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Barbara Franci, Filippo Fabiani and Lorenzo Zino

***Abstract*— We consider generalized Nash equilibrium (GNE) problems under a partial-decision information regime, in which each agent typically reconstructs the opponents' strategies through a linear averaging dynamics. In contrast, we consider a state-dependent, nonlinear susceptibility term within the communication mechanism, thereby modelling possible biases on the part of agents in processing information. By including such a term in a relaxed forward-backward iteration scheme, we design a distributed algorithm possessing convergence guarantees to a GNE. Simulation results illustrate how the susceptibility term affects the GNE computation.**

I. Introduction

In the realm of multi-agent decision-making, generalized Nash equilibrium problems (GNEPs) [1], [2] denote a well-established paradigm able to model self-interested agents with mutually coupled decisions. Several research efforts have concentrated on the design of distributed algorithms for generalized Nash equilibrium (GNE) seeking, which consist on iterative procedures alternating computation and communication steps among the agents, ultimately leading to a scenario in which none of the agents can further decrease its cost function, given what the other participants are currently doing [3]–[5].

To devise the underlying GNE seeking schemes, a widely employed assumption requires agents to have access to all the decision variables of their peers, i.e., a *full-decision information* setup [3], [4]. Such a requirement is however strong in many real-world applications where agents have little information on the others. This limitation has been overcome with the introduction of a *partial-decision information* setup [6]–[8], where agents communicate with few peers to reconstruct the strategies of the whole population.

A key aspect characterizing partial-decision information regimes is the possibility to reach consensus on the estimates across the agents in a distributed fashion. Inspired by standard linear averaging algorithms [9], a Laplacian-type constraint is usually imposed within the distributed algorithm to reach a GNE [8]. Such a Laplacian constraint implicitly induces a DeGroot-like opinion dynamics [10], [11], in which the agents average on the estimates of their neighbours to update their own, ultimately yielding consensus under reasonable connectivity assumptions. On the other hand, such a mechanism tacitly assumes that the agents' trust in each other is unbiased, and the way they process information does not depend on their current opinion. This could however be a stretch in several real-world applications of GNEPs involving humans-in-the-loop, such as charging electric vehicles and the energy market [7], [12]. In fact, it has been empirically observed in many different scenarios —spanning from risk perception to information sharing on social media [13], [14]— that the way humans process information is indeed not uniform and unbiased, but it is strongly dependent on their current opinions and the information already gathered.

More realistic extensions of the DeGroot dynamics have been proposed in, e.g., [15]. In [16], the agents' susceptibility has been included as a state-dependent diagonal matrix that pre-multiplies the Laplacian, yielding a nonlinear dynamic. This approach allows for capturing different behavioural attitudes, consistent with the social psychology literature [17], whereby an agent can be more or less susceptible to the information received from others —and consequently prone to change opinion— depending on their current opinion. Interestingly, [16] shows how including such a susceptibility term can still (under reasonable assumptions) lead the system to a consensus, which may however differ from the one reached through an unbiased dynamic.

Motivated by the approach and observations in [16], we bring such a *susceptibility to persuasion* term in the domain of GNEPs. Specifically, our contribution is four-fold. First, based on [16], we propose a discrete-time opinion dynamics that includes the agents' susceptibility, and we establish conditions under which such a dynamic is well-defined and converges to a consensus. Second, we encapsulate such an opinion dynamic in a distributed GNE seeking algorithm based on relaxed forward-backward (rFB) iterations [18], [19]. In this framework, the novel dynamics is used to estimate the decision variables of the other participants in a partial-decision information regime, and thus supporting the GNE seeking mechanism. Third, we prove that our distributed GNE seeking algorithm converges to a GNE. Finally, we consider a case study based on a Nash-Cournot game [7], [8] to illustrate our theoretical findings and discuss how susceptibility shapes the GNE reached by the algorithm.

Notation: The definitions are taken from [1], [20], [21].

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Operator theory: For a closed set $\Omega \subseteq \mathbb{R}^n$, $\mathrm{proj}_\Omega : \mathbb{R}^n \to \Omega$ denotes the standard projection onto Ω. The mapping $\mathrm{N}_\Omega : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator for the set Ω , i.e., $\mathrm{N}_\Omega(x) = \varnothing$ if $x \notin \Omega$, $\mathrm{N}_\Omega(x) = \{v \in \mathbb{R}^n \mid \sup_{z\in\Omega} v^\top(z-x) \leq 0\}$ otherwise. Id is the identity operator. A mapping $F : \mathrm{dom}\, F \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is ℓ-Lipschitz continuous if, for some $\ell > 0$, $\|F(x) - F(y)\| \leq \ell \|x-y\|$ $\forall x, y \in \mathrm{dom}(F)$; η-strongly monotone if, for some $\eta > 0$, $\langle F(x) - F(y), x-y\rangle \geq \eta\|x-y\|^2$ $\forall x, y \in \mathrm{dom}(F)$; (restricted) monotone if $\forall x \in \mathrm{dom}(F), y \in \mathrm{Fix}(F)$, $\langle F(x) - F(y), x-y\rangle \geq 0$; β-(restricted) cocoercive if, for some $\beta > 0$, $\forall x \in \mathrm{dom}(F), y \in \mathrm{Fix}(F)$, $\langle F(x) - F(y), x-y\rangle \geq \beta\|F(x) - F(y)\|^2$; maximally monotone if $\nexists$ monotone operator $G : C \to \mathbb{R}^n$ so that the graph of G contains that of F. The set of fixed points of F is $\mathrm{fix}(F) := \{x \in \mathbb{R}^n \mid x \in F(x)\}$. F is nonexpansive if it is 1-Lipschitz continuous. F is α-averaged for some $\alpha > 0$, if there exists a nonexpansive operator $H : \mathrm{dom}\, F \to \mathbb{R}^n$ such that $F = (1-\alpha)Id + \alpha H$.

