Exploring Algebraic Preconditioning of EFIE Matrices Arising from Higher Order Additive Singular Bases

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Bruno Carpentieri, Roberto D. Graglia, *IEEE, Life Fellow*, Paolo Petrini, and Maurizio Tavelli

***Abstract*—Currently there are no operator-dependent preconditioners (for example of the Calderón type) to handle matrices obtained using high-order singular vector bases. So this letter has a dual purpose. The first is to show that the electric field integral equation (EFIE) discretized by high-order singular bases can be quickly solved iteratively using special general-purpose algebraic preconditioners. The second is to demonstrate that the results obtained with the fast solver have the same accuracy as those obtained using classical direct solution methods. The algebraic preconditioner specifically considered here has been used elsewhere to efficiently solve problems with several million unknowns. Thus, in light of the dual purpose and without loss of generality, we use as benchmarks medium-sized test problems involving singular induced currents because, for these problems, the preconditioned solutions can be compared with those obtained by direct methods, which are notoriously unsuitable for solving very large, ill-conditioned problems. In particular, to demonstrate that our approach correctly models the singular behavior of fields in the near-field region, we report several numerical results for current components induced by plane waves on infinitely thin flat plates. On the edges of these plates the current component parallel to the edges can be unlimited (i.e. going to infinity), while the component normal to the edges must vanish. This behavior is correctly modeled by our singular bases when necessary and is not corrupted by the fast solver, which demonstrates the effectiveness and robustness of the singular bases and the preconditioner used.**

***Index Terms*—Integral equations, method of moments, higher order vector elements, preconditioners, fast solvers.**

I. Introduction

High-order additive singular bases for the electromagnetic (EM) analysis of three-dimensional metallic structures with the method of moments (MoM) improve the precision and convergence of the numerical solution, in particular when the induced current is unbounded near edges or corners [1]- [7]. In addition to reducing the number of unknowns needed to achieve a given accuracy, high-order singular basis functions eliminate the need for mesh refinement to estimate error, since this can be done by increasing the order of the used base; more importantly, they allow the use of p-adaptive techniques that increase the base's order locally, where suggested by the error estimation code. Nevertheless, when studying large structures, the computation time needed to solve the system of equations in the traditional way is usually greater than the computation time needed to fill the system matrix, even if parallel computers are used (let's say it is sufficient to have more than about 10,000 unknowns). In these cases, whatever the base used, it is necessary to resort to rapid iterative solution techniques, which require appropriate preconditioners. There are quite well-established operator-dependent preconditioning techniques suitable for using *polynomial* bases, such as the Calderón preconditioner for RWG bases [8]- [14]; however, these do not seem to be usable for bases involving singular functions or polynomial (i.e., non-singular) bases of locally varying order. In this regard, when studying structures with sharp edges and corners, the only alternative to using much denser meshes near the edges and basis functions of the lowest possible order, such as rooftop and RWG functions, is to introduce singular additive basis functions along the edges, which does not require the use of dense meshes and not even mesh refinement [1]. The lack of results on preconditioners for EFIE systems obtained with singular basis functions is also due to the available EM fast-solvers that typically use low-order basis functions (e.g. RWG basis functions) despite requiring a very large number of cells and extensive use of parallel computers to study non-smooth large bodies.

For these reasons this paper considers *purely algebraic* preconditioners such as the one summarized in Section II below, to show that the quality of the results obtained using singular basis functions does not deteriorate due to the preconditioner and/or the iterative solution methods adopted. This study is logically preparatory to the parallelization (still in progress) of the code that calculates the coefficients of the system matrix, which are very complicated MoM integrals with singular kernel, singular basis functions and singular testing functions (as we use Galerkin's testing method) [5]- [7]. Also, this paper does not present a *"new"* preconditioner, because the algebraic preconditioner specifically considered in Section II has been shown elsewhere to be able to efficiently handle problems with several million unknowns [15]–[18]. We could as well consider other algebraic preconditioners [19]–

