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Mean Distances and Dependence Structures for Lifetimes of Systems With Shared Components

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ABSTRACT

Distortion and copula functions represent powerful tools in the description of the reliability of some complex systems as functions of their components' reliability. On this aim, we study several pairs of reliability systems with one or more shared components, in the case when their lifetimes are independent and identically distributed or independent but not identically distributed. We focus on the dependence that arises from sharing components, often described by Marshall-Olkin copulas, making use of some distance measures related to the Gini's mean difference and its new recent generalizations. A special role is played by a new distortion function related to the ROC curve.

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1 | Introduction

The most interesting problems in reliability theory often involve complex systems, consisting of several components. In this context, a non-negative random variable, namely *random lifetime*, describes the failure time of a system or a component that starts to work at time $t = 0$. The study of the system's lifetime can be performed from its components' lifetimes. More in detail, the lifetime distribution of some reliability systems can be determined from the components' lifetimes distributions making use of suitable distortion functions, by taking into account the system structure (see Section 2.4 in Navarro [1] and references therein). In this sense, distortion functions and copulas (that are multivariate distortions) represent powerful tools for system reliability issues and survival analysis. For instance, they can be used to predict the failure times of systems from the early failure times of their components, see Navarro et al. [2]. Another option is to employ distortion functions for the preservation of several stochastic orders under the formation of reliability systems, see Arriaza and Sordo [3] and Navarro et al. [4].

An alternative approach in order to obtain the system's reliability from the components' reliabilities involves the notion of the system's signature, see Navarro [5] and references therein.

Comparing two or more systems is a relevant reliability task as well. Usually, the analysis and the comparisons between reliability systems are performed for independent subsystems. However, in many contexts, the case in which the systems share one or more components is also interesting (see, for instance, Ashrafi et al. [6] and Parsa et al. [7]). Such schemes are often adopted to design industrial machines, where the architecture of items is based on shared components, in order to save costs and production time. Another example can be found in computer architecture, where graphic chips can share the RAM with the CPU and other components.

Motivated by the need to study different kinds of systems, in this paper, we investigate various pairs of reliability systems with one or more shared components, whose lifetimes are independent and identically distributed (i.i.d.) or independent but not identically distributed (i.n.i.d.). See Navarro and Rychlik [8] for comparisons between expected systems' lifetimes having i.i.d. or i.n.i.d. components' lifetimes. The presence of shared components yields a dependence between the systems' lifetimes. Such dependence is described in terms of copulas, and in this framework, a special role is often played by a new distortion function based on the Receiver Operating Characteristic (ROC) curve (see Calì and Longobardi [9] and references therein). For example, if X_1 and X_2 denote the lifetimes of two series systems that do not share components, then the random vector (X_1, X_2) possesses the independence copula. Conversely, if the series systems having lifetimes X_1 and X_2 share a certain number of components, then the copula of (X_1, X_2) belongs to the Marshall-Olkin family of copulas, defined as

$$C_{\gamma,\delta}(u,v) = \min(u^{1-\gamma}v, uv^{1-\delta}), \qquad u, v, \gamma, \delta \in [0,1] \tag{1.1}$$

We address the reader to Li and Pellerey [10] for other details about the Marshall-Olkin family of copulas and Generalized Marshall-Olkin copulas.

In these configurations, a natural tool for the evaluation of the distance between the marginal systems' time to failures is represented by the expected absolute lifetime distance. In particular, given a random vector (X, Y) with non-degenerate marginal lifetimes, we use the absolute mean difference $\mathsf{E}|X - Y|$ as a distance measure between X and Y, for both cases in which the corresponding systems share, or not, one or more components. We also discuss the various types of distance reductions that can be gained, and also the strengthness of such distance reductions in terms of the Gini's mean difference (see, for instance, Yitzhaki and Schechtman [11]) and its new bivariate version introduced recently in Capaldo and Navarro [12]. Specifically, we provide sufficient conditions aiming to compare the bivariate Gini's mean difference of (X, Y) with the respective univariate Gini's mean difference of X and Y. Different generalizations of the Gini's mean difference based on distortion functions can be found in Capaldo et al. [13] and Capaldo et al. [14], while in Furman et al. [15] for further versions in the context of risk measures. Moreover, see Asadi and Finkelstein [16] for an interpretation of the Gini's mean difference in terms of the residual lifetime.

Optimization problems based on Gini's mean differences are considered too. In particular, given a certain number of i.i.d. components' lifetimes, we provide sufficient conditions for the choice of the suitable structure between series and parallel systems in order to increase (or decrease) the absolute mean distance between the lifetimes of systems having shared components. A similar result is also proved for series-parallel and parallel-series systems' structures. This is useful for managing replacement policies of the single failed component (increasing distance) or of the whole system structure (decreasing distance).

We also investigate allocation problems regarding systems having different components' ages. For this aim, we consider a dynamic version of the bivariate Gini's mean difference and we discuss about its monotonicity properties. For further optimization problems in system reliability theory see, for instance, Di Crescenzo and Pellerey [17] and Navarro et al. [18].

The paper is organized as follows. In Section 2 we recall some useful notions. We also define a new distortion function based on the ROC curve. In Section 3 we study various pairs of reliability systems having one or more shared components, such as series systems, parallel systems, parallel-series systems, and series-parallel systems. For these systems, in the case of components with i.i.d. lifetimes, we show that the Marshall-Olkin copula given in Equation (1.1) is the natural tool to describe the dependence within the marginal systems' lifetimes. We also provide sufficient conditions such that the positive (negative) dependence between the marginal systems' lifetimes yields a decreasing (increasing) of the lifetime distance in the sense of the absolute mean difference. An optimization problem based on systems' structures is also considered, aiming to reduce or increase the absolute mean distance between the systems' lifetimes. In Section 4 we provide some analogous results for components with i.n.i.d. lifetimes. A specific analysis, based on suitable comparisons, is performed when the ages of the system's components are different. In Section 5 we study the dependence of pairs of mixed systems with shared components when the first is a series system and the second is a parallel system. Both cases of i.i.d. and i.n.i.d. components' lifetimes are analyzed along the same procedure discussed in Sections 3 and 4, respectively. Finally, some concluding remarks are given in Section 6.

Throughout the paper, the terms increasing and decreasing are used in a non-strict sense, and $\mathbb{N}$ denotes the set of positive integers.

2 | Background and Preliminary Notions

In this section, we recall some useful notions, such as stochastic orders, aging classes, quantile functions, distortion, and copula functions. We also define a new distortion function based on the ROC curve.

Let X be a random lifetime with cumulative distribution function (c.d.f.) $F(t) = \mathsf{P}(X \leq t)$, for $t \geq 0$, and survival function (s.f.), or *reliability function*, denoted as $\overline{F} = 1 - F$. For our aims, we consider only random lifetimes, even if some results of this paper can be extended to more general supports. If X is absolutely continuous, then its probability density function is denoted as $f = F'$.

In the following, we recall some stochastic orders. Here $a/0$ is taken to be equal to $+\infty$ whenever $a > 0$. Moreover, the subscript refers to the random variable.

Definition 1. We say that X is smaller than Y in the

- usual stochastic order, denoted by $X \leq_{st} Y$, if $\overline{F}_X(t) \leq \overline{F}_Y(t)$, for all $t \geq 0$;
- hazard rate order, denoted by $X \leq_{hr} Y$, if $\overline{F}_X(t)/\overline{F}_Y(t)$ is decreasing in $t \geq 0$;

TABLE 1 | Some examples of d.q.d.f. $\tilde{q}(u)$ defined in Equation (2.2), for $u \in (0,1)$, with indications about the validity of Assumptions A1 and A2.

Random lifetime X	s.f. $\overline{F}(t)$	d.q.d.f. $\tilde{q}(u)$	A1	A2
(i) $U(0,r)$, $r \geq 0$	$1-\frac{t}{r}$, $t \in (0,r)$	r	Yes	Yes
(ii.a) Power(λ), $0 \leq \lambda < 1$	$1-t^{\lambda}$, $t \in [0,1]$	$\left(\lambda(1-u)^{\frac{\lambda-1}{\lambda}}\right)^{-1}$	Yes	No
(ii.b) Power(λ), $\lambda > 1$	$1-t^{\lambda}$, $t \in [0,1]$	$\left(\lambda(1-u)^{\frac{\lambda-1}{\lambda}}\right)^{-1}$	No	Yes
(iii) $Exp(\lambda)$, $\lambda > 0$	$e^{-\lambda t}$, $t \geq 0$	$(\lambda u)^{-1}$	Yes	No
(iv) Weibull$\left(\frac{1}{\lambda};k\right)$, $\lambda > 0$, $k > 0$	$e^{-(\lambda t)^k}$, $t \geq 0$	$\left(k\lambda u(-\ln u)^{\frac{k-1}{k}}\right)^{-1}$	Yes $\|_{0<k<3}$	No

- reversed hazard rate order, denoted by $X \leq_{rhr} Y$, if $F_X(t)/F_Y(t)$ is decreasing in $t \geq 0$;
- likelihood ratio order, denoted by $X \leq_{lr} Y$, if $f_X(t)/f_Y(t)$ is decreasing in $t \geq 0$, provided that X and Y are absolutely continuous.

If X and Y are identically distributed (i.d.), then we write $X =_{st} Y$. For more details about stochastic orders, we refer to Belzunce et al. [19] and Shaked and Shanthikumar [20].