Graph theory: A graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$ is defined by a node set $\mathcal{I}$ and an edge set $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{I}\}$. Pair $(i,j) \in \mathcal{E}$ if agent i communicates with j. The set of neighbors of agent i is indicated with $\mathcal{N}_i = \{j \in \mathcal{I} \mid (i,j) \in \mathcal{E}\}$. We call $W \in \mathbb{R}^{N\times N}$ the weighted adjacency matrix of $\mathcal{G}$, with entry $w_{ij} > 0$ if $(i,j) \in \mathcal{E}$, $w_{ij} = 0$ otherwise, and $w_{ii} = 0$ $\forall i \in \mathcal{I}$. Let $d_i = \sum_{j=1}^N w_{ij}$ be the degree of node i, $D = \mathrm{diag}\{d_1, \dots, d_N\}$, and $d_{\max} = \max_{i\in\mathcal{I}} d_i$. The graph Laplacian is $L = D - W$. Graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$ is undirected if $(i,j) \in \mathcal{E}$ and $(j,i) \in \mathcal{E}$, and it is connected if there is a path between every pair of vertices. We indicate the consensus subspace of dimension N as $\mathcal{K}_N = \{\kappa \mathbf{1}_N \mid \kappa \in \mathbb{R}\}$. If the graph is connected, $\mathrm{null}(L) = \mathcal{K}_N$. Moreover, 0 is a simple eigenvalue of L, while all other eigenvalues are positive and can be ordered as $0 < s_2(L) \leq \dots \leq s_N(L) \leq 2d_{\max}$.

II. Problem formulation

We consider a GNEP involving a set of N agents, indexed by $\mathcal{I} := \{1, \dots, N\}$, where each agent controls a decision variable $x_i \in \mathbb{R}^{n_i}$, and aims at minimizing a cost function $J_i(x_i, \boldsymbol{x}_{-i})$ subject to both local and coupling constraints, thus resulting in a collection of optimization problems:

$$\forall i \in \mathcal{I} : \begin{cases} \min_{x_i \in \Omega_i} & J_i(x_i, \boldsymbol{x}_{-i}) \\ \text{s.t.} & C\boldsymbol{x} \leq d. \end{cases} \tag{1}$$

Each cost function $J_i : \mathbb{R}^n \to \mathbb{R}$, $n := \sum_{i\in\mathcal{I}} n_i$, depends on the i-th agent's decision variable $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$ and on the decisions of the other agents $\boldsymbol{x}_{-i} = \mathrm{col}((x_j)_{j\in\mathcal{I}\setminus\{i\}})$, collectively stacked into $\boldsymbol{x} = \mathrm{col}((x_i)_{i\in\mathcal{I}}) \in \Omega := \prod_{i\in\mathcal{I}} \Omega_i$.

Standing Assumption 1: For all $i \in \mathcal{I}$ and $\boldsymbol{x}_{-i} \in \prod_{j\in\mathcal{I}\setminus\{i\}} \Omega_j$, $J_i(\cdot, \boldsymbol{x}_{-i})$ is convex and of class $\mathcal{C}^1$. □

The agents are subject to both local, i.e., $x_i \in \Omega_i$, and coupling constraints $C\boldsymbol{x} \leq d$, for $C \in \mathbb{R}^{m\times n}$ and $d \in \mathbb{R}^m$. The collective feasible set of the GNEP in (1) is then given by $\mathcal{X} := \{\boldsymbol{y} \in \Omega \mid C\boldsymbol{y} - d \leq 0\}$, and the feasible decision set for agent $i \in \mathcal{I}$, parametric in $\boldsymbol{x}_{-i}$, is $\mathcal{X}_i(\boldsymbol{x}_{-i}) = \{y_i \in \Omega_i \mid C_i y_i \leq d - \sum_{j\in\mathcal{I}\setminus\{i\}} C_j x_j\}$ where $C_i \in \mathbb{R}^{m\times n_i}$.

Standing Assumption 2: For all $i \in \mathcal{I}$, $\Omega_i = [a_i, b_i]^{n_i}$. Moreover, the set $\mathcal{X}$ is nonempty, compact and convex, and it satisfies the Slater's constraint qualification. □

A widely employed solution concept for the GNEP in (1) refers to a GNE, which is formally defined as follows.

Definition 1: A collective decision vector $\boldsymbol{x}^* \in \mathcal{X}$ is a GNE of the GNEP in (1) if, for all $i \in \mathcal{I}$, $J_i(x_i^*, \boldsymbol{x}_{-i}^*) \leq \min_{y_i \in \mathcal{X}_i(\boldsymbol{x}_{-i}^*)} J_i(y_i, \boldsymbol{x}_{-i}^*)$. □

As established in [2], the problem of finding a GNE can be equivalently recast as an inclusion problem:

$$0 \in \mathcal{T}(\boldsymbol{x}, \lambda) := \begin{bmatrix} \mathrm{N}_\Omega(\boldsymbol{x}) + F(\boldsymbol{x}) + C^\top \lambda \\ \mathrm{N}_{\mathbb{R}^m_{\geq 0}}(\lambda) - (C\boldsymbol{x} - d) \end{bmatrix}, \tag{2}$$

where $F(\boldsymbol{x}) := \mathrm{col}((\nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i}))_{i\in\mathcal{I}})$ is the pseudo-gradient mapping, $\lambda \in \mathbb{R}^m_{\geq 0}$ represents the dual variables associated to the coupling constraints, while the normal cone $\mathrm{N}_\Omega(\boldsymbol{x})$ guarantees the satisfaction of the local constraints. Note that the constraint qualification in Standing Assumption 2 ensures bounded dual variables λ [22, §5.2.3].

Standing Assumption 3: The pseudogradient mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is η-strongly monotone and ℓ_F-Lipschitz continuous, for some constants η, $\ell_F > 0$. □

Remark 1: Standing Assumption 3 is standard in the partial-decision information regime [7], [8] as it guarantees some monotonicity properties of the operators needed for the analysis (see §III-B and Lemma 3). It also certifies the existence and uniqueness of a solution [1, Th. 2.3.3]. The work in [6] considers a weaker assumption, although the coupling constraints as in our GNEP are not included. □

The reformulation in (2) concerns the problem of finding a zero of the set-valued mapping $\mathcal{T} : \mathcal{X} \times \mathbb{R}^m_{\geq 0} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$, and can be obtained via a primal-dual characterization of the equilibria. In fact, it holds that a collective decision $\boldsymbol{x}^*$ is a so-called variational generalized Nash equilibrium (v-GNE) of the GNEP in (1) if and only if the Karush-Kuhn-Tucker (KKT) conditions associated to the collection of optimization problems in (1) are satisfied with consensus on the dual variables, i.e., $\lambda_i = \lambda$ for all $i \in \mathcal{I}$ [23, Th. 3.1]. Here, $\lambda_i \in \mathbb{R}^m_{\geq 0}$ is the local vector of dual variables associated to the coupling constraints, for each agent $i \in \mathcal{I}$.