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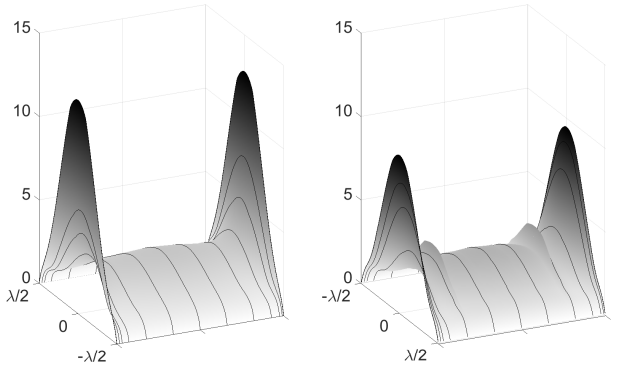

Fig. 1. $1\lambda \times 1\lambda$ square plate on the $z = 0$ plane. Magnitude of the (copolar) current component J_x induced by a normally incident plane wave with $H_y = 1$ A/m. The quality of the results at left obtained with the singular base of order $p = 2, s = 1, m = 0$ is definitely superior to that obtained with the regular base of order $p = 2$ (at right). Note that the higher quality of the results is achieved with a relatively small increase in the number of unknowns [5]. In fact, the number of degrees of freedom is 1,740 and 1,816 for the regular and singular base, respectively.

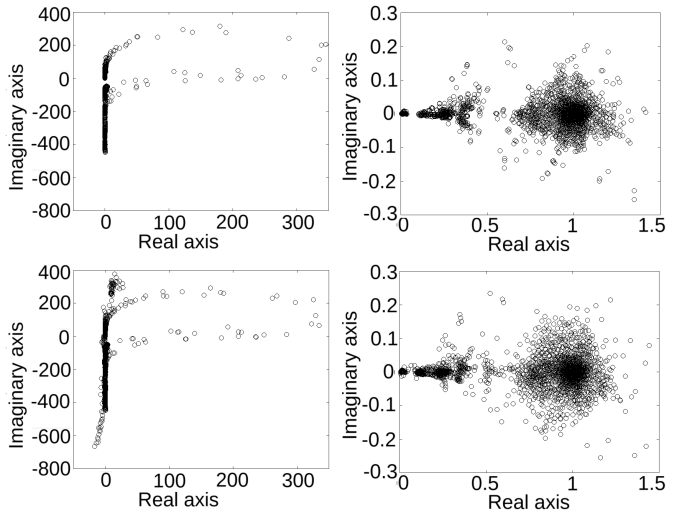

Fig. 2. Spectra of the MoM matrix before (at left) and after preconditioning (at right) for the case illustrated in Fig. 1. The top figures are obtained using a purely polynomial base, the bottom ones with a singular base. Note that the scales used in the right column are dramatically different from the left.

[22], which we do not do here for brevity. That said, Section III presents some results for test problems involving sharp-edged structures, focusing on medium-sized problems because direct solution methods, whose results we take as benchmarks, are unsuitable for solving very large and ill-conditioned problems. Preliminary results of this work have been presented in [23].

II. Matrix Fast-Solver

We solve the dense linear system arising from MoM discretization of the pertinent integral equation using the generalized minimum residual (GMRES) iterative Krylov subspace solver [24], [25]. A Krylov method solves an $n \times n$ matrix problem in N_{iter} $\mathcal{O}$(M-V) flops per right-hand side, where M-V indicates the matrix-vector product operation, $\mathcal{O}$(M-V) represents the computational cost of the M-V product operation, and N_{iter} is the number of iterations required to achieve a certain solution accuracy, depending on the characteristics of the integral formulation and on the problem geometry. Combined with fast algorithms for the M-V multiplication, such as MLFMA [26]–[28], panel clustering [29] and $\mathcal{H}$-matrices [30]–[32], iterative solvers can overcome the memory constraints of direct methods for solving dense linear systems.

