Action, time and entropy: The mechanical origin

DOI: 10.1478/AAPP.1031RA1

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 103, No. 1, RA1 (2025)

ACTION, TIME AND ENTROPY: THE MECHANICAL ORIGIN

UMBERTO LUCIA [a*], GIULIA GRISOLIA [b] AND ALEXANDER L. KUZEMSKY [c]

(communicated by Paolo V. Giaquinta)

ABSTRACT. In this review, we discuss the interrelation of advanced interdisciplinary concepts such as time, motion, and change in the context of modern physics and their connection to the idea of action. We mainly address such aspects that permit us to expose the specificity of these concepts. Specifically, we examine the modern paradigm about time and action and its relation to the notions of space, energy, and entropy in the context of classical, quantum, and relativistic physics. We have touched briefly on the logical character of dimensional analysis in the appropriate contexts. The review also includes synthesising past and present research and a methodology survey. Finally, we we briefly discuss some consequences to the science of complexity that these circumstances may cause for building explanatory and predictive theories.

1. Introduction

In order to penetrate into the inner and further recesses of nature,
it is necessary that both notions and axioms be derived
from things by a more sure and guarded way.
Francis Bacon, *Novum Organum*, 1620

According to Einstein (1940), "Science is the century-old endeavour to bring together using systematic thought the perceptible phenomena of this world into as thorough-going an association as possible. To put it boldly, it is the attempt at the posterior reconstruction of existence by the process of conceptualization". The development of physics and closely related natural sciences demonstrates this conceptualisation process (Hesse 1962; Cohen 1985; Whittaker 1989; D'Agostino 2000; Varvoglis 2014; Longair 2020; Carrol 2022; Kuzemsky 2023a,b).

Indeed, the natural sciences elaborated numerous concepts to describe and interpret different phenomena and complex natural processes (Einstein and Infeld 1954; Harman 1982; Holton and Brush 2001; Peat 2002; Buchwald and Fox 2013). These are: motion (Synge 1960; Truesdell 1968; Whittaker 1988; Barbour 2001), inertia (Giufolini and Wheeler 1995), energy (Bell 1943; Lindsay 1971; Tribus and McIrvine 1971; Lindsay 1975; Muller 2007; Coopersmith 2015), power (Muller 2007), force (Wilczek 2007; Jammer 2011; Wilczek 2021), mass (Jammer 2000; Okun 2009), space (Sklar 1977; Janner 2012), time

(Carrol 2010; Weinert 2013; G.'t Hooft and Vandoren 2015; Kuzemsky 2023a,b), action (Barrow and Tipler 1988; Siburg 2004; Toms 2007; Lawrie 2015; Kuzemsky 2023b), field (Whittaker 1989; Zee 2013; Lawrie 2015), quantum (Jammer 1966; Holton and Brush 2001), interaction (Holton and Brush 2001; Wilczek 2021), heat (Muller 2007), entropy (Muller 2007; Kuzemsky 2023b), information (Tribus and McIrvine 1971; Gitt 2006; Gleick 2011; Eigen 2013; Kuzemsky 2016), symmetry (Kuzemsky 2010, 2017; Schwichtenberg 2017), emergence (Laughlin and Pines 2000; Kuzemsky 2010), complexity (Nicolis and Prigogine 1989; Gell-Mann 1994; Mainzer 2007) and many others. Many notions and concepts are interrelated with others, e.g., motion and inertia, energy and entropy, time and space, force and interaction, entropy and information, mass and energy, etc. Discussion of the phenomenon of time and certain aspects of the concept of time from an interdisciplinary perspective was carried out by Kuzemsky (2023a,b).

In comparison with other notions, the studies of the basic concepts and the mathematical and logical structure of equilibrium thermodynamics were developed extensively (Kuzemsky 2023b). Various aspects of the subject were formulated consistently and rigorously in an axiomatic form. Now it may be considered as a branch of mathematical physics (Arens 1963; Giles 1964; Rulle 1978; Wehrl 1978; Owen 1984; Emch and Liu 2002). Energy E and entropy S are considered the most important core concepts of natural sciences, particularly in concerning physics. A function S representing the degree of energy spreading and sharing was developed using a set of acceptable properties. Many properties of entropy, e.g., invariance, additivity, concavity, sub-additivity, continuity, etc., were developed in axiomatic form concerning their consequences in statistical mechanics (Arens 1963; Giles 1964; Rulle 1978; Wehrl 1978; Owen 1984; Emch and Liu 2002).

It was established that Caratheodory's formulation of classical thermodynamics might be reformulated into a mathematically rigorous form, in which such well-known results as the existence of entropy and (positive) absolute temperature, the principle of increase of entropy and the positivity of isometric-specific heats were shown to follow logically from a set of axioms closely related to the traditional laws of thermodynamics and more transparent than those of the earlier formulation. The basic tools were differential geometry and topology, the latter being needed to eliminate those flaws in the previous arguments that arose from the erroneous application of local results in a global context (Kuzemsky 2023b). It is worth mentioning that this approach formulated the relatively systematic and contradiction-free scheme of thermodynamics based on Pfaff's forms. In addition, the development of measure theory helped to improve the development of thermodynamics and the physics of continual matter, where extensive quantities are measured and intensive quantities are densities. The attention was attracted to measure theory, but not to topology, as a natural approach for the comprehension of the difficulties (ergodicity, approach to equilibrium, irreversibility) based on statistical mechanics. The measure-theoretic proof of the Poincaré recurrence theorem was formulated. The approach stimulated Birkhoff to point out the ergodicity as metric transitivity. Later, Koopman and von Neumann developed the spectral analysis of dynamical systems in Hilbert spaces (Moore 2015). However, mixing explained equilibrium but not the too-complicated notion of irreversibility. The consequent development of the spectral theory of dynamical systems to locally convex spaces revealed the nontrivial time asymmetric spectral decompositions for unstable and non-integrable systems. Thus, irreversibility can be explained in a more sophisticated way (Kuzemsky 2018). Hence the axiomatic approach

provided thermodynamics with a strong design that clarified the role of the binary relation of adiabatic accessibility between equilibrium states.

Another important axiomatic approach to physical problems was elaborated and formulated by Bogolubov and his school (Bogolubov, Logunov, and Todorov 1975; Bogolubov *et al.* 2012) in the framework of relativistic quantum field theory. The previous basic results of relativistic quantum theory were obtained in terms of the local quantum field approach. The development of the basic principles of the local theory and its mathematical structure were the marked streams in this area. The axiomatic approach has appeared from attempts to give a mathematical meaning to the quantum field theory of strong interactions. In this theory, the fields were represented by operators in the Hilbert space with a positive Poincaré-invariant scalar product. This semi-classical part of the axiomatic approach attained its modern form decades ago. Nevertheless, its significance is preserved today, even if the main description prospects of the electro-weak and strong interactions are related to the theory of gauge fields. In fact, from the point of view of the quark model, the theory of strong interactions of the Wightman type was obtained by restricting attention to just the *physical* local operators (such as hadronic fields consisting of fundamental quark fields) acting in a Hilbert space of physical states. In principle, there are enough such physical fields to describe of hadronic physics, although this means that one must reject the traditional local Lagrangian formalism. The connection may be restored in the approximation of low-energy "phenomenological" Lagrangian. The logical aspects of the related approaches were discussed and clarified. This approach is under some development today on the new grounds.

It should be said that the conceptualisation and adaptation processes of other great ideas and concepts were difficult and even dramatic in part (Peat 2002; Weinert 2013; Kuzemsky 2023a,b). There are some concepts, e.g., vacuum, space, time, information, emergence, etc., the conceptualisation process of for which is still in progress and is characterized by the hottest debates and polemics (Barrow 2001; Mittelstaedt and Weingartner 2005; Mittelstaedt 2011; Kuzemsky 2023b).

The present study aims to improve the fundamental understanding of the concepts of space, time, action, and entropy by placing them in a suitable context. By confronting the concepts of time and action, it is possible to emphasize transparently the dynamical aspect of both notions (Truesdell 1968; Barbour 2001). These points are so basic to how science works that it is worth looking at the rationale for it more carefully. We will discuss and critically examine various physical and mathematical accounts of these core concepts and their relevance in various fields.

2. Time and action

When describing the notions of space and time, Davies (1977) remarked: "In common understanding the space is empty but time is fulfilled with action". Hence space, time, and action have their specificity (Kuzemsky 2023b). In common usage, the term *action* means a *doing* or *making*. It is the *process* of doing something, or something done, especially for a particular purpose. The time of action can be the moment in time or period when an action takes place. That means that the notions of time and action are interrelated intimately, even on the linguistic level. These problems were discussed in the context of representations for natural language systems, problem-solving systems, and work in linguistics and philosophy

(Allen 1984). In physics, time is considered a change associated with the second law of thermodynamics. In other words, it may be treated as the physical result of increasing entropy. Time is also a spacetime dimension with the opposite metric signature to space dimensions. Hence it is the fourth dimension. Einstein (1947) formulated this statement in a semi-axiomatic way: "Physical space was thus increased to a four-dimensional which also included the dimension of time. The four-dimensional space of the special theory of relativity is just as rigid and absolute as Newton's space".

These matters were reformulated within constructive axiomatic space-time geometry combined with some elements of quantum theory (Lammerzahl and Macias 1993). It was speculated that by using the specificity of the dynamical behaviour of general matter fields in a geometry-free way, it may be possible to give plausible arguments for the spacetime dimension to be four. In addition, the low-dimensional manifolds were also seen to possess special mathematical properties. For a space with more than three dimensions, it is more difficult. There can be no atoms in the usual sense and, consequently, no stable structures. In addition, a space with less than three dimensions prevents gravitational force. It also may be too primitive to contain observers. However, it should be stressed that space-time is a tool for explaining one aspect of reality and, in this sense, is a model of reality. It is worth mentioning that there is a detailed analysis of the special theory of relativity kinematics and dynamics as well as the general theory of relativity under the aspect of logic (Tornebohm 1952; Prokhovnik 1967; Andréka *et al.* 2010; Ma 2013; Kanda, Prunescu, and Wong 2020; Engler 2021). In these studies, the axiomatic logical analysis of relativity theories was described. In this context, scientific arguments are usually highlighted as rational descriptions of certain scientific ideas and the evidence for or against them. A scientific argument uses a logical approach to establish a rigid background for whether a scientific idea is relevant or irrelevant (Baker 2007).