Throughout this paper we assume to work in a *partial-decision information* regime. Given a communication network describing the interconnections among the agents, mathematically captured by a graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$, we thus suppose that the agents cannot access the decision variables of the others, but they keep an estimate of them [7], [8]. In particular, $\hat{x}_{i,j}$ is the estimate that agent i has of the opponent j, collected in $\hat{x}_i = \mathrm{col}((\hat{x}_{i,j})_{j\in\mathcal{I}}) \in \mathbb{R}^n$ for each $i \in \mathcal{I}$ with $\hat{x}_{i,i} = x_i$ and, collectively, in $\hat{\boldsymbol{x}} = \mathrm{col}((\hat{x}_i)_{i\in\mathcal{I}}) \in \mathbb{R}^{nN}$.

Standing Assumption 4: The communication graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$ is undirected and connected. □

To achieve a GNE, the agents then need to learn the decision variables of their opponents, and therefore the goal under a partial-decision information regime is to exploit several communication rounds to reach consensus on the estimates, i.e., eventually $\hat{x}_i = \hat{x}_j$, for all pairs $(i,j) \in \mathcal{I}^2$.

III. GNE SEEKING WITH AGENTS' SUSCEPTIBILITY

We now discuss how to design a distributed GNE seeking algorithm for the GNEP described in §II, under a partial-decision information regime, by incorporating agents' susceptibility in the information communication mechanism.

To reach consensus on the primal and dual variables with partial-decision information, available works [5], [8] usually impose Laplacian-type constraints of the form $0 = (L \otimes I_n)\hat{\boldsymbol{x}} =: L_n \hat{\boldsymbol{x}}$ and $0 = (L \otimes I_m)\boldsymbol{\lambda} =: L_m \boldsymbol{\lambda}$, where $\boldsymbol{\lambda} = \mathrm{col}((\lambda_i)_{i \in \mathcal{I}})$ and $L \in \mathbb{R}^{N \times N}$ denotes the Laplacian matrix associated to $\mathcal{G}$, and implicitly assume a DeGroot-like mechanism [10]. In this paper, we propose a different approach, in which we drop the simplistic assumption that agents have a uniform and constant trust in their neighbors. We therefore consider a different nonlinear dynamics to reproduce the behavior of the agents involved in the GNEP.

A. Discrete-time opinion dynamics with susceptibility

Inspired by [16], we consider a dynamics in which different levels of agents' stubbornness are introduced thorough a state-dependent diagonal matrix $A(\hat{\boldsymbol{x}}) \in \mathrm{diag}([0,1]^{nN})$. Here, the generic n-dimensional diagonal block $A_i(\hat{x}_i)$ denotes the *susceptibility to persuasion* of agent i, which depends on the local estimate only, with entry $a_{ij}(\cdot) := [A_i(\cdot)]_{jj}$ referring to the j-th entry of the estimate vector. Specifically, this model assumes that susceptibility is not (necessarily) a constant property of an agent, but rather a state-dependent property that may change with their opinion. This assumption captures different agents' behavioral attitudes [17], e.g., agents with more extreme opinions can be more stubborn or more prone to conform with others, depending on the considered application.

The inclusion of such state-dependent multiplicative term ultimately leads to a nonlinear dynamics for the estimates, which are updated according to the following iterative rule:

$$\hat{\boldsymbol{x}}^{k+1} = (I_{nN} - \kappa A(\hat{\boldsymbol{x}}^k) L_n)\hat{\boldsymbol{x}}^k, \tag{3}$$

where $\kappa > 0$ is the discretization step of the dynamics.

The model in (3) consists of two main terms: the averaging component L_n, which steers the agents toward consensus, and the susceptibility term $A(\cdot)$ that modulates such process. In particular, [16] considered few examples for $A(\hat{\boldsymbol{x}})$: $(I_{nN} - \mathrm{diag}(\hat{\boldsymbol{x}})^2)$, $\frac{1}{2}(I_{nN} - \mathrm{diag}(\hat{\boldsymbol{x}}))$ or $\mathrm{diag}(\hat{\boldsymbol{x}})^2$, which are able to model different scenarios of behavioral attitudes, such as stubborn extremist (SE), stubborn positive (SP), and stubborn neutral (SN) agents, respectively (see also §V).

Under some reasonable assumptions on the matrix function A and the discretization step κ, we can prove that the dynamics in (3) is well-defined and converges to a consensus.

Standing Assumption 5: The matrix function $A : \mathbb{R}^{nN} \to \mathbb{R}^{nN \times nN}$ is so that $\sigma = \min_{i,j,\hat{\boldsymbol{x}} \in \Omega^N} \; a_{ij}(\hat{x}_i) > 0$, $\max_{i,j,\hat{\boldsymbol{x}} \in \Omega^N} a_{ij}(\hat{x}_i) < \infty$, $\kappa < \max_{i,j,\hat{\boldsymbol{x}} \in \Omega^N} \frac{1}{a_{ij}(\hat{x}_i) d_{\max}}$. □

Lemma 1: The domain Ω^N is invariant under dynamics in (3), and the sequence $\{\hat{\boldsymbol{x}}^k\}_{k \in \mathbb{N}}$, generated by (3) from initial condition $\hat{\boldsymbol{x}}^0 \in \Omega^N$, converges to some $\bar{\boldsymbol{x}} \in \mathrm{null}(L_n)$. □

Proof: According to (3), the i-th update reads as:

$$\hat{x}_i^{k+1} = (I_n - \kappa d_i A_i(\hat{x}_i^k))\hat{x}_i^k + \textstyle\sum_{j \in \mathcal{I} \setminus \{j\}} \kappa A_i(\hat{x}_i^k) w_{ij} \hat{x}_j^k. \tag{4}$$

