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Some developments in balanced and SKT Geometry

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Abstract

This thesis is focused on the study of curvature properties of balanced and SKT metrics.

More in details, as regards balanced metrics, we use gluing techniques to show that the blow-up of a compact Chern-Ricci flat orbifold at a finite number of smooth points admits constant Chern scalar curvature balanced metrics, even obtaining a control on the balanced class of the constructed metrics.

In the case of SKT metrics, we proceed in a systematic study of the generalized Ricci flow with symmetries, adapting the bracket flow technique by Lauret in the context of generalized Geometry. This allows us to prove long-time existence of the homogeneous generalized Ricci flow on any solvmanifold. Using then the equivalence between said flow and the pluriclosed flow, we are able to deduce long-time existence of the pluriclosed flow on any SKT solvmanifold.

Finally, we focus our attention on hyperHermitian Geometry. In this setting, we prove an incompatibility result between strong HKT metrics and balanced hyperHermitian ones, providing an evidence of the validity of the Fino-Vezzoni conjecture in the hyperHermitian setting.

Contents

Introduction

Motivated by theoretical physics and by the advances in the understanding of the connection between *canonical* Kähler metrics and *stability* conditions, the interest in the study of special non-Kähler metrics has grown enormously throughout the last decades.

Since the Kähler condition is a strong condition per se, imposing many restrictions, for instance, on the topology of the manifold, the rise of special Kähler metrics happened naturally. The main idea is to combine the Kähler condition with some other conditions on the curvature of the metric. Among the latter, the most famous and well-understood is the Einstein condition, giving rise to Kähler-Einstein metrics. The existence problem for Kähler-Einstein metrics, starting from the works by Aubin [34] and Yau [338], was concluded recently in the works by Chen, Donaldson and Sun, see [74, 75, 76], characterizing the existence of such metrics in the Fano case in terms of K-polystability of the anti-canonical bundle.

On the other hand, Kähler-Einstein metrics are a particular instance of the so-called *extremal Kähler metrics* introduced by Calabi in [67], defined as critical points of the *Calabi functional.* The latter is the functional associating to any Kähler metric in a fixed Kähler class the L^2-norm of the scalar curvature. As a matter of fact, extremal Kähler metrics are those Kähler metrics whose scalar curvature has holomorphic $(1,0)$-part of the gradient. Stemming from this characterization, *constant scalar curvature Kähler metrics*, or *cscK*, attracted the attention of many authors and they are now regarded as canonical, being minimizers of the Calabi functional. The importance of cscK metrics has then increased due to the moment map formulation of the scalar curvature by Fujiki and Donaldson, see [137, 90], and to the Yau, Tian, Donaldson conjecture, see [91, 316, 339] and [293] for the proof of one implication. Great effort was put to construct examples of extremal Kähler metrics throughout the last years, we refer to [31, 88, 108, 109, 311, 313, 340] for some constructions.

On the other hand, in the last decades, many special geometries, such as Sasakian or G_2 Geometry, turned out to be relevant in the study of string theory and black holes. Focusing our attention on the complex side, non-Kähler models were proposed by many theoretical physicists, highlighting and promoting the need for a better mathematical understanding of non-Kähler Geometry. A remarkable example is the Hull-Strominger system, see [190, 308], originally arising in heterotic string theory, which has nowadays attracted great attention, mainly due to the belief that its solutions may geometrize the non-Kähler Calabi-Yau condition. We refer to [143] for an introductory account of the system and to [15, 20, 21, 66, 83, 102, 103, 104, 106, 107, 114, 134, 136, 135, 145, 147, 148, 252, 261, 262, 263, 264, 265] for explicit solutions and properties of the system.

Other interesting non-Kähler geometries arise from the study of $(2,0)$ and $(4,0)$-supersymmetric sigma models with Wess-Zumino term, see [150, 189], giving rise to the so-called *Kähler with torsion* and *hyperKähler with torsion* geometries. Other physical situations where these geometries appeared can be found in [101, 162, 176, 180, 181, 241].

