Abstract

Continued fractions have been studied in mathematics for centuries and have been generalized in several ways for different reasons. The study of p–adic continued fractions has started around 1940 with a question of Mahler and the first algorithms have been defined by Ruban, Schneider and Browkin around the 1970. The main goal is to replicate, inside the field of p–adic numbers $\mathbb{Q}_p$, all the optimal properties that continued fractions share in the field of real numbers. Several results on continued fractions regarding convergence, finiteness, approximation and periodicity have been proved in $\mathbb{Q}_p$. However, some of the nice properties of real continued fractions do not generalize very well, due to the very different structure of $\mathbb{Q}_p$ and the different properties of the p–adic absolute value. The main still open problem is to find an algorithm that provides a periodic p–adic continued fraction for any quadratic irrational, i.e. an analogue of the famous Lagrange's Theorem. The results contained in this thesis move toward this direction. In particular, we study the convergence, the finiteness and the periodicity of many algorithms and we search for other new algorithms sharing better properties than the existent ones. We start by studying the properties of periodicity of *Browkin II*, that is the algorithm that seemingly provides more periodic representations for quadratic irrationals. We highlight that it does not share good properties of pure periodicity and, unlike continued fractions in the real field, the length of the pre-period is variable and it is not possible to predict where the period starts. Therefore, we define some new algorithms to improve these periodicity properties. In order to do that, we prove some effective characterizations for the convergence of continued fractions in $\mathbb{Q}_p$. These results provide the explicit shape of the partial quotients for defining convergent p–adic continued fractions, therefore they open the way for the exploration of new algorithms. We use these

results to define a new algorithm, obtained as a modification of *Browkin II*, that shares better properties of periodicity from both a theoretical and a computational point of view. Finally, we propose a novel approach to disprove Lagrange's Theorem for *Browkin-type* continued fractions, i.e. those allowing negative partial quotients, that is a well studied open problem. We prove that a necessary condition for the periodicity of p–adic continued fractions is the convergence of the continued fraction in $\mathbb{R}$ to the real embedding of the same quadratic irrational. We use this necessary condition to develop a probabilistic argument for the non-periodicity of *Browkin-type* continued fractions, reasoning on the expected size of the partial quotients in Euclidean absolute value. In the final section, we briefly summarize some results about other research topics, namely universal quadratic forms over number fields, linear recurrence sequences and some cryptographic methods for blockchain applications.