Then, in view of Standing Assumptions 2 and 5 it holds that i) $(I_n - \kappa d_i A_i(\hat{x}_i^k))$ is diagonal and nonnegative and ii) $\sum_{j \in \mathcal{I} \setminus \{j\}} \kappa A_i(\hat{x}_i^k) w_{ij} \le I$. Therefore, the updated state of each node at iteration $k+1$ is in the convex hull of the states of the nodes at iteration k [24]. Hence, if the dynamics in (3) is initialized within Ω^N, i.e., $\hat{\boldsymbol{x}}^0 \in \Omega^N$, then $\hat{\boldsymbol{x}}^k \in \Omega^N$ for any $k \ge 0$. Finally, to prove that (4) converges to a consensus state, we note that (4) amounts to a weighted averaging process with time-varying weights, where all nonzero off-diagonal entries are greater than $\kappa \sigma \min_{(i,j)} w_{ij} > 0$, while diagonal entries are strictly larger than 0 and bounded away from it, being $\kappa d_i A_i(\hat{x}_i^k) < \mathbf{1}_n$. Hence, all entries of $\tilde{W}^k$, if positive, are greater than some constant, which guarantees that opinions converge to a consensus by [25, Lemma 4]. ■

Remark 2: With the current definition of $A : \mathbb{R}^{nN} \to \mathbb{R}^{nN \times nN}$ we assume that the agent's stubbornness depends on each single estimate $\hat{x}_{i,j}$, while one could redefine $A : \mathbb{R}^{nN} \to \mathbb{R}^{N \times N}$ to obtain an agent-wise stubbornness, thus turning (3) into $\hat{\boldsymbol{x}}^{k+1} = (I_{nN} - \kappa(A(\hat{\boldsymbol{x}}^k) L \otimes I_n))\hat{\boldsymbol{x}}^k$. □

B. Including susceptibility in the zero-finding problem

To show how the communication dynamic affects the GNE seeking problem, we introduce some notation first. Since the agents use the estimates to compute a GNE of the game in (1), the pseudogradient mapping is modified to:

$$F_p(\hat{\boldsymbol{x}}) = \mathrm{col}((\nabla_{x_i} J_i(x_i, \hat{\boldsymbol{x}}_{i,-i}))_{i \in \mathcal{I}}). \tag{5}$$

The latter is called *extended pseudogradient*, and we note that, in view of Standing Assumption 3, it is ℓ_p-Lipschitz continuous with constant $0 < \ell_p \le \ell_F$ [26, Lemma 3].

In the spirit of [8], for all $i \in \mathcal{I}$ we define matrices:

$$\begin{aligned} \mathcal{R}_i &:= \begin{bmatrix} \mathbf{0}_{n_i \times n_{<i}} & I_{n_i} & \mathbf{0}_{n_i \times n_{>i}} \end{bmatrix}, \\ \mathcal{S}_i &:= \begin{bmatrix} I_{n_{<i}} & \mathbf{0}_{n_{<i} \times n_i} & \mathbf{0}_{n_{<i} \times n_{>i}} \\ \mathbf{0}_{n_{>i} \times n_{<i}} & \mathbf{0}_{n_{>i} \times n_i} & I_{n_{>i}} \end{bmatrix}, \end{aligned} \tag{6}$$

with $n_{<i} := \sum_{j<i, j \in \mathcal{I}} n_j, n_{>i} := \sum_{j>i, j \in \mathcal{I}} n_j$. In particular, $\mathcal{R}_i$ allows one to select the i-th n_i-dimensional component from a vector in $\mathbb{R}^n$, while $\mathcal{S}_i$ to remove it [8].

Following classic operator splitting results [5], [8], [19] and including the consensus constraints, the inclusion (2) can be reformulated by considering the sum of two operators:

$$\begin{aligned} \mathcal{A} &: \begin{bmatrix} \hat{\boldsymbol{x}} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{R}^\top F_p(\hat{\boldsymbol{x}}) + cA(\hat{\boldsymbol{x}}) L_n \hat{\boldsymbol{x}} \\ \mathbf{0} \\ L_m \boldsymbol{\lambda} + \boldsymbol{d} \end{bmatrix} + \begin{bmatrix} \mathcal{R}^\top \mathbf{C}^\top \boldsymbol{\lambda} \\ -L_m \boldsymbol{\lambda} \\ -\mathbf{C}\mathcal{R}\hat{\boldsymbol{x}} + L_m \boldsymbol{z} \end{bmatrix} \\ \mathcal{B} &: \begin{bmatrix} \hat{\boldsymbol{x}} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{R}^\top \mathrm{N}_\Omega(\mathcal{R}\hat{\boldsymbol{x}}) \\ \mathbf{0} \\ \mathrm{N}_{\mathbb{R}^{mN}_{\ge 0}}(\boldsymbol{\lambda}) \end{bmatrix} \end{aligned} \tag{7}$$

where $\mathbf{C} = \mathrm{diag}\{C_1, \ldots, C_N\} \in \mathbb{R}^{mN \times n}$, and $\boldsymbol{d} = \mathbf{1}_N \otimes d \in \mathbb{R}^{mN}$ split the coupling constraints in $\mathcal{X}$ between the agents. In (7) we are actually considering the estimates $\hat{\boldsymbol{x}}$ as decision variables, and we use $\mathcal{R} = \mathrm{diag}((\mathcal{R}_i)_{i \in \mathcal{I}})$ to select the variables corresponding to each agent. In particular,

Algorithm 1: Relaxed Forward-Backward (rFB)

Initialization: For all $i \in \mathcal{I}$, set $\hat{x}_i^0, \bar{y}_i^{-1} \in \Omega^N$, $x_i^0 \in \Omega_i, \lambda_i^0, \bar{\lambda}_i^{-1} \in \mathbb{R}^m_{\geq 0}$, and $z_i^0, \bar{z}_i^{-1} \in \mathbb{R}^m$.