Besides their physical interest, both the examples are highlighting an important difference, from the mathematical point of view, between Kähler and non-Kähler Geometry. This discrepancy is the presence of multiple connections preserving the Hermitian structure. In particular, the *anomaly cancellation equation* in the Hull-Strominger system involves the curvature of the *Chern connection* associated to the metric, namely the unique Hermitian connection whose torsion has vanishing $(1,1)$-part. On the other hand, targets of $(2,0)$ and $(4,0)$-supersymmetric sigma models with torsion are naturally endowed with a connection preserving the Hermitian structure with totally skew-symmetric torsion, which, nowadays, is known as the *Strominger* or *Bismut connection*, see [52, 308].

Since they have non-vanishing torsion, these two connections identify naturally two types of special Hermitian metrics by imposing conditions on their torsion. More in details, *balanced* metrics are those Hermitian metrics whose Chern connection has traceless torsion. On the other side, *Strong Kähler with Torsion*, or *SKT* for short, metrics are those Hermitian metrics whose Bismut connection has d-closed torsion. Remarkably, these two conditions on the torsion can be equivalently written as cohomological conditions on the fundamental form associated to the Hermitian metric. More precisely, given (M^n, J, ω) a Hermitian manifold, the metric ω is balanced if and only if

$$d\omega^{n-1} = 0$$

while ω is SKT if and only if

$$\partial\bar{\partial}\omega = 0\,.$$

Furthermore, hyperKähler with torsion, or *HKT* for short, metrics sit in the stricter framework of hyperHermitian Geometry. More specifically, given (M, I, J, K, g) a hyperHermitian manifold, i.e. I, J, K are integrable complex structures such that

$$K = IJ = -JI$$

and g is Hermitian with respect to I and J, g is called HKT if the Bismut connections associated to (I, g), (J, g) and (K, g) coincide. Equivalently, this condition can be rephrased as

$$\partial\Omega = 0\,, \qquad \Omega := \frac{\omega_J + \sqrt{-1}\omega_K}{2}\,, \quad \omega_J(\cdot,\cdot) := g(J\cdot,\cdot)\,,\ \omega_K(\cdot,\cdot) := g(K\cdot,\cdot)\,,$$

where ∂ is induced by the splitting of d with respect to I. Moreover, we will say that a HKT metric g is strong HKT if ω_I is SKT.

It is now fairly easy to see that both the balanced and the SKT condition are weaker than the Kähler one while the HKT condition is straightforwardly generalizing the hyperKähler one, which is equivalent to $d\Omega = 0$. Moreover, as in the Kähler case, both balanced, SKT and HKT metrics define natural cohomology classes in appropriate cohomology rings. For instance, the *balanced class* of a balanced metric ω is defined as:

$$[\omega^{n-1}]_{\mathrm{BC}} \in H^{n-1,n-1}_{\mathrm{BC}}(M) := \frac{\ker\partial \cap \ker\bar{\partial} \cap \Lambda^{n-1,n-1}M}{\operatorname{Im}\partial\bar{\partial} \cap \Lambda^{n-1,n-1}M}\,,$$

while the *HKT class* of a HKT metric Ω is

$$[\Omega]_{\mathrm{qBC}} \in H^{2,0}_{\mathrm{qBC}}(M) := \frac{\ker\partial \cap \ker\partial_J \cap \Lambda^{2,0}_I M}{\operatorname{Im}\partial\partial_J \cap \Lambda^{2,0}_I M}\,,$$

where $\partial_J := J^{-1}\bar{\partial}J$ and $\Lambda^{2,0}_I M$ is the set of $(2,0)$-forms with respect to I.

Besides being the building blocks for some physical models, balanced and SKT metrics have also attracted great attention due to the so-called Fino-Vezzoni conjecture, see [125, Problem 3]. In [14], it was showed that a Hermitian metric cannot be both balanced and SKT, unless Kähler. The Fino-Vezzoni conjecture is a natural generalization of the aforementioned result, stating that the co-existence of balanced and SKT metrics forces the manifold to be Kähler.

