,$$

respectively. In particular, G is not unimodular.

We put $q_j = \dim \mathfrak{n}_j$, $1 \le j \le S$, and consider a basis $\{\check{X}_{j,i} : 1 \le i \le q_j\}$ of $\mathfrak{n}_j$. We fix a scalar product on $\mathfrak{n}$ that makes $\{\check{X}_{j,i} : 1 \le j \le S, 1 \le i \le q_j\}$ an orthonormal basis of $\mathfrak{n}$. Consequently, $\{\check{X}_{1,1}, \dots, \check{X}_{1,q_1}\}$ is an orthonormal basis of $\mathfrak{n}_1$ and provides a subbundle $HN \subset TN$ that is called horizontal. We say that a curve $\gamma_N : [0, 1] \to N$ of N is horizontal if $\dot{\gamma}_N(t) \in HN$ for every $t \in (0, 1)$. The Carnot–Carathéodory distance $d_N(n, n')$ between two elements $n, n' \in N$ is given by the infimum of the lengths of the horizontal curves joining n and n'. Since the horizontal distribution that makes $\check{X}_{1,1}, \dots, \check{X}_{1,q_1}$ into an orthonormal basis is bracket-generating, the distance d_N is finite and induces on N the usual topology. Moreover, the distance d_N is left-invariant and homogeneous with respect to the automorphic dilations D_a, namely $d_N(D_a(n), D_a(n')) = a^M d_N(n, n')$, for every $n, n' \in N$ and $a \in A$.

The vector fields $\check{X}_{j,i} \in \mathfrak{n}$ introduced before can be lifted to left-invariant vector fields on G by the formula

$$X_{j,i}|_{(n,a)} := a\check{X}_{j,i}|_n \qquad \text{for} \quad j = 1, \dots, S, \quad i = 1, \dots, q_j.$$

Let $\check{X}_0 = a\frac{\mathrm{d}}{\mathrm{d}a}$ be the canonical basis on $\mathfrak{a}$. We lift it to G by

$$X_0|_{(n,a)} := \check{X}_0|_a.$$

The system $\{X_0, X_{1,1}, \dots, X_{1,q_1}\}$ generates the Lie algebra $\mathfrak{g}$ and defines a sub-Riemannian structure on G with associated horizontal distribution HG, sub-Riemannian metric g and left-invariant Carnot–Carathéodory distance d_G. The following relation between the Carnot–Carathéodory distances on G and N is proved in [28, Proposition 2.7].

Proposition 2.1. *For all* $(n, a), (n', a') \in G$,

$$\cosh\left(d_G((n, a), (n', a'))\right) = \cosh\left(\log \frac{a}{a'}\right) + \frac{1}{2aa'} d_N(n, n')^2. \tag{1}$$

Now, we consider the scalar product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ that makes

$$\{X_0\} \cup \{X_{j,i} : 1 \le i \le q_j, 1 \le j \le S\}$$

an orthonormal basis of $\mathfrak{g}$. We consider the vector fields with nonvanishing, hence normalized, vertical component, namely

$$\mathcal{Z} := \{Z \in \mathfrak{g} : \langle Z, X_0 \rangle = 1\}.$$

Observe that for every $Z \in \mathcal{Z}$,

$$Z - X_0 \in \{X_0\}^{\perp} = \mathrm{span}\{X_{j,i} : 1 \leq i \leq q_j, 1 \leq j \leq S\} = \mathfrak{n} \oplus \{0\} \subset \mathfrak{g}.$$

Since for every $t \in \mathbb{R}$, $\exp(tX_0) = (1_N, e^t)$, there exists $n(t) \in N$ such that

$$(n(t), 1) = \exp(t(Z - X_0)) = \exp(tZ)\exp(-tX_0),$$

namely,

$$\exp(tZ) = (n(t), e^t), \quad t \in \mathbb{R}.$$

Definition 2.2. Given $Z \in \mathcal{Z}$, we say that a Borel measure μ on G is a Z-*flow measure* if it is absolutely continuous with respect to the right Haar measure ρ, and the Radon–Nikodym derivative $\varphi : N \times A \to [0, +\infty)$ is such that

$$\varphi(n, a) = \varphi((n, a)\exp(tZ)), \quad \text{for every } (n, a) \in G, \ t \in \mathbb{R}. \tag{2}$$

Clearly, if μ is a Z-flow measure, then

$$\mu(E\exp(tZ)) = \mu(E), \qquad \text{for every Borel set } E \subset G, \ t \in \mathbb{R}. \tag{3}$$

We associate to μ a measure on N given by

$$\mu_N(F) = \int_F \varphi(n, 1)dn, \quad \text{for every Borel set } F \subset N.$$

In this paper, we will only consider Z-flow measures μ such that (N, d_N, μ_N) is a doubling metric measure space. Therefore, we put

$$\mathcal{F}_Z := \{Z\text{-flow measures } \mu : (N, d_N, \mu_N) \text{ is doubling}\}.$$

We recall that a measure ν on a metric space X is doubling if for every $C > 1$ there exists a constant $D(\nu, C) > 1$ such that

$$\nu(B(x, Cr)) \leq D(\nu, C)\nu(B(x, r)), \quad \text{for every } x \in X, r > 0. \tag{4}$$

Observe that the right Haar measure ρ is in $\mathcal{F}_Z$ with respect to any vector field $Z \in \mathcal{Z}$, since $\varphi \equiv 1$ satisfies Equation (2). Furthermore, in such case, ρ_N is a Haar measure of N, which is doubling with respect to a Carnot–Carathéodory metric on N.

Remark 2.3. Observe that, for any $Z \in \mathcal{Z}$, *any* doubling measure μ_N on N which is absolutely continuous with respect to dn, can be associated with some Z-flow in the sense of the above definition. Indeed, if $\psi : N \to [0, +\infty)$ is the density of $d\mu_N$ with respect to dn, then it is not difficult to see that the function $\varphi : G \to [0, +\infty)$ defined by $\varphi(n, a) = \psi(nn(\log a)^{-1})$ is such that $\varphi(n, 1) = \psi(n)$ and satisfies Equation (2), so that the measure on G having φ as a density is in $\mathcal{F}_Z$ and μ_N is its associated measure on N. See, for example, the case $N = \mathbb{R}$. Fix the generator $X = \partial_x \in \mathfrak{n}$. If we consider the vector field $Z = X + Z_0 \in \mathfrak{g}$, then $\exp(tZ) = (e^t - 1, e^t)$, hence $n(t) = e^t - 1$. The Haar measure on N is the Lebesgue measure $dx = dn$. Given a doubling measure $d\mu_N(x) = \psi(x)dx$ on $\mathbb{R}$, we have

$$\varphi(x, a) = \psi(x - n(\log a)) = \psi(x - a + 1).$$

Hence, the value of φ at a point of the upper half-plane only depends on its projection to $N \times \{1\} = \mathbb{R} \times \{1\}$ through the lines of the vector field Z. This implies that

$$\mathcal{F}_Z = \left\{ d\mu(x,a) = \psi(x - a + 1)\frac{dxda}{a}, \, (\mathbb{R}, d_{\mathbb{R}}, \psi dx) \text{ is doubling} \right\},$$

where $\mathrm{d}_{\mathbb{R}}$ is the Euclidean distance on $\mathbb{R}$.

3 | ADMISSIBLE CYLINDERS AND DYADIC PARTITIONS

In this section, we first define a family of sets in G which we call *cylinders* and we discuss a number of useful properties they enjoy. Then, we introduce the subfamily of the *admissible cylinders*. Finally, we prove the existence of a family $\mathcal{D}^Z$ of dyadic partitions of G made of admissible cylinders, which leads to a CZ decomposition for functions in $L^1(\mu)$.

3.1 | Cylinders

Definition 3.1. Let E be any subset of N, $r > 1$ and $a \in A$. The cylinder $P_{r,E}(a)$ is defined by

$$P_{r,E}(a) = \left\{ (n,1)\exp(tZ) \, : \, n \in E, \; t \in U_r(a) \right\},$$

where

$$U_r(a) = \left(\log\left(\frac{a}{r}\right), \log(ar) \right) \subset \mathbb{R}. \tag{5}$$

We say that E is the base set of $P_{r,E}(a)$.

The next proposition collects some properties enjoyed by cylinders which will be useful in the following.

Proposition 3.2. *For any $Z \in \mathcal{Z}$, every $s \in \mathbb{R}$, $r, r_1, r_2 > 1$, $a, a_1, a_2 \in A$, $E, E_1, E_2 \subset N$, $m \in N$, the following hold:*

(i) $P_{r,E}(a)\exp(sZ) = P_{r,E}(ae^s)$;
(ii) $\exp(sZ)P_{r,E}(a) = P_{r,\psi_s(E)}(ae^s)$, *where* $\psi_s(m) := n(s)D_{e^s}(m)n(s)^{-1}$;
(iii) $(m,1)P_{r,E}(a) = P_{r,mE}(a)$;
(iv) two cylinders $P_i = P_{r_i,E_i}(a_i)$, $i = 1,2$, *intersect if and only if* $E_1 \cap E_2 \neq \emptyset$ *and* $U_{r_1}(a_1) \cap U_{r_2}(a_2) \neq \emptyset$;
(v) let $P_i = P_{r_i,E_i}(a_i)$, $i = 1,2$. *Then,*

$$P_1P_2 \supset P_{r_1r_2, E_1\cdot\Psi_{r_1,a_1}(E_2)}(a_1a_2), \quad \textit{where} \quad \Psi_{r,a}(E') := \bigcap_{t\in U_r(a)} \psi_t(E');$$

(vi) if μ is a Z-flow measure, then

$$\mu(P_{r,E}(a)) = 2\mu_N(E)\log r.$$

Proof. Property (i) is immediate since

$$P_{r,E}(a)\exp(sZ) = \left\{ (n,1)\exp((t+s)Z) \, : \, n \in E, t + s \in U_r(a) + s \right\}$$

and

$$U_r(a) + s = U_r(ae^s). \tag{6}$$

To prove (ii), first observe that

$$\exp(sZ)(n,1) = (n(s)D_{e^s}n, e^s) = (\psi_s(n)n(s), e^s) = (\psi_s(n),1)\exp(sZ).$$

Clearly $n \in E$ if and only if $\psi_s(n) \in \psi_s(E)$. Hence we have, by Equation (6)

$$\begin{aligned}\exp(sZ)P_{r,E}(a) &= \left\{ \exp(sZ)(n,1)\exp(tZ) : n \in E,\ t \in U_r(a)\right\}\\ &= \left\{(\psi_s(n),1)\exp((s+t)Z) : n \in E,\ s+t \in U_r(ae^s)\right\}\\ &= \left\{(n',1)\exp(t'Z) : n' \in \psi_s(E),\ t' \in U_r(ae^s)\right\}\\ &= P_{r,\psi_s(E)}(ae^s).\end{aligned}$$

It is straightforward to get (iii):

$$\begin{aligned}(m,1)P_{r,E}(a) &= \left\{(m,1)(n,1) : n \in E,\ t \in U_r(a)\right\}\\ &= \left\{(mn,1) : mn \in mE,\ t \in U_r(a)\right\} = P_{r,mE}(a).\end{aligned}$$

We now prove (iv). Two cylinders $P_i = P_{r_i,E_i}(a_i)$, $i = 1,2$, intersect if and only if there exist $n_i \in E_i$ and $t_i \in U_{r_i}(a_i)$ such that

$$(n_1,1) = (n_2,1)\exp((t_2-t_1)Z) = (n_2 n(t_2-t_1), e^{t_2-t_1}).$$

This is possible if and only if $t_1 = t_2$ and $n_1 = n_2$. Therefore, $P_1 \cap P_2 \neq \emptyset$ if and only if $E_1 \cap E_2 \neq \emptyset$ and $U_{r_1}(a_1) \cap U_{r_2}(a_2) \neq \emptyset$, which is (iv).