Iteration $k \in \mathbb{N}_0$: For all $i \in \mathcal{I}$, **do**

(1) Update variables:

$$\begin{aligned} \bar{y}_i^k &= (1-\delta)\hat{x}_i^k + \delta \bar{y}_i^{k-1} \\ \bar{z}_i^k &= (1-\delta) z_i^k + \delta \bar{z}_i^{k-1} \\ \bar{\lambda}_i^k &= (1-\delta)\lambda_i^k + \delta \bar{\lambda}_i^{k-1} \end{aligned}$$

(2) Receive $\hat{x}_j^k$, z_j^k, and λ_j^k from all $j \in \mathcal{N}_i$, update:

$$\begin{aligned} x_i^{k+1} &= \mathrm{proj}_{\Omega_i}[\bar{y}_i^k - \alpha_i(F_p(x_i^k, \hat{\boldsymbol{x}}_{i,-i}^k) + C_i^\top \lambda_i^k \\ &\qquad + c\textstyle\sum_{j\in\mathcal{N}_i} a_{ii}(x_i) w_{ij}(x_i^k - \hat{x}_{j,i}^k))] \\ \hat{\boldsymbol{x}}_{i,-i}^{k+1} &= \bar{y}_{i,-i}^k - \alpha_i c \textstyle\sum_{j\in\mathcal{N}_i} a_{i,-i}(\hat{x}_i) w_{ij}(\hat{\boldsymbol{x}}_{i,-i}^k - \hat{\boldsymbol{x}}_{j,-i}^k) \\ z_i^{k+1} &= \bar{z}_i^k + \nu_i \textstyle\sum_{j\in\mathcal{N}_i} w_{i,j}(\lambda_i^k - \lambda_j^k) \\ \lambda_i^{k+1} &= \mathrm{proj}_{\mathbb{R}^m_{\geq 0}}\{\bar{\lambda}_i^k + \tau_i C_i x_i^k - \tau_i \textstyle\sum_{j\in\mathcal{N}_i} w_{i,j}(z_i^k - z_j^k) \\ &\qquad - \tau_i \textstyle\sum_{j\in\mathcal{N}_i} w_{i,j}(\lambda_i^k - \lambda_j^k) - \tau_i d_i\} \end{aligned}$$

$\mathrm{N}_\Omega(\mathcal{R}\hat{\boldsymbol{x}}) = \mathrm{N}_\Omega(\boldsymbol{x}) = \prod_{i\in\mathcal{I}} \mathrm{N}_{\Omega_i}(x_i)$, and $\mathrm{N}_{\mathbb{R}^{mN}_{\geq 0}}(\boldsymbol{\lambda}) = \prod_{i\in\mathcal{I}} \mathrm{N}_{\mathbb{R}^{mN}_{\geq 0}}(\lambda_i)$, while $\boldsymbol{z} \in \mathbb{R}^{mN}$ is an auxiliary variable. The scaling constant c is properly in the proof of Lemma 3.

The term $A(\hat{\boldsymbol{x}})L_n\hat{\boldsymbol{x}}$ is the communication dynamics for the agents to share their estimates with (some of) the other participants. By taking into account their susceptibility, these terms drive the agents towards consensus of the estimates (Lemma 1) while the (extended) pseudogradient serves to minimize the agents' costs (via the first line of the operators); the second and third lines enforce the coupling constraints.

Note that the consensus constraints on the dual variables is not affected by the susceptibility matrix. The reason for this choice is that, while the agents might want to be stubborn on the estimates and primal variables that directly affect their cost function, they still need to satisfy the coupling constraints and be more lenient with the other participants.

Next, we establish the equivalence between the zeros of the operator $\mathcal{T}$ in (2) and those of $\mathcal{A}+\mathcal{B}$ in (7) :

Proposition 1: Let $\mathrm{col}(\hat{\boldsymbol{x}}^*, \boldsymbol{z}^*, \boldsymbol{\lambda}^*) \in \mathrm{zer}(\mathcal{A}+\mathcal{B})$. Then, $\hat{\boldsymbol{x}}^* = \mathbf{1}_N \otimes \boldsymbol{x}^*$ and $\boldsymbol{\lambda}^* = \mathbf{1}_N \otimes \lambda^*$, where $\mathrm{col}(\boldsymbol{x}^*, \lambda^*) \in \mathrm{zer}(\mathcal{T})$, and $\boldsymbol{x}^*$ is a v-GNE of the game in (1). □

Proof: The proof follows the same steps as the ones for the proof of [8, Th. 1], [19, Lemma 1], which allow us to conclude that, in view of Lemma 1, $\hat{\boldsymbol{x}}^* \in \mathrm{null}(L_n)$ and $\boldsymbol{\lambda}^* \in \mathrm{null}(L_m)$, i.e., $\hat{\boldsymbol{x}}^* = \mathbf{1}_N \otimes \boldsymbol{x}^*$ and $\boldsymbol{\lambda}^* = \mathbf{1}_N \otimes \boldsymbol{\lambda}^*$, with $\mathrm{col}(\boldsymbol{x}^*, \lambda^*) \in \mathrm{zer}(\mathcal{T})$, thus satisfying the KKT equivalent conditions characterizing a v-GNE for (2). ■

C. Algorithm design with susceptibility to persuasion

Once shown that a zero of the sum of the two operators in (7) coincides with a solution to (1), we can now leverage their structure to design an iterative GNE seeking scheme accounting for agents' susceptibility in the communication.

We use an rFB scheme, whose sequence of instructions is reported in Algorithm 1 [18], [19]. Note that, however, as long as the required monotonicity properties on the operators in (7) are satisfied, other algorithms can be considered, such as forward-backward, extragradient, or forward-backward-forward iterations [3], [8], [27]. More details on the properties our operators should possess are discussed in §IV.