The residual lifetime of X is defined as

$$X_t = [X - t | X > t], \qquad \forall t \geq 0 : \overline{F}(t) > 0 \tag{2.1}$$

having s.f. $\overline{F}_t(x) = \overline{F}(x+t)/\overline{F}(t)$ for $x \geq 0$, which describes the remaining lifetime of a system that has worked without failures in the interval $[0,t]$.

Hereafter we define some useful aging properties. Such definitions are based on Equation (2.1). Further properties of aging notions can be found, for instance, in Section 4 in Navarro [1].

Definition 2. We say that X is

- new better (worse) than used (NBU or NWU, respectively) if $X_t \leq_{st} (\geq_{st}) X$, for all $t \geq 0$;
- increasing (decreasing) in the failure rate (IFR or DFR, respectively) if $X_t \leq_{hr} (\geq_{hr}) X_s$, for all $t > s \geq 0$;
- increasing (decreasing) in likelihood ratio (ILR or DLR, respectively) if $X_t \leq_{lr} (\geq_{lr}) X_s$, for all $t > s \geq 0$.

Often quantile functions are used as alternatives to distribution functions. In particular, given a c.d.f. F, we denote the right-continuous version of its inverse as $F^{-1}(u) = \sup\{t : F(t) \leq u\}$, for all $u \in [0,1]$, also named *quantile function* in statistical framework. We also define $\overline{F}^{-1}(u) = F^{-1}(1-u)$, for all $u \in [0,1]$. Moreover, the *quantile-density function* of X is given by

$$q(u) := \frac{1}{f(F^{-1}(u))} = \frac{\mathrm{d}}{\mathrm{d}u}F^{-1}(u), \qquad u \in (0,1)$$

see, for instance, Sunoj and Sankaran [21]. Referring to s.f.'s instead of c.d.f.'s, in Capaldo et al. [14] the *dual quantile-density function* (d.q.d.f.) of X has been defined as

$$\tilde{q}(u) := \frac{1}{f\left(\overline{F}^{-1}(u)\right)} = -\frac{\mathrm{d}}{\mathrm{d}u}\overline{F}^{-1}(u), \qquad u \in (0,1) \tag{2.2}$$

and the following assumptions regarding $\tilde{q}(u)$ have been used.

Assumption 1. (A1) One has $\tilde{q}(u) - \tilde{q}(1-u) \geq 0$, for all $u \in (0,1/2)$.

Assumption 2. (A2) One has $\tilde{q}(u) - \tilde{q}(1-u) \leq 0$, for all $u \in (0,1/2)$.

We remark that $\tilde{q}(u) = q(1-u)$, for all $u \in (0,1)$, and therefore A1 (A2) is equivalent to require $\tilde{q}(u) \geq (\leq) q(u)$, for all $u \in (0,1/2)$. Note that such assumptions are related to the location of the median of the considered distribution. In Table 1 we show a few examples of the d.q.d.f. for some remarkable distributions.

2.1 | Distortion and Copula Functions

An increasing continuous function $h : [0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$ is called *distortion function*. Given a c.d.f. F, then the function $h(F)$ is a proper (distorted) c.d.f. The *dual distortion function* related to h is defined as $\tilde{h}(u) := 1 - h(1-u)$, for all $u \in [0,1]$. Then $\tilde{h}(\overline{F})$ is the (distorted) s.f. of $h(F)$. Some distortions have an interpretation in terms of well-known hazard models. Indeed, for a real parameter $\beta > 0$ and $u \in [0,1]$, the distortion $h(u) = 1-(1-u)^{\beta}$ is related to the proportional hazards model (see Kumar and Klefsjö [22]), while $h(u) = u^{\beta}$ comes from the proportional reversed hazards model (see Di Crescenzo [23] and Gupta and Gupta [24]). Further details about distortion functions can be found in Section 2.4 in Navarro [1].

Throughout the paper, we deal with a distortion that allows us to switch from one reliability function to another. Specifically, if X and Y are absolutely continuous random lifetimes having interval supports, with s.f. $\overline{F}$ and $\overline{G}$, respectively, then the *ROC distortion* between X and Y is defined as

$$\begin{aligned} ROC_{\overline{G},\overline{F}} &: [0,1] \to [0,1] \\ u &\mapsto ROC_{\overline{G},\overline{F}}(u) := \overline{G}\left(\overline{F}^{-1}(u)\right) \end{aligned} \tag{2.3}$$

This distortion function is related to the ROC curve representation of Proposition 2 in Calì and Longobardi [9]. We mention that Schumacher [25] provides a relation between distortion functions and ROC curves in the context of risk measures. Clearly, the distortion given in Equation (2.3) acts as a transformation function, in the sense that it allows to map the s.f. of X in that of Y, since $ROC_{\overline{G},\overline{F}}(\overline{F}(t)) = \overline{G}(t)$, for all $t \geq 0$. We remark that, if $\overline{G} = \tilde{h}(\overline{F})$, with $\tilde{h}$ dual of a given distortion h, then $ROC_{\overline{G},\overline{F}}(u) = \tilde{h}(u)$ for all $u \in [0,1]$.

From the Sklar's theorem, given a random vector (X,Y) having marginal c.d.f.'s F_X, F_Y, its joint c.d.f. can be written as $F_{(X,Y)}(x,y) = \mathsf{P}(X \le x, Y \le y) = C(F_X(x), F_Y(y))$, for $x, y \in \mathbb{R}$, where C is the *copula function* of (X, Y). Similarly, the joint s.f. of (X, Y) can be written as

$$\overline{F}_{(X,Y)}(x,y) = \mathsf{P}(X > x, Y > y) = \hat{C}(\overline{F}_X(x), \overline{F}_Y(y)), \qquad x, y \in \mathbb{R} \tag{2.4}$$

where $\hat{C}$ is the *survival copula function* of (X, Y), and $\overline{F}_X, \overline{F}_Y$ are the marginal s.f.'s. Every copula function is a (multivariate) distortion function, cf. Navarro et al. [4]. The reverse is not true in general.

Definition 3. Let (X, Y) be a random vector with copula C. We say that X and Y are positively quadrant dependent (PQD), or equivalently C is PQD, if $C(u,v) \ge uv$, for all $u, v \in [0,1]$. If the inequality is reversed we say that X and Y are negative quadrant dependent (NQD), or equivalently C is NQD.

Since

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v), \qquad u, v \in [0,1] \tag{2.5}$$

it follows that C is PQD (NQD) if and only if $\hat{C}$ is PQD (NQD). For these and further details about copulas, we refer the reader to Durante and Sempi [26] and Nelsen [27].

3 | Reliability Systems With I.I.D. Components' Lifetimes

In this section, we study different pairs of reliability systems with one or more shared components, whose lifetimes are independent copies of a given random lifetime. We focus on the copulas of these systems, discussing about situations in which sharing one or more components reduces the absolute mean distance between their lifetimes.

Let X' be an independent copy of a non-degenerate random lifetime X, having c.d.f. F and s.f. $\overline{F}$. The *Gini's mean difference* of X was defined as

$$\mathsf{GMD}(X) \ := \ \mathsf{E}|X - X'| = 2\int_0^{+\infty} F(t)\overline{F}(t)\mathrm{d}t \tag{3.1}$$

see Yitzhaki and Schechtman [11]. We see such a measure as the absolute mean distance between the i.d. lifetimes of two systems that do not share components (and, thus, they are also independent).

Along the same line adopted in Capaldo et al. [14], below we generalize the Gini's mean difference given in Equation (3.1) to the case in which the involved random lifetimes are i.d. and possibly dependent. Therefore, the following new measure refers to the case in which the considered systems have i.d. random lifetimes and they share a certain number of components.

Definition 4. Let $(X, \tilde{X})$ be a random vector with survival copula $\hat{C}_\theta(u,v)$ in a parametric family for $u, v \in [0,1]$ and $\theta \in \Theta$, where Θ is a given ordered set. Suppose that X and $\tilde{X}$ are i.d. according to $\overline{F}$. The copula-based Gini's mean difference of X is defined as

$$\begin{aligned}\mathsf{GMD}_\theta(X) &= \mathsf{E}_{\hat{C}_\theta}|X - \tilde{X}| \\ &= 2\int_0^{+\infty} \left\{ \overline{F}(t) - \hat{C}_\theta(\overline{F}(t), \overline{F}(t)) \right\} \mathrm{d}t, \qquad \forall \theta \in \Theta\end{aligned} \tag{3.2}$$

Clearly, for $\theta_I \in \Theta$ such that $\hat{C}_{\theta_I}(u,v) = uv$ for all $u, v \in [0,1]$, one has $\mathsf{GMD}_{\theta_I}(X) = \mathsf{GMD}(X)$.

In the next result, we provide a sufficient condition on the copula of two random lifetimes in order to compare the Gini's mean differences given in Equations (3.1) and (3.2).