We now turn to (v). First, observe that by means of (ii) and (iii) we have

$$(n,1)\exp(tZ)P_2 = (n,1)P_{r_2,\psi_t(E_2)}(a_2e^t) = P_{r_2,n\psi_t(E_2)}(a_2e^t).$$

Therefore, we can write

$$\begin{aligned}P_1P_2 &= \bigcup_{x\in P_1} xP_2 = \bigcup_{n\in E_1}\bigcup_{t\in U_{r_1}(a_1)} (n,1)\exp(tZ)P_2\\ &= \bigcup_{n\in E_1}\bigcup_{t\in U_{r_1}(a_1)} P_{r_2,n\psi_t(E_2)}(a_2e^t) = \bigcup_{t\in U_{r_1}(a_1)} P_{r_2,E_1\psi_t(E_2)}(a_2e^t).\end{aligned}$$

But for any $t \in U_{r_1}(a_1)$ we have $E_1\psi_t(E_2) \supset E_1\Psi_{r_1,a_1}(E_2)$, and hence

$$\begin{aligned}P_1P_2 &= \bigcup_{t\in U_{r_1}(a_1)} P_{r_2,E_1\psi_t(E_2)}(a_2e^t)\\ &= \bigcup_{t\in U_{r_1}(a_1)} \left\{(m,1)\exp(sZ) : m \in E_1\psi_t(E_2),\ s \in U_{r_2}(a_2e^t)\right\}\\ &\supset \left\{(m,1)\exp(sZ) : m \in E_1\Psi_{r_1,a_1}(E_2),\ s \in \bigcup_{t\in U_{r_1}(a_1)} U_{r_2}(a_2e^t)\right\},\end{aligned}$$

where $\Psi_{r,a}$ is defined as in the statement. Hence, (v) follows from the fact that

$$\bigcup_{t\in U_{r_1}(a_1)} U_{r_2}(a_2e^t) = U_{r_1r_2}(a_1a_2).$$

Finally, by (i), Equation (3), and the change of variables $(n,a) = (n',1)\exp(tZ)$, one has

$$\begin{aligned}\mu(P_{r,E}(a)) &= \mu(P_{r,E}(1)\exp(\log aZ)) = \mu(P_{r,E}(1))\\ &= \int_{-\log r}^{\log r}\int_E \varphi((n,1)\exp(tZ))dndt\\ &= \int_{-\log r}^{\log r} dt\int_E \varphi(n,1)dn\\ &= 2\log r\mu_N(E),\end{aligned}$$

which gives (vi) and completes the proof. □

3.2 | Admissible cylinders

In order to introduce the family of admissible cylinders, we need to recall a celebrated result by Christ, which guarantees the existence of systems of dyadic cubes in any doubling metric space. We will apply the result to the doubling space (N, d_N, μ_N).

Theorem 3.3 [4]. *Let (N, d_N, μ_N) be a doubling metric space. There exist a family $\mathcal{Q} := \{Q \in \mathcal{Q}_k : k \in \mathbb{Z}\}$ of open sets of N and constants $\delta \in (0,1)$, $C_1, c > 0$, such that for each $k \in \mathbb{Z}$, $\mathcal{Q}_k$ consists of countably many pairwise disjoint subsets of N enjoying the following properties:*

(i) $\mu_N\left(N \setminus \bigsqcup_{Q\in\mathcal{Q}_k} Q\right) = 0$, *for every* $k \in \mathbb{Z}$;
(ii) *for every* $Q \in \mathcal{Q}_k$, *there exists a unique set* $p_N(Q) \in \mathcal{Q}_{k-1}$ *such that* $Q \subset p_N(Q)$, *while* $Q \cap Q' = \emptyset$ *for any other* $Q' \in \mathcal{Q}_{k-1}$;
(iii) *for each* $Q \in \mathcal{Q}_k$ *there exists a point* $n_Q \in Q$ *such that*

$$B_N(n_Q, c\delta^k) \subset Q \subset B_N(n_Q, C_1\delta^k);$$

(iv) *for each* $Q \in \mathcal{Q}$, $\mu_N(p_N(Q)) \le C_1\mu_N(Q)$ *and if we put* $s_N(Q) := \{Q' \in \mathcal{Q} : Q = p_N(Q')\}$, *then* $\#s_N(Q) \le C_1$.

We refer to the properties (iii) and (iv) as *eccentricity condition* and *volume control condition*, respectively. It is immediate that (iv) is a consequence of (iii), and $\#s_N(Q) \le C_1$ follows by $\mu_N(p_N(Q)) \le C_1\mu(Q)$. For simplicity and without loss of generality, we shall assume $C_1 \ge 3$ and $\#s_N(Q) \ge 2$ for every Q. Furthermore, if Q_1, Q_2 are such that $p_N(Q_i) = Q$, $i = 1,2$, then, by (iv) we have $\mu_N(Q_1) \le \mu_N(Q) \le C_1\mu_N(Q_2)$. Hence,

$$\frac{\mu_N(Q)}{\mu_N(Q_1)} = 1 + \sum_{\substack{Q'\in s_N(Q)\\ Q'\neq Q_1}} \frac{\mu_N(Q')}{\mu_N(Q_1)} \ge 1 + \frac{\#s_N(Q) - 1}{C_1} \ge 1 + \frac{1}{C_1},$$

which gives us

$$\mu_N(Q) \ge \left(1 + \frac{1}{C_1}\right)\mu_N(Q'), \qquad Q' \in s_N(Q). \tag{7}$$

In short, we say that N admits a Christ decomposition, which from now on will be intended to be the family of sets $\mathcal{Q}$ prescribed by Theorem 3.3. We refer to these sets as *Christ cubes* and we say that $Q \in \mathcal{Q}_k$ has *generation* $k \in \mathbb{Z}$. We will compare this result with the dyadic decomposition we provide in Theorem 3.9.

Definition 3.4. Fix $\gamma \geq 5$, and $\lambda > e^3/\delta$. We say that the cylinder $P = P_{r,Q}(a)$ is an admissible cylinder if Q is a Christ cube of N of generation k, for some $k \in \mathbb{Z}$, and one of the following holds:

(1) $r > e$ and $ar^2 \leq \delta^k \leq \lambda a r^\gamma$, and in this case we say that P is a *large* admissible cylinder;
(2) $1 < r \leq e$ and $ae^2 \log r \leq \delta^k \leq \lambda a e^2 \log r$, and in this case we say that P is a *small* admissible cylinder.

We now provide a canonical way to partition an admissible cylinder as the disjoint union of smaller admissible cylinders of comparable measure.

Definition 3.5. Given an admissible cylinder $P = P_{r,Q}(a)$, we define the associated cylinders

$$P^\vee = P_{\sqrt{r},Q}\left(\frac{a}{\sqrt{r}}\right), \quad P^\wedge = P_{\sqrt{r},Q}(a\sqrt{r}).$$

Then, we define the set $s(P)$ of the *sons* of P as follows:

(i) $s(P) = \{P^\vee, P^\wedge\}$, if $P^\vee$ and $P^\wedge$ are simultaneously admissible;
(ii) $s(P) = \{P_{r,Q'}(a) \,:\, Q' \in s_N(Q)\}$, otherwise.

Proposition 3.6. *Let $Z \in \mathcal{Z}$, $\mu \in \mathcal{F}_Z$ and P be an admissible cylinder. Every $P' \in s(P)$ is admissible and*

$$\left(1 + \frac{1}{C_1}\right)\mu(P') \leq \mu(P) \leq C_1 \mu(P'). \tag{8}$$

Proof. Let $P = P_{r,Q}(a)$ be an admissible cylinder. We need to analyze separately different cases.

CASE 1: P is small admissible. In this case, $P^\vee$ and $P^\wedge$ cannot be large admissible. On the other hand, it is easy to see that they are simultaneously small admissible unless

$$(i)\ \delta^k < \frac{\sqrt{r}}{2} a e^2 \log r \quad \text{or} \quad (ii)\ \delta^k > \frac{1}{2\sqrt{r}} \lambda a e^2 \log r.$$

But (i) cannot hold, since it contradicts the small admissibility of P. It remains to check that if (ii) holds, then $P_{r,Q'}(a)$ is small admissible for some (hence, for any) $Q' \in s_N(Q)$, that is, that

$$ae^2 \log r \leq \delta^{k+1} \leq \lambda a e^2 \log r.$$

The inequality on the right is implied by P being small admissible (since $\delta^{k+1} < \delta^k$). For the other, one can use (ii), $\delta\lambda > e^3$ and $r \leq e$ to see that

$$\delta^{k+1} > \frac{\delta\lambda}{2\sqrt{r}} a e^2 \log r > \frac{1}{2} a e^{9/2} \log r > ae^2 \log r.$$

CASE 2: P large admissible. In this case, we need to distinguish two sub-cases.

CASE 2.1: $r > e^2$. In this case, $P^\vee$ and $P^\wedge$ cannot be small admissible. On the other hand, it is easy to see that they are simultaneously large admissible unless

$$(i)'\ \delta^k < ar^{3/2} \quad \text{or} \quad (ii)'\ \delta^k > \lambda a r^{(\gamma-1)/2}.$$

But $(i)'$ cannot hold, since it contradicts the large admissibility of P. It remains to check that if $(ii)'$ holds, then $P_{r,Q'}(a)$ is large admissible for some (hence, for any) $Q' \in s_N(Q)$, that is, that

$$ar^2 \le \delta^{k+1} \le \lambda a r^\gamma. \tag{9}$$

The inequality on the right is implied by P being large admissible (since $\delta^{k+1} < \delta^k$). For the other, one can use $(ii)'$, $\delta\lambda > e^3$ and $\gamma > 5$ to see that

$$\delta^{k+1} > \delta\lambda a r^{(\gamma-1)/2} > e^3 ar^2 > ar^2.$$

CASE 2.2: $e < r \le e^2$. In this case, $P^\vee$ and $P^\wedge$ cannot be large admissible, and we know from CASE 1 that they are simultaneously small admissible unless either (i) or (ii) hold. But once again, (i) cannot hold. Indeed, since $\sup_{(e,e^2]} r^{-3/2}\log r = e^{-3/2}$, we have

$$\frac{\sqrt{r}}{2} ae^2 \log r = \frac{e^2}{2}\left(r^{-3/2}\log r\right) ar^2 \le \frac{\sqrt{e}}{2} ar^2 \le ar^2,$$

which makes it clear that (i) contradicts the large admissibility of P. It remains to check that if (ii) holds, then $P_{r,Q'}(a)$ is large admissible for some (hence, for any) $Q' \in s_N(Q)$, that is, that Equation (9) holds true.

The inequality on the right of Equation (9) is implied by P being large admissible (since $\delta^{k+1} < \delta^k$). For the left inequality, one can use (ii), $\delta\lambda > e^3$ and the fact that $\inf_{(e,e^2]} r^{-5/2}\log r = 2e^{-5}$ to get

$$\delta^{k+1} > \frac{\delta\lambda}{2\sqrt{r}} ae^2 \log r > \frac{1}{2\sqrt{r}} ae^5 \log r = \frac{e^5}{2}\left(r^{-5/2}\log r\right) ar^2 \ge ar^2.$$

Finally, Equation (8) follows by (vi) in Proposition 3.2, Equation (7), Theorem 3.3 (iv), and the fact that $C_1 \ge 3$. □

Definition 3.7. Given an admissible cylinder $P = P_{r,Q}(a)$, we define the associated cylinders

$$p^\downarrow(P) = P_{r^3,Q}\left(\frac{a}{r^2}\right), \quad p^\uparrow(P) = P_{r^2,Q}(ar), \quad p^\leftrightarrow(P) = P_{r,p_N(Q)}(a).$$

Proposition 3.8. *Let $P = P_{r,Q}(a)$ be an admissible large cylinder, $r > e$, $Q \in \mathcal{Q}_k$, $a \in A$. Then,*

(1) if $ar^2 \le \delta^k \le \delta\lambda a r^\gamma$, then $p^\leftrightarrow(P)$ is a large admissible cylinder;
(2) if $\delta\lambda a r^\gamma < \delta^k \le \lambda a r^\gamma$, then $p^\downarrow(P)$, $p^\downarrow(P)\setminus P$, $p^\uparrow(P)$, $p^\uparrow(P)\setminus P$ are large admissible cylinders.