Conjecture A. Let (M, J) be a compact complex manifold admitting a balanced and a SKT metric. Then, M is Kähler.

Evidences of the validity of Conjecture A can be found in the works of many authors, see [70, 79, 80, 113, 119, 125, 126, 128, 133, 167, 234, 266]. A counterexample in the non-compact case was found by Freibert and Swann in [128]. Even if Conjecture A was proved to hold in many classes of manifolds, the proofs of most of the results are investigating structural restrictions on the manifold imposed by the existence of balanced and SKT metrics, for instance conditions on the structural equations of the Lie algebra or the existence of specific currents, ultimately obstructing the co-existence of them, in the non-Kähler case.

The *fil rouge* of the present thesis is the study of Conjecture A from a more curvature-based point of view, trying to extrapolate information on the manifold from the study of canonical balanced or SKT metrics.

From a more analytic perspective, the nature itself of the balanced and SKT conditions highlights a substantial difference in the approaches adapted to study curvature conditions on such metrics. While in the Kähler case, the elliptic approach, with the study of *Monge-Ampère equations* and the cscK equation, and the parabolic one, with the *Kähler-Ricci flow* and the *Calabi flow*, see [67, 77], are intimately related and proved to be both successful, in the non Kähler setting, these two approaches turned out to be more well-suited when used on particular types of Hermitian metrics.

Focusing on the elliptic approach, balanced metrics are the main objects in the so-called *balanced Gauduchon conjecture*, see [314, 320], stating that on a compact balanced manifold the first Chern-Ricci form can always be prescribed within the balanced class of any balanced metric. Later, in [130], the conjecture was equivalently reformulated in terms of the so-called *Form-Type Calabi-Yau equation*, which is a Monge-Ampère type equation for $(n-1,n-1)$-forms. Moreover, the latter equation has the natural purpose to produce, on suitable compact manifolds, balanced Chern-Ricci flat metrics within any balanced class, promoting balanced Chern-Ricci flat metrics as canonical representatives of the balanced class, as Kähler Ricci-flat metrics are in the Kähler Calabi-Yau case.

On the other hand, due to a result by Angella, Calamai and Spotti in [24], we cannot impose a balanced metric to be first Chern-Einstein with non-zero Chern scalar curvature in a non-Kähler manifold. Motivated by this and following the Kähler case, one may wonder if balanced constant Chern scalar curvature metrics might be good candidates to be canonical representatives of a fixed balanced class.

One of the main theorems of this thesis proceeds in this direction. An outstanding result by Arezzo and Pacard in [31] ensures that the blow-up in a finite number of smooth points of a cscK orbifold with isolated singularities and no non-trivial holomorphic vector fields admits cscK metrics in Kähler classes which make the volume of the exceptional divisors small. The following theorem is a generalization of the result by Arezzo and Pacard in the balanced Chern-Ricci flat case.

Theorem B. Let $(M^n, \tilde{\omega})$ be a compact balanced Chern-Ricci flat manifold or orbifold with isolated singularities. Then, given $p_1, \ldots, p_k \in M$ and $a_1, \ldots, a_k > 0$, there exists $\varepsilon_0 > 0$ such that the blow-up of M at $p_1, \ldots, p_k$ admits a balanced negative constant Chern scalar curvature metric

$$\omega_\varepsilon^{n-1} \in \pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} - \varepsilon^{2n-2} \sum_{i=1}^{k} a_i^{n-1} [E_i]_{\mathrm{BC}}^{n-1} ,$$

where $[E_i]_{\mathrm{BC}}$ is the first Bott-Chern class of the line bundle associated to the exceptional divisor E_i of the blow-up at p_i and $\varepsilon \in (0, \varepsilon_0)$.