By letting $\boldsymbol{\omega} := \mathrm{col}(\hat{\boldsymbol{x}}, \boldsymbol{z}, \boldsymbol{\lambda})$ and $\bar{\boldsymbol{\omega}} := \mathrm{col}(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}})$, the rFB algorithm in compact form reads as follows:

$$\begin{cases} \bar{\boldsymbol{\omega}}^k = (1-\delta)\boldsymbol{\omega}^k + \delta\bar{\boldsymbol{\omega}}^{k-1} \\ \boldsymbol{\omega}^{k+1} = (\mathrm{Id} + \Phi^{-1}\mathcal{B})^{-1} \circ (\bar{\boldsymbol{\omega}}^k - \Phi^{-1}\mathcal{A}(\boldsymbol{\omega}^k)), \end{cases} \tag{8}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined in (7) and the diagonal matrix $\Phi = \mathrm{diag}(\alpha^{-1}, \nu^{-1}, \tau^{-1})$, where $\alpha := \mathrm{diag}((\alpha_i I_{n_i})_{i\in\mathcal{I}}) \in \mathbb{R}^{n\times n}$ (similarly ν and τ), is a block-diagonal matrix collecting the step-sizes of the scheme, yielding the following expanded form for (8):

$$\begin{cases} \boldsymbol{x}^{k+1} = \mathrm{proj}_\Omega[\mathcal{R}\bar{\boldsymbol{y}}^k - \alpha(F_p(\hat{\boldsymbol{x}}^k) + \mathbf{C}^\top\boldsymbol{\lambda}^k \\ \qquad\qquad + c\mathcal{R}A(\hat{\boldsymbol{x}}^k)L_n\hat{\boldsymbol{x}})], \\ \mathcal{S}\hat{\boldsymbol{x}}^{k+1} = \mathcal{S}\bar{\boldsymbol{y}}^k - \alpha c\mathcal{S}A(\hat{\boldsymbol{x}}^k)L_n\hat{\boldsymbol{x}} \\ \boldsymbol{z}^{k+1} = \bar{\boldsymbol{z}}^k + \nu L_m\boldsymbol{\lambda}^k, \\ \boldsymbol{\lambda}^{k+1} = \bar{\boldsymbol{\lambda}}^k + \tau(-L_m\boldsymbol{\lambda}^k - d + \mathbf{C}\mathcal{R}\hat{\boldsymbol{x}}^k - L_m\boldsymbol{z}^k). \end{cases} \tag{9}$$

In particular, the first two lines of (9) are obtained by premultiplying the first line of the equation obtained expanding (8) by $\mathcal{R}$ and by $\mathcal{S}$, respectively [8, Lemma 1]. The relaxation parameter δ is properly defined in Standing Assumption 6.

IV. Convergence analysis

To show convergence of Algorithm 1 to a GNE, we need to ensure that $\mathcal{A}$ and $\mathcal{B}$ in (7) satisfy some monotonicity properties. Specifically, we need to show that $\mathcal{A}$ is monotone and Lipschitz-continuous and that $\mathcal{B}$ is maximally monotone.

To this end, we introduce a number of results on the properties of the operators composing $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2: Let $G(\hat{\boldsymbol{x}}) = A(\hat{\boldsymbol{x}})L_n\hat{\boldsymbol{x}}$, where A is the susceptibility matrix and L is the Laplacian matrix. Then, $G(\cdot)$ is an $\frac{1}{2d_{\max}}$-cocoercive operator. □

Proof: From the Baillon-Haddad theorem [20, Cor. 18.16], the L is a $\frac{1}{2d_{\max}}$-cocoercive operator, which is equivalent to the $\frac{1}{2}$-averagedness of operator $\frac{1}{2d_{\max}}L$ [20, Rem. 4.24.(iv)]. From [28, Lemma 4], the latter coincides with the following conditions on the spectrum of $\frac{1}{2d_{\max}}L$:

$$\begin{cases} \Lambda\big(\frac{1}{2d_{\max}}L\big) \subset \mathbb{D}_{1/2}, \\ \forall \lambda \in \Lambda\big(\frac{1}{2d_{\max}}L\big) \cap \mathrm{bdry}\left(\mathbb{D}_{1/2}\right),\ \lambda \text{ semi-simple}, \end{cases} \tag{10}$$

where $\mathbb{D}_{1/2} := \left\{ z \in \mathbb{C} \;\middle|\; |z - \frac{1}{2}| \leq \frac{1}{2} \right\}$ denotes the disk of radius $\frac{1}{2}$ centered in $(\frac{1}{2}, 0)$. Since $A(\hat{\boldsymbol{x}})$ is diagonal with entries in $[0,1]$, it performs a positive scaling on the eigenvalues of $\frac{1}{2d_{\max}}L$ that does alter neither their inclusion in $\Lambda(\frac{1}{2d_{\max}}L)$, i.e., if $\lambda \in \Lambda(\frac{1}{2d_{\max}}L)$ then $\bar{\lambda} \in \Lambda(\frac{1}{2d_{\max}}L)$ with $\bar{\lambda}$ obtained through scaling, nor their geometric multiplicity. Thus, also matrix $\frac{1}{2d_{\max}}A(\hat{\boldsymbol{x}})L$ meets the conditions in (10) for all x, and in view of the equivalences above, $\frac{1}{2d_{\max}}A(\hat{\boldsymbol{x}})L$ is an $\frac{1}{2}$-averaged operator, i.e., G is $\frac{1}{2d_{\max}}$-cocoercive. ■

As a consequence of Lemma 2, note that G is ℓ_G-Lipschitz continuous [20]. Next, we prove monotonicity of $\mathcal{A}$ and $\mathcal{B}$:

Lemma 3: Let $\mathcal{A}$ and $\mathcal{B}$ be defined as in (7). We have: i) $\mathcal{A}$ is restricted monotone on $Z = \mathcal{K}_n \times \mathbb{R}^{mN} \times \mathbb{R}^{mN}$; ii) $\mathcal{A}$ is $\ell_\mathcal{A}$-Lipschitz continuous; and iii) $\mathcal{B}$ is maximally monotone.