3. Time, change and dynamics

Time and dynamics are highly linked to one another. The fundamental property of dynamical systems is the dependence of their state changes upon time (Synge 1960; Whittaker 1988; Barbour 2001). The interpretations of time and change are complementary in mechanics. Indeed, "change consists in a single object having a given property at a given time and a distinct and incompatible property at a different time" (Belot 2007). Belot analysed the formulation of time and change in the general theory of relativity, based on Hamiltonian and Lagrangian mechanics, highlighting the role of dynamics in the comprehension of the evolution of physical quantities in time. Indeed, the aim of dynamics is the description of the trajectories of particles under the action of external forces. He showed that "it is a fact of primary importance that for well-behaved theories the space of initial data and the space of solutions share a common geometric structure - these spaces are isomorphic as symplectic manifolds" (Belot 2007).

Guillemin and Sternberg (1990) proved that symplectic geometry is a very useful tool for a clear and concise formulation of some classical physics topics, e.g., the importance of symmetries, analogies between classical dynamics and optics, the importance of symplectic techniques in classical variational theory, symplectic features of classical field theories, and the principle of general covariance. They used the uncluttered, coordinate-free approach to

symplectic geometry and classical mechanics. Concerning the fundamental principles in physics, Guillemin and Sternberg (1990) conjectured three principles (or axioms). These axioms differ substantially from the other, but all give rise to the same systems of Hamilton equations. The first is the variational principle. The second is the method of high-frequency approximation. The third is the principle of general covariance (Kuzemsky 2023a).

The first fundamental principle emphasizes the variational approach (Lanczos 1986). The Hamilton equations of motion are the Euler-Lagrange equations for the integral $\int \mathscr{L}(q,\dot{q})ds$ along curves. Here $\mathscr{L}(q,\dot{q}) = 1/2\|\dot{q}\|^2$ and $\dot{q}$ denotes the tangent vector to the curvature. In this approach, Fermat's principle, or Maupertuis's principle states itself as being the core principle of mechanics. Hence in certain cases, the problem of finding the basic laws of physics may be formulated as the problem of finding the true Lagrangian.

Following the Lagrangian formalism, Belot (2007) analysed the two groups of solutions: the solutions that represent change and changeless solutions "which are invariant under the action of a group of time translations" (Guillemin and Sternberg 1990; Sternberg 1995; Belot 2007). Thus, Belot developed a workable formalism of the representation of time and change, concerning the implementation of time translation and evolution. In addition, Belot studied the problem of time in general relativity (Schwinger 1986), reconsidering the general covariance of general relativity with related consequences. So, he developed a workable approach for time and change in general relativity.

Thébault (2012) introduced a complementary approach to symplectic reduction and to the problem of time in non-relativistic mechanics, describing a deep connection between the theories invariant under local symmetry transformations (for example the gauge theories) and the concepts of space and time. As in general relativity, the Hamiltonian constraints are also introduced in the canonical formulation of these theories. Thébault (2012) conjectured that the physical foundation of the problem is the symplectic reduction. Indeed, if reduction is avoided, two viable alternative strategies could be introduced to deal with Hamiltonian constraints. Consequently, a new re-conception of time and change emerged in non-relativistic mechanics. Both these approaches and the failure of reduction determine important consequences on the relational or absolute status of time in physics.

4. Dynamics and action

Action in mechanics is the movement using force or power for some purpose; how a body moves. In this context, it is worth stressing that action is one of the key concepts in classical and quantum physics (Landau and Lifshitz 1976; Barrow and Tipler 1988; Kibble and Berkshire 2004; Toms 2007). The concept of action is in the row of other key concepts, such as time, space, energy, force, field, etc. It interconnects in a specific way with the notion of time (Kuzemsky 2020; Lucia, Grisolia, and Kuzemsky 2020).

A short historical account of the development of the concept of action will not be out of place here. The notion of the action was introduced by Leibnitz about 1699 (Kuzemsky 2023b). He called it *actio formalis*, i.e., *formative action*. Maupertuis introduced an analytical definition for the action as $\mathscr{A}_M = m \cdot v \cdot s$, where m is the mass, v is the velocity, and s is the path length. Considering that $s = vt$ it follows that $\mathscr{A}_M = 2E_{\text{kin}} \cdot t$. We indicate the Maupertuis action by $\mathscr{A}_M$ to avoid confusion with the Hamilton action $\mathscr{A}_H$ (Kuzemsky 2023b). More precisely, action may be defined as an integral $\mathscr{A}$ associated with the trajectory

of a system in configuration space, equal to the sum of the integrals of the generalized momenta of the system over their canonically conjugate coordinates (Landau and Lifshitz 1976; Kibble and Berkshire 2004). This is usually known as the action integral (Kibble and Berkshire 2004). This terminology does not accord with the original, in which the term *action* was used for a related but different quantity (the integral of twice the kinetic energy), which is relatively little used (Kibble and Berkshire 2004). However, it is more convenient, and it is usual nowadays, to call it the action integral (Kuzemsky 2023b).

The scientific validity of the concept of action is closely linked to the fundamental law (or axiom) of physics, i.e., the principle of least action formulated by Maupertuis and Euler (Kuzemsky 2023b), whose basis is the principle of Pierre de Fermat, who conjectured that *light travels between two given points along the path of shortest time*, today named the principle of least time or Fermat's principle. This can be analytically expressed in modern notation as $\int ds/v = \min$. We wish to highlight that Euler was the first to replace the path s with the element of the path ds in the variational problem. Maupertuis (1698-1759) developed the principle of least action (Kuzemsky 2023b) as an improvement for Leibnitz's ideas of $mvs = \min$. This expression is known as Maupertuis' principle (Landau and Lifshitz 1976) - an integral equation that determines the path followed by a physical system. We stress that Maupertuis proposed this principle as a metaphysical idea that underlies all the laws of mechanics. The principle of least action states that in all natural phenomena, a quantity called *action* tends to be minimised (Landau and Lifshitz 1976; Lanczos 1986), an idea that Maupertuis stated (Kuzemsky 2023b):

> *Nature, in the production of its effects, does so always by the simplest means.*

We highlight that in optics the quantity that presents the minimum is the *time* (Kuzemsky 2023b). Hence, what should be done, is to arrange a possibility for the measurement of time's intervals and to establish the relations for it: equal, more than, and less than.

On the contrary, mechanical problems are very different from the optical ones. Indeed, it is not obvious which is the quantity that presents the extremum (minimum or maximum). Following Leibnitz and Maupertuis, this quantity is quite non-trivial: the *action*. Maupertuis writes the *action* as the product of the body mass , the distance travelled and its velocity (Kuzemsky 2023b). Moreover, he pointed out that a system of bodies at rest tends to reach a position where any change determines the smallest possible change in the *action*. The fundamental property of this principle is the great number of applications so it is universal. Moreover, Maupertuis admitted that the major arguments advanced to prove God, from the wonders of nature or the apparent regularity are all open to objection. But a universal principle of wisdom provides an *undeniable proof* of the shaping of the universe by a wise Creator.

5. The basic notions of mechanics and time

To clarify the various mechanical and quantum mechanical base concepts and the notions discussed above we summarise here the relevant information very briefly for the sake of a self-contained presentation (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b).

Classical mechanics describes the evolution of mechanical systems over time. Thus, the state of the system under consideration is identified by a set of physical quantities concerning the physical condition of the system itself at any time, $\{\text{state}\} = \{q, \dot{q}\} = \{\text{position}, \text{velocity}\}$. From a classical mechanics viewpoint, the description of a many-particle system composed of N particles (molecules) needs the knowledge of several variables of the order of the total number of molecules. Consequently, if the system has n degrees of freedom then the identification of $2n$ variables $\{q_1, q_2, \dots q_k, \dots q_n\}$ and $\{p_1, p_2, \dots p_k, \dots p_n\}$ is required, being the set $\{q(t)\}$ of the generalised coordinates and the set $\{p(t)\}$ of the particles' associated conjugate momenta, defined as:

$$p_k = \frac{\partial \mathscr{L}}{\partial \dot{q}_k}, \quad k = 1, 2, \dots, n; \quad \dot{q}_k = \frac{dq_k}{dt} \tag{1}$$

where $\dot{q}_k = dq_k/dt$ are the generalised velocities, and $\mathscr{L} = T - U$ represents the Lagrange function, with $T = E_{kin}(q, \dot{q})$ the kinetic energy and $U = E_{pot}(q)$ the potential energy. The quantities q_k and $p_k$are named canonically conjugate. For a N-particles system, the Lagrangian results:

$$\mathscr{L} = \mathscr{L}(q_k, \dot{q}_k, t) = \frac{1}{2} \sum_k m_k \dot{q}_k^2 - U(q_1, q_2, \dots q_k, \dots q_N, t) \tag{2}$$

The generalised coordinates can represent different physical quantities (length, angle, etc.), but the Lagrange function always has the dimension of energy. The number of generalised coordinates is always the same as the number of the system's degrees of freedom. We wish to highlight that in classical mechanics the time t and the Hamiltonian H are never canonically conjugate. In classical mechanics, the Lagrangian approach allows us to obtain the equations of motion and the related conserved quantities (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b), reformulating Newton's second law by a system of n second order differential equations concerning time (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b):

$$\frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} - \frac{\partial \mathscr{L}}{\partial q_k} = 0 \tag{3}$$

Their solutions require $2n$ integration constants.

The Lagrange equations and Newton's second law are two equivalent formulations of classical mechanics. However, the Lagrangian approach presents some limitations; indeed, the Lagrangian $\mathscr{L}$ that brings to a set of equations of motion is not unique. For example, both the Lagrangian $\mathscr{L}(q_k, \dot{q}_k, t)$ and $\tilde{\mathscr{L}}(q_k, \dot{q}_k, t) = \mathscr{L}(q_k, \dot{q}_k, t) + dF/dt$, where dF/dt is a total time derivative of a function $F(q,t)$, yields the same equations of motion since dF/dt satisfies Lagrange's equations identically for arbitrary $F(q,t)$. Moreover, the system of the Lagrangian equations does not consider dissipation. For example, a Lagrangian that describes a damped simple harmonic oscillator has been shown to describe a completely different physical system than the harmonic oscillator (Landau and Lifshitz 1976).