Regarding the parabolic approach, a flow of balanced metrics was proposed by Bedulli and Vezzoni in [48], generalizing the Calabi flow. As far as the author knows, very few results are known concerning this flow. In [48], the authors proved short-time existence and studied the behaviour of the flow on the Iwasawa manifold. Later, the flow was studied by Fino and Paradiso in [119] on 6-dimensional almost abelian solvmanifolds. Analytically speaking, the flow introduced by Bedulli and Vezzoni is of fourth order which makes it really hard to work with, due to a poor understanding of fourth order elliptic equations.

On the other hand, the parabolic approach turned out to be successful for SKT metrics. Streets and Tian in [302] introduced the *pluriclosed flow* as the following evolution equation:

$$\frac{\partial}{\partial t} g = -S + Q , \quad g(0) = g_0 ,$$

where S is the *second Chern-Ricci tensor* while Q is an appropriate quadratic expression in the torsion of the Chern connection. It was proved in [302] that the pluriclosed flow preserves the SKT condition and it can be rewritten as an evolution equation of $(1,1)$-forms as follows:

$$\frac{\partial}{\partial t}\omega = -(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}\,, \quad \omega(0) = \omega_0\,,$$

where $\mathrm{Ric}^{\mathrm{B}}(\omega)$ is the *Bismut-Ricci form* of ω, which closely resembles the structure of the Kähler-Ricci flow. Besides not having the same good analytic properties of the Kähler-Ricci flow, such as the reduction to a flow of potentials, many results concerning the pluriclosed flow are known, see [32, 33, 37, 38, 54, 99, 111, 118, 122, 125, 146, 198, 272, 294, 295, 296, 297, 298, 299, 300, 304, 305, 307, 341, 342]. Among these, one of the most important is the gauge-equivalence of the pluriclosed flow with the *generalized Ricci flow*, which is the following coupled system of evolution equations:

$$\begin{cases} \frac{\partial}{\partial t}g = -2\mathrm{Ric}(g) + \frac{1}{2}H^2 & g(0) = g_0\,, \\ \frac{\partial}{\partial t}H = \Delta_g H & H(0) = H_0\,, \end{cases}$$

where H_0 is a closed 3-form, $\mathrm{Ric}(g)$ is the classical Ricci tensor and the symmetric $(2,0)$-tensor H^2 is defined by:

$$H^2(X,Y) := g(\iota_X H, \iota_Y H)\,, \quad X, Y \in \Gamma(TM)\,.$$

Besides being a straightforward generalization of the Ricci flow, the generalized Ricci flow arises naturally in the context of Hitchin's generalized Geometry, see [84, 82, 144, 149, 177, 208, 236, 281, 282, 283], as a flow of generalized metrics.

Having the equivalence between the pluriclosed flow and the generalized Ricci flow in hand, we focused our study on the behaviour of the generalized Ricci flow to obtain information on the pluriclosed flow, especially on *solvmanifolds*, i.e. quotients of solvable Lie groups by a discrete subgroup.

Our main theorem in this direction is stating that the homogeneous generalized Ricci flow has long-time existence on any solvmanifold.

Theorem C. Any invariant solution to the generalized Ricci flow on a solvmanifold exists for all positive times.

More generally, we characterise the maximal existence time for the invariant generalized Ricci flow on any Lie group G in terms of the blow-up behaviour of the *generalized scalar curvature*, see Theorem 3.2.33. An *invariant solution* is simply a solution to the generalized Ricci flow on G/Γ that lifts to a left-invariant solution on G.

As a straightforward corollary of Theorem C, we obtain the long-time existence of the homogeneous pluriclosed flow on any solvmanifold endowed with a left-invariant SKT structure.

Corollary D. Any invariant solution to the pluriclosed flow on a solvmanifold endowed with a left-invariant SKT structure exists for all positive times.