Proposition 1. *Let $(X, \tilde{X})$ be a random vector with i.d. marginal lifetimes, having survival copula $\hat{C}_\theta(u,v)$ for $u, v \in [0,1]$ and $\theta \in \Theta$. If $\hat{C}_\theta$ is PQD (NQD) for some $\theta \in \Theta$, then*

$$\mathsf{GMD}_\theta(X) \le (\ge) \mathsf{GMD}(X) \tag{3.3}$$

Proof. Let us denote by $\tilde{q}(u)$, for $u \in [0,1]$, the d.q.d.f. of X. From Equations (3.1) and (3.2), by setting $u = \overline{F}(t)$ one has

$$\mathsf{GMD}(X) - \mathsf{GMD}_\theta(X) = 2\int_0^1 \tilde{q}(u)\left(\hat{C}_\theta(u,u) - u^2\right)\mathrm{d}u$$

Since $\hat{C}_\theta$ is PQD (NQD), it follows $\hat{C}_\theta(u,u) \ge (\le) u^2$ for all $u \in [0,1]$, hence the thesis. □

As an application of the previous result, consider now the Farlie-Gumbel-Morgenstern (FGM) family of survival copulas, defined as

$$\hat{C}_\theta(u,v) = uv(1 + \theta(1-u)(1-v)), \qquad u, v \in [0,1], \ \theta \in [-1,1] \tag{3.4}$$

that is PQD for all $\theta \in [0,1]$ and NQD for all $\theta \in [-1,0]$. Therefore, if $(X, \tilde{X})$ is a random vector with i.d. marginal lifetimes, having survival copula given in Equation (3.4), then $\mathsf{GMD}_\theta(X) \le \mathsf{GMD}(X)$ for all $\theta \in [0,1]$, while $\mathsf{GMD}_\theta(X) \ge \mathsf{GMD}(X)$ for all $\theta \in [-1,0]$.

In Capaldo et al. [13] the authors defined the *cumulative information generating function* of X as

$$\mathsf{G}_X(\beta_1, \beta_2) \ := \ \int_0^{+\infty} [F(t)]^{\beta_1}[\overline{F}(t)]^{\beta_2}\ \mathrm{d}t \tag{3.5}$$

for all $(\beta_1, \beta_2) \in \mathbb{R}^2$ such that $\mathsf{G}_X(\beta_1, \beta_2)$ is finite. Note that, from Equations (3.1) and (3.5), one has $\mathsf{GMD}(X) = 2\mathsf{G}_X(1,1)$. Moreover, under the FGM copula, the measure given in Equation (3.2) becomes

$$\mathsf{GMD}_\theta(X) = \mathsf{GMD}(X) - 2\theta\mathsf{G}_X(2,2), \qquad \theta \in [-1,1]$$

Let us denote by Y and $\tilde{Y}$ the random lifetimes of two series systems with one shared component having respectively two i.i.d. components' lifetimes, according to $\overline{F}$. An example of this

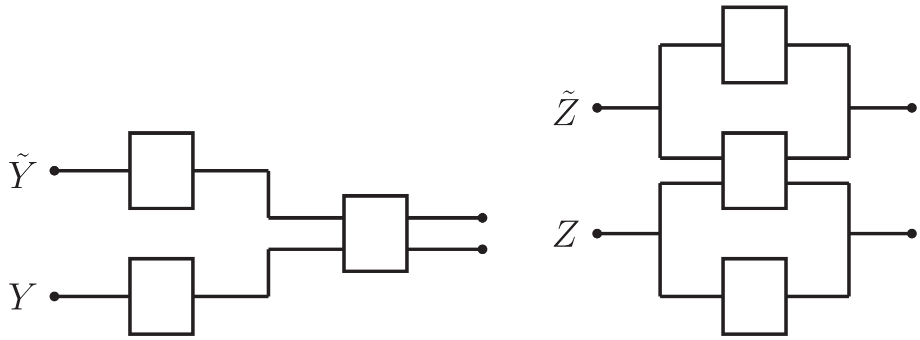

FIGURE 1 | Schematic representation of two series systems with one shared component (left-hand side) and two parallel systems with one shared component (right-hand side).

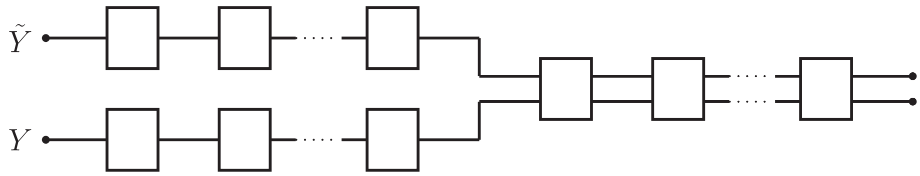

FIGURE 2 | Schematic representation of two series systems with shared components.

kind of system's structure is shown in the left-hand side of Figure 1. The joint s.f. of $(Y, \tilde{Y})$ is

$$\overline{F}_{(Y,\tilde{Y})}(s,t) = \begin{cases} \overline{F}(s)(\overline{F}(t))^2, & s \le t, \\ \overline{F}(t)(\overline{F}(s))^2, & s > t \end{cases}$$

and, from Equation (2.4), the survival copula of $(Y, \tilde{Y})$ is expressed by

$$\hat{C}_{\frac{1}{2}}(u,v) = \begin{cases} uv^{\frac{1}{2}}, & u < v, \\ u^{\frac{1}{2}}v, & u \ge v \end{cases} \tag{3.6}$$

for $u, v \in [0,1]$, that is the Marshall-Olkin copula defined in Equation (1.1) for the case $\gamma = \delta = 1/2$. We remark that the copula in Equation (3.6) is singular, in the sense that $\mathsf{P}(Y = \tilde{Y}) > 0$, due to the presence of a shared component. Since the copula in Equation (3.6) is PQD, from Proposition 1 it follows that $\mathsf{GMD}_{\frac{1}{2}}(Y) < \mathsf{GMD}(Y)$.

By taking into account the duality between series and parallel systems, it is easy to verify the same for the case in which Z and $\tilde{Z}$ describe the random lifetimes of two parallel systems with one shared component having respectively two i.i.d. components' lifetimes. We denote by F and $\overline{F}$ the c.d.f. and s.f. of such components' lifetimes. An example of this kind of system's structure is shown in the right-hand side of Figure 1. In this case, the joint s.f. of $(Z, \tilde{Z})$ is

$$\overline{F}_{(Z,\tilde{Z})}(s,t) = \begin{cases} 1 - (F(t))^2 - \overline{F}(t)(F(s))^2, & s \le t, \\ 1 - (F(s))^2 - \overline{F}(s)(F(t))^2, & s > t \end{cases}$$

and, from Equation (2.4), the survival copula of $(Z, \tilde{Z})$ is expressed by

$$\hat{C}_{\frac{1}{2}}(u,v) = \begin{cases} u - (1-v)[1-(1-u)^{\frac{1}{2}}], & u < v, \\ v - (1-u)[1-(1-v)^{\frac{1}{2}}], & u \ge v \end{cases} \tag{3.7}$$

for $u, v \in [0,1]$. From Equation (2.5) it follows that Equation (3.7) represents the survival copula of the Marshall-Olkin copula given in Equation (3.6); therefore one has $\mathsf{GMD}_{\frac{1}{2}}(Z) < \mathsf{GMD}(Z)$.

In the rest of this section, we show other pairs of reliability systems with shared components for which Equation (3.3) holds. We remark that these reliability systems can be also described in terms of subsystems that do not share any component, namely *modules*. For more details about redundancy studies of systems formed by modules see Torrado et al. [28].

3.1 | Series and Parallel Systems

Let Y and $\tilde{Y}$ be the random lifetimes of two series systems having n shared components, respectively with $s+n$ and $k+n$ i.i.d. components' lifetimes, according to $\overline{F}$. An example of this kind of system's structure is shown in Figure 2. Therefore, for $s, k, n \in \mathbb{N}$ one has

$$\overline{F}_Y(t) = [\overline{F}(t)]^{s+n}, \quad \overline{F}_{\tilde{Y}}(t) = [\overline{F}(t)]^{k+n}, \qquad t \ge 0$$

Clearly, if $k \le s$, then $Y \le_{st} \tilde{Y}$. From Equation (2.4), the survival copula of $(Y, \tilde{Y})$, for $\theta = (s,k,n) \in \Theta$ and for all $u, v \in [0,1]$, is expressed by

$$\hat{C}_\theta(u,v) = \begin{cases} uv^{\frac{k}{k+n}}, & u < v^{\frac{s+n}{k+n}}, \\ u^{\frac{s}{s+n}}v, & u \ge v^{\frac{s+n}{k+n}} \end{cases} \tag{3.8}$$

that is the Marshall-Olkin copula in Equation (1.1) for $\gamma = \frac{s}{s+n}$ and $\delta = \frac{k}{k+n}$. Since the copula in Equation (3.8) is PQD, recalling Proposition 1, it follows that $\mathsf{GMD}_\theta(Y) < \mathsf{GMD}(Y)$, for all θ.

If $s = k$, that is $Y =_{st} \tilde{Y}$, then Equation (3.8) becomes

$$\hat{C}_\tau(u, v) = \begin{cases} uv^\tau, & u < v, \\ u^\tau v, & u \geq v \end{cases}$$

in which the dependence changes as the ratio $\tau = \frac{k}{k+n}$, i.e., as the number of non-shared components over the total number of components for a single system. In addition, for $s = k = n$, Equation (3.8) reduces to Equation (3.6), and therefore the survival copula of $(Y, \tilde{Y})$ does not depend on the values of s, k and n. In other words, when the number of shared components coincides with the non-shared ones for both Y and $\tilde{Y}$, the dependence of $(Y, \tilde{Y})$ does not change when we add (or remove) the same number of components in shared and non-shared positions.

Similar remarks hold for the parallel setting. Let Z and $\tilde{Z}$ be the random lifetimes of two parallel systems having n shared components, respectively with $s + n$ and $k + n$ i.i.d. components' lifetimes. An example of this kind of system's structure is shown in Figure 3. From Equation (2.4), the survival copula of $(Z, \tilde{Z})$ is expressed by

$$\hat{C}_\theta(u, v) = \begin{cases} u - (1-v)[1 - (1-u)^{\frac{s}{s+n}}], & u < 1 - (1-v)^{\frac{s+n}{k+n}}, \\ v - (1-u)[1 - (1-v)^{\frac{k}{k+n}}], & u \geq 1 - (1-v)^{\frac{s+n}{k+n}} \end{cases} \tag{3.9}$$

for $\theta = (s, k, n) \in \Theta$ and for all $u, v \in [0, 1]$. We remark that, from Equation (2.5), the function given in Equation (3.9) is the survival copula of the Marshall-Olkin copula given in Equation (3.8).