A basis for approximation methods of classical mechanics is *the variational principle*, which was introduced in physics and applied mathematics as a useful tool for the analysis of mechanics, hydrodynamics, theory of elasticity, etc., (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b). Let $\tilde{q}_k(t)$ be a trajectory

between two different positions $q_k(t_1)$ and $q_k(t_2)$, slightly deviated from the real pathway $q_k(t)$, i.e., $\tilde{q}_k(t) = q_k(t) + \delta q_k(t)$, where $\delta q_k(t)$ is the virtual displacement and $\tilde{q}_k(t)$ is the virtual trajectory. Hamilton's variational principle within the Lagrangian formulation represents an efficient approach in classical dynamics because it involves the action function $\mathscr{A}$ defined as (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b)

$$\mathscr{A} = \int_{t_1}^{t_2} \mathscr{L}(q_k, \dot{q}_k, t) dt \tag{4}$$

To summarise, the action functional $\mathscr{A}[q(t)]$, whose dimensions are energy times time ($\mathrm{Dim}[\mathscr{A}] = \mathrm{J} \times \mathrm{s}$), is an integral associated with the trajectory of a system in configuration spaces (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b), which presents a symmetry concerning time because the Lagrangian $\mathscr{L}$ is not an explicit function of t: time symmetry implies energy conservation. In this context, a fundamental role is played by Hamilton's variational principle which is the principle of minimum action

$$\delta \int_{t_1}^{t_2} \mathscr{L} dt = \delta \mathscr{A}[q(t)] = 0 \tag{5}$$

We highlight that the Hamilton principle is independent of the choice of coordinates. An extremum principle is equivalent to the equations of motions of Newton or Lagrange (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b).

Hamiltonian mechanics introduces the Hamiltonian function defined as (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b):

$$H = \Sigma_{k=1}^{n} \dot{q}_k p_k - \mathscr{L}(q_k, \dot{q}_k, t) \tag{6}$$

where Hamilton introduced the generalized momenta $p_k = \partial \mathscr{L} / \partial \dot{q}_k$, with the consequence that $\mathscr{L} \neq H$. Legendre transform allows the transition from the Lagrange to the Hamilton approach. The Legendre transformation also plays a fundamental role in thermodynamics to transform state variables into other ones. The Hamiltonian gives the energy value along a solution (that is preserved for time-independent systems) whereas the Lagrangian describes the action. Hamilton's equations are equivalent to the Euler-Lagrange equations for the Lagrangian. These equations express the variational character of solutions of the Lagrangian system (Siburg 2004).

Garrido (1961) in a few papers analysed complementary aspects of the action principle for classical mechanics. He considered a differential action principle for classical mechanics from which the equations of motion can be deduced and also yields an easy method for treating perturbations in classical mechanics. The equations of motion determine the time development of the system. They are a set of differential equations that relate quantities evaluated at the same instant. All mechanics involve two different sets of hypotheses, corresponding to the kinematic description of the system at a time instant and to the specific dynamical time evolution of the same. The action principle considered by Garrido (1961) refers to the second aspect when one uses sets of canonical conjugate variables to describe the system, that is, in phase space. Perturbations have so far been treated in classical mechanics in an elementary way. It was assumed that the solution to the

perturbed equations of motion could be written as a power series of a certain parameter, the coupling parameter between the unperturbed and perturbed system; substituting this solution into the perturbed equations of motion author got, as a consequence of equating to zero the different coefficients of the successive powers of the coupling parameter, a set of equations whose solutions generated the perturbed solution up to any desired degree of approximation in the coupling parameter provided that the convergence of the series was assumed. The full complexity of classical perturbation theory was revealed in a detailed treatment that goes as far as treating the adiabatic invariance of action variables. Garrido formulated a differential principle for systems whose motion can be deduced from the variation of an action integral; the Lagrangian may depend on time explicitly. A differential action principle was obtained using variations of the action integral. It yielded Hamilton's equations of motion, provided a general method to treat perturbations in classical mechanics, and corresponds to Schwinger's principle of quantum mechanics (Watson 1989). Garrido considered also a general action principle for mechanics written using the elements of a Lie algebra. He studied the physical reasons for which we must choose a Lie algebra to write the action principle. Using such an action principle author worked out the equations of motion and a technique to evaluate perturbations in general mechanics that is equivalent to a general *interaction picture*. Classical or quantum mechanics come out as a particular case when one makes realizations of the Lie algebra by derivations into the algebra of products of functions or operators, respectively. Later, the author developed the applications of the action principle to classical and quantum mechanics, seeing that in this last case, it agrees with Schwinger's action principle (Watson 1989). The main contribution (Garrido 1961) was to introduce a perturbation theory and an interaction picture of classical mechanics on the same footing as in quantum mechanics.

The Maupertuis' principle was discussed clearly by Landau and Lifshitz (Landau and Lifshitz 1976). According to their approach, the motion of a mechanical system is entirely determined by the principle of least action. Indeed, they stressed that by solving the equations of motion which follow from that principle, we can find both the form of the path and the position on the path as a function of time.

If the problem is the more restricted one of determining only the path, without reference to time, a simplified form of the principle of least action may be used. They assumed that the Lagrangian, and therefore the Hamiltonian, does not involve the time explicitly so that the system energy is conserved: $H(p,q) = E = \text{constant}$. According to the principle of least action, the variation of the action, for given initial and final coordinates and times (t_0 and t, say), is zero. If, however, we allow a variation of the final time t, the initial and final coordinates remaining fixed, we have

$$\delta \mathscr{A} = -H \delta t \tag{7}$$

They compared only the system motion that satisfies the law of energy conservation. For such paths, we can replace H in the above equation with a constant E, which gives

$$\delta \mathscr{A} + E \delta t = 0 \tag{8}$$

Next, they rewrite this equation in the form

$$\mathscr{A} = \int \sum_i p_i dq_i - E(t - t_0) \tag{9}$$

The first term on the right side of this equation is sometimes called the abbreviated action $\mathscr{A}_0$. A combination of the last two equations gives the equality $\delta\mathscr{A}_0 = 0$. Thus, the abbreviated action has a minimum concerning all paths that satisfy the law of conservation of energy and pass through the final point at any instant. Using the differential dt in terms of the coordinates q and their differentials dq it is possible to express the momenta in terms of q and dq, with the energy E as a parameter. The variational principle so obtained determines the path of the system and is usually called Maupertuis' principle, although its precise formulation is due to Euler and Lagrange (Landau and Lifshitz 1976). For the abbreviated action, Landau and Lifshitz (Landau and Lifshitz 1976) write down the following equation

$$\frac{\partial \mathscr{A}_0}{\partial E} = t - t_0 \tag{10}$$

which manifests an inherent interconnection of time and action (Lucia, Grisolia, and Kuzemsky 2020). Hence, the principle of least action states that, for sufficiently short times, trajectories of a Lagrangian system minimise the action amongst all paths in configuration space with the same endpoints. However, when the time interval becomes larger, the Euler-Lagrange equations describe just critical points of the action functional; they may well be saddle points. This is why the right term should be the *principle of critical action*.

Hamilton equations, formulated by Hamilton (1805-1865) as ordinary canonical first-order differential equations of motion for holonomic mechanical systems, are equivalent to Lagrangian ones (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b); their analytical expression is:

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \tag{11}$$

If there are the non-potential *generalized forces* $\mathscr{F}_k$ acting on the system, they must be added to the right-hand side of the second equation. A variational principle for non-holonomic systems can be obtained (Kuzemsky 2023b) from a different variational approach in which the comparison paths do not satisfy the velocity-dependent constraint conditions: this result cannot be obtained by the Lagrangian approach.

It is worth mentioning that classical analytical mechanics may be connected with entropy production. Indeed, the usual canonical Hamiltonian or Lagrangian formalism of classical mechanics for macroscopic systems expresses energy conservation for adiabatic motions, but, if irreversible diabatic processes must be considered, then the law of increasing entropy must also be introduced (Silverberg and Widom 2007). In this way, the concept of entropy is introduced into the classical mechanical formalism.

When the Hamiltonian is time-independent, it represents the total energy which is a conserved quantity (or *integral*) of the motion:

$$\frac{\partial H}{\partial t} = \frac{dH}{dt} = 0, \quad \frac{\partial H}{\partial t} = -\frac{\partial \mathscr{L}}{\partial t} \tag{12}$$

i.e., $H = T + U = E =$ constant. In classical dynamics, the time derivative of a quantity is obtained as:

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t} \tag{13}$$

where A is some function of p and q and $\{A,H\}$ is the Poisson bracket:

$$\{A,H\} = \sum_{k=1}^{n} \Big[\frac{\partial A}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial q_k} \Big] \tag{14}$$

A function of phase $f(p,q,t)$ will evolve in time according to the equation

$$\frac{df(p,q,t)}{dt} = \{f,H\} + \frac{\partial f}{\partial t} \tag{15}$$

Constants of motion are important quantities in classical mechanics and statistical mechanics (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b), because any integrable dynamical system has additional (to the energy) constants of motion, which commute with the Hamiltonian under the Poisson bracket. Let $f(p,q)$ be a constant of motion, thus $(p(t),q(t))$ is a trajectory or solution to the Hamilton equations of motion, so, $\mathrm{d}f/\mathrm{d}t = 0$ along that trajectory:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(p,q) = \{f,H\} + \frac{\partial f}{\partial t} = 0 \tag{16}$$

where, as above, the intermediate step follows by applying the equations of motion. Let us consider the density of representative points of an ensemble, $\rho = \rho(p,q,t)$. The total number of representative points in the ensemble is a conserved quantity. It implies that their density ρ must obey a conservation equation in the $2N-$dimensional phase space, named Liouville's theorem:

$$\{\rho,H\} + \frac{\partial \rho}{\partial t} = 0; \quad \frac{d\rho(p,q,t)}{dt} = 0 \tag{17}$$

If ρ is time-dependent, the equation for ρ results:

$$\frac{\partial \rho}{\partial t} = \{\rho,H\} = \{H,\rho\} \tag{18}$$

We highlight that this equation is not a precise equation of motion ($d\rho(p,q,t)/dt \neq \{H,\rho\}$). A Hamiltonian system is completely integrable if all the constants of motion are in mutual involution (Landau and Lifshitz 1976; Lanczos 1986; Kibble and Berkshire 2004; Kuzemsky 2017, 2023b).

In classical mechanics every solution of the equations of motion is a state of definite energy; consequently, position, momentum and energy are well-defined and evaluated in any instant. This is not the case in quantum mechanics.

6. Uncertainty and action

Quantum mechanics (Birkhoff and Von Neumann 1936; Heisenberg 1949; Mackey 1963; Blokhintsev 1964, 1968; Piron 1976; Von Neumann 2018) formulates two basic axioms, in terms of the notions of the states and the observables. The important aspect of this approach is the reconstruction of quantum theory based on a formal language of quantum mechanical axioms and propositions (Mackey 1963; Von Neumann 2018). These concepts are related tightly to the notion of measurement (Blokhintsev 1964, 1968; Von Neumann 2018). This section aims to clarify some of the issues.