Proof: Let $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ according to (7) and let us introduce some quantities, adjusted from [8, Lemmas 3, 4]. First, we show that $\mathcal{A}_1$ is β-restricted cocoercive, hence restricted monotone, with constant $\beta \in (0, \min\{\mu/\theta^2, 1/2d_{\max}\})$, where $\mu = s_{\min}(\Upsilon)$, $\theta = \ell_p + 2\kappa c d$ and $\Upsilon = \begin{bmatrix} \frac{\eta}{N} & -\frac{\ell_p+\ell_F}{2\sqrt{N}} \\ -\frac{\ell_p+\ell_F}{2\sqrt{N}} & c\sigma\kappa s_2(L)-\ell_p \end{bmatrix}$, κ and σ chosen as in Standing Assumption 5. The constant c is such that $c > c_{\min}$ and $c_{\min}\sigma\kappa s_2(L) = \frac{(\ell_p+\ell_F)^2}{4\eta} + \ell_p$. The proof follows [8, Lemma 3-4] by showing that $\|\mathcal{R}^\top F_p(\hat{\boldsymbol{x}}) + cA(\hat{\boldsymbol{x}})L_n\hat{\boldsymbol{x}} - (\mathcal{R}^\top F_p(\hat{\boldsymbol{x}}') + cA(\hat{\boldsymbol{x}}')L_n\hat{\boldsymbol{x}}')\| \leq \theta\|\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}'\|$ which leads to $\langle \mathcal{R}^\top F_p(\hat{\boldsymbol{x}}) + cA(\hat{\boldsymbol{x}})L_n\hat{\boldsymbol{x}} - (\mathcal{R}^\top F_p(\hat{\boldsymbol{x}}') + cA(\hat{\boldsymbol{x}}')L_n\hat{\boldsymbol{x}}'), \hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}'\rangle \geq \frac{\mu}{\theta^2}\|\mathcal{R}^\top F_p(\hat{\boldsymbol{x}}) + cA(\hat{\boldsymbol{x}})L_n\hat{\boldsymbol{x}} - \mathcal{R}^\top F_p(\hat{\boldsymbol{x}}') + cA(\hat{\boldsymbol{x}}')L_n\hat{\boldsymbol{x}}'\|$, which together with cocoercivity of $G(\hat{\mathrm{x}})$ (Lemma 2) contribute to $\langle \mathcal{A}_1(\boldsymbol{\omega}) - \mathcal{A}_1(\boldsymbol{\omega}'), \boldsymbol{\omega} - \boldsymbol{\omega}'\rangle \geq \beta\|\mathcal{A}_1(\boldsymbol{\omega}) - \mathcal{A}_1(\boldsymbol{\omega}')\|$. The operator $\mathcal{A}_2$ is skew symmetric, hence monotone, yielding i).

Concerning ii), we use the fact that F_p is ℓ_p-Lipschitz, G is ℓ_G-Lipschitz continuous, L is ℓ_L-Lipschitz continous and $\|\mathcal{R}\| = 1$. Then, it follows that $\mathcal{A}_1$ is $\ell_{\mathcal{A}_1} = \ell_p + \ell_G + \ell_L$ and $\mathcal{A}_2$ is $\ell_{\mathcal{A}_2} = 2\|C\| + 2\ell_L$. Then, $\mathcal{A}$ is $\ell_{\mathcal{A}} = \ell_{\mathcal{A}_1} + \ell_{\mathcal{A}_2}$-Lipschitz continuous.

Finally, iii) holds true by [20, Ex. 20.26] because the normal cone is maximally monotone. ■

By relying on Lemma 3, we finally show that Algorithm 1 converges to a v-GNE. Next, we postulate bounds on the step-sizes and then state the convergence results [18], [19].

Standing Assumption 6: The averaging parameter δ in (8) is such that $\frac{1}{\varphi} \leq \delta \leq 1$ where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The step-size is such that $0 < \left\|\Phi^{-1}\right\| \leq \frac{1}{2\delta(2\ell_{\mathcal{A}}+1)}$ where $\ell_{\mathcal{A}}$ is the Lipschitz constant of $\mathcal{A}$ as in Lemma 3, and such that Standing Assumption 5 is satisfied. □

Theorem 1: The sequence $\{\boldsymbol{\omega}^k\}_{k\in\mathbb{N}}$ generated by (8) converges to $\boldsymbol{\omega}^* \in \mathrm{zer}(\mathcal{A}+\mathcal{B})$. In particular, $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ converges to the v-GNE of the game in (1). □

Proof: Given the properties of $\mathcal{A}$ and $\mathcal{B}$ established in Lemma 3, we can follow the steps of the convergence proofs in [18], [19], where [19, Eq. (37)] holds when $\boldsymbol{\omega}^* \in Z$. ■

V. Illustrative Example

We now test our algorithm on a standard Nash–Cournot game under a partial-decision information regime, as also considered, e.g., in [7], [8]. Specifically, we aim at emphasizing two aspects here: i) confirm our theoretical findings numerically, and ii) investigate the effects the susceptibility to persuasion term may have on the GNE seeking dynamics.

A Nash–Cournot game considers N agents that are involved in the production of a homogeneous commodity, and compete over m markets with capacity constraints, thereby resulting in shared constraints coupling the agents' strategies. Each agent $i \in \mathcal{I}$ decides to produce and deliver $x_i \in [a_i, b_i]^{n_i} \subset \mathbb{R}^{n_i}$ amount of products to the markets it connects with. Each i has a local matrix $C_i \in \mathbb{R}^{m\times n_i}$, with $0/1$ entries specifying which markets it participates in, i.e., the k-th entry of the j-th column of C_i is 1 if i delivers the amount of production associated to the j-th

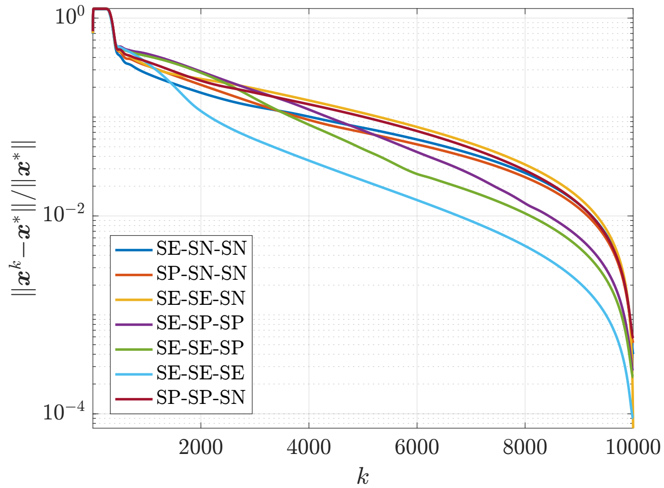

Fig. 1. Average convergence behaviour of Algorithm 1.