Hereafter we show that Equation (3.5) provides a bound for the copula-based Gini's mean difference of such systems' lifetimes. For bounds of the Gini's mean difference in Equation (3.1) see, for instance, Cerone and Dragomir [29].

Remark 1. Let us recall Equations (3.2) and (3.5). From Equations (3.8) and (3.9), if $s = k = n$, then one respectively gets

$$\begin{aligned} \mathsf{GMD}_{\frac{1}{2}}(Y) &= 2\int_0^{+\infty} \left(1 - (\overline{F}(t))^n\right)(\overline{F}(t))^{2n}\mathrm{d}t \\ &\geq 2\mathsf{G}_X(1, 2n), \qquad \forall n \in \mathbb{N} \end{aligned}$$

and

$$\begin{aligned} \mathsf{GMD}_{\frac{1}{2}}(Z) &= 2\int_0^{+\infty} (F(t))^{2n}(1 - (F(t))^n)\mathrm{d}t \\ &\geq 2\mathsf{G}_X(2n, 1), \qquad \forall n \in \mathbb{N} \end{aligned}$$

where X denotes the random lifetime of the i.i.d. components' lifetimes, having c.d.f. F and s.f. $\overline{F}$.

Given a certain number of components having i.i.d. lifetimes, it is interesting to choose the suitable structure between two series or two parallel systems with shared components in order to reduce, or to increase, the lifetime distance in the sense of Equation (3.2). On this aim, in the next result we obtain sufficient conditions on the d.q.d.f. of the single component's lifetime in order to compare $\mathsf{GMD}_{\frac{1}{2}}(Y)$ with $\mathsf{GMD}_{\frac{1}{2}}(Z)$, i.e., the two systems above when the numbers of shared and non-shared components are equal for both Y and Z. The following result is based on Assumptions 1 and 2 listed in Section 2.

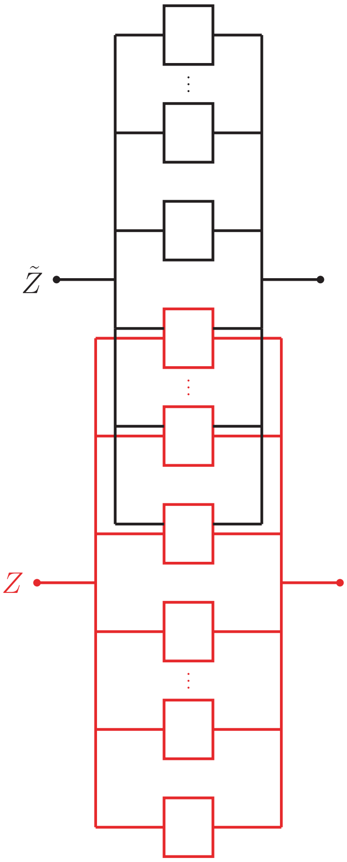

FIGURE 3 | Schematic representation of two parallel systems with shared components.

Proposition 2. *Let Y and $\tilde{Y}$ be the random lifetimes of two series systems having n shared components, respectively with $2n$ i.i.d. components' lifetimes, according to $\overline{F}$, for $n \in \mathbb{N}$. Let Z and $\tilde{Z}$ be the random lifetimes of two parallel systems having n shared components, respectively with $2n$ i.i.d. components' lifetimes, according to $\overline{F}$, for $n \in \mathbb{N}$. If A1 (A2) holds, then*

$$\mathsf{GMD}_{\frac{1}{2}}(Y) \leq (\geq)\ \mathsf{GMD}_{\frac{1}{2}}(Z)$$

Proof. By setting $u = \overline{F}(t)$, with few calculations one has

$$\mathsf{GMD}_{\frac{1}{2}}(Z) - \mathsf{GMD}_{\frac{1}{2}}(Y) = 2\int_0^{\frac{1}{2}} b(u)[\tilde{q}(u) - \tilde{q}(1-u)]\mathrm{d}u \tag{3.10}$$

where $b(u) := (1-u)^{2n}(1 - (1-u)^n) - u^{2n}(1 - u^n) \geq 0$ for all $u \in [0, 1/2]$ and for all $n \in \mathbb{N}$. Therefore, the thesis follows from A1 (A2). □

Below we provide two examples related to Proposition 2.

Example 1. If the components' reliability is uniformly distributed over $[0, 1]$, then $\tilde{q}(u) = 1$, for all $u \in [0, 1]$, and thus both A1 and A2 are satisfied, as shown in (i) of Table 1. Therefore, from Equation (3.10) one has $\mathsf{GMD}_{\frac{1}{2}}(Y) = \mathsf{GMD}_{\frac{1}{2}}(Z)$. Hence, in this case, the distance in the sense of Equation (3.2) is the same for both series and parallel structures.

Example 2. If the components' reliability is standard exponentially distributed, then $\tilde{q}(u) = 1/u$, for all $u \in [0, 1]$, so that A1 holds, as shown in (iii) of Table 1. Therefore, from Equation (3.10) one has $\mathsf{GMD}_{\frac{1}{2}}(Y) \leq \mathsf{GMD}_{\frac{1}{2}}(Z)$. Hence, in this case, in order to reduce the distance in the sense of Equation (3.2) it is better to use the series structure instead of the parallel one.

3.2 | Parallel-Series and Series-Parallel Systems

Let Y and $\tilde{Y}$ be the random lifetimes of two parallel-series systems having n shared components, respectively with $s+n$ and $k+n$ i.i.d. components' lifetimes, according to F. An example of this kind of system's structure is shown in Figure 4. For $s, k, n \in \mathbb{N}$ one has

$$\overline{F}_Y(t) = (1 - [F(t)]^s)(1 - [F(t)]^n)$$
$$\overline{F}_{\tilde{Y}}(t) = (1 - [F(t)]^k)(1 - [F(t)]^n), \qquad t \geq 0 \tag{3.11}$$

since, with obvious notation, one has

$$Y =_{st} \min\{\max\{X_1, \dots, X_s\}, \max\{X_{s+1}, \dots, X_{s+n}\}\}$$

Clearly, if $s \leq k$, then $Y \leq_{st} \tilde{Y}$. From Equation (2.4), the survival copula of $(Y, \tilde{Y})$, for $\theta = (s,k,n) \in \Theta$ and for all $u, v \in [0,1]$, can be expressed by

$$\hat{C}_\theta(u,v) = \begin{cases} \frac{uv}{1-\left(1-ROC_{\overline{F},\overline{F}_Y}(u)\right)^n}, & \overline{F}_Y^{-1}(u) < \overline{F}_{\tilde{Y}}^{-1}(v), \\ \frac{uv}{1-\left(1-ROC_{\overline{F},\overline{F}_{\tilde{Y}}}(v)\right)^n}, & \overline{F}_Y^{-1}(u) \geq \overline{F}_{\tilde{Y}}^{-1}(v) \end{cases} \tag{3.12}$$

where the dependence on s and k is expressed by the involved ROC distortions. It is easy to see that the copula in Equation (3.12) is PQD for all θ. By recalling Proposition 1, it follows that $\mathsf{GMD}_\theta(Y) < \mathsf{GMD}(Y)$, for all θ.

If $s = k = n$, then from Equation (3.11) one has $\overline{F}_Y(t) = (1 - [F(t)]^n)^2$. Under this assumption, one gets $1 - \left(1 - ROC_{\overline{F},\overline{F}_Y}(u)\right)^n = u^{\frac{1}{2}}$ for all $u \in [0,1]$, and therefore Equation (3.12) reduces to Equation (3.6). Hence, in the i.i.d. case, given a pair of parallel-series systems having the same number of shared and non-shared components, the dependence coincides with that of a pair of series systems having two respective components, one of these being shared.

Analogous remarks follow for the series-parallel system's structure, due to its duality with the parallel-series one. Indeed, let Z and $\tilde{Z}$ be the random lifetimes of two series-parallel systems having n shared components, respectively with $s+n$ and $k+n$ i.i.d. components' lifetimes. Note that, with obvious notation, it holds

$$Z =_{st} \max\{\min\{X_1, \dots, X_s\}, \min\{X_{s+1}, \dots, X_{s+n}\}\}$$

see Figure 5 for an example of this kind of system's structure. Then, the survival copula of $(Z, \tilde{Z})$, for $\theta = (s,k,n) \in \Theta$ and for all $u, v \in [0,1]$, is expressed by

$$\hat{C}_\theta(u,v) = \begin{cases} v - (1-u)\left[1 - \frac{1-v}{1-(ROC_{\overline{F},\overline{F}_{\tilde{Z}}}(v))^n}\right], & \overline{F}_Z^{-1}(u) < \overline{F}_{\tilde{Z}}^{-1}(v), \\ u - (1-v)\left[1 - \frac{1-u}{1-(ROC_{\overline{F},\overline{F}_Z}(u))^n}\right], & \overline{F}_Z^{-1}(u) \geq \overline{F}_{\tilde{Z}}^{-1}(v) \end{cases} \tag{3.13}$$

where the dependence on s and k is indicated by the involved ROC distortions.