The measure of a physical quantity is an eigenvalue of the Hamiltonian and the state is the related eigenvector, so any observable is a self-adjoint operator acting in the Hilbert space $\mathscr{H}$. Only in the particular case of the simple physical systems can the state be identified by a single observable, while in the usual real systems, a greater number of observables are required. Consequently, the set of possible measures of an observable X is identified by the spectrum of its representative operator. In quantum mechanics, time t determines the evolution of any system, even if it is not represented by a dynamical variable (operator), so, while any measurement is developed *at an instant of time*, histories cannot be measured (Kuzemsky 2020, 2023b). Observables are represented by operators identified based on specific commutation relations, which are the basic axioms of quantum mechanics: as axioms, they cannot be deduced or proven. They are the cornerstones of quantum physics on the whole (Birkhoff and Von Neumann 1936; Heisenberg 1949; Mackey 1963; Blokhintsev 1964, 1968; Piron 1976; Von Neumann 2018). The Poisson brackets of classical physics in quantum mechanics are replaced by the commutator of two quantities which represent observables

$$\{A,B\}_{PB} \to \frac{1}{i\hbar}[AB-BA] = \frac{1}{i\hbar}[A,B]_{-} \tag{19}$$

For the position x and momentum p, it gives

$$[xp-px] = \left[x, -i\hbar\frac{\partial}{\partial x}\right] = i\hbar \tag{20}$$

where $\hbar = h/2\pi$, with h the Planck's constant.

A fundamental difference between quantum mechanics and classical physics is the introduction of Heisenberg uncertainty relations (Heisenberg 1949) for the canonical variables, e.g., position x and momentum p:

$$\Delta x \Delta p \geq h \tag{21}$$

In general case

$$\Delta A \Delta B \geq \frac{1}{2}|\langle AB-BA\rangle|, \tag{22}$$

where ΔA is the standard deviation $(\Delta A)^2 = \langle (A-\langle A\rangle)^2\rangle$. Hence we can write that

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{h^2}{4}; \quad \Delta x\Delta p \geq \frac{\hbar}{2} = 5.3\times 10^{-35}\mathrm{J}\cdot\mathrm{s}. \tag{23}$$

Uncertainty relations result in canonical variables because they are operators in the Hilbert space. A Hilbert space $\mathscr{H}$ (or complex Hilbert space) is a complex vector space with an inner product that is a complete metric space concerning the norm associated with the inner product. So, from the mathematical viewpoint, quantum mechanics is the implementation of the representation of commutation relations by operators in the Hilbert space (Piron 1976; Kuzemsky 2017). The measures of some particular quantities are required to form a basis for the Hilbert space imposing the observables to be compatible, i.e., their related operators must all commute. This set of variables is usually called a complete set of commuting observables.

The state of a system in quantum mechanics is fully described by a function $\psi(\mathbf{r},t)$ (or $\Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots,t)$), the wave function, which is a basic notion of quantum physics. It is defined so that the probability of finding the particle in the interval x to $x+dx$ results

$P(x)dx = |\psi|^2 dx = \psi^* \psi dx$. It contains all the measurable information about the system (particle). Thus, while in classical physics a particle follows a definite trajectory $(x(t), p(t))$, in quantum mechanics the state of a particle is described in terms of a wave function $\psi(\mathbf{r},t)$, whose probabilistic interpretation is to obtain the probability to find the particle at position $\mathbf{r}$ at time t as $P(\mathbf{r}) = |\psi(\mathbf{r},t)|^2$, being $P(\mathbf{r}) = |\psi(\mathbf{r},t)|^2 d^3r$ the probability to find the particle in an infinitesimal volume d^3r centred at $\mathbf{r}$ at time t. Consequently, in classical physics, the position and momentum of a particle are precisely determined, while in quantum physics the *probabilistic description* of phenomena at the atomic level emerges (Heisenberg 1949; Blokhintsev 1964, 1968), whose quantum state is completely determined by the complex wave function $\psi(\mathbf{r},t)$. The Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = H\psi(\mathbf{r},t) \tag{24}$$

is a fundamental equation which demarcates and determines the temporal behaviour of quantum objects (Kuzemsky 2017), with H the Hermitian operator, called the system's Hamiltonian, depending on the properties of the interaction both inside the system itself and outside with its environment. Remarkably Schrödinger's equation includes the universal constant $\hbar$ that defines all the specificity of the quantum world.

The Schrödinger equation, a basic reference of quantum mechanics (Kuzemsky 2017), is a second-order partial differential equation that allows us to obtain the spatial shape and the temporal evolution of a wave function in a given potential V under definite boundary conditions. Its solutions $\psi(\mathbf{r},t)$ are complex functions. Concerning quantum dynamics, the state vector $\psi(\mathbf{r},t)$ at time t is related to the initial state at time t_0 by unitary transformations. In quantum dynamics, time is not considered an observable depending on the state, but it is used as a parameter, useful for obtaining an ordered sequence of events. The axiom of continuous unitary transformation for time evolution is introduced.

The Schrödinger equation presents a symmetry (Kuzemsky 2017); indeed, the wave function is arbitrary up to a global phase and changing $\psi \to \exp(i\alpha)\psi$ does not modify the probability distribution $P(\mathbf{r})$. It is mathematically realised by multiplying the wave function by $\exp(i\varphi)$, with φ phase represented by a real number, constant over all of space, with the consequence of the invariance of the squared modulus of the wave function: *a symmetry emerges about the global phase factor*. The action of a free particle, in the range of coordinate position [**0**,**r**] over time t results (Kuzemsky 2023b):

$$\mathscr{A} = -Et + \mathbf{p}\cdot\mathbf{r}. \tag{25}$$

If the free particle is in rest, the action becomes:

$$\mathscr{A} = -E_0 t = -mc^2 t. \tag{26}$$

where E_0 is the energy at rest and c is the velocity of light, and the wave function results:

$$\Psi \sim \exp\Big(\frac{i\mathscr{A}}{\hbar}\Big). \tag{27}$$

with $\exp\Big(i\mathscr{A}/\hbar\Big)$ amplitude and $\mathscr{A}$ the action for the path considered, $\mathscr{A}/\hbar$ is the phase angle, also considering that an amplitude phase is a complex number. Both action $\mathscr{A}$ and Planck's constant $\hbar$ have the same dimensions (energy times time), so, $\mathscr{A}/\hbar$ is a

dimensionless quantity. Thus, the equation is non-linear in the new quantities, even if Ψ obeys a linear equation. The path integral formalism allows us to interpret the term $\exp(i\mathscr{A}/\hbar)$ considering a particle described by a Lagrangian $\mathscr{L}(r,\dot{r},t)$ and providing a set of formal rules to establish how the probability of observing such particle at some spacetime position (r,t) can be obtained in terms of the quantum theory. The variations of $\mathscr{A}_{\rm cl}$ lead to strong oscillations of the terms $\exp(i\mathscr{A}/\hbar)$ to the path integral, also determining a destructive interference between these contributions. Only the paths close to the classical one is the interference ruled out, because of the extremal property of the action integral, so near the classical path the value of the action integral changes so little that destructive interference cannot occur. On the contrary, concerning microscopic particles, it results in $10 \gtrapprox \mathscr{A}/\hbar$, so, the variations of the exponent result in the order of unity and the destructive interference is still present but less destroyable than in the macroscopic case.

So, two domains result in the parameters concerning two pictures - classical and quantum. Classical systems are characterised by the values $(v/c) \ll 1$ and $(\mathscr{A}/\hbar) \gg 1$, while quantum systems are characterised by the values $(v/c) \ll 1$ and $(\mathscr{A}/\hbar) \ll 1$, with v velocity of the particle and c the speed of light. In quantum mechanics (Blokhintsev 1964) two kinds of states occur: for instance, the pure state, represented by a wave function $\psi(\mathbf{r},t)$, and the mixed state, represented by a density matrix ρ_{nm}. A mixed state may be regarded as a probability distribution over a set of pure states. Similarly, in classical mechanics, a pure state is represented by a point in phase space and a mixed state by a probability distribution over phase space. A basis of quantum theory is the time-dependent Schrödinger equation which determines the quantum evolution (Bayfield 1999) and temporal behaviour of quantum systems; it governs all phenomena of the microscopic world.

Field (2011) showed how the time-dependent Schrodinger equation can be derived reasonably from the dynamical postulate of Feynman's path integral formulation of quantum mechanics (Derbes 1996) and the Hamilton-Jacobi equation of classical mechanics. Field (Field 2011) analysed the Schrödinger original derivation of quantum wave equations, based on the Hamilton-Jacobi equation. The Field's derivation of the time-dependent equation was based on a certain postulate which is equivalent in his opinion to Feynman's dynamical postulate. de Broglie's concepts of 'matter waves' and their phase and group velocities were also critically discussed.

Schleich *et al.* (2013) also clarified the origin of the Schrödinger equation also. They deduced the Schrödinger equation from a mathematical identity by a general formulation of classical statistical mechanics based on the Hamilton-Jacobi equation. This approach elucidated that the linearity of quantum mechanics is intimately connected to the strong coupling between the amplitude and phase of a quantum wave.

Briggs (2015) conjectured some complementary approach. Beginning with the principle that a closed mechanical composite system is timeless, he speculated that "time may be defined by the regular changes in a suitable position coordinate (clock) in the observing part when one part of the closed composite observes another. Translating this scenario into classical and quantum mechanics allows a transition from time-independent mechanics for the closed composite to a time-dependent description of the observed part alone. The Hamilton-Jacobi theory yields a very close parallel between the derivations of classical and quantum mechanics. The time-dependent equations, Hamilton-Jacobi or Schrödinger, appear as approximations since no observed system is truly closed. The quantum case

has an additional feature in the condition that the observing environment must become classical to define a real classical time variable. This condition leads to the removal of the entanglement begotten by the interaction between the observed system and the observing environment. The comparison was also made to the similar emergence of time in quantum gravity theory".

It should be stressed that an important aspect of quantum theory, which has attracted the most general attention, is the novelty of the logical notions which it presumes (Birkhoff and Von Neumann 1936; Beltrametti and Cassinelli 2010). It states that even a complete mathematical description of a physical system Φ does not in general enable one to predict with certainty the result of an experiment on Φ, and that in particular, it is difficult to predict with certainty both the position and the momentum of Φ (Heisenberg's uncertainty principle). The observation and the measurement themselves produce disturbances in the phenomenon under observation (Blokhintsev 1964, 1968).