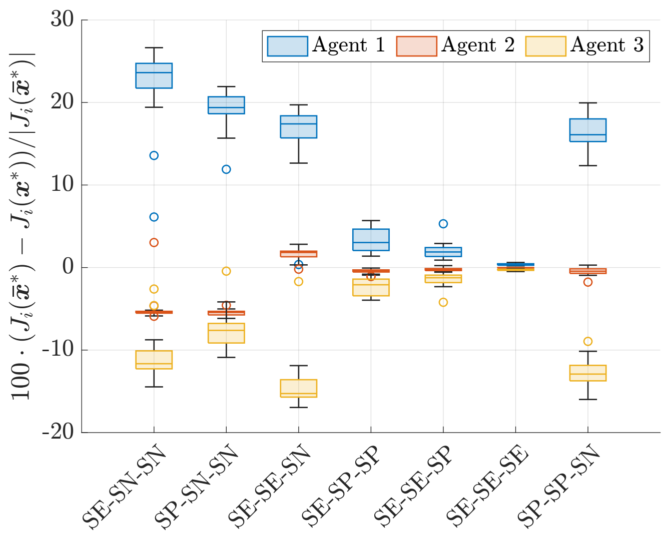

Fig. 2. Average cost variation at equilibrium, for each agent, w.r.t. the "nominal" one, i.e., $\bar{\boldsymbol{x}}^*$, obtained by setting the matrix function $A(\cdot) = I$.

component of x_i to market k. Then, each market k has capacity $d_k > 0$, so that $\sum_{i\in\mathcal{I}} C_i x_i \leq d$, with $d \in \mathbb{R}^m_{>0}$. For the cost function in (1), we use the form in [8]: $J_i(x_i, \boldsymbol{x}_{-i}) = x_i^\top Q_i x_i + q_i^\top x_i - (\bar{p} - \Xi C\boldsymbol{x})^\top C_i x_i$ involving a linear inverse demand with $\bar{p} \in \mathbb{R}^m_{>0}$ and diagonal $\Xi \in \mathbb{R}^{m\times m}_{>0}$.

For illustrative purposes, we set $N = 3$, $m = 3$, with $n_i = 3$, $C_i = I_3$, $a_i = 0$, and $b_i = 1$, for all $i \in \mathcal{I}$. The maximal capacity of the markets $d \in \mathbb{R}^3_{>0}$ are uniformly drawn from $(0.85, 1.5)$. To define each J_i, we have preliminary set $Q_i = 4 \cdot I_3$, while each q_i is uniformly drawn from $(1, 2)$, whereas $\bar{p}$ from $(10, 20)$, and the diagonal entries of Ξ from $(1, 3)$. To consider a monotone instance featuring multiple equilibria, in view of the quadratic structure characterizing the game mapping $F(\boldsymbol{x}) = Q\boldsymbol{x} + h$, where $Q \in \mathbb{R}^{n\times n}$ and $h \in \mathbb{R}^n$ are obtained by rearranging the terms of the stack of the pseudogradients, we have successively subtracted $\lambda_{\min}(Q)I_9$ to Q, and then re-adjusted the parameters of each cost function J_i, accordingly. This example shows that Algorithm 1 returns a v-GNE (verified ex-post through a round of best-responses) despite lacking of strong monotonicity. The latter assumption, however, cannot be weakened [7].

Once chosen $\delta = 0.75$, $c = 1$, $\alpha_i = 0.03$, $\nu_i = 0.5$, and $\tau_i = 0.25$, for all $i \in \mathcal{I}$, Algorithm 1 has been tested by considering several combinations for the matrix function $A(\cdot)$ as in §III-A, i.e., $\left(I - \mathrm{diag}(\cdot)^2\right)$ models SE agents, $\frac{1}{2}\left(I - \mathrm{diag}(\cdot)\right)$ SP, and $\mathrm{diag}(\cdot)^2$ SN ones. For instance, the first combination "SE-SN-SN" in Fig. 1 (or Fig. 2) means that agent 1 is an SE, while agents 2 and 3 are SNs. The numerical results have then been obtained by averaging over 20 runs of Algorithm 1 with random initial conditions.

In Fig. 1 we show convergence of Algorithm 1 in all the combinations considered for the matrix function $A(\cdot)$. In particular, it seems that the presence of SE agents allows one to achieve a faster convergence to a v-GNE of the game.

In Fig. 2, instead, we illustrate the effect the agents' attitude has on the computed GNE. Here, we compare the cost at equilibrium obtained in a specific configuration with the "nominal" one, $\bar{\boldsymbol{x}}^*$, computed by considering the same (random) initial condition with $A(\cdot) = I$. Our simulations confirm what one would intuitively expect: when SE agents are mixed with other type of behaviours, they can strongly drive the v-GNE computation, thereby achieving noticeable performance improvements (in term of cost minimization). When all the agents are SEs, instead, the cost variation is negligible for anyone. While SN agents appear irrelevant to the v-GNE computation, and thus always obtain a performance deterioration, SP agents can finally experience mixed results, depending on the configuration considered.

VI. Conclusion

We have proposed a novel, distributed GNE seeking algorithm for GNEPs in a partial-decision information regime, which is typically addressed by incorporating a linear averaging term in the iterative scheme. In contrast, we have considered a scenario in which biases affect the way the agents process information, and we modelled them by incorporating a state-dependent susceptibility term in the averaging process. As a main result, we have proven that the proposed GNE seeking scheme converges to a v-GNE, even in case one considers such a nonlinear averaging process.

Our results offer several insights for future investigations. From our numerical results, indeed, it seems that the obtained equilibrium can be "shaped" by the state-dependent susceptibility term. Providing an explicit characterization of such a dependence is a key objective for future research. Besides considering different available GNE seeking schemes attached to the nonlinear dynamics for the estimates of $\boldsymbol{x}$, one could also investigate how the susceptibility to persuasion terms affect the estimate of the dual variable λ, thereby complicating the analysis of the operator $\mathcal{A}$ in (7). Finally, testing the proposed methodology on real-world applications would definitely shed further light on the results achievable through the flexibility offered by our technique.

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