In the following, we show suitable bounds in terms of Equation (3.5) for the copula-based Gini's mean difference of the respective systems' lifetimes considered above.

Remark 2. Let us recall Equations (3.2) and (3.5). Under the assumption $s = k = n$, from Equation (3.12) one has

$$\mathsf{GMD}_{\frac{1}{2}}(Y) = 2\int_0^{+\infty} (F(t))^n (1 - (F(t))^n)^2 \mathrm{d}t$$
$$\geq 2\mathsf{G}_X(n,2), \qquad \forall n \in \mathbb{N}$$

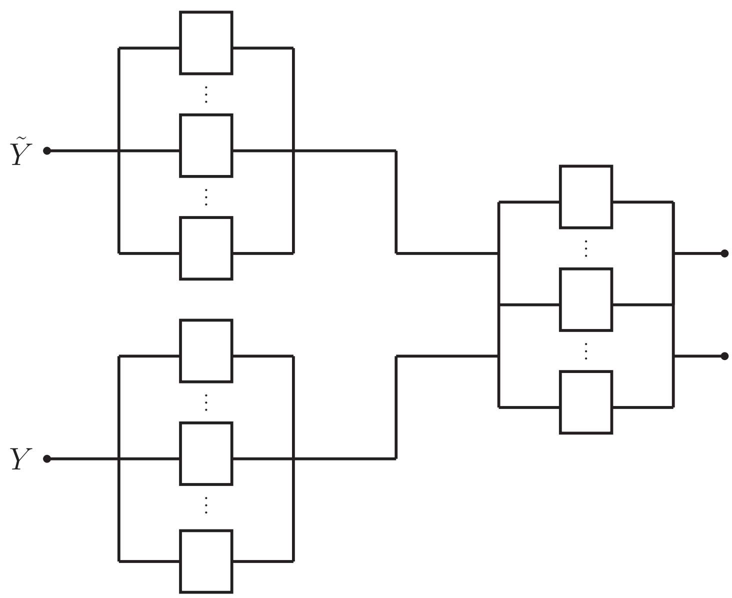

FIGURE 4 | Schematic representation of two parallel-series systems with shared components.

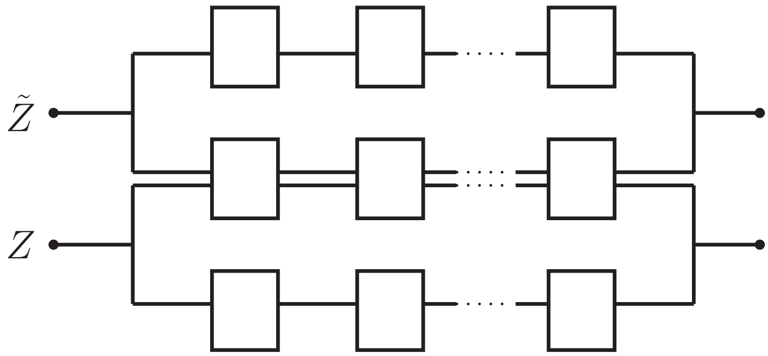

FIGURE 5 | Schematic representation of two series-parallel systems with shared components.

and, from Equation (3.13), it holds

$$\mathsf{GMD}_{\frac{1}{2}}(Z) = 2\int_0^{+\infty}\left(1-(\overline{F}(t))^n\right)^2(\overline{F}(t))^n \geq 2\mathsf{G}_X(2,n), \qquad \forall n \in \mathbb{N}$$

where X denotes the random lifetime of the i.i.d. components, having c.d.f. F and s.f. $\overline{F}$.

As in Proposition 2, hereafter we consider an optimal allocation problem. In particular, given a stock of components having i.i.d. lifetimes, we are interested in choosing the suitable structure between two parallel-series or two series-parallel systems with shared components in order to reduce, or to increase, the lifetime distance in the sense of Equation (3.2). We assume that the numbers of shared and non-shared components are equal to n for both structures. Note that, we consider $n \geq 2$ since for $n = 1$ the parallel-series structure reduces to the series one, whereas the series-parallel structure becomes the parallel one.

Proposition 3. *Let Y and $\tilde{Y}$ be the random lifetimes of two parallel-series systems having n shared components, respectively with $2n$ i.i.d. components' lifetimes, according to $\overline{F}$, for $n \in \mathbb{N} \setminus \{1\}$. Let Z and $\tilde{Z}$ be the random lifetimes of two series-parallel systems having n shared components, respectively with $2n$ i.i.d. components' lifetimes, according to $\overline{F}$, for $n \in \mathbb{N} \setminus \{1\}$. If A1 (A2) holds, then*

$$\mathsf{GMD}_{\frac{1}{2}}(Y) \geq (\leq)\ \mathsf{GMD}_{\frac{1}{2}}(Z)$$

Proof. By setting $u = \overline{F}(t)$, with few calculations one has

$$\mathsf{GMD}_{\frac{1}{2}}(Z) - \mathsf{GMD}_{\frac{1}{2}}(Y) = 2\int_0^{\frac{1}{2}} b(u)[\tilde{q}(u) - \tilde{q}(1-u)]\mathrm{d}u \qquad (3.14)$$

where $b(u) := (1-u^n)^2u^n - (1-(1-u)^n)^2(1-u)^n \leq 0$ for all $u \in [0, \frac{1}{2}]$ and for all $n \in \mathbb{N}$ such that $n \geq 2$. Therefore, the thesis follows from A1 (A2). □

Below we provide two examples related to Proposition 3.

Example 3. If the components' reliability is uniformly distributed over $[0,1]$, then $\tilde{q}(u) = 1$, for all $u \in [0,1]$, and thus both A1 and A2 are satisfied, as shown in (i) of Table 1. Therefore, from Equation (3.14) one has $\mathsf{GMD}_{\frac{1}{2}}(Y) = \mathsf{GMD}_{\frac{1}{2}}(Z)$. Hence, in this case, the distance in the sense of Equation (3.2) is the same for both parallel-series and series-parallel structures.

Example 4. If the components' reliability is standard exponentially distributed, then $\tilde{q}(u) = 1/u$, for all $u \in [0,1]$, so that A1 holds, as shown in (iii) of Table 1. Therefore, from Equation (3.14) one has $\mathsf{GMD}_{\frac{1}{2}}(Y) \geq \mathsf{GMD}_{\frac{1}{2}}(Z)$. Hence, in this case, in order to reduce the distance in the sense of Equation (3.2) it is better to use the series-parallel structure instead of the parallel-series one.

4 | Reliability Systems With i.n.i.d. Components' Lifetimes

Now we face the same problem under the hypothesis of i.n.i.d. components' lifetimes. We first consider optimal allocation problems regarding the instance in which systems' components have different ages, by analysing absolute mean distances. Moreover, we investigate conditions such that sharing components reduces the absolute mean distance between their lifetimes. Finally, under the stated i.n.i.d. assumption, we focus on the copulas of different pairs of reliability systems with one or more shared components.

In Capaldo and Navarro [12] the authors introduced and studied new multivariate versions of the Gini's mean difference given in Equation (3.1). The formal definition for the bivariate case is stated as follows.

Definition 5. Let (X, Y) be a random vector. The bivariate Gini's mean difference of (X, Y) is defined as

$$\mathsf{GMD}(X,Y) = \mathsf{E}|X - Y| \qquad (4.1)$$

provided that X and Y have finite means.

Such a measure can be viewed as the absolute mean distance between the not necessarily i.d. lifetimes of two systems that share a certain number of components.

If $\overline{F}_{(X,Y)}$ denotes the joint s.f. of (X,Y), while $\overline{F}_X$ and $\overline{F}_Y$ are the s.f.'s of X and Y, respectively, then from Equation (4.1) one has

$$\begin{aligned}\mathsf{GMD}(X,Y) &= \int_0^{+\infty}\left\{\overline{F}_X(t) + \overline{F}_Y(t) - 2\overline{F}_{(X,Y)}(t,t)\right\}\mathrm{d}t \\ &= \int_0^{+\infty}\left\{\overline{F}_X(t) + \overline{F}_Y(t) - 2\,\hat{C}\left(\overline{F}_X(t), \overline{F}_Y(t)\right)\right\}\mathrm{d}t \qquad (4.2)\end{aligned}$$

where in the last equality we use Equation (2.4). Note that, if $X =_{st} Y$, then from Equations (3.2) and (4.2) one has $\text{GMD}(X, Y) = \text{GMD}_{\theta'}(X)$, for a fixed $\theta' \in \Theta$.

Hereafter we investigate the absolute mean distance between the lifetimes of two series systems with a shared component. We remark that, due to the duality between series and parallel systems, a similar study can be performed by considering a pair of parallel systems.