Uncertainty relations, which were heuristically created (or postulated) by Heisenberg (1949), are the fundamental notions of quantum mechanics. They state limits of precision for simultaneous measurements of some pairs of physical quantities. These relations are the internal properties of quantum systems. The important fact is that time is not observable in quantum mechanics (Hilgevoord 2002; Briggs, Boonchui, and Khemmani 2007; Arce 2012; Bauer 2017; Kuzemsky 2023b). Contrary to the coordinate-momentum uncertainty relation where the uncertainty in momentum and coordinate at a given moment is emphasised, in the energy-time uncertainty relation, just opposite to that, the uncertainty in energy and time at a given value of the coordinate is considered. As a result the time-energy uncertainty relation

$$\Delta H \Delta t \geq h \quad \text{or} \quad \Delta E \Delta t \geq h \tag{28}$$

has a different character from the uncertainty relation involving any pair of non-commuting observables (J. B. M. Uffink and Hilgevoord 1985; Busch 1990a,b; Hilgevoord and J. Uffink 1991). Moreover, the method for its derivation and its interpretation depends on the physical context (Briggs 2008; Busch, Lahti, and Werner 2014). Hilgevoord (1996, 1998) claimed that an uncertain relation between energy and time does not exist in the usual sense. It was also conjectured that there is a general uncertainty relation of a different type between energy and time which provides a workable expression of the relation between the lifetime and the energy spread of a quantum state; that relation was discussed and elucidated.

Briggs (2008) discussed some of the difficulties associated with the derivation of a time-energy uncertainty relation. In particular, the construction of the uncertainty relations was analysed in the context of the measurement of two observables (Busch, Lahti, and Werner 2014). That approach clarified to some extent the notions of the state and observable and uncertainty relations which were derived heuristically by Heisenberg (1949). It was demonstrated also that "the quantum uncertainties can be derived directly from the limits imposed on the action, $Et = p\lambda = h$, without introducing further, more complex premises (Antonopoulos 2004). The weird predictions entailed by such premises, namely the Fourier treatment of $E = h\nu$ leading to $\Delta E \Delta t \geq h$, which have caused the scepticism of all concerned since the first appearance of the Aharonov-Bohm attack on $\Delta E \Delta t \geq h$, were completely avoided and this uncertainty, as presently interpreted, complies fully with ordinary experience. A distinction was drawn between a *particle* associated with a wave and

the action thus associated, rendering wave-particle duality idle as a basis of complementary reasoning. Reciprocity, a concept introduced by Bohr but kept behind the scenes, was reinstated and understood as a duet of complementary 'counter-happenings' at either end of a quantized unit of action, the E, p end, on the one hand, the t, q end, on the other, and was shown to be the mediating link between quantum wholeness and complementarity". Indeed, the important thing is that the pairs xp and Et have the dimension of action Dim:$[\mathscr{A}]$. Hence their values have the lower bound, namely the Planck constant h.

7. Planck constant and Planck scale

It will be instructive to briefly discuss the dimension of the basic physical notions. The dimensional analysis is a conventional and workable tool. In sciences, dimensional analysis (Bridgman 1963; Gibbings 2011) aims to establish the relationships between different physical quantities by identifying their base quantities (such as length, mass, time, and electric charge) and units of measure (such as second, meter and kilogram) and tracking these dimensions as calculations or comparisons are performed. The importance of a full and consistent system of units was recognized long ago. The derived units depend on the arbitrary fundamental units; they are functions of them. The function which corresponds to a given derived unit is called its dimensions. Thus, if l a length, t a time, and m a mass, the dimensions of a velocity, for instance, are l/t or lt^{-1}.

The most convenient base is the metric system. The International System of Units (SI) is based (Bordé 2005; Gibbings 2011; Bich 2019; Gupta 2020) on precise and definite standards and uses base 10, just like our number system. SI units are interrelated in such a way that one unit is derived from other units without conversion factors. Any physically meaningful equation (and any inequality) will have the same dimensions on its left and right sides, a property known as dimensional homogeneity.

The dimension of a physical quantity can be expressed as a product of the basic physical dimensions such as length, mass and time, each raised to a rational power. In symbolic form, the dimension of the quantity X can be written as Dim: $\mathrm{X} = \mathrm{L}^a\ \mathrm{M}^b\ \mathrm{T}^c\ \mathrm{I}^d\ \Theta^e\ \mathrm{N}^f\ \mathrm{J}^g$, where a, b, c, d, e, f, g are the dimensional exponents. Other physical quantities could be defined as the base quantities, as long as they form a linearly independent basis (Table 1).

The dimensional analysis may be considered as the other language of physics. It is of use as a method for the systematic study and analysis of physical concepts and phenomena at numerous levels of the physical sciences. It is worth mentioning that Mach devoted to dimension analysis a special section of his book (Mach 2014): *The Formulae and Units of Mechanics*. He examined thoroughly the mechanical units of his time and discussed such terms as *force*, *living force*, *momentum*, *impulse*, and *energy*. Mach (2014) systematized carefully the mechanical units: "Thus, as the unit of area and the unit of volume, the square and cube of the unit of length are always employed. According to this, we assume then, that by unit velocity, unit length is described in unit time, that by unit acceleration, unit velocity is gained in unit time, that by unit force, unit acceleration is imparted to unit mass, and so on". He selected the nine basic mechanical notions in the form of a table similar to Table 1. Consequently, it is possible to think of a unit as a convenient reference physical quantity with the aid of which it is possible to characterize other quantities of the same kind (Bordé 2005; Bich 2019; Gupta 2020).

TABLE 1. Some units and their dimension

Unit Name	Unit Symbol	Quantity Name	Dimension Symbol
kilogram	kg	mass	M
meter	l	length	L
second	t	time	T
joule	J	energy	$\mathrm{M\,L^2\,T^{-2}}$
energy × time	$\mathscr{A}$	action	$\mathrm{M\,L^2\,T^{-1}}$
action $\times$time^{-2}	P	power	$\mathrm{M\,L^2\,T^{-3}}$
kelvin	K	thermodynamic temperature	Θ
joule/kelvin	J/K	entropy	$\mathrm{M\,L^2\,T^{-2}\,\Theta^{-1}}$

Planck introduced the constant h in his efforts to fit the classical theory of radiation to the emission spectrum of a blackbody (Jammer 1966). It is by this fundamental constant namely, and by the choice of that constant as an action, Dim:$[\mathscr{A}] = \mathrm{M\,L^2\,T^{-1}}$, the Planck quantum radiation law

$$\rho(\nu,T)d\nu = \frac{8\pi h\nu^3}{c^3(\exp(h\nu/k_BT)-1)}d\nu$$

and the Sommerfeld's quantization conditions

$$\oint p\,dq = nh$$

were deduced (Jammer 1966). Here ν is the frequency of radiation. The Planck constant may be written as

$$h \cong 6.626\times10^{-34}\frac{\mathrm{J}}{\mathrm{Hz}} \quad \text{or} \quad \hbar = \frac{h}{2\pi} \cong 1.055\times10^{-34}\mathrm{J\cdot s}$$

Using dimensional analysis, Epstein (1954) has shown that both the Stefan-Boltzmann law and Wien's displacement law imply the existence of Planck's constant of action h or an equivalent constant. Maslov (2008) carried out a complimentary consideration of these aspects. Carati and Galgani (2004) re-analysed some aspects of Planck's derivation in the context of the equipartition axiom. As the second step, Planck constructed a system of 'natural' units (Lesche 2014) for the fundamental quantities of mass, length, and time, which could be constructed from the three fundamental constants $\hbar$, c, and G, where $\hbar$ is the reduced Planck constant $h/2\pi$, c = speed of light, and G = Newton's gravitation constant.

Faraoni (2017) examined the three new roads to the Planck scale (Barrow 1983, 2009). He presented the three new heuristic derivations of the Planck scale. They are based on basic principles or phenomena of relativistic gravity and quantum physics. The Planck scale quantities thus obtained are within one order of magnitude of the 'standard' ones. The

author contemplated the pair creation of causal bubbles so small that they can be treated as particles, the scattering of a matter wave of the background curvature of spacetime that it induces, and the Hawking evaporation of a black hole in a single burst at the Planck scale. It should be stressed that temperature and entropy are the non-mechanical notions (Kuzemsky 2023b). Its dimensions depend on the units selected for the purpose. Thermodynamic entropy has the dimensions of energy per temperature, in SI units, it is joules per kelvin (J/K), i.e., coincides with the dimension of the Boltzmann constant k_B. The Boltzmann constant is equal to the ratio of the gas constant R to Avogadro number N_A

$$k_B = \frac{R}{N_A} = 1.38 \cdot 10^{-23} \frac{\mathrm{J}}{\mathrm{K}} = 8.617 \cdot 10^{-5} \frac{\mathrm{eV}}{\mathrm{K}} \tag{29}$$

Hence the Boltzmann constant is a *derived* physical constant since its value is determined by other physical constants. It should be stressed that the Boltzmann constant is very important but not a fundamental constant. It is the conversion between average translational kinetic energy per particle and temperature units. As a consequence, its value depends on the size of the temperature unit. In the case of SI its value follows from the water-based centigrade scale. The Boltzmann constant would have an arduous numerical value if SI had been based on the different (e.g., Rankin) temperature scale, or some other scale derived from the properties of a substance other than water. Thermodynamic entropy is the dynamical units of measurement has dimensionality (Bridgman 1963; Gibbings 2011) which coincides with the dimensionality of energy divided by temperature: $[S] = [E/\Theta] = \mathrm{L}^2\ \mathrm{T}^{-2}\ \mathrm{M}\ \Theta^{-1}$. In principle, it would be practical to have a special unit for entropy. It is not so clear why the entropy does not obtain a separate unit name. One reason for this may be the fact that entropy's dimensions are linked to the definition of the Kelvin temperature scale (Kuzemsky 2023b). Entropy can be defined as dimensionless when temperature Θ is defined as an energy.

When using natural units (Bridgman 1963; Bordé 2005; Gibbings 2011; Bich 2019) one sets each of the constants $\hbar$, k_B and c equal to unity. It is still possible to check the dimensions of a given equation, but one then has to understand that quantities such as those associated with velocity are dimensionless and that things like length and time have the same dimensions. One can write every quantity in terms of the powers of a single unit, e.g., GeV(= 10^9 eV). The conversion factors to SI units are: Energy : 1 GeV = 1.6 10^{-10} J; Temperature : 1 GeV = 1.16 10^{13} K. It is worth noting that the action function has the dimension energy times time. The simplest analysis of dimension (Bridgman 1963; Gibbings 2011) gives the relation:

$$t \sim \frac{\text{Action}}{\text{Energy}}.$$

As a simplest illustration we take the minimal possible action, the Planck constant h, and obtain

$$t \sim \frac{h}{h\nu} \sim \frac{h}{\hbar\omega} \sim \frac{2\pi}{\omega}.$$

8. Concluding remarks

In this review, we formulated a conceptual framework of the key notions: action and time. The science and logic of studies of time and action do not allow us to ignore the deep interconnection of these core concepts. The study enlightens the problem in some new way and might help gain insight into the statement (Barrow and Tipler 1988): "Physical theories have their most fundamental expression as action integrals". In this connection, it was important to show and underline that at the heart of the physical and astrophysical sciences and, more generally, natural sciences, lies the smallest possible action, i.e., the Planck constant h. It was stressed that the Planck constant makes rules for a wide variety of physical, chemical, astrophysical and, in part, biological processes (Kuzemsky 2023a,b).