Furthermore, whenever $\tilde{P} \in \{p^\leftrightarrow(P), p^\downarrow(P), p^\uparrow(P)\}$,

$$\left(1+\frac{1}{C_1}\right)\mu(P) \le \mu(\tilde{P}) \le C_1\mu(P).$$

Proof. Let $P = P_{r,Q}(a)$ be an admissible large cylinder, $Q \in \mathcal{Q}_k$. If $ar^2 \le \delta^k \le \delta\lambda a r^\gamma$, then $p^\leftrightarrow(P) = P_{r,p_N(Q)}(a)$, with $p_N(Q) \in \mathcal{Q}_{k-1}$, is large admissible since

$$ar^2 \le \delta^k < \delta^{k-1} \le \lambda a r^\gamma.$$

This proves (1). If $\delta\lambda a r^\gamma < \delta^k \le \lambda a r^\gamma$, then the fact that $p^\downarrow(P)$ and $p^\uparrow(P)$ are large admissible follows from the fact that $\delta\lambda > e^3$ and $\gamma \ge 5$. Indeed, we have

$$ar^4 < \delta\lambda a r^\gamma < \delta^k \le \lambda a r^\gamma < \lambda a r^{3\gamma-2}, \quad ar^5 < \delta\lambda a r^\gamma < \delta^k \le \lambda a r^\gamma < \lambda a r^{2\gamma+1}.$$

Note that

$$p^{\downarrow}(P) \setminus P = P_{r^2,Q}\left(\frac{a}{r^3}\right), \qquad p^{\uparrow}(P) \setminus P = P_{r,Q}(ar^2),$$

and, as above, they are large admissible because, by $\delta\lambda > e^3$ and $\gamma \geq 5$

$$ar < \delta\lambda ar^{\gamma} < \delta^k \leq \lambda ar^{\gamma} < \lambda ar^{2\gamma-3}, \quad ar^5 < \delta\lambda ar^{\gamma} < \delta^k \leq \lambda ar^{\gamma} < \lambda ar^{\gamma+2}.$$

Hence, we proved (2). The statement on the measures follows by Proposition 3.2 (vi), Equation (7), and Theorem 3.3 (iv). Indeed,

$$\left(1 + \frac{1}{C_1}\right)\mu(P) \leq \mu(p^{\leftrightarrow}(P)) = 2\log r\mu_N(p_N(Q)) \leq C_1\mu(P),$$

$$\mu(p^{\downarrow}(P)) = 3\mu(P), \quad \mu(p^{\uparrow}(P)) = 2\mu(P).$$

□

3.3 | Dyadic and Calderón–Zygmund decompositions

We start by constructing a dyadic decomposition of the measure space (G, μ) made of admissible sets.

Theorem 3.9. *For any $Z \in \mathcal{Z}$ and any $\mu \in \mathcal{F}_Z$, there exists a family $\mathcal{D}^Z = \{P \in \mathcal{D}^Z_k : k \in \mathbb{Z}\}$ such that for each $k \in \mathbb{Z}$, $\mathcal{D}^Z_k$ consists of pairwise disjoint admissible cylinders enjoying the following properties:*

(i) $\mu\left(G \setminus \bigsqcup_{P\in\mathcal{D}^Z_k} P\right) = 0$, *for every* $k \in \mathbb{Z}$*;*
(ii) for every $P \in \mathcal{D}^Z_k$, there exists a unique cylinder $p(P) \in \mathcal{D}^Z_{k-1}$ such that $P \subset p(P)$, while $P \cap P' = \emptyset$ for any other $P' \in \mathcal{D}^Z_{k-1}$;
(iii) for almost every $x \in G$ there is a, necessarily unique, cylinder $P^x_k \in \mathcal{D}^Z_k$ which contains x for any $k \in \mathbb{Z}$ and for any such x

$$\lim_{k\to-\infty} \mu(P^x_k) = +\infty, \qquad \lim_{k\to+\infty} \mu(P^x_k) = 0;$$

(iv) for every $P \in \mathcal{D}^Z$, $\mu(p(P)) \leq C_1\mu(P)$ and $\#\{P' \in \mathcal{D}^Z : P = p(P')\} \leq C_1$.

Observe that (iv) above essentially extends the volume control condition ((iv) of Theorem 3.3) from the group N to the group G. However, while in doubling metric spaces the volume control condition is a trivial consequence of the eccentricity condition ((iii) of Theorem 3.3), this is not the case, in general, in nondoubling metric spaces and, a fortiori, in our case where the eccentricity condition, and more in general any metric condition, is not even available at all. Hence, having the volume control condition (iv) in Theorem 3.9 is noteworthy and it has to be proved from scratch. Our construction is inspired by the one developed for $ax + b$ groups equipped with a right Haar measure in [26].

Proof. Fix a large admissible cylinder $P_0 = P_{r_0,Q_0}(1)$ and let $P_{k+1} := p(P_k)$. Here, we are defining $p(P_k)$ as follows: $p(P_k) = p^{\leftrightarrow}(P_k)$ whenever admissible, then for the smallest k such that the latter is not admissible we set $p(P_k) = p^{\uparrow}(P_k)$, for the next value of k such that $p^{\leftrightarrow}(P_k)$ is not admissible we set $p(P_k) = p^{\downarrow}(P_k)$, and then we keep on alternating $p^{\uparrow}$ and $p^{\downarrow}$ in this way.

We set $P_k = P_{r_k,Q_k}(a_k)$ for some $r_k > e$, $j(k) \in \mathbb{Z}$, $Q_k \in \mathcal{Q}_{j(k)}$, $a_k \in A$. By Proposition 3.8, it can neither happen that $p^n(P_k) = p^{\leftrightarrow}(p^{n-1}(P_k))$ for all $n \in \mathbb{N}$ nor that $p^n(P_k) \neq p^{\leftrightarrow}(p^{n-1}(P_k))$ for all $n \in \mathbb{N}$. Hence, there is an alternation of all the three choices that makes $r_k \to +\infty$ and $j(k) \to -\infty$ as $k \to +\infty$, so that

$$G = \bigcup_{k\in\mathbb{N}} P_k.$$

Now, we introduce the following notation: if $P = P_{r,Q}(a)$, where $Q \in \mathcal{Q}_j$, then the family of its *siblings* is

$$\mathcal{S}(P) = \{P_{r,Q'}(a) : Q' \in \mathcal{Q}_j\}.$$

For every $k \in \mathbb{N}$, we define the families $\mathcal{S}_k := \mathcal{S}(P_k)$ and

$$\widetilde{\mathcal{S}}_k := \begin{cases} \mathcal{S}(P_{k+1} \setminus P_k), & \text{if } P_{k+1} \in \{p^{\uparrow}(P_k), p^{\downarrow}(P_k)\}, \\ \emptyset, & \text{if } P_{k+1} = p^{\leftrightarrow}(P_k). \end{cases}$$

By Proposition 3.8, all the cylinders in $\mathcal{S}_k$ and $\widetilde{\mathcal{S}}_k$ are large admissible. Furthermore, the sets in $\mathcal{S}_k \cup (\bigcup_{\ell \geq k} \widetilde{\mathcal{S}}_\ell)$ are disjoint and their union has full measure in G with respect to μ.

Fix $k \in \mathbb{N}$ and take $\ell > k$ such that $\widetilde{\mathcal{S}}_\ell \neq \emptyset$. For every $P \in \widetilde{\mathcal{S}}_\ell$, we iterate Definition 3.5, by putting $s^1(P) = s(P)$ and $s^{m+1}(P) := s(s^m(P))$, for every $m \in \mathbb{N}$, $m \geq 1$, and we put

$$\widetilde{\mathcal{S}}_\ell^m := \{P' \in s^m(P) : P \in \widetilde{\mathcal{S}}_\ell\}.$$

Clearly, the disjoint union of the sets in $\widetilde{\mathcal{S}}_\ell^m$ has full measure in the union of the sets in $\widetilde{\mathcal{S}}_\ell$ with respect to μ. We define

$$\mathcal{D}_{-k}^Z := \mathcal{S}_k \cup \widetilde{\mathcal{S}}_k \cup \Big(\bigcup_{\ell > k} \widetilde{\mathcal{S}}_\ell^{\ell-k}\Big).$$

By construction, the sets in $\mathcal{D}_{-k}^Z$ are disjoint and their union has full measure in G with respect to μ. For $P \in \mathcal{S}_k \subset \mathcal{D}_{-k}^Z$, we define $p(P)$ to be $p^{\leftrightarrow}(P)$, $p^{\uparrow}(P)$ or $p^{\downarrow}(P)$, when $p(P_k)$ is given by $p^{\leftrightarrow}(P_k)$, $p^{\uparrow}(P_k)$ or $p^{\downarrow}(P_k)$, respectively. Observe that $p(P) \in \mathcal{S}_{k+1} \subset \mathcal{D}_{-k-1}^Z$. When $P \in \widetilde{\mathcal{S}}_k$, there is $P' \in \mathcal{S}_k$ such that $P = p(P') \setminus P'$ and then we put $p(P) := p(P') \in \mathcal{S}_{k+1} \subset \mathcal{D}_{-k-1}^Z$. In both cases $\mu(p(P)) \leq C_1 \mu(P)$ by Proposition 3.8. Finally, if $P \in \widetilde{\mathcal{S}}_\ell^{\ell-k}$ for some $\ell > k$, then we write $p(P)$ for the unique set in $\widetilde{\mathcal{S}}_\ell^{\ell-k-1} \subset \mathcal{D}_{-k-1}^Z$ such that $P \in s(p(P))$. The control of the measures follows by Proposition 3.6. The properties (*i*), (*ii*), and (*iv*) have been proved to be satisfied by sets in $\{\mathcal{D}_{-k}^Z\}_{k\in\mathbb{N}}$, since the last inequality in (*iv*) follows by the control of the measure, as already observed. Coming to the "positive generations," we define inductively

$$\mathcal{D}_k^Z := \{P' \in s(P) : P \in \mathcal{D}_{k-1}^Z\}, \qquad k \in \mathbb{N}.$$

For every $P' \in \mathcal{D}_k^Z$, we put $p(P') = P$ where $P \in \mathcal{D}_{k-1}^Z$ is such that $P' \in s(P)$. Note that (*i*), (*ii*) and (*iv*) follow by the construction above and by Proposition 3.6.

We showed that

$$\mu(V_k) = 0, \qquad V_k := G \setminus \bigcup_{P \in \mathcal{D}_k^Z} P, \qquad k \in \mathbb{Z}.$$

Furthermore, $V_k \subset V_{k+1}$ and then

$$\mu(V) = \lim_{k\to+\infty} \mu(V_k) = 0, \qquad V := \bigcup_{k\in\mathbb{Z}} V_k.$$

It remains to prove (*iii*). Let $x \in G \setminus V$. For every $k \in \mathbb{Z}$, there exists $P_k^x \in \mathcal{D}_k^Z$ such that $x \in P_k^x$. Clearly, $\{P_k^x\}_{k\in\mathbb{N}}$ is a sequence of elements of $\mathcal{D}^Z$ such that $p(P_k^x) = P_{k-1}^x$, then by Propositions 3.6 and 3.8, we have that

$$\mu(P_{k-1}^x) \geq \Big(1 + \frac{1}{C_1}\Big)\mu(P_k^x).$$

Hence, since $1 + \frac{1}{C_1} > 1$, by iterating the inequality we prove (*iii*). □

A natural consequence of Theorem 3.9 is that $L^1(\mu)$ admits a CZ decomposition with respect to the family $\mathcal{D}^Z$. The proof follows classical lines, but we include it here for the convenience of the reader.

Theorem 3.10. *For any $Z \in \mathcal{Z}$ and any $\mu \in \mathcal{F}_Z$, the space $L^1(\mu)$ admits a CZ decomposition with respect to the family $\mathcal{D}^Z$. Namely, for every $f \in L^1(\mu)$ and $\alpha > 0$ there exists a family of disjoint sets $\mathcal{E}(f,\alpha) = \{P_j\}$, with $P_j \in \mathcal{D}^Z$ and functions g, b_j such that $f = g + \sum_j b_j$ and*

(a) $|g| \leq C_1\alpha$ *μ-almost everywhere;*
(b) $b_j = 0$ *on* $G \setminus P_j$ *and* $\int_{P_j} b_j \, d\mu = 0$;
(c) $\sum_j \|b_j\|_1 \leq 2C_1\|f\|_1$.