Let Y be the random lifetime of a series system with two independent components' lifetimes X_1 and X_3, having s.f.'s $\overline{F}_1(t)$ and $\overline{F}_3(t)$, for $t \geq 0$, respectively. Let $\tilde{Y}$ be the random lifetime of another series system with two independent components' lifetimes X_2 and X_3, having s.f.'s $\overline{F}_2(t)$ and $\overline{F}_3(t)$, for $t \geq 0$, respectively. Suppose that Y and $\tilde{Y}$ share the component distributed as X_3. An example of this kind of system's structure is shown in the left-hand side of Figure 1. The random vector $(Y, \tilde{Y})$ has joint s.f.

$$\overline{F}_{(Y,\tilde{Y})}(s,t) = \begin{cases} \overline{F}_1(s)\overline{F}_2(t)\overline{F}_3(t), & s \leq t, \\ \overline{F}_1(s)\overline{F}_2(t)\overline{F}_3(s), & s > t \end{cases} \tag{4.3}$$

and, from Equations (4.2) and (4.3), it follows

$$\text{GMD}(Y, \tilde{Y}) = \int_0^{+\infty} \overline{F}_3(t)\left[\overline{F}_1(t) + \overline{F}_2(t) - 2\overline{F}_1(t)\overline{F}_2(t)\right]\mathrm{d}t \tag{4.4}$$

Let $(Y_1, \tilde{Y}_1)$ represent another pair of systems with the same structure of $(Y, \tilde{Y})$, but with shared component distributed as X_4 and having s.f. $\overline{F}_4$. We remark that if $X_3 \leq_{st} X_4$, then $\text{GMD}(Y, \tilde{Y}) \leq \text{GMD}(Y_1, \tilde{Y}_1)$, i.e., the absolute mean distance between the systems increases when the lifetime of the shared component increases in the usual stochastic sense. Moreover, if $X_1 =_{st} X_2 =_{st} X$, and $X_3 =_{st} \tilde{X}$, then $Y =_{st} \tilde{Y}$. We denote by $\overline{F}$ and $\overline{H}$ the s.f.'s of X and $\tilde{X}$, respectively. In this case Equation (4.4) becomes

$$\text{GMD}(Y, \tilde{Y}) = 2\int_0^{+\infty} \overline{H}(y)\overline{F}(y)F(y)\mathrm{d}y \tag{4.5}$$

Under these assumptions, by recalling Equation (2.1), we denote

$$\text{GMD}_{(s,t)}(Y, \tilde{Y}) = 2\int_0^{+\infty} \overline{H}_s(y)\overline{F}_t(y)F_t(y)\mathrm{d}y \tag{4.6}$$

where $\overline{F}_t$ is the s.f. of the residual lifetime X_t for $t \geq 0$, while $\overline{H}_s$ is the s.f. of the residual lifetime $\tilde{X}_s$ for $s \geq 0$. Note that the measure given in Equation (4.6) can be viewed as dynamic bivariate Gini's mean difference. According to such interpretation, in the following, we provide some results regarding distance lifetime reductions for systems made up of components with different ages.

Proposition 4. *Under the hypothesis of Equation (4.6), suppose that X is exponentially distributed. If $\tilde{X}$ is IFR (DFR), then $\text{GMD}_{(s,t)}(Y, \tilde{Y})$ is decreasing (increasing) in $s \geq 0$ for any $t \geq 0$.*

Proof. Since X is exponentially distributed, one has $\overline{F}_t(x) = \overline{F}(x)$, for all $t \geq 0$ and $x \geq 0$. If $\tilde{X}$ is IFR (DFR), then $\overline{H}_{s_2}(x) \leq (\geq) \overline{H}_{s_1}(x)$, for all $0 \leq s_1 \leq s_2$ and $x \geq 0$. Therefore, from Equation (4.6) one has

$$\text{GMD}_{(s_1,t)}(Y, \tilde{Y}) \geq (\leq) \, \text{GMD}_{(s_2,t)}(Y, \tilde{Y})$$

for all $0 \leq s_1 \leq s_2$ and the proof is completed. □

For the following results, we recall that a function $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ is said to be totally positive of order 2 (TP_2) in its arguments if and only if $\phi(t_1, s)/\phi(t_2, s)$ is decreasing in s for any $t_1 \leq t_2$ (see Karlin [30]).

Lemma 1. *Let X be a random lifetime with c.d.f. F and s.f. $\overline{F}$. If X is DLR, then the function*

$$\Phi(t, x) = \overline{F}_t(x)F_t(x)$$

is TP_2 in $(t, x) \in (\mathbb{R}^+)^2$.

Proof. If X is DLR, then $X_{t_1} \leq_{lr} X_{t_2}$ for all $t_1 \leq t_2$, and from Theorem 1.C.1 in Shaked and Shanthikumar [20] one has $X_{t_1} \leq_{hr} X_{t_2}$ and $X_{t_1} \leq_{rhr} X_{t_2}$. Hence

$$\frac{\Phi(t_1, x)}{\Phi(t_2, x)} = \frac{\overline{F}_{t_1}(x)F_{t_1}(x)}{\overline{F}_{t_2}(x)F_{t_2}(x)}$$

is decreasing in x for all $t_1 \leq t_2$, i.e., $\Phi(t, x)$ is TP_2 in $(t, x) \in (\mathbb{R}^+)^2$. □

Proposition 5. *Let X and $\tilde{X}$ be random lifetimes with s.f.'s $\overline{F}$ and $\overline{H}$, respectively. If X is DLR and $\tilde{X}$ is IFR, then $\text{GMD}_{(s,t)}(Y, \tilde{Y})$ is TP_2 in $(s, t) \in (\mathbb{R}^+)^2$.*

Proof. If $\tilde{X}$ is IFR, then $\tilde{X}_{s_2} \leq_{hr} \tilde{X}_{s_1}$ for all $0 \leq s_1 \leq s_2$. Therefore the ratio $\overline{H}(s + y)/\overline{H}(y)$ is decreasing in y, i.e., $\overline{H}_s(y)$ is TP_2 in $(s, y) \in (\mathbb{R}^+)^2$. By recalling Equation (4.6), from Lemma 1 and the Basic Composition Formula (see Karlin [30], p. 98) one has the thesis. □

Let us suppose in Equation (4.4) that $X_1 =_{st} X_3 =_{st} X$ and $X_2 =_{st} \tilde{X}$. We denote by $\overline{F}$ and $\overline{H}$ the s.f.'s of X and $\tilde{X}$, respectively. If $\hat{Y}$ denotes the random lifetime of the series system having i.i.d. components' lifetimes according to $\overline{F}$, then

$$\text{GMD}(\hat{Y}, \tilde{Y}) = \int_0^{+\infty} \overline{F}(y)[\overline{F}(y) + \overline{H}(y)(1 - 2\overline{F}(y))]\mathrm{d}y \tag{4.7}$$

We now provide sufficient conditions in order to compare Equation (4.5) with (4.7).

Proposition 6. *Under the assumptions for Equations (4.5) and (4.7), if $X \leq_{st} \tilde{X}$, then*

$$\text{GMD}(\hat{Y}, \tilde{Y}) \leq \text{GMD}(Y, \tilde{Y})$$

Proof. Recalling Equations (2.3), (4.5), and (4.7), by setting $u = \overline{F}(t)$, with some calculations one has

$$\text{GMD}(Y, \tilde{Y}) - \text{GMD}(\hat{Y}, \tilde{Y}) = \int_0^1 \tilde{q}(u)u\left(ROC_{\overline{H},\overline{F}}(u) - u\right)\mathrm{d}u$$

The thesis follows by noting that, since $\overline{F}(t) \leq \overline{H}(t)$ for all t, one has $ROC_{\overline{H},\overline{F}}(u) \geq u$ for all $u \in [0, 1]$. □

We remark that the conditions given in Proposition 6 allow to choose the suitable position, between shared or not shared, for the component having a lifetime with different distribution, in order to increase or decrease the absolute mean distance between systems' lifetimes.

Corollary 1. *Under the assumptions for Equations (4.5) and (4.7), let us suppose $\tilde{X} =_{st} X_t$. If X is NBU (NWU), then*

$$\mathsf{GMD}(\hat{Y},\tilde{Y}) \geq (\leq)\mathsf{GMD}(Y,\tilde{Y})$$

Let us consider an example for the bivariate Gini's mean difference given in Equation (4.7).

Example 5. Under the assumptions of Equation (4.7), we assume that X and $\tilde{X}$ are exponentially distributed, having s.f.'s $\overline{F}(t)=e^{-t}$ and $\overline{H}(t)=e^{-3t}$, for $t\geq 0$, respectively. Then, from Equation (4.7) and with straightforward calculations, one has $\mathsf{GMD}(\hat{Y},\tilde{Y})=0.35$. By recalling Equation (3.1), it follows $\mathsf{GMD}(\tilde{Y})=0.25$. Therefore, in this case, it holds $\mathsf{GMD}(\tilde{Y}) < \mathsf{GMD}(\hat{Y},\tilde{Y})$.

We remark that Example 5 shows a configuration in which the expected absolute distance between the random lifetime $\tilde{Y}$ and its independent copy (i.e., two series systems with i.d. lifetimes and non-shared components) is less than the one between $\hat{Y}$ and $\tilde{Y}$ (i.e., two series systems with not i.d. lifetimes and one shared component).

Along this line, in the next proposition, we provide sufficient conditions on the random lifetimes in order to compare the bivariate Gini's mean difference defined in Equation (4.2) with the univariate Gini's mean difference introduced in Equation (3.1) of the corresponding random lifetimes. In this sense, we obtain conditions aiming to reduce the systems' lifetimes distance in the above sense when their components' lifetimes are i.n.i.d.