Our purpose was to address the basic problems of finding foundations for the conceptualisation and organisation of the key notions used for the description of natural processes. We demonstrated also that there is an intricate interconnection between these two core concepts, action and time. The arguments were formulated earlier (Kuzemsky 2023a,b) about the necessity of the study and careful reconsideration of the problem of time and temporality from a broad perspective in the context of natural processes. And the first of all processes is the flow of physical time. These ideas may reveal possible fundamental principles of how the logical and axiomatic approaches work in natural phenomena. We have highlighted the perspective of finding global principles and tried to explicate the rules governing the emergence of these global principles. These matters are of great importance for cosmological (Ridderbos 2003; Calcagni 2017; Kuzemsky 2023a,b), biological (Kuzemsky 2023b; Velazquez, Mateos, and Guevara 2023) and other processes studies (Lucia 2023). We discussed the main features of quantum mechanics that distinguish it from classical physics. It is a fact that canonical variables are related to each other by the Heisenberg uncertainty principle and are considered operators in the Hilbert space. We emphasized the central role of the Planck constant, which is the universal constant that defines the quantum nature of energy and relates the energy of a photon to its corresponding frequency. Planck constant determines the amount of energy that a photon can carry, in accordance with the frequency of the wave by which it propagates.

To summarize, we considered and justified the thesis that action and time are the foundational concepts in classical and quantum physics and surveyed certain selected advances in the theoretical understanding of the interrelation of these two core concepts concerning other basic notions of the science of complex systems. Now, we wish to suggest some considerations on recent advances in research on time because time remains a big problem in physics. Annila (2021) proved that the flow of time is associated with the flow of quanta and it pointed out how the equation of time can be derived from statistical mechanics of open systems: time and energy change are linked together. Annila (2021) proved that the evolution of any substance can be understood so that the quanta redistribute through all kinds of events until the most likely state has been attained. Following the results of Annila and the physical-mathematical rigorous approach of Barbour (2009) and Kon (2011), modified to introduce the laws of thermodynamics, we consider the Second Law of Thermodynamics (Bejan 1997):

$$\frac{dS}{dt} = \frac{\phi}{T} + \Sigma \tag{30}$$

where S is the entropy [J K^{-1}], ϕ is the heat power [W], T is the temperature [K], $\Sigma = d\sigma/dt$ is the entropy production rate [W K^{-1}], with σ entropy production [J K^{-1}] and t time [s]. For a stationary system $dS/dt = 0$, so it follows (Lucia and Grisolia 2024):

$$\frac{\phi}{T} + \frac{d\sigma}{dt} = 0 \tag{31}$$

obtaining (Lucia and Grisolia 2024):

$$dt = -\frac{d\sigma}{\phi/T} \tag{32}$$

from which a time interval, τ [s], can be defined as (Lucia and Grisolia 2024):

$$\tau = -\frac{T\,\sigma}{\phi} \tag{33}$$

Considering the experimental evidence (Doyle 1968; Batalhão *et al.* 2015; Doyle 2016, 2023) a thermodynamic approach to irreversibility in quantum systems has been developed (Lucia 2016, 2023) focusing the causes of irreversibility on the continuous interaction between the environmental electromagnetic waves and the matter (Planck 1959; Jammer 1966; Surdin, Braffort, and Taroni 1966; Rueda 1974; Fonseca, Gomes, and Moura-Melo 2009; Heitler 2010; Beretta and Gyftopoulos 2015; Boyer 2018). Consequently, considering that the radiation inflow is an inflow of energy that in thermodynamics can be considered as the heat inflow from an reservoir, it follows (Beretta and Gyftopoulos 2015):

$$\phi = \frac{A}{2}\varepsilon_0 c E_{el}^2 + \frac{A}{2\mu_0} c B_m^2 \tag{34}$$

where E_{el} is the electric field, B_m is the magnetic field, $c = 299792458$ m s^{-1} is the speed of light, $\varepsilon_0 = 8.8541878128(13) \times 10^{-12}$ F m^{-1} is the electric permittivity in vacuum and $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is the magnetic permeability in vacuum, A is the area of the border of the thermodynamic control volume, and T_0 is the environmental temperature, while the entropy production has been evaluated with respect to the semi-classical analysis of the photon-bound electron interaction (Lucia 2016):

$$T_0\,\sigma = \frac{m_e}{M} E_\gamma \tag{35}$$

where T_0 is the environmental temperature, m_e and M are the mass of the electron and the atomic nucleus respectively, and E_γ is the energy of the incoming photon. Time interval results from Equation (33) (Lucia and Grisolia 2024):

$$\tau = \frac{2\,m_e}{M\,c\,A}\,\frac{E_\gamma}{\varepsilon_0 E_{el}^2 + \mu_0^{-1} B_m^2} \tag{36}$$

This result is based on the principle of increasing entropy, which is a universal principle derived from the statistical physics of open systems assuming that quanta of actions embody everything. Consequently, all systems, atoms included, evolve from one state to another either by acquiring quanta from their surroundings or by discarding quanta to the surroundings in ubiquitous scale-free patterns (Annila 2016). Moreover, this result is in accordance

with the statistical notion of flows of energy that diminishes differences between energy densities that form space so that the flow of energy can be identified with the flow of time. Consequently, following Tuisku, Pernu, and Annila (2009), the curved space–time is in evolution when energy is flowing down along gradients and levelling the density differences.

References

Allen, J. F. (1984). "Towards a general theory of action and time". *Artificial Intelligence* **23**, 123–154. DOI: 10.1016/0004-3702(84)90008-0.

Andréka, H., Madarász, J., Németi, I., and Székely, G. (2010). "On logical analysis of relativity theories". *Hungarian Philosophical Review* **54**(4), 204–222. URL: https://real.mtak.hu/33982/.

Annila, A. (2016). "Natural thermodynamics". *Physica A: Statistical Mechanics and its Applications* **444**, 843–852. DOI: 10.1016/j.physa.2015.10.105.

Annila, A. (2021). "The matter of time". *Entropy* **23**, 943. DOI: 10.3390/e23080943.

Antonopoulos, C. (2004). "Reciprocity, complementarity and minimal action". *Annales Fondation Louis de Broglie* **29**(3), 427–446. URL: https://fondationlouisdebroglie.org/AFLB-293/aflb293m131.htm.

Arce, J. C. (2012). "Unification of the conditional probability and semiclassical interpretations for the problem of time in quantum theory". *Physical Review A* **85**, 042108. DOI: 10.1103/PhysRevA.85.042108.

Arens, R. (1963). "An axiomatic basis for classical thermodynamics". *Journal of Mathematical Analysis and Applications* **6**, 207–229. DOI: 10.1016/0022-247X(63)90005-2.

Baker, A. (2007). "Occam's Razor in science: a case study from biogeography". *Biology and Philosophy* **22**, 193–215. DOI: 10.1007/s10539-006-9027-9.

Barbour, J. B. (2001). *The Discovery of Dynamics. A Study from a Machian Point of View of the Discovery and the Structure of Dynamical Theories*. Oxford: Oxford University Press.

Barbour, J. B. (2009). "The nature of time". DOI: 10.48550/arXiv.0903.3489. arXiv: 0903.3489v1 [gr-qc].

Barrow, J. D. (1983). "Dimensionality". *Philosophical Transactions of the Royal Society London A* **310**, 337–346. DOI: 10.1098/rsta.1983.0095.

Barrow, J. D. (2001). *The Book of Nothing*. London: Vintage.

Barrow, J. D. (2009). *The Constants of Nature: The Numbers that Encode the Deepest Secrets of the Universe*. London: Vintage.

Barrow, J. D. and Tipler, F. J. (1988). "Action principle in nature". *Nature* **331**, 31–34. DOI: 10.1038/331031a0.

Batalhão, T. B., Souza, A. M., Sarthour, R. S., Oliveira, I. S., Paternostro, M., Lutz, E., and Serra, R. M. (2015). "Irreversibility and the arrow of time in a quenched quantum Ssystem". *Physical Review Letters* **115**, 190601. DOI: 10.1103/PhysRevLett.115.190601.

Bauer, M. (2017). "On the problem of time in quantum mechanics". *European Journal of Physics* **38**, 035402. DOI: 10.1088/1361-6404/aa6025.

Bayfield, J. E. (1999). *Quantum Evolution: An Introduction to Time-Dependent Quantum Mechanics*. New York (U.S.A.): John Wiley and Sons.

Bejan, A. (1997). *Advanced Engineering Thermodynamics*. 2nd ed. New York: Wiley.

Bell, A. E. (1943). "The concept of energy". *Nature* **151**, 519–523. DOI: 10.1038/151519a0.

Belot, G. (2007). "The Representation of Time and Change in Mechanics". In: *Handbook of the Philosophy of Physics*. Ed. by J. Butterfield and J. Earman. London (U.K.): North-Holland, pp. 133–228.

Beltrametti, E. G. and Cassinelli, G. (2010). "The Logic of Quantum Mechanics". In: *Encyclopedia of Mathematics and its Applications*. Ed. by G.-C. Rota. Vol. 15. Cambridge: Cambridge University Press.

Beretta, G. P. and Gyftopoulos, E. P. (2015). "Electromagnetic radiation: a carrier of energy and entropy". *Journal of Energy Resources Technology* **137**, 021005. DOI: 10.1115/1.4026381.

Bich, W. (2019). "The third-millennium International System of Units". *La Rivista del Nuovo Cimento* **42**(2), 49–102. DOI: 10.1393/ncr/i2019-10156-2.

Birkhoff, G. and Von Neumann, J. (1936). "The logic of quantum mechanics". *Annals of Mathematics* **37**, 823–843. DOI: 10.2307/1968621.

Blokhintsev, D. I. (1964). *Quantum Mechanics*. Dordrecht: D. Reidel Publ.

Blokhintsev, D. I. (1968). *The Philosophy of Quantum Mechanics*. Dordrecht: D. Reidel Publ.

Bogolubov, N. N., Logunov, A. A., Oksak, A. I., and Todorov, I. T. (2012). *General Principles of Quantum Field Theory*. Berlin: Springer.