Conversely, any set $P \in \mathcal{D}^Z$ belongs to $\mathcal{E}(f,\alpha)$ for some $f \in L^1(\mu)$ and some $\alpha > 0$.

Proof. Let $f \in L^1(\mu)$, and consider the dyadic Hardy–Littlewood maximal operator associated with $\mathcal{D}^Z$ applied to f, that is,

$$\mathcal{M}_Z^{\mathcal{D}} f(x) = \sup_{P \in \mathcal{D}^Z, P \ni x} \frac{1}{\mu(P)} \int_P |f| d\mu, \qquad x \in G.$$

Let $\alpha > 0$ and $E_\alpha = \{x \in G : \mathcal{M}_Z^{\mathcal{D}} f(x) > \alpha\}$. Let $V \subset G$ be the null measure set as in the proof of Theorem 3.9. For every $x \in E_\alpha \cap V$, let P^x be the set of maximal measure in $\mathcal{D}^Z$ among those containing x such that

$$\frac{1}{\mu(P^x)} \int_{P^x} |f| d\mu > \alpha.$$

Such set exists because, by Theorem 3.9 (*iii*), if $x \in P_k^x \in \mathcal{D}_k^Z$ then $\mu(P_k^x) \to +\infty$ as $k \to -\infty$. The family $\{P^x\}_{x \in E_\alpha}$ is at most countable and we denote it by $\{P_j\}$. We then have that $\bigsqcup_j P_j$ has full measure in E_α and

$$\mu(E_\alpha) = \sum_j \mu(P_j) \leq \frac{1}{\alpha} \sum_j \int_{P_j} |f| d\mu \leq \frac{1}{\alpha}\|f\|_1. \tag{10}$$

Moreover, for every j,

$$\frac{1}{\mu(P_j)} \int_{P_j} |f| d\mu > \alpha, \quad \text{and} \quad \frac{1}{\mu(p(P_j))} \int_{p(P_j)} |f| d\mu \leq \alpha. \tag{11}$$

We define now

$$g(x) = \begin{cases} \dfrac{1}{\mu(P_j)} \displaystyle\int_{P_j} f \, d\mu, & \text{if } x \in P_j, \\ f(x), & \text{else,} \end{cases}$$

$$b_j(x) = \left(f(x) - \frac{1}{\mu(P_j)} \int_{P_j} f \, d\mu \right) \chi_{P_j}(x).$$

Clearly (b) holds. Moreover by Equation (11) and Theorem 3.9 (*iv*),

$$\|b_j\|_1 \leq 2 \int_{P_j} |f| \, d\mu \leq 2 \int_{p(P_j)} |f| \, d\mu \leq 2\alpha\mu(p(P_j)) \leq 2C_1\alpha\mu(P_j). \tag{12}$$

Item (c) follows from Equations (10) and (12). To prove (a), assume first that $x \in E_\alpha$. Then, by Equation (11) and Theorem 3.9 we have

$$|g(x)| \leq \frac{1}{\mu(P_j)} \int_{P_j} |f| \, d\mu \leq \frac{C_1}{\mu(p(P_j))} \int_{p(P_j)} |f| d\mu \leq C_1\alpha.$$

On the other hand, since for almost every x we have the pointwise bound $|f| \le \mathcal{M}_Z^{\mathcal{D}} f$, it follows that for almost every $x \notin E_\alpha$,

$$|g(x)| = |f(x)| \le \mathcal{M}_Z^{\mathcal{D}} f(x) \le \alpha.$$

This proves that for any $f \in L^1(\mu)$ and any $\alpha > 0$ we can choose $\mathcal{E}(f,\alpha) \subset \mathcal{D}^Z$ with the desired properties, and therefore $\mathcal{E} = \{\mathcal{E}(f,\alpha) : f \in L^1(\mu), \alpha > 0\} \subset \mathcal{D}^Z$.

Now, we show that, conversely, for every $P \in \mathcal{D}^Z$ there exists a function $f \in L^1(\mu)$ and a number $\alpha > 0$ such that $P \in \mathcal{E}(f,\alpha)$, which proves that $\mathcal{E} = \mathcal{D}^Z$. To see this, fix an arbitrary set $P_0 \in \mathcal{D}^Z$, let $S \in s(P_0)$ and let R be the set of minimal measure in $\mathcal{D}^Z$ among those properly containing P_0. Set $f = \chi_S$ and $\alpha = \mu(S)/\mu(R)$. Then,

$$M_Z^{\mathcal{D}} f(x) = \sup_{P \in \mathcal{D}^Z, P \ni x} \frac{\mu(P \cap S)}{\mu(P)}.$$

Hence, if $x \in S$ we have $M_Z^{\mathcal{D}} f(x) = 1 > \alpha$, while for $x \notin S$, denoting by P' the smallest element of $\mathcal{D}^Z$ containing both x and S, we have $M_Z^{\mathcal{D}} f(x) = \mu(S)/\mu(P')$. But according to Theorem 3.9, the smallest set in $\mathcal{D}^Z$ containing S is P_0. It follows that $M_Z^{\mathcal{D}} f(x) = \mu(S)/\mu(P_0) > \alpha$ if $x \in P_0 \setminus S$, while $M_Z^{\mathcal{D}} f(x) \le \mu(S)/\mu(R) = \alpha$ if $x \notin P_0$. It follows that $E_\alpha = \{x \in G : \mathcal{M}_Z^{\mathcal{D}} f(x) > \alpha\} = P_0$. Following verbatim the above construction, one obtains a CZ decomposition for f where the family $\{P_j\}$ consists of a single element, which is P_0. □

4 | THE MAXIMAL HARDY–LITTLEWOOD OPERATOR

Along the lines of the proof of Theorem 3.10 in the previous section, it is proved that the dyadic Hardy–Littlewood maximal function is of weak-type (1,1). In this section, we provide a covering lemma for admissible cylinders which allows us to deduce that also the Hardy–Littlewood maximal function associated with the family of all admissible cylinders (not necessarily dyadic) is of weak-type (1,1).

This result fits into a quite active line of research, since in the last years many authors investigated the L^p and weak-type (1,1) boundedness of Hardy–Littlewood maximal operators in nondoubling metric measure spaces, such as Lie groups and manifolds of exponential growth [1, 12–14, 18, 38, 41], nondoubling infinite graphs [9, 17, 23, 25, 33, 35], and, more generally, nondoubling measure metric spaces [19, 20, 31, 37].

Definition 4.1. Given a constant $C > 1$ and a cylinder $P = P_{r,Q}(a)$ its *C-envelope* is defined as

$$P^{(C)} := P_{r^C,Q}(a).$$

Fix $Z \in \mathcal{Z}$ and $\mu \in \mathcal{F}_Z$. Observe that by Proposition 3.2 (vi)

$$\mu(P^{(C)}) = C\mu(P). \tag{13}$$

It is well known that in a metric measure space of homogeneous type the Vitali covering Lemma implies the weak-type (1,1) boundedness of the Hardy–Littlewood maximal function, see, for example, [5, Theorem 2.1]. Hence, it is natural in our context to check whether one can obtain appropriate covering lemmas for admissible cylinders which, together with Equation (13), imply the weak-type (1,1) boundedness for the Hardy–Littlewood maximal function

$$\mathcal{M}_Z f(x) = \sup_{\substack{P \in \mathcal{P}^Z \\ P \ni x}} \frac{1}{\mu(P)} \int_P |f| d\mu, \qquad f \in L^1_{\mathrm{loc}}(\mu),\ x \in G,$$

where $\mathcal{P}^Z$ denotes the family of all admissible cylinders.

In this section, we carry out this program and we are able to prove the following maximal theorem.

Theorem 4.2. *For any $Z \in \mathcal{Z}$, $\mu \in \mathcal{F}_Z$, $\alpha > 0$ and $f \in L^1(\mu)$,*

$$\mu(\{x \in G : \mathcal{M}_Z f(x) > \alpha\}) \le \frac{C_2}{\alpha} \|f\|_1,$$

with $C_2 = 3 \max\{\gamma + 1 + \log \lambda, \lambda e^3\}$.

The next result is a fundamental step in obtaining a covering lemma for admissible cylinders, from which Theorem 4.2 will follow as a rather direct consequence.

Lemma 4.3. *Let $P_j = P_{r_j,Q_j}(a_j)$, $j = 1,2$, be two admissible cylinders such that $P_1 \cap P_2 \neq \emptyset$. Denote by k_j the generation of Q_j. If $k_1 \leq k_2$, then $P_2 \subset P_1{}^{(C_2)}$.*

Proof. Let $I_j := U_{r_j}(a_j)$, defined as in Equation (5). By Proposition 3.2 (*iv*), Q_1 and Q_2 must intersect, hence due to the properties of dyadic sets and $k_2 \geq k_1$ it must be $Q_2 \subset Q_1$, and $I_1 \cap I_2 \neq \emptyset$, hence

$$\left(\frac{a_1}{r_1}, a_1 r_1\right) \cap \left(\frac{a_2}{r_2}, a_2 r_2\right) \neq \emptyset, \quad \text{that is,} \quad a_1 < a_2 r_1 r_2, \tag{14}$$

$$a_2 < a_1 r_1 r_2. \tag{15}$$

For every interval $I = (t-R, t+R) \subset \mathbb{R}$ and $C > 0$ we put $I^C := (t - CR, t + CR)$. Hence, it is enough to prove that $I_2 \subset I_1^{C_2} = U_{r_1^{C_2}}(a_1)$, or equivalently, that

$$\log(ar_2) \leq C_2 \log r_1, \quad a := \max\left\{\frac{a_1}{a_2}, \frac{a_2}{a_1}\right\}.$$

If $r_1 \geq r_2$ the result is easily proved: by Equations (14) and (15), we have

$$\log(ar_2) < \log(r_1 r_2^2) \leq 3\log(r_1).$$

Hereinafter, let $r_1 < r_2$. By the same argument we have $\log(ar_2) \leq 3\log(r_2)$. To prove the result, it is therefore sufficient to show that $3\log r_2 \leq C_2 \log r_1$. In order to do that, we distinguish three cases. If P_2 is small then also P_1 is, and

$$a_2 e^2 \log r_2 \leq \delta^{k_2} \leq \delta^{k_1} \leq \lambda a_1 e^2 \log r_1,$$

which together with Equation (14) gives

$$\log r_2 < \lambda r_1 r_2 \log r_1 \leq \lambda e^2 \log r_1.$$

If P_1 and P_2 are both large, then

$$a_2 r_2^2 \leq \delta^{k_2} \leq \delta^{k_1} \leq \lambda a_1 r_1^{\gamma},$$

which together with Equation (14), and the fact that, since $r_1 > e$, $\lambda < r_1^{\log\lambda}$, gives

$$r_2 < \lambda r_1^{\gamma+1} < r_1^{\gamma+1+\log\lambda},$$

which in turn implies

$$\log r_2 \leq (\gamma + 1 + \log\lambda)\log r_1.$$

Finally, if P_2 is large and P_1 is small, then

$$a_2 r_2^2 \leq \delta^{k_2} \leq \delta^{k_1} \leq \lambda a_1 e^2 \log r_1,$$

which together with Equation (14) gives

$$\log r_2 \leq r_2 < \lambda r_1 e^2 \log r_1 \leq \lambda e^3 \log r_1.$$

Summing up, in any case, if P_1 and P_2 are admissible then

$$3\log r_2 \leq C_2 \log r_1,$$

where $C_2 = 3\max\{\gamma + 1 + \log\lambda, \lambda e^3\}$, since $\lambda e^3 > 1$, and this concludes the proof. □

From this technical geometric result, a Vitali-like covering lemma for admissible cylinders follows.