Proposition 7. *Let (X,Y) be a random vector with survival copula $\widehat{C}_\theta(u,v)$, for $u,v\in[0,1]$ and $\theta\in\Theta$. Let us denote by $\tilde{q}_X(u)$ and $\tilde{q}_Y(u)$, for $u\in[0,1]$, the d.q.d.f.'s of X and Y, respectively. Suppose that $\widehat{C}_\theta$ is PQD for some $\theta\in\Theta$. If A1 (A2) holds for $\tilde{q}_X$ and if*

$$u - ROC_{\overline{G},\overline{F}}(u) \geq (\leq) 1-u-ROC_{\overline{G},\overline{F}}(1-u), \qquad \forall u\in\left(0,\frac{1}{2}\right) \tag{4.8}$$

then

$$\mathsf{GMD}(X,Y) \leq \mathsf{GMD}(X) \tag{4.9}$$

If A1 (A2) holds for $\tilde{q}_Y$ and if

$$u - ROC_{\overline{F},\overline{G}}(u) \geq (\leq) 1-u-ROC_{\overline{F},\overline{G}}(1-u), \qquad \forall u\in\left(0,\frac{1}{2}\right)$$

then

$$\mathsf{GMD}(X,Y) \leq \mathsf{GMD}(Y) \tag{4.10}$$

Proof. From Equations (3.1) and (4.2), since $\widehat{C}_\theta$ is PQD for some $\theta\in\Theta$, with few calculations it follows

$$\begin{aligned}\mathsf{GMD}(X)-\mathsf{GMD}(X,Y) &\geq \int_0^1 \tilde{q}(u)\left[u-ROC_{\overline{G},\overline{F}}(u)\right](1-2u)\mathrm{d}u\\ &= \int_0^{\frac{1}{2}}(1-2u)[\tilde{q}(u)-\tilde{q}(1-u)]a(u)\mathrm{d}u\end{aligned}$$

where we have set

$$a(u) := 2u-1+ROC_{\overline{G},\overline{F}}(1-u)-ROC_{\overline{G},\overline{F}}(u), \quad u\in\left(0,\frac{1}{2}\right) \tag{4.11}$$

From Equation (4.8), one gets $a(u)\geq(\leq)0$, for all $u\in(0,1/2)$. Therefore, Equation (4.9) follows from A1 (A2) satisfied by $\tilde{q}_X$. Equation (4.10) can be proved in a similar way. This completes the proof. □

A result similar to Proposition 7 can be obtained when $\widehat{C}_\theta$ is NQD for some $\theta\in\Theta$. In the following, we provide two examples where the assumptions of Proposition 7 are satisfied. Moreover, in the rest of the paper, we show PQD copulas for which Proposition 7 holds under suitable assumptions on the corresponding marginal lifetimes.

Example 6. Let (X,Y) be a random vector having a PQD copula. Let us suppose $X\sim Power(\alpha)$ and $Y\sim Power(\beta)$, for $\alpha,\beta>0$. By recalling Equation (2.3), with few calculations one has

$$ROC_{\overline{G},\overline{F}}(u)=1-(1-u)^{\beta/\alpha}, \qquad u\in[0,1]$$

If $0<\beta/\alpha\leq 1$ or $\beta/\alpha\geq 2$, then Equation (4.11) becomes

$$a(u)=2u-1+(1-u)^{\beta/\alpha}-u^{\beta/\alpha}\geq 0, \qquad \forall u\in\left(0,\frac{1}{2}\right)$$

while for $1\leq\beta/\alpha\leq 2$ one has $a(u)\leq 0$ for all $u\in(0,1/2)$. Under these assumptions, for example if $\alpha=1$ and $\beta\geq 2$, by recalling (i) of Table 1, since $\tilde{q}_X$ satisfies A1, then Equation (4.9) follows.

Example 7. Let us consider the Kumaraswamy (Kw) distribution introduced in Kumaraswamy [31] with s.f. $\overline{H}(t)=(1-t^\gamma)^\lambda$, for $t\in[0,1]$ and $\lambda>0,\gamma>0$. Such distribution does not satisfy Assumptions 1 and 2 for general γ and λ, but for certain fixed choices of these parameters. Indeed, let (X,Y) be a random vector having PQD copula, with $X\sim \mathrm{Kw}\left(\frac{1}{2};2\right)$ and $Y\sim \mathrm{Kw}\left(\frac{1}{3};2\right)$. By recalling Equation (2.3), with few calculations one has $ROC_{\overline{G},\overline{F}}(u)=u^{\frac{2}{3}}$, for $u\in[0,1]$. Then Equation (4.11) becomes

$$a(u)=2u-1+(1-u)^{\frac{2}{3}}-u^{\frac{2}{3}}\leq 0, \qquad \forall u\in\left(0,\frac{1}{2}\right)$$

Under the stated hypothesis, $\tilde{q}_X$ satisfies A2 and therefore Equation (4.9) follows.

In the rest of this section, we specify the copulas of different pairs of reliability systems with one or more shared components having i.n.i.d. lifetimes. Specifically, we consider series and parallel-series systems. Due to the duality between the structures, similar studies can be undertaken for the parallel and series-parallel systems.

4.1 | Series Systems

Let Y be the random lifetime of a series system with $s+n$ independent components' lifetimes, where $X_{11},\ldots,X_{1s}$ have s.f. $\overline{F}_1$, while $X_{31},\ldots,X_{3n}$ have s.f. $\overline{F}_3$. Let $\tilde{Y}$ be the random lifetime of another series system with $k+n$ independent components' lifetimes, where $X_{21},\ldots,X_{2k}$ have s.f. $\overline{F}_2$, while the other n components' lifetimes are distributed as $X_{31},\ldots,X_{3n}$ (i.e., Y and $\tilde{Y}$ share the n components with s.f. $\overline{F}_3$). Therefore, for $s,k,n\in\mathbb{N}$ one has

$$\overline{F}_Y(t)=\left[\overline{F}_1(t)\right]^s\left[\overline{F}_3(t)\right]^n, \qquad \overline{F}_{\tilde{Y}}(t)=\left[\overline{F}_2(t)\right]^k\left[\overline{F}_3(t)\right]^n, \qquad t\geq 0$$

From Equation (2.4), the survival copula of $(Y, \tilde{Y})$, for $\theta = (s, k, n)$ and for all $u, v \in [0, 1]$, is expressed by

$$\hat{C}_\theta(u, v) = \begin{cases} \frac{uv}{\left(ROC_{\overline{F}_3,\overline{F}_Y}(u)\right)^n}, & \overline{F}_Y^{-1}(u) < \overline{F}_{\tilde{Y}}^{-1}(v), \\ \frac{uv}{\left(ROC_{\overline{F}_3,\overline{F}_{\tilde{Y}}}(v)\right)^n}, & \overline{F}_Y^{-1}(u) \geq \overline{F}_{\tilde{Y}}^{-1}(v) \end{cases} \tag{4.12}$$

We remark that the ROC distortions appearing in the right-hand side of Equation (4.12) depend on s, k and n. Since the components' lifetimes are not i.d., providing assumptions on the values of s, k, n could be not sufficient in order to simplify the expression of the copula given in Equation (4.12). In this case, it is useful to consider proportional hazards models. For instance, if

$$\overline{F}_3(t) = \left(\overline{F}_1(t)\right)^\alpha, \qquad \overline{F}_2(t) = \left(\overline{F}_1(t)\right)^\beta, \qquad t \geq 0,\ \alpha > 0,\ \beta > 0$$

then Equation (4.12) reduces to

$$\hat{C}_\theta(u, v) = \begin{cases} u v^{\frac{\beta k}{\beta k + \alpha n}}, & u \leq v^{\frac{s + \alpha n}{\beta k + \alpha n}}, \\ u^{\frac{s}{s + \alpha n}} v, & u > v^{\frac{s + \alpha n}{\beta k + \alpha n}} \end{cases} \tag{4.13}$$

with $\theta = (s, k, n, \alpha, \beta)$. We remark that, if $\beta = 1$ and $\alpha n = m$, then Equation (4.13) is the copula given in Equation (3.8) when the number of shared components is equal to m.

4.2 | Parallel-Series Systems

Let Y be the random lifetime of a parallel-series system with $s + n$ independent components' lifetimes, where $X_{11}, \dots, X_{1s}$ have c.d.f. F_1, while $X_{31}, \dots, X_{3n}$ have c.d.f. F_3. Let $\tilde{Y}$ be the random lifetime of another parallel-series system with $k + n$ independent components' lifetimes, where $X_{21}, \dots, X_{2k}$ have c.d.f. F_2, while the other n components' lifetimes are distributed as $X_{31}, \dots, X_{3n}$ (i.e., Y and $\tilde{Y}$ share all the n components with s.f. $\overline{F}_3 = 1 - F_3$). Therefore, for $s, k, n \in \mathbb{N}$ one has

$$\overline{F}_Y(t) = \left(1 - [F_1(t)]^s\right)\left(1 - [F_3(t)]^n\right),$$
$$\overline{F}_{\tilde{Y}}(t) = (1 - [F_2(t)]^k)(1 - [F_3(t)]^n), \qquad t \geq 0$$

From Equation (2.4), the survival copula of $(Y, \tilde{Y})$, for $\theta = (s, k, n)$ and for all $u, v \in [0, 1]$, is expressed by

$$\hat{C}_\theta(u, v) = \begin{cases} \frac{uv}{1-\left(1-ROC_{\overline{F}_3,\overline{F}_Y}(u)\right)^n}, & \overline{F}_Y^{-1}(u) < \overline{F}_{\tilde{Y}}^{-1}(v), \\ \frac{uv}{1-\left(1-ROC_{\overline{F}_3,\overline{F}_{\tilde{Y}}}(v)\right)^n}, & \overline{F}_Y^{-1}(u) \geq \overline{F}_{\tilde{Y}}^{-1}(v) \end{cases} \tag{4.14}$$

We remark that the ROC distortions appearing on the right-hand side of Equation (4.14) depend on s, k and n. In order to simplify the expression of the copula given in Equation (4.14), one can use proportional reversed hazards models, under suitable assumptions on the parameters. For instance, if

$$F_3(t) = \left(F_1(t)\right)^\alpha, \qquad F_2(t) = \left(F_1(t)\right)^\beta, \qquad t \geq 0$$

for $\alpha > 0$ and $\beta > 0$ such that $\alpha n = s$ and $\beta k = s$, then Equation (4.14) reduces to Equation (3.6).