Bogolubov, N. N., Logunov, A. A., and Todorov, I. T. (1975). *Introduction to Axiomatic Quantum Field Theory*. New York (U.S.A.): W. A. Benjamin.

Bordé, C. J. (2005). "Base units of the SI, fundamental constants and modern quantum physics". *Philosophical Transactions of the Royal Society A* **363**, 2177–2201. DOI: 10.1098/rsta.2005.1635.

Boyer, T. H. (2018). "Blackbody radiation in classical physics: A historical perspective". *American Journal of Physics* **86**, 495–509. DOI: 10.1119/1.5034785.

Bridgman, P. W. (1963). *Dimensional Analysis*. New Haven (U.S.A.): Yale University Press.

Briggs, J. S. (2008). "A derivation of the time-energy uncertainty relation". *Journal of Physics: Conference Series* **99**, 012002. DOI: 10.1088/1742-6596/99/1/012002.

Briggs, J. S. (2015). "Equivalent emergence of time dependence in classical and quantum mechanics". *Physical Review A* **91**, 052119. DOI: 10.1103/PhysRevA.91.052119.

Briggs, J. S., Boonchui, S., and Khemmani, S. (2007). "The derivation of time-dependent Schrödinger equations". *Journal of Physics A: Mathematical and Theoretical* **40**, 1289–1302. DOI: 10.1088/1751-8113/40/6/007.

Buchwald, J. Z. and Fox, R., eds. (2013). *The Oxford Handbook of the History of Physics*. Oxford: Oxford University Press. DOI: 10.1093/oxfordhb/9780199696253.001.0001.

Busch, P. (1990a). "On the energy-time uncertainty relation. Part I: Dynamical time and time indeterminacy". *Foundations of Physics* **20**, 1–32. DOI: 10.1007/BF00732932.

Busch, P. (1990b). "On the energy-time uncertainty relation. Part II: Pragmatic time versus energy indeterminacy". *Foundations of Physics* **20**, 33–43. DOI: 10.1007/bf00732933.

Busch, P., Lahti, P., and Werner, R. F. (2014). "Measurement uncertainty relations". *Journal of Mathematical Physics* **55**, 042111. DOI: 10.1063/1.4871444.

Calcagni, G. (2017). *Classical and Quantum Cosmology*. Berlin: Springer.

Carati, A. and Galgani, L. (2004). "A critical remark on Planck's model of black-body". *International Journal of Modern Physics B* **18**, 549–553. DOI: 10.1142/S0217979204024161.

Carrol, S. M. (2010). *From Eternity to Here: The Quest for the Ultimate Theory of Time*. New York (U.S.A.): Dutton.

Carrol, S. M. (2022). *The Biggest Ideas in the Universe: Space, Time, and Motion*. New York (U.S.A.): Dutton.

Cohen, I. B. (1985). *The Birth of a New Physics*. London: Penguin Books.

Coopersmith, J. (2015). *Energy, the Subtle Concept: The Discovery of Feynman's Blocks from Leibnitz to Einstein*. London: Routledge.

D'Agostino, S. (2000). *A History of the Ideas of Theoretical Physics. Essays on the Nineteenth and Twentieth Century Physics*. Dordrecht: Springer. DOI: 10.1007/978-94-010-9034-6.

Davies, P. C. W. (1977). *Space and Time in the Modern Universe*. Cambridge: Cambridge University Press.

Derbes, D. (1996). "Feynman's derivation of the Schrödinger equation". *American Journal of Physics* **64**, 881–884. DOI: 10.1119/1.18114.
Doyle, R. O. (1968). "The continuous spectrum of the hydrogen quasi-molecule". *Journal of Quantitative Spectroscopy and Radiative Transfer* **8**(9), 1555–1569. DOI: 10.1016/0022-4073(68)90050-2.
Doyle, R. O. (2016). *Great Problems in Philosophy & Physics – Solved?* Cambridge, MA, USA: I-Phi Press. URL: https://philpapers.org/archive/DOYGPI.pdf.
Doyle, R. O. (2023). *The Origin of Irreversibility*. Information Philosopher. URL: https://informationphilosopher.com/problems/reversibility/Irreversibility.pdf.
Eigen, M. (2013). *From Strange Simplicity to Complex Familiarity: A Treatise on Matter, Information, Life and Thought*. Oxford: Oxford University Press.
Einstein, A. (1940). "Science and Religion". *Nature* **146**, 605–607. DOI: 10.1038/146605a0.
Einstein, A. (1947). "The problem of space, ether, and the field in physics". In: *Man and the Universe. The Philosophers of Science*. Ed. by S. Commins and R. N. Linscott. Vol. 4. The Great Thinkers. New York (U.S.A.): Random House.
Einstein, A. and Infeld, L. (1954). *The Evolution of Physics. The Growth of Ideas from Early Concepts to Relativity and Quanta*. New York (U.S.A.): Simon and Shuster.
Emch, G. G. and Liu, C. (2002). *The Logic of Thermostatistical Physics*. Berlin: Springer.
Engler, F. O. (2021). "Einstein and logical empiricism". In: *The Routledge Handbook of Logical Empiricism*. Ed. by T. Uebel and C. Limbeck-Lilienau. London (U.K.): Routledge. Chap. 9.
Epstein, P. S. (1954). "On Planck's quantum of action". *American Journal of Physics* **22**, 402–405. DOI: 10.1119/1.1933756.
Faraoni, V. (2017). "Three new roads to the Planck scale". *American Journal of Physics* **85**, 865–869. DOI: 10.1119/1.4994804.
Field, J. H. (2011). "Derivation of the Schrödinger equation from the Hamilton-Jacobi equation in Feynman's path integral formulation of quantum mechanics". *European Journal of Physics* **32**, 63–87. DOI: 10.1088/0143-0807/32/1/007.
Fonseca, J. M., Gomes, A. H., and Moura-Melo, W. A. (2009). "Emission and absorption of photons and the black-body spectrum in Lorentz-odd electrodynamics". *Physics Letters B* **671**(2), 280–283. DOI: 10.1016/j.physletb.2008.12.029.
G.'t Hooft and Vandoren, S. (2015). *Time in Power Ten. Natural Phenomena and Their Timescales*. Singapore: World Scientific.
Garrido, L. M. (1961). "Action principle for classical mechanics". *Journal of Mathematical Analysis and Applications* **3**, 295–314. DOI: 10.1016/0022-247X(61)90056-7.
Gell-Mann, M. (1994). *The Quark and the Jaguar. Adventures in the Simple and the Complex*. New York (U.S.A.): Freeman and Co.
Gibbings, J. C. (2011). *Dimensional Analysis*. Berlin: Springer.
Giles, R. (1964). *Mathematical Foundations of Thermodynamics*. London: Pergamon Press.
Gitt, W. (2006). *In the Beginning Was Information*. Green Forest, Arkansas: Master Books.
Giufolini, I. and Wheeler, J. A. (1995). *Gravitation and Inertia*. Princeton: Princeton University Press.
Gleick, J. (2011). *The Information: A History, A Theory, A Flood*. London: Vintage.
Guillemin, V. and Sternberg, S. (1990). *Symplectic Techniques in Physics*. Cambridge: Cambridge University Press.
Gupta, S. V. (2020). *Units of Measurement. History, Fundamentals and Redefining the SI Base Units*. Berlin: Springer.
Harman, P. M. (1982). *Energy, Force and Matter: The Conceptual Development of Nineteenth-Century Physics*. Cambridge: Cambridge University Press.
Heisenberg, W. (1949). *The Physical Principles of the Quantum Theory*. New York (U.S.A.): Dover.
Heitler, W. (2010). *The Quantum Theory of Radiation*. New York: Dover Publications.

Hesse, M. B. (1962). *Forces and Fields. The Concept of Action at a Distance in the History of Physics*. New York (U.S.A.): Dover.
Hilgevoord, J. (1996). "The uncertainty principle for energy and time". *American Journal of Physics* **64**, 1451–1456. DOI: 10.1119/1.18410.
Hilgevoord, J. (1998). "The uncertainty principle for energy and time. II." *American Journal of Physics* **66**, 396–402. DOI: 10.1119/1.18880.
Hilgevoord, J. (2002). "Time in quantum mechanics". *American Journal of Physics* **70**, 301–306. DOI: 10.1119/1.1430697.
Hilgevoord, J. and Uffink, J. (1991). "Uncertainty in prediction and in inference". *Foundations of Physics* **21**, 323–341. DOI: 10.1007/bf01883638.
Holton, G. and Brush, S. G. (2001). *Physics, the Human Adventure. From Copernicus to Einstein and Beyond*. New Brunswick: Rutgers University Press.
Jammer, M. (1966). *The Conceptual Development of Quantum Mechanics*. New York: McGraw Hill. URL: https://archive.org/details/conceptualdevelo0000jamm/mode/2up.
Jammer, M. (2000). *Concepts of Mass in Contemporary Physics and Philosophy*. Princeton: Princeton University Press.
Jammer, M. (2011). *Concepts of Force*. New York (U.S.A.: Dover.
Janner, M. (2012). *Concepts of Space: The History of Theories of Space in Physics*. New York (U.S.A.): Dover.
Kanda, A., Prunescu, M., and Wong, R. (2020). "Logical analysis of relativity theory". DOI: 10.48550/arXiv.2006.15289. arXiv: 2006.15289 [physics.hist-ph].
Kibble, T. W. B. and Berkshire, F. H. (2004). *Classical mechanics*. London: Imperial College Press.
Kon, M. (2011). "A conceptual analysis of Julian Barbour's time". PhD thesis. The University of Leeds. URL: https://etheses.whiterose.ac.uk/id/eprint/2288/.
Kuzemsky, A. L. (2010). "Bogoliubov's vision: quasiaverages and broken symmetry to quantum protectorate and emergence". *International Journal of Modern Physics B* **24**, 835–935. DOI: 10.1142/S0217979210055378.
Kuzemsky, A. L. (2016). "Probability, information and statistical physics". *International Journal of Theoretical Physics* **55**, 1378–1404. DOI: 10.1007/s10773-015-2779-8.
Kuzemsky, A. L. (2017). *Statistical Mechanics and the Physics of Many-Particle Model Systems*. Singapore: World Scientific.
Kuzemsky, A. L. (2018). "Temporal evolution, directionality of time and irreversibility". *La Rivista del Nuovo Cimento* **41**, 513–574. DOI: 10.1393/ncr/i2018-10152-0.
Kuzemsky, A. L. (2020). "In search of time lost: Asymmetry of time and irreversibility in natural processes". *Foundations of Science* **25**, 597–645. DOI: 10.1007/s10699-020-09658-0.
Kuzemsky, A. L. (2023a). "Temporal behavior of complex systems: From microworld to macroworld". *Physics of Particles and Nuclei* **54**, 843–868. DOI: 10.1134/S1063779623050155.
Kuzemsky, A. L. (2023b). *The Mystery of Time. Asymmetry of Time and Irreversibility in the Natural Processes*. Singapore: World Scientific.
Lammerzahl, C. and Macias, A. (1993). "On the dimensionality of space-time". *Journal of Mathematical Physics* **34**, 4540–4553. DOI: 10.1063/1.530355.
Lanczos, C. (1986). *The Variational Principles of Mechanics*. New York (U.S.A.): Dover.
Landau, L. D. and Lifshitz, E. M. (1976). *Mechanics*. Oxford: Butterworth-Heinemann.
Laughlin, R. B. and Pines, D. (2000). "The Theory of Everything". *Proceedings of the National Academy of Sciences* **97**, 28–31. DOI: 10.1073/pnas.97.1.28.
Lawrie, I. D. (2015). *A Unified Grand Tour of Theoretical Physics*. London: Taylor and Francis.
Lesche, B. (2014). "The $c = \hbar = G = 1-$ question". *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **47**, 107–116. DOI: 10.1016/j.shpsb.2014.06.002.