The number of iterations N_{iter} can be significantly reduced by *preconditioning*, which transforms the initial linear system

$$Ax = b \tag{1}$$

into an equivalent one with a well clustered spectrum around point one of the complex plane and guarantees rapid convergence [33], [34]. The preconditioned system has the form

$$MAx = Mb \tag{2}$$

or

$$A\,My = b, \text{ with } x = My \tag{3}$$

depending on whether preconditioning from the left or the right. (We usually do preconditioning using (3)). Preconditioning plays a crucial role in the iterative solution process, particularly for certain integral formulations like EFIE, in which the number of iterations required by Krylov solvers can scale as $\mathcal{O}(n^{0.5})$. The development of efficient, inherently parallel and scalable preconditioners for boundary integral equations is an important research topic; see, e.g., [35] and references therein.

We consider purely algebraic preconditioning methods that compute matrix M only using the coefficient matrix of system (1). Algebraic preconditioners can be applied to various integral operators and complex geometries by changing a few parameters, although they may not be the optimal method for every case. We compute the preconditioner as the sparse matrix M that minimizes the Frobenius-norm of the error matrix

$$\|I - \widehat{A}M\|_F^2 = \sum_{j=1}^{n} \|e_j - \widehat{A}m_{\bullet j}\|_2^2, \tag{4}$$

where $\widehat{A}$ is a sparse approximation of A which incorporates the most relevant contribution to the singular integrals, e_j is the jth vector of the standard basis of $\mathbb{C}^n$ and $m_{\bullet j}$ is the column vector representing the jth column of M. A similar relation can be established when left preconditioning is used [15]. In this study, the sparsity pattern of $\widehat{A}$ is predetermined by selecting the k largest entries in each column of A [36]. The results in the next section were obtained with a system equipped with an Intel i9 12900ks and an NVIDIA RTX A4000.

III. Numerical Experiments

First, let us consider the results for the square plate of size $(1\lambda \times 1\lambda)$ already studied in [5]. By solving this problem with singular and regular bases (this time with 10×10 square cells) we obtain the results of Fig. 1. Observing the spectra of the MoM matrices, we notice that eigenvalues with a negative real part appear when singular additive basis functions are used (Fig. 2 at bottom left), while these are not

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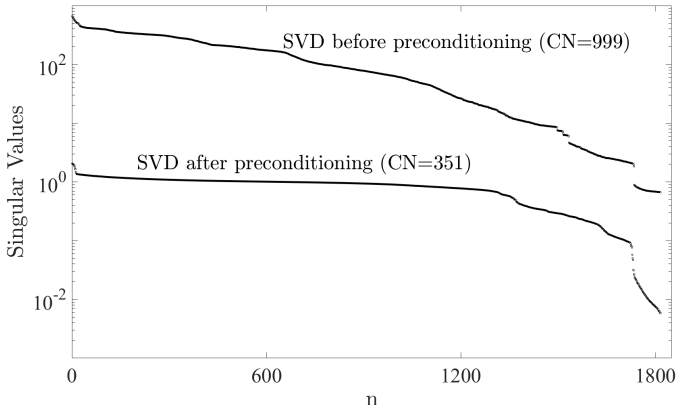

Fig. 3. SVD values before and after preconditioning for the square plate of Fig. 1 obtained using the singular base [2,1,0] and 1.816 DOFs. The condition number CN is the ratio of the maximum to minimum singular value.

there when purely polynomial bases are used (Fig. 2 at top left). Since the preconditioners try to cluster the eigenvalues near the (1,0) point of the complex plane, it is possible that some get trapped near the origin. This is more likely to happen if there are eigenvalues with a large negative real part and, as said, the EFIE system matrix has eigenvalues with a negative real part when using singular basis functions. Furthermore, eigenvalues close to 0, or scattered, require a large number of iterations. These problems are solved by using the preconditioner proposed above in Section II, as shown in Fig. 2 at right.