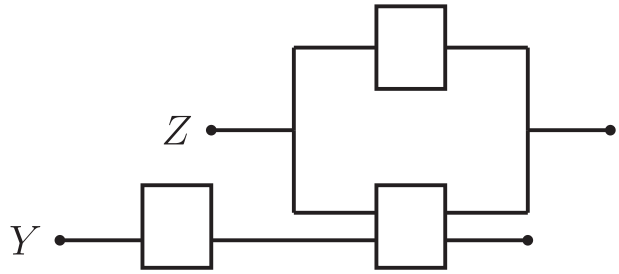

FIGURE 6 | Schematic representation of a series system and a parallel system with one shared component.

5 | Mixed Systems With Shared Components

In this section, we examine reliability issues that deal with more than one system structure. These schemes can be described through the concept of a mixed system, namely a system that works under a certain reliability structure i with probability p_i, for $i = 1, \dots, n$ (cf. Section 1.3.3 in Navarro [1]).

In particular, let us consider two mixed systems with shared components, where the structure of both systems is built in series or in parallel, respectively, according to Bernoulli's trials. Note that the dependence that arises in the special case when both systems work in series (or in parallel) corresponds to that described by the copulas given in the previous sections. Hereafter we go further by studying the dependence in terms of copulas regarding the specific case in which one system works as a series system, while the other is a parallel system. An example of this kind of system's structure is given in Figure 6. As in the previous sections, we distinguish between the case of i.i.d. and i.n.i.d. components' lifetimes.

5.1 | I.I.D. Case

Let Y describe the lifetime of a series system with $\ell + m$ i.i.d. components' lifetimes. Let Z describe the lifetime of a parallel system with $j + m$ i.i.d. components' lifetimes. All the components' lifetimes are distributed as X with c.d.f. F and s.f. $\overline{F}$. Suppose that Y and Z share m components. Therefore, for $\ell, j, m \in \mathbb{N}$ one has

$$\overline{F}_Y(t) = [\overline{F}(t)]^{\ell+m}, \qquad \overline{F}_Z(t) = 1 - [F(t)]^{j+m}, \qquad t \geq 0$$

From Equation (2.4), the survival copula of (Y, Z), for $\theta = (\ell, j, m) \in \Theta$ and for all $u, v \in [0, 1]$, is expressed by

$$\hat{C}_\theta(u, v) = \begin{cases} u - u^{\frac{\ell}{\ell+m}}(1-v)^{\frac{j}{j+m}} \\ \times\left[u^{\frac{1}{\ell+m}} + (1-v)^{\frac{1}{j+m}} - 1\right]^m, & u \leq \left(1 - (1-v)^{\frac{1}{j+m}}\right)^{\ell+m}, \\ u, & u > \left(1 - (1-v)^{\frac{1}{j+m}}\right)^{\ell+m} \end{cases} \tag{5.1}$$

With a few calculations, it holds that the copula in Equation (5.1) is PQD, for all θ. In particular, for $\ell = j = m = 1$, the survival copula of (Y, Z) is expressed by

$$\hat{C}_1(u,v) = \begin{cases} u, & u \le v, \\ u - (u(1-v))^{\frac{1}{2}}\left(u^{\frac{1}{2}} - (1-v)^{\frac{1}{2}} + 1\right), & u > v \end{cases} \tag{5.2}$$

Therefore, by recalling Equations (3.1) and (4.2), making use of Equation (5.2) it follows

$$\mathrm{GMD}(Y,Z) = \mathrm{GMD}(X)$$

Finally, we remark that the copula given in Equation (5.1) can be extended to other types of k-out-of- n systems.

5.2 | i.n.i.d. Case

Let Y be the random lifetime of a series system with $\ell + m$ independent components' lifetimes, where $X_{11}, \dots, X_{1\ell}$ have s.f. $\overline{F}_1$, while $X_{31}, \dots, X_{3m}$ have s.f. $\overline{F}_3$. Let Z be the random lifetime of a parallel system with $j + m$ independent components' lifetimes, where $X_{21}, \dots, X_{2j}$ have c.d.f. F_2 and s.f. $\overline{F}_2$, while the other m components' lifetimes are distributed as $X_{31}, \dots, X_{3m}$ (i.e., Y and Z share the m components with c.d.f. $F_3 = 1 - \overline{F}_3$). Therefore, for $\ell, j, m \in \mathbb{N}$ one has

$$\overline{F}_Y(t) = \left[\overline{F}_1(t)\right]^{\ell}\left[\overline{F}_3(t)\right]^{m},$$
$$\overline{F}_Z(t) = 1 - \left[F_2(t)\right]^{j}\left[F_3(t)\right]^{m}, \qquad t \ge 0$$

From Equation (2.4), the survival copula of (Y,Z), for $\theta = (\ell, j, m)$ and for all $u, v \in [0,1]$, is expressed by

$$\hat{C}_\theta(u,v) = \begin{cases} u, & \overline{F}_Y^{-1}(u) \le \overline{F}_Z^{-1}(v), \\ u - R_\theta(u,v), & \overline{F}_Y^{-1}(u) > \overline{F}_Z^{-1}(v) \end{cases} \tag{5.3}$$

where

$$R_\theta(u,v) := \left[ROC_{\overline{F}_1,\overline{F}_Y}(u)\right]^{\ell}\left[1 - ROC_{\overline{F}_2,\overline{F}_Z}(v)\right]^{j} \times \left[ROC_{\overline{F}_3,\overline{F}_Y}(u) - ROC_{\overline{F}_3,\overline{F}_Z}(v)\right]^{m}$$

In order to simplify the expression of the copula given in Equation (5.3), one can use both the proportional hazards model and the proportional reversed hazards model. For example, suppose that

$$\overline{F}_3(t) = \left(\overline{F}_1(t)\right)^{\alpha}, \qquad \overline{F}_2(t) = 1 - \left(F_3(t)\right)^{\beta}, \qquad t \ge 0,\ \alpha > 0,\ \beta > 0$$

Therefore $\overline{F}_2$ is a distorted s.f. from $\overline{F}_1$ through the following dual distortion

$$\tilde{h}_{\alpha,\beta}(u) = 1 - (1 - u^{\alpha})^{\beta}, \qquad u \in [0,1], \alpha > 0, \beta > 0$$

which represents a composition between the distortions related to both the proportional hazards model and the proportional reversed hazards model, having possibly different parameters. Under these assumptions Equation (5.3) reduces to

$$\hat{C}_\theta(u,v) = \begin{cases} u - u^{\frac{\ell}{\ell+\alpha m}}(1-v)^{\frac{\beta j}{\beta j + m}} \\ \times\left[u^{\frac{\alpha}{\ell+\alpha m}} + (1-v)^{\frac{1}{\beta j+m}} - 1\right]^{m}, & u < \left(1 - (1-v)^{\frac{1}{\beta j+m}}\right)^{\frac{\ell+\alpha m}{\alpha}}, \\ u, & u \ge \left(1 - (1-v)^{\frac{1}{\beta j+m}}\right)^{\frac{\ell+\alpha m}{\alpha}} \end{cases} \tag{5.4}$$

with $\theta = (\ell, j, m, \alpha, \beta)$. We remark that, if $\alpha = \beta = 1$, then Equation (5.4) coincides with the copula given in Equation (5.1).

6 | Conclusions

In this paper, we have investigated about the importance of distortion and copula functions in the analysis of some systems' reliability from their components' reliabilities. In this sense, a new distortion function related to the ROC curve has been defined, aiming to switch from one reliability to another. Moreover, several pairs of reliability systems with shared components have been studied, under i.i.d. and i.n.i.d. configurations for their respective components' lifetimes. It is remarkable that the copula of these kinds of systems turned out to possess all the information for the description of their systems' reliability. A lifetime distance analysis has been provided by using the Gini's mean difference and its new recent generalizations, also by considering optimization problems for a given components' stock having i.i.d. lifetimes. An optimal allocation problem for the case of components having different ages is studied as well. Note that such distance analysis allows to specify the goodness of the whole system not only in terms of its duration.

Future developments may concern the determination of these copulas when the components' lifetimes are dependent since they share the same environment. Reliability issues regarding systems composed by units having dependent lifetimes have been also studied in the literature (see, for instance, Di Crescenzo and Pellerey [32] and Zhang et al. [33]). Further complex systems' structures can be considered as well, such as k-out-of- n systems. Note that, similarly to Remarks 1 and 2, the absolute mean difference between the systems' lifetimes is still related with the cumulative information generating function of the i.i.d. components' lifetimes. For example, let $Y_{2:3}$ and $\tilde{Y}_{2:3}$ denote the random lifetime of two 2-out-of-3 systems, having i.i.d. components' lifetimes according to X. If $Y_{2:3}$ and $\tilde{Y}_{2:3}$ share one component, then $\mathrm{GMD}(Y_{2:3}, \tilde{Y}_{2:3}) = 6\mathrm{G}_X(2,2)$. Note that, due to the complexity of the systems' structure, the copula of $(Y_{2:3}, \tilde{Y}_{2:3})$ has a more elaborate expression.

Finally, the analysis of lifetimes' distance through different kinds of information measures is also of interest.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

No data was used for the research described in the article.

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