Lindsay, R. B. (1971). "The concept of energy and its early historical development". *Foundations of Physics* **1**, 383–393. DOI: 10.1007/BF00708586.
Lindsay, R. B., ed. (1975). *Energy. Historical Development of the Concept*. Vol. 1. Benchmark Papers on Energy. New York (U.S.A.): John Wiley & Sons, Inc.
Longair, M. S. (2020). *Theoretical Concepts in Physics: An Alternative View of Theoretical Reasoning in Physics*. Cambridge: Cambridge University Press.
Lucia, U. (2016). "Macroscopic irreversibility and microscopic paradox: A Constructal law analysis of atoms as open systems". *Scientific Reports* **6**, 35796. DOI: 10.1038/srep35796.
Lucia, U. (2023). "Irreversible and quantum thermodynamic considerations on the quantum Zeno effect". *Scientific Reports* **13**, 10763. DOI: 10.1038/s41598-023-38040-w.
Lucia, U., Grisolia, G., and Kuzemsky, A. L. (2020). "Time, irreversibility and entropy production in nonequilibrium systems". *Entropy* **22**, 887. DOI: 10.3390/e22080887.
Lucia, U. and Grisolia, G. (2024). "Definition of time: From the Second to Constructal Law". In: *Proceedings of the 14th Constructal Law Conference. Design in Nature and Evolution. 11-14 October 2024*. Ed. by A. Morega, Ü. Guneş, and A. Bejan. Bucharest: Academia Romana. DOI: 10.59277/CLC.2024.08.
Ma, Q.-P. (2013). *Theory of Relativity: Principles, Logic and Experimental Foundation*. New York (U.S.A.): Nova Science Publishers, Inc.
Mach, E. (2014). *The Science of Mechanics: A Critical and Historical Account of Its Development*. Cambridge: Cambridge University Press.
Mackey, G. W. (1963). *The Mathematical Foundations of Quantum Mechanics*. New York (U.S.A.): W. A. Benjamin.
Mainzer, K. (2007). *Thinking in Complexity. The Computational Dynamics of Matter, Mind, and Mankind*. Berlin: Springer.
Maslov, V. P. (2008). "Quasithermodynamics and a correction to the Stefan-Boltzmann law". *Mathematical Notes* **83**, 72–79. DOI: 10.1134/S0001434608010094.
Mittelstaedt, P. (2011). *Rational Reconstructions of Modern Physics*. Berlin: Springer.
Mittelstaedt, P. and Weingartner, P. A. (2005). *Laws of Nature*. Berlin: Springer.
Moore, C. C. (2015). "Ergodic theorem, ergodic theory, and statistical mechanics". *Proceedings of the National Academy of Sciences* **112**, 1907–1911. DOI: 10.1073/pnas.1421798112.
Muller, I. (2007). *A History of Thermodynamics. The Doctrine of Energy and Entropy*. Berlin: Springer.
Nicolis, G. and Prigogine, I. (1989). *Exploring Complexity: An Introduction*. New York (U.S.A.): Freeman and Co.
Okun, L. B. (2009). *Energy and Mass in Relativity Theory*. Singapore: World Scientific.
Owen, D. R. (1984). *A First Course in the Mathematical Foundations of Thermodynamics*. Berlin: Springer.
Peat, F. D. (2002). *From Certainty to Uncertainty: The Story of Science and Ideas in the Twentieth Century*. New York (U.S.A.): Joseph Henry Press.
Piron, C. (1976). *Foundations of Quantum Physics*. New York (U.S.A.): W. A. Benjamin.
Planck, M. (1959). *The Theory of Heat Radiation*. New York: Dover Publications.
Prokhovnik, S. J. (1967). *The Logic of Special Relativity*. Cambridge: Cambridge University Press.
Ridderbos, K. (2003). "The thermodynamic arrow of time in quantum cosmology". In: *Philosophical Dimensions of Logic and Science*. Ed. by A. Rojszczak, J. Cachro, and G. Kurczewski. Vol. 320. Synthese Library. Dordrecht: Springer, pp. 179–194. DOI: 10.1007/978-94-017-2612-2_14.
Rueda, A. (1974). "On the irreversible thermophysics of radiative processes". *Foundations of Physics* **4**, 215–226. DOI: 10.1007/BF00712688.
Rulle, D. (1978). *Thermodynamic Formalism: The Mathematical Structures of Classical Equilibrium Statistical Mechanics*. New York (U.S.A.): Addison-Wesley.

Schleich, W. P., Greenberger, D. M., Kobe, D. H., and Scully, M. O. (2013). "Schrödinger equation revisited". *Proceedings of the National Academy of Sciences* **110**, 5374–5379. DOI: 10.1073/pnas.1302475110.

Schwichtenberg, J. (2017). *Physics From Symmetry*. Berlin: Springer.

Schwinger, J. (1986). *Einstein's Legacy: The Unity of Space and Time*. New York (U.S.A.): Scientific American Books.

Siburg, K. F. (2004). *The Principle of Least Action in Geometry and Dynamics*. Berlin: Springer.

Silverberg, J. and Widom, A. (2007). "Classical analytical mechanics and entropy production". *American Journal of Physics* **75**, 993–996. DOI: 10.1119/1.2772279.

Sklar, L. (1977). *Space, Time and Spacetime*. California (U.S.A.): University of California Press.

Sternberg, S. (1995). *Group Theory and Physics*. Cambridge: Cambridge University Press. URL: https://www.cambridge.org/it/universitypress/subjects/mathematics/algebra/group-theory-and-physics?format=PB&isbn=9780521558853.

Surdin, M., Braffort, P., and Taroni, A. (1966). "Black-body radiation law deduced from stochastic electrodynamics". *Nature* **210**, 405–406. DOI: 10.1038/210405a0.

Synge, J. L. (1960). "Classical Dynamics". In: *Principles of Classical Mechanics and Field Theory / Prinzipien der Klassischen Mechanik und Feldtheorie*. Ed. by S. Flügge. Berlin, Heidelberg: Springer. DOI: 10.1007/978-3-642-45943-6_1.

Thébault, K. P. Y. (2012). "Symplectic reduction and the problem of time in nonrelativistic mechanics". *The British Journal for the Philosophy of Science* **63**(4), 789–824. DOI: 10.1093/bjps/AXR045.

Toms, D. (2007). *The Schwinger Action Principle and Effective Action*. Cambridge: Cambridge University Press.

Tornebohm, H. (1952). *A Logical Analysis of the Theory of Relativity*. Stockholm: Almqvist-Wiksell.

Tribus, M. and McIrvine, E. C. (1971). "Energy and information". *Scientific American* **225**(3), 179–190. URL: https://www.jstor.org/stable/24923125.

Truesdell, C. (1968). *Essays in the History of Mechanics*. Berlin: Springer. DOI: 10.1007/978-3-642-86647-0.

Tuisku, P., Pernu, T. K., and Annila, A. (2009). "In the light of time". *Proceedings of the Royal Society A* **465**, 1173–1198. DOI: 10.1098/rspa.2008.0494.

Uffink, J. B. M. and Hilgevoord, J. (1985). "Uncertainty principle and uncertainty relations". *Foundations of Physics* **15**, 925–944. DOI: 10.1007/BF00739034.

Varvoglis, H. (2014). *History and Evolution of Concepts in Physics*. Cham: Springer. DOI: 10.1007/978-3-319-04292-3.

Velazquez, J. L. P., Mateos, D. M., and Guevara, R. (2023). "Is the tendency to maximize energy distribution an optimal collective activity for biological purposes? A proposal for a global principle of biological organization". *Heliyon* **9**, e15005. DOI: 10.1016/j.heliyon.2023.e15005.

Von Neumann, J. (2018). *Mathematical Foundations of Quantum Mechanics*. Princeton (U.S.A.): Princeton University Press.

Watson, D. K. (1989). "Schwinger variational methods". *Advances in Atomic and Molecular Physics* **25**, 221–250. DOI: 10.1016/S0065-2199(08)60088-3.

Wehrl, A. (1978). "General properties of entropy". *Reviews of Modern Physics* **50**, 221–260. DOI: 10.1103/RevModPhys.50.221.

Weinert, F. (2013). *The March of Time. Evolving Conceptions of Time in the Light of Scientific Discoveries*. Berlin: Springer.

Whittaker, E. T. (1988). *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*. Cambridge: Cambridge University Press.

Whittaker, E. T. (1989). *A History of the Theories of Aether and Electricity*. New York (U.S.A.): Dover.

Wilczek, F. (2007). "Fundamental Constants". DOI: 10.48550/arXiv.0708.4361. arXiv: 0708.4361 [hep-ph].
Wilczek, F. (2021). *Fundamentals: Ten Keys to Reality*. London: Penguin Press.
Zee, A. (2013). *Quantum Field Theory in a Nutshell*. Princeton: Princeton University Press.

---

[a] Politecnico di Torino,
Dipartimento Energia "Galileo Ferraris",
Corso Duca degli Abruzzi 24, 10129 Turin, Italy

[b] Politecnico di Torino,
Dipartimento di Ingegneria dell'Ambiente, del Territorio e delle Infrastrutture,
Corso Duca degli Abruzzi 24, 10129 Turin, Italy

[c] Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
141980 Dubna, Moscow Region, Russia

* To whom correspondence should be addressed | email: umberto.lucia@polito.it

Manuscript received 27 February 2024; communicated 6 May 2024; published online 11 March 2025