Lemma 4.4 (Covering lemma for admissible cylinders). *Let $\mathcal{C}$ be a family of admissible cylinders such that*

$$k_0 := \min\{k \in \mathbb{Z} : P_{r,Q}(a) \in \mathcal{C},\ Q \in \mathcal{Q}_k\} > -\infty.$$

Then, there exists a countable subfamily $\mathcal{G} \subset \mathcal{C}$ such that the cylinders of $\mathcal{G}$ are pairwise disjoint and for each $P \in \mathcal{C}$ there exists $R \in \mathcal{G}$ with $P \cap R \neq \emptyset$ and $P \subset R^{(C_2)}$. In particular,

$$\bigcup_{P\in\mathcal{C}} P \subset \bigcup_{R\in\mathcal{G}} R^{(C_2)}.$$

Proof. Choose a cylinder $P_0 \in \mathcal{C}$ with base set of generation k_0. For every $i \geq 1$, among all the sets of $\mathcal{C}$ not intersecting P_j for any $j < i$ (if any exists), let P_i be one with base set of minimal generation, say k_i. The elements of the resulting family $\mathcal{G} = \{P_i\}_{i=0}^{\infty}$ are pairwise disjoint. Moreover, for any $P \in \mathcal{C}$ there exists an index i such that $P \cap P_i \neq \emptyset$. In particular, if P_i is a cylinder with base set of minimal generation among those of $\mathcal{G}$ intersecting P, then the generation of the base set of P must be $\geq k_i$. In fact, either $P \in \mathcal{G}$, in which case it must be $P = P_i$, or $P \notin \mathcal{G}$, and since P does not intersect elements of $\mathcal{G}$ having base sets of generation $< k_i$, the generation of its base set cannot be $< k_i$, since otherwise P would belong to $\mathcal{G}$. Hence, by Proposition 4.3, $P \subset {P_i}^{(C_2)}$. □

We are now in a position to prove Theorem 4.2.

Proof of Theorem 4.2. Let $x \in E_\alpha = \{x \in G : \mathcal{M}_Z f(x) > \alpha\}$, and choose an admissible cylinder P_x containing x such that

$$\int_{P_x} |f| \mathrm{d}\mu > \alpha\mu(P_x).$$

Let K be a compact subset of E_α. Then, $\{P_x : x \in K\}$ is a covering of K. Since K is compact, there exists a subcovering $\mathcal{C}$ of K made by a finite family of admissible cylinders. Since the family $\mathcal{C}$ is finite, we can extract from it a subfamily $\mathcal{G}$ with the properties prescribed by Lemma 4.4. Then,

$$\|f\|_1 \geq \sum_{P\in\mathcal{G}} \int_P |f| d\mu \geq \alpha \sum_{P\in\mathcal{G}} \mu(P) = \frac{\alpha}{C_2} \sum_{P\in\mathcal{G}} \mu(P^{(C_2)}) \geq \frac{\alpha}{C_2}\mu(K).$$

By the inner regularity of the measure, passing to the supremum over all compact subsets K of E_α we obtain the desired result. □

5 | THE CZP FOR THE DISTANCE d_Z

In this section, we prove our main result, Theorem 1.2, by showing that for any $Z \in \mathcal{Z}$ and any $\mu \in \mathcal{F}_Z$ the family $\mathcal{D}^Z$ of dyadic admissible cylinders constructed in Theorem 3.9 is a CZ family for (G, d_Z, μ). Here, d_Z denotes a metric on G which takes into account the action of the vector field Z, defined by

$$d_Z((n,a),(n',a')) := d_G\Big(\big(nn(\log a)^{-1}, a\big), \big(n'n\big(\log a'\big)^{-1}, a'\big)\Big), \tag{16}$$

for every $(n,a),(n',a') \in G$. We call this metric a Z-flow metric.

Observe that the symmetry of d_Z follows from the symmetry of d_G, as well as the triangular inequality. If $d_Z((n,a),(n',a')) = 0$, then $a = a'$ and

$$nn(\log a)^{-1} = n'n\big(\log a'\big)^{-1} = n'n(\log a)^{-1},$$

which implies $n = n'$.

We shall denote by $B^Z_G(x,R)$ the ball centered at $x \in G$ with radius $R > 0$ with respect to d_Z.

Lemma 5.1. *There exists a constant $C_3 > 0$ such that for every admissible cylinder $P = P_{r,Q}(a)$,*

$$P \subset B^Z_G\big((n_Q,1)\exp((\log a)Z), C_3 \log r\big),$$

where n_Q is defined in Theorem 3.3 (iii).

Proof. Let $x = (n, 1)\exp(tZ)$ be a point in P. By Equation (1) and the fact that $e^t \in (a/r, ar)$, we get

$$\cosh\Big(d_Z\big(x, (n_Q, 1)\exp((\log a)Z)\big)\Big) = \cosh\Big(d_G\big((n, e^t), (n_Q, a)\big)\Big)$$

$$= \cosh\left(\log\frac{e^t}{a}\right) + \frac{1}{2ae^t} d_N\big(n, n_Q\big)^2 < \cosh(\log r) + \frac{d_N\big(n, n_Q\big)^2}{2a^2} r.$$

If $r > e$, since P is admissible $d_N\left(n, n_Q\right) \le C_1\delta^k \le C_1\lambda a r^\gamma$, by Theorem 3.3 (*iii*). It follows that there exists $C > 0$ such that for every $r > e$

$$\cosh\Big(d_Z\big(x, (n_Q, 1)\exp((\log a)Z)\big)\Big) < \cosh(\log r) + \frac{C_1^2\lambda^2}{2} r^{2\gamma+1} \le C r^{2\gamma+1},$$

and then

$$d_Z\big(x, (n_Q, 1)\exp((\log a)Z)\big) < \operatorname{arcosh}(C r^{2\gamma}) \le \log(2C) + (2\gamma + 1)\log r \lesssim \log r.$$

If $1 < r \le e$, since P is admissible $d_N\left(n, n_Q\right) \le C_1\delta^k \le C_1\lambda a e^2 \log r$, by Theorem 3.3 (*iii*), there exists $C > 0$ such that

$$\cosh\Big(d_Z\big(x, (n_Q, 1)\exp((\log a)Z)\big)\Big) < \cosh(\log r) + \frac{C_1^2 e^4\lambda^2}{2}(\log r)^2 r$$

$$\le \cosh(C\log r).$$

So, we proved

$$d_Z\big(x, (n_Q, 1)\exp((\log a)Z)\big) \lesssim \log r,$$

as desired. □

Lemma 5.2. *Let $P = P_{r,Q}(a)$ be an admissible cylinder. We have that*

$$P^* := \{x \in G : d_Z(x, P) < \log r\} \subset P_{r^2, B_N(n_Q, C^*\delta^k)}(a), \tag{17}$$

where $C^* = C_1 + \sqrt{2}$. In particular, if $D(\mu_N, C^*/c) > 0$ is as in Equation (4), then

$$\mu(P^*) \le C_4\mu(P), \qquad \text{where} \quad C_4 = 2D\left(\mu_N, \frac{C^*}{c}\right).$$

Proof. Let $x = (n, 1)\exp(tZ) \in P^*$ and $y = (m, 1)\exp(sZ) \in P$ such that $d_Z(x, y) < \log r$. Then,

$$\cosh(\log r) > \cosh\left(d_Z(x, y)\right) = \cosh\Big(d_G\big((n, e^t), (m, e^s)\big)\Big)$$

$$= \cosh\left(|t - s|\right) + \frac{1}{2e^{t+s}} d_N(n, m)^2. \tag{18}$$

On the one hand, this gives $|t - s| < \log r$, which implies $t \in U_{r^2}(a)$. On the other hand, if P is large admissible, from Equation (18) one gets

$$d_N(n, m)^2 < 2e^{t+s}\cosh(\log r) < 2a^2 r^4 \le 2\delta^{2k},$$

while if P is small admissible, again starting from Equation (18) but using a finer estimate we get

$$d_N(n, m)^2 < 2e^{t+s}(\cosh(\log r) - 1) = 4e^{t+s}\left(\sinh\left(\frac{\log r}{2}\right)\right)^2$$

$$< 4a^2 r^3 \log^2 r \le \frac{4}{e}(ae^2\log r)^2 \le \frac{4}{e}\delta^{2k}.$$

Hence, for any admissible P, $d_N(n, m) \leq \sqrt{2}\delta^k$. Since $m \in B_N(n_Q, C_1\delta^k)$, it follows that $n \in B_N(n_Q, (C_1 + \sqrt{2})\delta^k)$. This proves Equation (17). The inequality involving the measures now simply follows from Proposition 3.2 (vi), Equation (4), and Theorem 3.3 (iii). Indeed

$$\begin{aligned}\mu(P^*) &\leq 4\log r\,\mu_N(B_N(n_Q, C^*\delta^k)) \\ &\leq 4\log r D\left(\mu_N, \frac{C^*}{c}\right)\mu_N(B_N(n_Q, c\delta^k)) \\ &\leq 4\log r D\left(\mu_N, \frac{C^*}{c}\right)\mu_N(Q) = 2D\left(\mu_N, \frac{C^*}{c}\right)\mu(P).\end{aligned}$$

□

We are now ready to complete the proof of our main result.

Proof of Theorem 1.2. Set $C_0 = \max\{C_1, C_3, C_4\}$ and let $P_j = P_{r_j,Q_j}(a_j)$ be the sets in $\mathcal{D}^Z$ for which, according to Theorem 3.10, items (a), (b), and (c) of Definition 1.1 hold. By means of Lemma 5.1, we know that

(d) $P_j \subset B_G^Z(x_j, C_0R_j)$,

with $x_j = (n_{Q_j}, 1)\exp((\log a)Z)$ and $R_j = \log r_j$. Now, let $P_j^* = \{x : d_Z(x, P_j) < R_j\}$. By Lemma 5.2 and the construction of P_j's in Theorem 3.10, we get

$$\sum_j \mu(P_j^*) \leq C_4 \sum_j \mu(P_j) \leq \frac{C_4}{\alpha}\sum_j \int_{P_j} |f|\mathrm{d}\mu \leq \frac{C_4}{\alpha}\|f\|_1,$$

which is item (e) in Definition 1.1. □

It is immediate to see that, for any $Z \in \mathcal{Z}$ and any $\mu \in \mathcal{F}_Z$, the space (G, d_Z, μ) with the family $\mathcal{D}^Z$ satisfies what in [28, Definition 3.3] is called condition (C) (take $\mathcal{R}' = \mathcal{R} = \mathcal{D}^Z$ in that definition). This implies that one can introduce a suitable Hardy space $H^1(\mu)$ and a corresponding space $\mathrm{BMO}(\mu)$ as in [43].

Definition 5.3. An atom is a function $a \in L^1(\mu)$ such that

(1) a is supported in an admissible cylinder P;
(2) $\|a\|_2 \leq \mu(P)^{-1/2}$;
(3) $\int_P a\mathrm{d}\mu = 0$.

The Hardy space $H^1(\mu)$ is the Banach space

$$H^1(\mu) := \{f \in L^1(\mu) : f = \sum_j \lambda_j a_j,\ a_j \text{ atoms}, \lambda_j \in \mathbb{C}, \sum_j |\lambda_j| < \infty\}$$

endowed with the norm

$$\|f\|_{H^1} := \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j,\ a_j \text{ atoms}, \lambda_j \in \mathbb{C}\right\}.$$

Analogously, we introduce $\mathrm{BMO}(\mu)$ in the natural way that makes it the dual space of $H^1(\mu)$, as expected (see [5] for details). By [17, Theorem 1.2] see also [28, Theorem 3.2] and [43, Theorem 3.10]), as an immediate consequence of Theorem 1.2 we have the following result concerning the boundedness of a class of integral operators on (G, d_Z, μ).

Theorem 5.4. *Let $Z \in \mathcal{Z}$ and $\mu \in \mathcal{F}_Z$. Let $T = \sum_{j\in\mathbb{Z}} T_j$ be a linear operator bounded on $L^2(\mu)$, where the T_j's are integral operators with kernel K_j and the series converges in the strong operator topology on $L^2(\mu)$. Assume that there exist positive* *constants b, B, ε and $C > 1$ such that*

$$\int_G |K_j(x,y)|(1 + C^j d_Z(x,y))^\varepsilon d\mu(x) \le B, \quad y \in G,$$

$$\int_G |K_j(x,y) - K_j(x,z)| d\mu(x) \le B(C^j d_Z(y,z))^b \quad y, z \in G.$$

Then, T extends from $L^1(\mu) \cap L^2(\mu)$ to an operator of weak-type (1,1), bounded on $L^p(\mu)$, for $1 < p \le 2$ and bounded from $H^1(\mu)$ to $L^1(\mu)$.