Fig. 3 shows the SVD values before and after preconditioning obtained by studying the test case of Fig. 1. In this experiment, the preconditioner M exhibits a degree of sparsity that is below 3% in comparison to A. Using the GMRES method, the initial residual norm is reduced by six orders of magnitude after 74 iterations when using the regular base, while 108 are needed when using the singular base. Convergence worsens when using the singular base because it leads to a more scattered spectrum than that obtained using the regular base, as can be seen in Fig. 2.

Having clarified once again the greater accuracy obtainable using additive singular bases (*cfr.* [1]- [6]), we now consider in Fig. 4 three other test cases relating to rectangular plates of larger dimensions, solved with the bases of order $[p, s, m]$ discussed in [5]. For these and other cases, Table I reports the size in wavelengths of the plate, the base's order, the size of the system matrix A, and the percentage of singular basis functions over the total number of basis functions. All matrices are represented using double precision complex values. In simulations involving dense coefficient matrices, constructing a preconditioner directly from the full matrix A is computationally expensive, especially as the problem size increases. To mitigate this, as suggested in [36], a sparse approximation $\widehat{A}$ of the A matrix can be used, significantly reducing memory usage and execution time. While this approach may degrade the preconditioner's quality in general cases, in boundary element method (BEM) problems, even sparse approximations often preserve the most relevant contributions

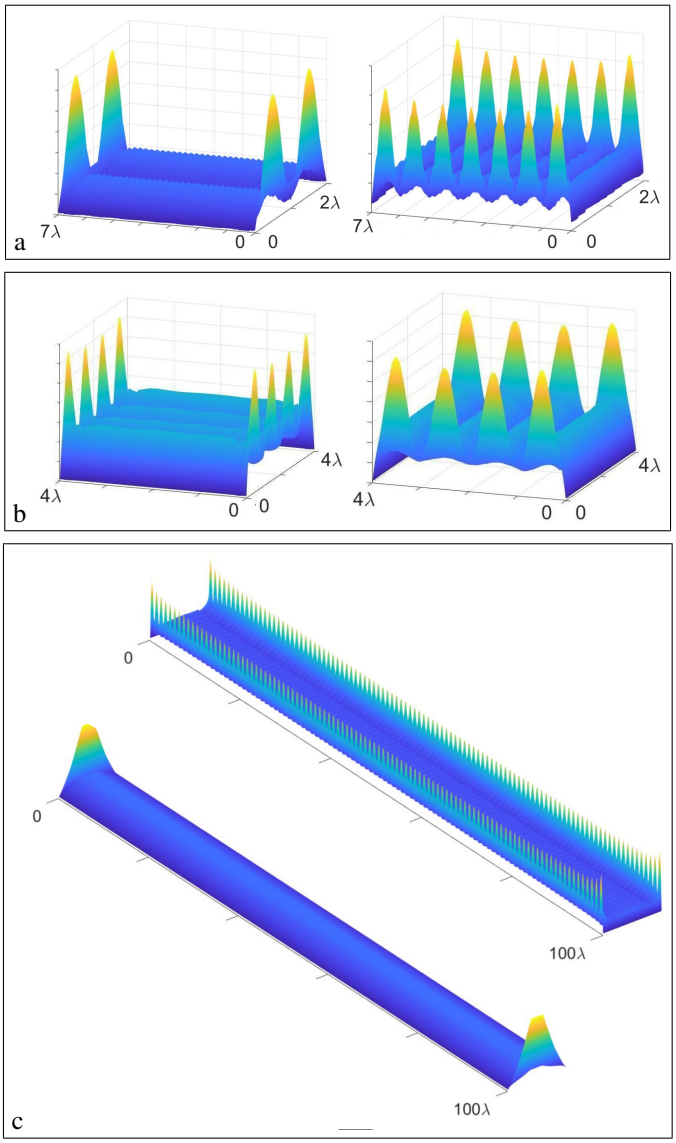

Fig. 4. Magnitude of the copolar current component induced on rectangular plates of different dimensions by a plane wave normally incident on the plate with magnetic field $H = 1$ A/m. The results for a plate of size ($7\lambda \times 2\lambda$), ($4\lambda \times 4\lambda$), and ($100\lambda \times 1\lambda$) are shown in subfigures (a), (b) and (c), respectively. The results on the left of these subfigures are for the incident magnetic field H parallel to the long side of the plate, those on the right for H parallel to the short side. Figure a, b and c, were obtained with bases of order [2,1,0], [0,1,0] and [0,1,0], respectively (see also Table I).