By duality, one can also obtain boundedness on $L^p(\mu)$ for $2 \le p < \infty$ and from L^∞ to $\mathrm{BMO}(\mu)$ for a class of operators whose kernels satisfy a dual version of the integral Hörmander condition appearing in the above theorem.

6 | THE CZP AND THE d_G METRIC

This section is devoted to discussing the relationship between the metric d_Z introduced in the previous section and the Carnot–Carathéodory metric d_G. In the first subsection, we will see that, in the general case of N nonabelian, $d_G = d_Z$ if $Z = X_0$, and if N is Abelian, d_Z is equivalent to d_G for all $Z \in \mathcal{Z}$. This implies that for $Z = X_0$ (and any $\mu \in \mathcal{F}_{X_0}$) the metric d_Z can be always substituted by d_G in Theorem 1.2. If N is Abelian, such a substitution can be made for any $Z \in \mathcal{Z}$. One may wonder if, indeed, also when N is nonabelian d_Z can be substituted by d_G for any $Z \in \mathcal{Z}$. In the second subsection we prove that the answer to the above question is negative, even for $\mu = \rho$. This, somehow, justifies our use of the new metric d_Z, which turns out to be more adapted to the context.

6.1 | Comparison with known results

Let $Z = X_0$ be the vertical vector field. In this case, $n(t) = 1_N$ for every $t \in \mathbb{R}$ and then $d_Z = d_G$. Then, as a special case of Theorem 1.2, we immediately have the following result.

Theorem 6.1. *The measure metric space (G, d_G, μ) has the CZP with respect to the family $\mathcal{D}^{X_0}$ for every $\mu \in \mathcal{F}_{X_0}$.*

This result was previously known only for $\mu = \rho$ [28, Theorem 3.20]. Hence, Theorem 6.1 can be considered, at the same time, a generalization and a new proof of the result by Martini et al.

The rest of the subsection is devoted to showing that if N is Abelian then the vector field X_0 in Theorem 6.1 can be substituted by any $Z \in \mathcal{Z}$.

Theorem 6.2. *If $N = \mathbb{R}^m$, then the measure metric space (G, d_G, μ) has the CZP with respect to $\mathcal{D}^Z$ for every $Z \in \mathcal{Z}$ and every $\mu \in \mathcal{F}_Z$.*

This result was previously known only for $Z = X_0$ and for $\mu = \rho$ [17, Lemma 5.1]. Hence, Theorem 6.2 can be considered, at the same time, a generalization and a new proof of the result by Hebisch and Steger.

We begin the discussion which will lead to the proof of Theorem 6.2. The semidirect group $G = \mathbb{R}^m \rtimes A$ is the affine group of $\mathbb{R}^{m+1}$ obtained from translations by vectors of $\mathbb{R}^m$ and by homogeneous dilations. The group G can be realized as group of matrices in $\mathrm{GL}(m+1)$ as

$$G = \left\{ g(n,a) = \begin{bmatrix} A & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{R}^m, A = \mathrm{diag}(a, \dots, a) \in \mathrm{GL}(m), a \in A \right\},$$

indeed the semidirect product is preserved by the matrix multiplication. The Lie algebra $\mathfrak{g}$ of G is then

$$\mathfrak{g} = \left\{ \begin{bmatrix} D & X \\ 0 & 0 \end{bmatrix} : X \in \mathbb{R}^m, D = \mathrm{diag}(s, \dots, s) \in M(m), s \in \mathbb{R} \right\}.$$

Furthermore, if we fix the canonical orthonormal basis $\check{X}_1, \dots, \check{X}_m$ of $\mathfrak{n}_1 = \mathfrak{n} \simeq \mathbb{R}^m$, then the Carnot–Carathéodory metric on N coincides with the Euclidean metric and the lifting X_ℓ of $\check{X}_\ell$ is such that

$$(X_\ell)_{ij} = \begin{cases} 1 & \text{if } (i,j) = (\ell, m+1), \\ 0 & \text{otherwise,} \end{cases}$$

for every $1 \le \ell \le m$. Let d_G be the Carnot–Carathéodory induced by the vector fields $X_0, X_1, \dots, X_m$, where $X_0|_{(n,a)} = X_0|_a$, $X_0 = \partial_a \in \mathfrak{a}$.

A vector field $Z \in \mathfrak{g}$ with $\langle Z, X_0 \rangle = 1$ has the form

$$Z = X_0 + \sum_{\ell=1}^{m} \beta_\ell X_\ell, \qquad \beta_1, \dots, \beta_m \in \mathbb{R}. \tag{19}$$

By direct computation, one can easily obtain that for every $t \in \mathbb{R}$

$$\exp(tZ) = \begin{bmatrix} \operatorname{diag}(e^t, \dots, e^t) & (e^t - 1)\beta \\ 0 & 0 \end{bmatrix},$$

where $\beta^T := (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$. In particular, this means that

$$n(t) = (e^t - 1)\beta. \tag{20}$$

Proposition 6.3. *If $G = \mathbb{R}^m \rtimes A$ and Z is as in Equation (19), then the distance d_Z is left-invariant and is equivalent to d_G.*

Proof. Let $(n,a), (n',a') \in G$. By Equations (16) and (20), we have that

$$\begin{aligned}
\cosh\Big(d_Z\big((n',a')^{-1}(n,a), 1_G\big)\Big) &= \cosh\left(d_Z\Big(\Big(\frac{n-n'}{a'}, \frac{a}{a'}\Big), 1_G\Big)\right) \\
&= \cosh\Big(\log\Big(\frac{a}{a'}\Big)\Big) + \frac{a'}{2a}\left|\frac{n-n'}{a'} - \Big(\frac{a}{a'} - 1\Big)\beta\right|^2 \\
&= \cosh\Big(\log\Big(\frac{a}{a'}\Big)\Big) + \frac{1}{2aa'}|n - n' + (a'-a)\beta|^2 \\
&= \cosh\Big(\log\Big(\frac{a}{a'}\Big)\Big) + \frac{1}{2aa'}|(n - \beta(a-1)) - (n' - \beta(a'-1))|^2 \\
&= \cosh\Big(d_Z\big((n,a),(n',a')\big)\Big),
\end{aligned}$$

which proves the left-invariance of d_Z.

Since both d_G and d_Z are left-invariant, to prove that they define equivalent metrics it is sufficient to check that there exists a constant $D \ge 1$ such that

$$D^{-1} d_Z\big((n,a), 1_G\big) \le d_G\big((n,a), 1_G\big) \le D d_Z\big((n,a), 1_G)\big),$$

for every $(n,a) \in G$. For simplicity, we apply the change of coordinate $t = \log a$. Observe that by Equation (20)

$$\frac{|n(t)|^2}{e^t} = \frac{|\beta|^2(e^t-1)^2}{e^t} = 2|\beta|^2(\cosh t - 1).$$

Then, by the definition of d_Z and Equation (1),

$$\begin{aligned}
\cosh\big(d_Z((n,e^t), 1_G)\big) = \cosh\big(d_G((n - n(t), e^t), 1_G)\big) &= \cosh(t) + \frac{|n - n(t)|^2}{2e^t} \\
&\le \cosh(t) + \frac{|n|^2}{e^t} + \frac{|n(t)|^2}{e^t} \le D\cosh\big(d_G((n,e^t), 1_G)\big),
\end{aligned}$$

where $D = \max\{2|\beta|^2 + 1, 2\}$. On the other hand,

$$\cosh\left(d_G((n, e^t), 1_G)\right) \le \cosh(t) + \frac{|n - n(t)|^2}{e^t} + \frac{|n(t)|^2}{e^t}$$
$$\le D\cosh\left(d_Z((n, e^t), 1_G)\right).$$

Since $D \ge 2$, we can choose a constant C such that $D\cosh x \le \cosh(Dx)$ for $x \ge C$. But if $|t| \ge C$, by Equation (1) we have that both $d_G((n, e^t), 1_G)$ and $d_Z((n, e^t), 1_G)$ are greater or equal to C, and therefore

$$\frac{1}{D} d_G((n, e^t), 1_G) \le d_Z((n, e^t), 1_G) \le D d_G((n, e^t), 1_G), \quad |t| \ge C.$$

It is also clear that

$$\lim_{|n|\to\infty} \frac{d_Z((n, e^t), 1_G)}{d_G((n, e^t), 1_G)} = 1, \quad |t| < C.$$

It remains to study the behavior of the ratio of the distances when (n, t) tends to (0,0). In this case, we have the asymptotic estimates

$$d_G((n, e^t), 1_G) = \operatorname{arccosh}\left(\cosh(t) + \frac{1}{2e^t}|n|^2\right)$$
$$\sim \sqrt{2(\cosh(t) - 1) + |n|^2} \sim \sqrt{t^2 + |n|^2},$$

and

$$d_Z((n, e^t), 1_G) = \operatorname{arccosh}\left(\cosh(t) + \frac{1}{2e^t}|n + \beta(e^t - 1)|^2\right) \sim \sqrt{t^2 + |n + \beta t|^2}.$$

Then, for $(n, t) \to (0, 0)$ we have

$$\frac{d_G((n, e^t), 1_G)^2}{d_Z((n, e^t), 1_G)^2} \sim \frac{t^2 + |n|^2}{t^2 + |n + \beta t|^2} = \begin{cases} 1 & t = 0, \\ \Phi_\beta\left(\frac{n}{t}\right) & t \ne 0, \end{cases} \tag{21}$$

where $\Phi_\beta : \mathbb{R}^m \to [0, +\infty)$ is defined by

$$\Phi_\beta(v) := \frac{1 + |v|^2}{1 + |v + \beta|^2}, \qquad v \in \mathbb{R}^m.$$

Since Φ_β is bounded from below and from above by uniform positive constants on $\mathbb{R}^m$, the same is true for the left-hand side in Equation (21) for (n, t) in a compact neighbor of (0,0). □

Proof of Theorem 6.2. Since the CZP is invariant for equivalent metrics, by Theorem 1.2 and Proposition 6.3 we immediately get the result. □

Remark 6.4. A similar discussion to that carried out above also applies to Theorem 5.4, which was previously known to hold only for $d_Z = d_G$ and $\mu_Z = \rho$, as consequences of [17, Lemma 5.1] (when N is Abelian) and [28, Theorem 3.20] (when N is nonabelian). Although we do not provide explicit examples here, we can conclude that Theorem 5.4 provides boundedness results for a wider class of integral operators than those covered by [17, 28].

6.2 | Optimality of Theorem 6.1

In this section, we show that in Theorem 6.1, in general X_0 cannot be substituted by an *arbitrary* $Z \in \mathcal{Z}$, even for $\mu = \rho$. To construct a counterexample, we will consider the extended Heisenberg group, where some recurrent quantities that were abstract so far, such as $n(t)$ and $d_N(n, 1_N)$, can be made explicit. In the first subsection, we prove some geometric lemmas, of possibly independent interest, holding true for vector fields $Z \in \mathcal{Z}_1$, the subclass of $\mathcal{Z}$ made of vector fields with nonzero components only in the first layer of $\mathfrak{g}$. In the second subsection, we recall some general facts about the extended Heisenberg group $\mathbb{H}^1_e$ and compute the relevant quantities in this setting. Then, we construct the counterexample on $(\mathbb{H}^1_e, d_{\mathbb{H}^1_e}, \rho)$, using a vector field $Z \in \mathcal{Z}_1$.

6.2.1 | Geometric lemmas for $\mathcal{Z}_1$

We introduce the subfamily of vector fields of G having no components in higher layers of $\mathfrak{n}$, that is,

$$\mathcal{Z}_1 := \{Z \in \mathcal{Z} : Z \in \mathfrak{n}_1 \oplus \mathfrak{a} \subset \mathfrak{g}\}.$$

In other words, for every $Z \in \mathcal{Z}_1$, there exist $\beta_1, \dots, \beta_{q_1} \in \mathbb{C}$ such that

$$Z = X_0 + \sum_{i=1}^{q_1} \beta_i X_{1,i}.$$

Remark 6.5. For every $Z \in \mathcal{Z}_1$ and $t \in \mathbb{R}$ it holds

$$d_G(\exp(tZ), 1_G) \leq |t| \|Z\|, \tag{22}$$

where $\|Z\| = \langle Z, Z\rangle^{1/2}$, and the inner product is defined in Section 2.

Indeed, consider the curve $\gamma : [0,t] \to G$ defined by $\gamma(s) := \exp(sZ)$, joining 1_G and $\exp(tZ)$ in G. Now,

$$\dot{\gamma}(s_0) = \frac{d}{ds}\Big|_{s=s_0} \exp(sZ) = Z|_{\exp(s_0 Z)} \in HG,$$

by Theorem 3.31 in [46] and the fact that $Z \in \mathcal{Z}_1$. Hence, γ is a horizontal curve of G. By the fact that $Z|_{\exp(sZ)} = (\mathrm{d}L_{\exp(sZ)})_{1_G}(Z)$ and the left-invariance of the metric tensor, we have that

$$d_G(1_G, \exp(tZ)) \leq \int_0^t \|\dot{\gamma}(s)\| \mathrm{d}s = \int_0^t \|Z\| \mathrm{d}s = |t| \|Z\|.$$

Lemma 6.6. *Let $Z \in \mathcal{Z}_1$. For every $R > 0$,*

$$P_{e^{R/(2\|Z\|)}, B_N(1_N, R/2)}(1) \subset B_G(1_G, R).$$

Proof. Let $\eta > 0$ and $x = (n,1)\exp(tZ)$, with $n \in B_N(1_N, \eta)$, $t \in U_r(1)$, be an arbitrary point in $P_{r, B_N(1_N,\eta)}(1)$. By the triangular inequality,

$$d_G(x, 1_G) \leq d_G(x, (n,1)) + d_G((n,1), 1_G).$$

Now, by the left-invariance of d_G, Equation (22), and the fact that $|t| < \log r$ for $t \in U_r(1)$,

$$d_G(x, (n,1)) = d_G(\exp(tZ), 1_G) \leq \|Z\| |t| < \|Z\| \log r,$$

while by Equation (1),

$$\cosh\left(d_G((n,1), 1_G)\right) = 1 + \frac{1}{2} d_N(n, 1_N)^2 \leq 1 + \frac{\eta^2}{2} \leq \cosh \eta.$$

Gluing all together and choosing $r = e^{R/(2\|Z\|)}$ and $\eta = R/2$ we get the desired result. ☐

Lemma 6.7. *Let $Z \in \mathcal{Z}_1$ and $P = P_{r,Q}(a)$, where $Q \in \mathcal{Q}$ is a Christ cube in N. For every $R > 0$,*

$$P_{re^{R/(2\|Z\|)}, n_Q \cdot \Psi_{r,a}(B_N(1_N, R/2))}(a) \subset \{x \in G : d_G(x, P) < R\},$$

where $\Psi_{r,a}$ is defined in Proposition 3.2 (v).

Proof. Let $k \in \mathbb{Z}$ be the generation of Q. Recalling that, by (*iii*) of Theorem 3.3, $n_Q B_N(1_N, c\delta^k) \subset Q$ and applying first Lemma 6.6 and then Proposition 3.2 (v), we get

$$\begin{aligned}\{x \in G : d_G(x,P) < R\} &= \bigcup_{x\in P} B_G(x,R) = \bigcup_{x\in P} xB_G(1_G,R) = PB_G(1_G,R)\\ &\supset P_{r,n_Q B_N(1_N,c\delta^k)} P_{e^{R/(2\|Z\|)},B_N(1_N,R/2)}(1)\\ &\supset P_{re^{R/(2\|Z\|)},n_Q B_N(1_N,c\delta^k)\cdot\Psi_{r,a}(B_N(1_N,R/2))}(a)\\ &\supset P_{re^{R/(2\|Z\|)},n_Q\cdot\Psi_{r,a}(B_N(1_N,R/2))}(a),\end{aligned}$$

where the last inclusion simply follows from the fact that $1_N \in B_N(1_N, c\delta^k)$. □

6.2.2 | The counterexample

We consider the Heisenberg group $\mathbb{H}^1$, that is, $\mathbb{R}^3$ endowed with the product

$$(q,p,\tau)\cdot_{\mathbb{H}^1}(q',p',\tau') := \left(q+q', p+p', \tau+\tau' - \frac{1}{2}(qp'-pq')\right).$$

The neutral element is then (0,0,0) and $(q,p,\tau)^{-1} = (-q,-p,-\tau)$, $\mathbb{H}^1$ is nilpotent, hence unimodular. A Haar measure is $dq\,dp\,d\tau$ and we denote by $|\cdot|$ the Haar measure of sets. The Heisenberg group admits the following realization inside $\mathrm{Sp}(2,\mathbb{R})$, namely

$$N := \left\{ n(q,p,\tau) := \begin{bmatrix} 1 & 0 & 0 & 0\\ p & 1 & 0 & 0\\ \tau & -q/2 & 1 & -p\\ -q/2 & 0 & 0 & 1\end{bmatrix} : q,p,\tau \in \mathbb{R}\right\} \subset \mathrm{Sp}(2,\mathbb{R}).$$

The group law inherited from the matrix multiplication coincides with the classical group law of the one-dimensional Heisenberg group

$$n(q,p,\tau)n(q',p',\tau') = n((q,p,\tau)\cdot_{\mathbb{H}^1}(q',p',\tau')).$$

The group A acts on $\mathbb{H}^1$ via the dilations $D_a : \mathbb{H}^1 \to \mathbb{H}^1$, defined for $a \in A$ by $D_a(q,p,\tau) := (aq, ap, a^2\tau)$. In $\mathrm{Sp}(2,\mathbb{R})$, such dilations coincide with the conjugation by the matrices $A_a := \mathrm{diag}(a^{-1},1,a,1) \in \mathrm{Sp}(2,\mathbb{R})$, that is

$$A_a n(q,p,\tau)A_a^{-1} = n(aq,ap,a^2\tau) = n(D_a(q,p,\tau)).$$

The extended Heisenberg group is the semidirect product $\mathbb{H}^1_e = \mathbb{H}^1 \rtimes A$, that is $\mathbb{R}^3 \times A$ endowed by the product

$$(q,p,\tau;a)\cdot_{\mathbb{H}^1_e}(q',p',\tau';a') := ((q,p,\tau)\cdot_{\mathbb{H}^1} D_a(q',p',\tau'); aa'),$$

and it is realized in $\mathrm{Sp}(2,\mathbb{R})$ by

$$\begin{aligned} G &:= \{g(q,p,\tau;a) := n(q,p,\tau)A_a^{-1} : q,p,\tau\in\mathbb{R}, a\in A\}\\ &= \left\{\begin{bmatrix} a^{-1} & 0 & 0 & 0\\ a^{-1}p & 1 & 0 & 0\\ a^{-1}\tau & -q/2 & a & -p\\ -q/(2a) & 0 & 0 & 1\end{bmatrix} : q,p,\tau\in\mathbb{R}, a\in A\right\} \subset \mathrm{Sp}(2,\mathbb{R}),\end{aligned}$$

endowed with the matrix multiplication. Indeed

$$\begin{aligned} g(q,p,\tau;a)g(q',p',\tau';a') &= n(q,p,\tau)A_a^{-1}n(q',p',\tau')A_aA_a^{-1}A_{a'}^{-1} \\ &= n((q,p,\tau)\cdot_{\mathbb{H}^1} D_a(q',p',\tau'))A_{aa'}^{-1} \\ &= g((q,p,\tau;a)\cdot_{\mathbb{H}_e^1}(q',p',\tau';a')). \end{aligned}$$

The group G coincides with the classical Shearlet group, see [10, 21]. The Lie algebra of $\mathrm{Sp}(2,\mathbb{R})$ is:

$$\mathfrak{sp}(2,\mathbb{R}) = \{X \in \mathfrak{gl}(2,\mathbb{R}) : XJ + JX = 0\},$$

where J is the standard symplectic form

$$J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}.$$

The Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ is semisimple and has Cartan involution $\Theta X = -{}^tX$, relative to which it has the Cartan decomposition $\mathfrak{sp}(2,\mathbb{R}) = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of Θ, respectively. The standard maximal Abelian subspace of $\mathfrak{p}$ is

$$\mathfrak{a} = \left\{ H_{a,b} := \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix} : a,b \in \mathbb{R} \right\}.$$

The two linear operators on $\mathfrak{a}$ given by $\alpha(H_{a,b}) = a - b$ and $\beta(H_{a,b}) = 2b$ provide a natural basis of simple roots. In fact,

$$X_\alpha := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_\beta := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix},$$

$$X_{\alpha+\beta} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \end{bmatrix}, \quad X_{2\alpha+\beta} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfy for every $H \in \mathfrak{a}$

$$\begin{aligned} [H, X_\alpha] &= \alpha(H)X_\alpha; \\ [H, X_\beta] &= \beta(H)X_\beta; \\ [H, X_{\alpha+\beta}] &= (\alpha+\beta)(H)X_{\alpha+\beta}; \\ [H, X_{2\alpha+\beta}] &= (2\alpha+\beta)(H)X_{2\alpha+\beta}. \end{aligned}$$

Furthermore, $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ and $[X_\alpha, X_{\alpha+\beta}] = X_{2\alpha+\beta}$.

The Heisenberg algebra is

$$\mathfrak{n} := \mathrm{span}\{X_\alpha, X_{\alpha+\beta}, X_{2\alpha+\beta}\} \subset \mathfrak{sp}(2,\mathbb{R}).$$

The extended Heisenberg algebra is

$$\mathfrak{g} := \mathrm{span}\{X_\alpha, X_{\alpha+\beta}, X_{2\alpha+\beta}, H_{1,0}\} \subset \mathfrak{sp}(2,\mathbb{R}).$$

The Lie algebra $\mathfrak{n}$ is stratified with two layers and it has homogeneous dimension 4. The first layer is generated by X_α and $X_{\alpha+\beta}$. The Carnot–Carathéodory metric associated with X_α and $X_{\alpha+\beta}$ is the left-invariant metric induced by the Koranyi norm

$$\|(q,p,\tau)\|_{\mathbb{H}^1}^4 = \frac{1}{16}(q^2+p^2)^2+\tau^2, \qquad (q,p,\tau)\in\mathbb{H}^1, \tag{23}$$

that is, $d_N((q,p,\tau),1_{\mathbb{H}^1}) = \|(q,p,\tau)\|_{\mathbb{H}^1}$. We refer to [2] for more details on this distance. From now on, we shall adopt a little abuse of notation, by renaming N and G in $\mathrm{Sp}(2,\mathbb{R})$ with $\mathbb{H}^1$ and $\mathbb{H}^1_e$, respectively, and by putting $(q,p,\tau)\in\mathbb{H}^1$ and $(q,p,\tau,a)\in\mathbb{H}^1_e$.

The following geometric lemma will be useful in the construction of our counterexample.

Lemma 6.8. *For any $R>0$ and $t\in\mathbb{R}$,*

$$B_{\mathbb{H}^1}(1_{\mathbb{H}^1},\tilde{c}e^tR)\subset\psi_t(B_{\mathbb{H}^1}(1_{\mathbb{H}^1},R)),$$

where ψ_t is defined in Proposition 3.2 and $\tilde{c}^4=1/20$.