to singular integrals, ensuring effectiveness. The performance of the preconditioner depends on balancing the sparsity of both $\widehat{A}$ and M. Increasing the density of M improves convergence but incurs a nearly cubic increase in computational cost. A more efficient strategy is to allow the sparse approximation $\widehat{A}$ to have more non-zero entries than the preconditioner, which enhances the preconditioner's effectiveness without significantly increasing construction time. This is supported by the quadratic complexity of the QR factorization required to solve the least-squares problems involved in constructing M. Table I shows the sparsity ratios used for $\widehat{A}$ and M. Our experiments demonstrate that increasing the density of $\widehat{A}$ for a fixed preconditioner density accelerates convergence,

TABLE I
BENCHMARK PROBLEMS SOLVED USING A GPU-PARALLEL IMPLEMENTATION OF THE BLOCK VERSION OF GMRES (BGMRES) [39]

Cells' size	Base order $[p,s,m]$	Plate size	Matrix size	Singular DOFs	Blocks	Streams	Density of $\widehat{A}$ 10%		Density of $\widehat{A}$ 15%	
							M 2.5%	M 5%	M 5%	M 10%
$\lambda/5$	$[0,0,0]$	$8\lambda \times 8\lambda$	3,120	0%	5	5	85 (0.8s)	78 (0.8s)	75 (1.0s)	66 (0.7s)
		$10\lambda \times 10\lambda$	4,900	0%	5	5	84 (1.9s)	73 (1.6s)	71 (1.8s)	69 (1.6s)
		$100\lambda \times 1\lambda$	4,495	0%	5	5	118 (2.3s)	82 (1.5s)	82 (1.6s)	58 (1.1s)
	$[0,1,0]$	$8\lambda \times 8\lambda$	3,436	9.20%	4	4	88 (1.0s)	79 (0.9s)	76 (1.1s)	72 (0.9s)
		$10\lambda \times 10\lambda$	5,296	7.48%	4	4	88 (2.3s)	79 (2.1s)	77 (2.3s)	78 (2.4s)
		$100\lambda \times 1\lambda$	6,511	30.96%	17	17	142 (5.5s)	102 (3.8s)	102 (4.0s)	79 (3.0s)
	$[2,1,0]$	$7\lambda \times 2\lambda$	6.341	2.8%	17	17	366 (14.3s)	312 (12.6s)	113 (4.0)	109 (4.1s)
$\lambda/20$	$[0,1,0]$	$4\lambda \times 4\lambda$	13,276	4.8%	4	4	91 (14.3s)	77 (11.9s)	75 (12.0s)	66 (10.6s)

The columns on the right report the number of iterations and, in brackets, the GPU time spent to solve the linear system (1) using the CUDA version of the BGMRES solver to reduce the initial residual norm by six orders of magnitude, with restart parameter equal to 200, *maxiter* = 2000 and final tolerance $\epsilon = 10^{-10}$. The number of blocks and streams is set as in the Table. The data are obtained with matrices $\widehat{A}$ and M of different densities. (The density is the fraction of non-zero elements in percentage value.) The number of unknowns and the size of the linear system (i.e., the size of A, $\widehat{A}$, and M) is given row by row in the 4th column of the Table. It depends on the size in wavelengths of the square cells used to mesh the plates (shown in the 1st column on the left), the order $[p,s,m]$ of the base used (2nd column), and the size in wavelengths of the plate (3rd column). The 5th column reports the percentage of singular basis functions used; i.e., the percentage of singular DOFs. The latter value increases as the perimeter in wavelengths of the flat plate increases and in fact, for the data in the Table, the maximum number of singular DOFs is reached for the rectangular plate $100\lambda \times 1\lambda$ studied with the base of order $[p,s,m] = [0,1,0]$ (recall that the bases with $s = 0$ do not contain singular basis functions). Despite the tests involve relatively small-sized problems, it is clear that the solution times obtained using the preconditioner are shorter than those required by the direct solution method. For example, five times shorter for the case with 13,276 unknowns, for which the direct solution takes 74.98 seconds (matrix filling time excluded). The values obtained with the base of order [0,0,0] are given only for comparison; in fact, with such a loose mesh, the error on the induced current obtained with the base of order [0,0,0] is too high.