Proof. For every $L,M>0$ we put

$$Q(L,M):=[-L,L]^2\times[-M,M]\subset\mathbb{H}^1,$$

and $Q(L):=Q(L,L^2)$. It is easy to see that

$$Q(2\tilde{c}L)\subset B_{\mathbb{H}^1}(1_{\mathbb{H}^1},L)\subset Q(2L). \tag{24}$$

Furthermore, for every $t\in\mathbb{R}$, by Equation (27), we have that if $(q,p,\tau)\in\mathbb{H}^1$, then

$$\psi_t(q,p,\tau)=n(t)D_{e^t}(q,p,\tau)n(t)^{-1}=(e^tq,e^tp,e^{2t}\tau+e^t(e^t-1)q),$$

from which it follows for every $L>0$

$$\psi_t(Q(L))=Q(e^tL,e^{2t}L^2+e^t|1-e^t|L)\supset Q(e^tL). \tag{25}$$

Therefore, by Equations (24) and (25),

$$\psi_t\big(B_{\mathbb{H}^1}(1_{\mathbb{H}^1},R)\big)\supset\psi_t\big(Q(2\tilde{c}R)\big)\supset Q(\tilde{2}ce^tR)\supset B_{\mathbb{H}^1}(1_{\mathbb{H}^1},\tilde{c}e^tR),$$

as desired. □

We are now ready to show that for a suitable vector field $Z\in\mathcal{Z}_1$ the family $\mathcal{D}^Z$ is not a CZ family for the extended Heisenberg group equipped with a right Haar measure and the Carnot–Carathéodory distance $d_{\mathbb{H}^1_e}$. We consider the vector field

$$Z=X_\alpha+H_{1,0}=\begin{bmatrix}-1&0&0&0\\1&0&0&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}\in\mathfrak{g}. \tag{26}$$

Clearly, $\langle Z,H_{1,0}\rangle=1\neq0$, so that $Z\in\mathcal{Z}_1$. Furthermore, $\|Z\|=\sqrt{2}$. Let $t\in\mathbb{R}$. By explicit computation, we have that

$$\exp(tZ)=\begin{bmatrix}e^{-t}&0&0&0\\1-e^{-t}&1&0&0\\0&0&e^t&1-e^t\\0&0&0&1\end{bmatrix}=(0,e^t-1,0,e^t)\in\mathbb{H}^1_e,$$

that means

$$n(t) = (0, e^t - 1, 0) \in \mathbb{H}^1. \tag{27}$$

We recall that ρ is a right Haar measure of $\mathbb{H}^1_e$, which is given by $d\rho = a^{-1} dq dp d\tau da$.

Theorem 6.9. *Let Z be the vector field given by Equation (26). Then, $\mathcal{D}^Z$ is not a CZ family for $(\mathbb{H}^1_e, d_{\mathbb{H}^1_e}, \rho)$.*

Proof. It is enough to exhibit a sequence of admissible cylinders $\{P^\ell\}$ such that

$$\sup_\ell \frac{\rho\Big(\{x : d_{\mathbb{H}^1_e}(x, P^\ell) < K \operatorname{diam} P^\ell\}\Big)}{\rho(P^\ell)} = +\infty, \quad \text{for every } K > 0. \tag{28}$$

Indeed, if (d) in Definition 1.1 holds, namely, if there exists $C > 0$ such that for every $P^\ell \in \mathcal{D}^Z$ there is $R_\ell > 0$ such that $\operatorname{diam} P^\ell \le C R_\ell$, then by Equation (28) one would get

$$\sup_\ell \frac{\rho\Big(\{x : d_{\mathbb{H}^1_e}(x, P^\ell) < R_\ell\}\Big)}{\rho(P^\ell)} \ge \sup_\ell \frac{\rho\Big(\{x : d_{\mathbb{H}^1_e}(x, P^\ell) < \frac{1}{C} \operatorname{diam} P^\ell\}\Big)}{\rho(P^\ell)} = +\infty,$$

which is a contradiction of property (e) in Definition 1.1. Hence, (d) and (e) cannot hold together and $\mathcal{D}^Z$ is not a CZ family for $(\mathbb{H}^1_e, d_{\mathbb{H}^1_e}, \rho)$.

We consider the family of dyadic sets $\mathcal{D}^Z$ build as in the proof of Theorem 3.9 starting from the family $\{P_k\}_{k\in\mathbb{N}}$ with $P_0 = P_{r_0,Q_0}(1)$ for some $r_0 > e$ and some Christ dyadic cube Q_0 in $\mathbb{H}^1$. We consider the subsequence of $\{P_k\}$ such that, for $\ell \ge 0$, $P_{k_\ell+1} = p^\downarrow(P_{k_\ell})$. By simplicity, we denote by $r : \mathcal{D}^Z \to (1, +\infty)$ the map defined by $r(P_{r',Q'}(a')) := r'$. An immediate computation shows that

$$\log_{r_0} r(P_{k_\ell+1} \setminus P_{k_\ell}) = 4 \cdot 6^\ell, \qquad \ell \in \mathbb{N}.$$

Now observe that for every $P' \in s(P)$, $\log_{r_0} r(P')$ is equal to $1/2 \log_{r_0} r(P)$ if $s(P)$ is as (i) in Definition 3.5, or to $\log_{r_0} r(P)$ if $s(P)$ is as (ii) in Definition 3.5. We denote by $m(\ell) \in \mathbb{N}$ the smallest number of iterations of the set-valued function s on $P_{k_\ell+1} \setminus P_{k_\ell}$ in which the case (i) in Definition 3.5 occurs exactly $\lfloor \ell \log_2 3 \rfloor + \ell + 2$ times. Then, $\log_{r_0} r(P') \in [1, 2]$ for every $P' \in s^{m(\ell)}(P_{k_\ell+1} \setminus P_{k_\ell})$. Indeed,

$$\log_{r_0} r(P') = \frac{\log_{r_0} r(P)}{2^{\lfloor \ell \log_2 3\rfloor + \ell + 2}} = \frac{4 \cdot 6^\ell}{4 \cdot 2^\ell \cdot 2^{\lfloor \ell \log_2 3\rfloor}} = \frac{3^\ell}{2^{\lfloor \ell \log_2 3\rfloor}},$$

and then

$$1 = \left(\frac{3}{2^{\log_2 3}}\right)^\ell \le \log_{r_0} r(P') = \frac{3^\ell}{2^{\lfloor \ell \log_2 3\rfloor}} \le \frac{3^\ell}{2^{\ell \log_2 3 - 1}} = 2.$$

Furthermore, by the construction of $\mathcal{D}^Z$, we have that $s^{m(\ell)}(P_{k_\ell+1} \setminus P_{k_\ell}) \subset \mathcal{D}^Z$.

Again by the construction of the P_k's, one can see that $(1_{\mathbb{H}^1}, {r_0}^{-4\cdot 6^\ell}) \in P_{k_\ell+1} \setminus P_{k_\ell}$. Then, fix a $P^\ell \in s^{m(\ell)}(P_{k_\ell+1} \setminus P_{k_\ell}) \subset \mathcal{D}^Z$ such that $(1_{\mathbb{H}^1}, {r_0}^{-4\cdot 6^\ell}) \in \overline{P^\ell}$. We put $P^\ell = P_{r(P^\ell),Q_\ell}(a_\ell)$ and $k(\ell) \in \mathbb{Z}$ such that $Q_\ell \in \mathcal{Q}_{k(\ell)}$. Clearly, $r(P^\ell) \in [r_0, {r_0}^2]$. From the definition of admissible cylinders, it immediately follows that

$$a_\ell \in \left[\frac{1}{{r_0}^{4\cdot 6^\ell} r(P^\ell)}, \frac{r(P^\ell)}{{r_0}^{4\cdot 6^\ell}}\right] \subset [{r_0}^{-4\cdot 6^\ell - 2}, {r_0}^{-4\cdot 6^\ell + 2}], \tag{29}$$

$$\delta^{k(\ell)} \in [a_\ell r(P^\ell)^2, \lambda a_\ell r(P^\ell)^\gamma] \subset [a_\ell {r_0}^2, \lambda a_\ell {r_0}^{2\gamma}]. \tag{30}$$

Then by Lemma 6.8, and observing that $U_{r(P^\ell)}(a_\ell) = (\log(a_\ell/r(P^\ell)), \log(a_\ell r(P^\ell)))$, we have that for every $R > 0$

$$\Psi_{r(P^\ell),a_\ell}\left(B_{\mathbb{H}^1}\left(1_{\mathbb{H}^1}, \frac{R}{2}\right)\right) = \bigcap_{t\in U_{r(P^\ell)}(a_\ell)} \psi_t\left(B_{\mathbb{H}^1}\left(1_{\mathbb{H}^1}, \frac{R}{2}\right)\right)$$

$$\supset \bigcap_{t\in U_{r(P^\ell)}(a_\ell)} B_{\mathbb{H}^1}\left(1_{\mathbb{H}^1}, \frac{\tilde{c}}{2}e^t R\right) = B_{\mathbb{H}^1}\left(1_{\mathbb{H}^1}, C' a_\ell R\right),$$

where $C' = \tilde{c}/2{r_0}^2$, because $r(P^\ell) \le {r_0}^2$. Hence, applying Lemma 6.7, recalling that $\|Z\| = \sqrt{2}$ and that $r(P^\ell) \ge r_0$, it follows that for any $R > 0$

$$P_{e^{R/(2\sqrt{2})}r_0, n_{Q_\ell}\cdot B_{\mathbb{H}^1}(1_{\mathbb{H}^1}, C' a_\ell R)}(a_\ell) \subset \{x \in \mathbb{H}^1_e : d_{\mathbb{H}^1_e}(x, P^\ell) < R\}.$$

Then, by means of Proposition 3.2 (vi), we get

$$\rho(\{x : d_{\mathbb{H}^1_e}(x, P^\ell) < R\}) \ge 2\left(\frac{R}{2\sqrt{2}} + \log r_0\right)|B_{\mathbb{H}^1}(1_{\mathbb{H}^1}, C' a_\ell R)| \approx a_\ell^4 R^5,$$

if $R \gg 1$. On the other hand,

$$\rho(P^\ell) = 2\log r(P^\ell)|Q_\ell| \approx |B_{\mathbb{H}^1}(n_{Q_\ell}, c\delta^{k(\ell)})| \approx \delta^{4k(\ell)} \approx a_\ell^4,$$

by Equation (30), from which it follows that

$$\frac{\rho\left(\{x : d_{\mathbb{H}^1_e}(x, P^\ell) < K \operatorname{diam} P^\ell\}\right)}{\rho(P^\ell)} \gtrsim (K \operatorname{diam} P^\ell)^5. \tag{31}$$

Now, we estimate the diameter of P^ℓ from below. Let

$$n_\ell := n_{Q_\ell} \cdot (c\delta^{k(\ell)}, 0, 0) \in n_{Q_\ell} \cdot B_{\mathbb{H}^1}(1_{\mathbb{H}^1}, c\delta^{k(\ell)}) \subset Q_\ell.$$

By Equations (1), (27), and (23), we have

$$\begin{aligned}
\cosh(\operatorname{diam} P^\ell) &\ge \cosh\left(d_{\mathbb{H}^1_e}\left(n_\ell n(\log a_\ell), a_\ell), (n_{Q_\ell} n(\log a_\ell), a_\ell)\right)\right) \\
&\ge \frac{1}{2a_\ell^2}\|n(\log a_\ell)^{-1} n_\ell^{-1} n_{Q_\ell} n(\log a_\ell)\|^2_{\mathbb{H}^1} \\
&= \frac{1}{2a_\ell^2}\|(c\delta^{k(\ell)}, 0, c\delta^{k(\ell)}(1 - a_\ell))\|^2_{\mathbb{H}^1} \\
&= \frac{1}{2a_\ell^2}\sqrt{\frac{c^4\delta^{4k(\ell)}}{16} + c^2\delta^{2k(\ell)}(1 - a_\ell)^2} \\
&\gtrsim \sqrt{\left(\frac{\delta^{k(\ell)}}{a_\ell}\right)^4 + \left(\frac{\delta^{k(\ell)}}{a_\ell}\right)^2\left(\frac{1 - a_\ell}{a_\ell}\right)^2} \\
&\gtrsim \frac{1 - a_\ell}{a_\ell} \gtrsim \frac{1}{a_\ell} \gtrsim {r_0}^{4\cdot 6^\ell},
\end{aligned}$$

since $\delta^{k(\ell)}/a_\ell \ge {r_0}^2$ by Equation (30), and $a_\ell \lesssim {r_0}^{-4\cdot 6^\ell}$ by Equation (29). This implies that $\operatorname{diam} P^\ell \gtrsim 6^\ell$. Then, by Equation (31) we get that for any $K > 0$,

$$\frac{\rho\left(\{x : d_{\mathbb{H}^1_e}(x, P^\ell) < K \operatorname{diam} P^\ell\}\right)}{\rho(P^\ell)} \gtrsim K^5 6^{5\ell} \longrightarrow \infty, \quad \ell \to \infty. \qquad \square$$

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