while maintaining a sparse preconditioner significantly reduces overall CPU time. In these numerical experiments, the linear systems are solved with a parallel implementation of the block GMRES (BGMRES) algorithm [37] such as presented in [38], [39]. For the GPU algorithm, the number of streams and blocks is set as indicated in Table I. In all these experiments we solve simultaneously the problem for the x and the y component of (3) using the multi-RHS strategy, i.e. $N_b = 2$, we set the restart of the BGMRES solver to 200 and a final tolerance of $\epsilon = 10^{-10}$.

Finally, it is interesting to note from Table I how the solution time when using the preconditioner is five times less than that required by the direct solution method, despite using only 13,276 unknowns. Furthermore, as can be seen for example from Fig. 5 relative to the $4\lambda \times 4\lambda$ plate, without preconditioning the iterative method does not achieve the required precision (i.e. it does not converge).

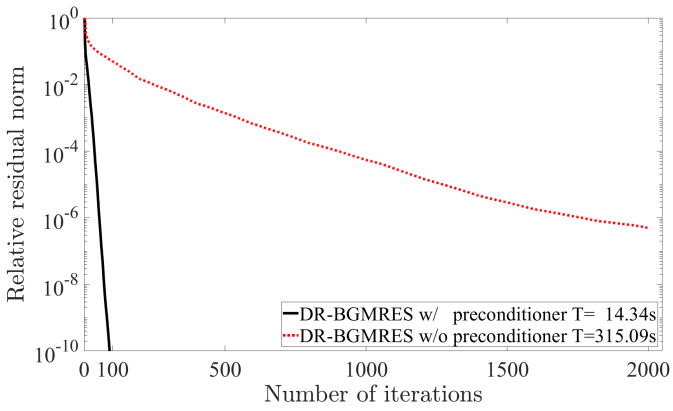

Fig. 5. Convergence history for solving the $(4\lambda \times 4\lambda)$ plate problem of Fig. 4b with 13,276 DOFs, and densities of $\widehat{A}$ and M of 10% and 2.5%, respectively.

IV. CONCLUSION

Calderón-type preconditioners are not able to handle problems using singular bases. This letter shows that the electric field integral equation discretized by high-order singular bases can be quickly solved iteratively using a special general-purpose algebraic preconditioner. The results thus obtained have the same accuracy as those obtained using classical direct solution methods, which are unsuitable for solving large problems. To demonstrate that our numerical approach does not influence the correct behavior in the near-field region, the letter reports numerical results for the current induced by plane waves on infinitely thin flat plates. In fact, the current density induced on infinitely thin plates can be unbounded (tend to infinity) at the edges of the plates and should be modeled with singular basis functions. It has been seen that in regions where the current is unbounded the relative distance (i.e. the relative difference) between the numerical solutions obtained with the fast solver and those obtained with the traditional solvers is always less than 10^{-8}. Similarly, in the regions of bounded currents, the difference between the solutions is always less than 10^{-8} in absolute value. The code used here for computing MoM matrices is being parallelized. Once that is done, we will use higher-order singular bases to tackle larger problems, explore other preconditioners, and combine everything into fast integral solvers.

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