Surprisingly, the elliptic and parabolic approach are both well-suited for the study of canonical HKT metrics. Nowadays, a great source of interest towards the HKT world is represented by the *quaternionic Calabi conjecture*, firstly formulated by Alesker and Verbitsky in [9]. Mimicking the Calabi conjecture in Kähler geometry, the quaternionic Calabi conjecture states that we can always prescribe the complex volume or, equivalently, the first Chern-Ricci form of the given manifold within the HKT class of a fixed HKT metric. This conjecture can also be reformulated as the problem of finding solutions to the so-called *quaternionic Monge-Ampère equation*. Even though the quaternionic Calabi conjecture is still open, partial results of the validity of the conjecture were proved in [7, 89, 157, 158, 159]. Furthermore, a parabolic approach, using the *parabolic quaternionic Monge-Ampère equation*, was used to prove the known results regarding the quaternionic Calabi conjecture, see [46, 47, 160]. Although both approaches are suitable, to prove the existence part of the quaternionic Calabi conjecture, some technical difficulties are arising in the proof of a priori estimates, mainly due to the geometric situation. A great source of complications is the fact that, in general, a hypercomplex manifold might not be locally isomorphic to open sets of $\mathbb{H}^n$, thus making it impossible to consider local quaternionic coordinates. Other issues are arising from the presence of the derivatives of complex structures which might not be vanishing.

Mainly due to these obstacles, we focus our study on more curvature-related aspects of HKT and other types of special hyperHermitian metrics. Specializing the attention on strong HKT metrics, we managed to prove the following theorem, providing a strong evidence of the validity of Conjecture A in the hypercomplex setting.

Theorem E. Let (M, H, Ω) be a compact strong HKT manifold, then there exists no balanced hyper-Hermitian metric compatible with H, unless the manifold is hyperKähler.

The main property used in the proof of Theorem E is the fact that the *J-anti-invariant part* of the first Chern-Ricci form of a strong HKT metric is positive semidefinite, but not identically vanishing, unless the metric is hyperKähler, see Proposition 4.5.2.

The present thesis is structured as follows.

Chapter 1 is meant to be an introduction to all the different Geometries we will treat throughout the thesis. In particular, we recall the basic knowledge in complex Geometry, together with the description of the main properties of balanced and SKT metrics. Then, we move to the broader framework of generalized Geometry, rewieving the preliminary definitions, necessary to introduce the generalized Ricci flow. Finally, we restrict our attention to hypercomplex and hyperHermitian Geometry, discussing the main concepts in these settings and the definitions of HKT and other types of hyperHermitian metrics.

Chapter 2 is focused on the proof of Theorem B, which consists in a classical gluing procedure. After setting up the constant Chern scalar curvature equation and comparing the problem with the one arising in Kähler Geometry, we perform a deformation argument which allows us to conclude. Moreover, we prove a similar result when a suitable $(n-2, n-2)$-form is available and discuss some examples on which such form exists.

Chapter 3 is divided in two main sections. In Section 3.1, we collect results on the behaviour of the pluriclosed flow on Oeljeklaus-Toma manifolds, which are particular solvmanifolds endowed with a left-invariant complex structure, generalizing the Inoue surfaces. In Section 3.2, we study the homogeneous generalized Ricci flow on Lie groups. In order to prove Theorem C and, consequently, Corollary D, we perform a systematic study of the generalized Ricci flow with symmetries by using the generalized bracket flow, a suitable flow in the space of Dorfman brackets on a fixed exact Courant algebroid. We moreover study the asymptotics of the generalized Ricci flow on nilmanifolds highlighting particular behaviours, called generalized nilsolitons.

In Chapter 4, we give a detailed treatment of the properties of special hyperHermitian metrics, providing necessary and sufficient conditions for their existence. Then, we focus on the study of curvature aspects, especially, in the case of strong HKT metrics, ultimately proving Theorem E. Finally, we define a relevant notion of hyperHermitian Einstein metric and discuss some examples.

Chapter 1

Preliminaries

The first chapter of the present thesis serves the purpose of introducing and discussing all the necessary and basic notions for the next chapters.

The starting point of all the treatment is the study of complex Geometry, in particular of special Hermitian structures on a given complex manifold. The presence of a complex structure on the fixed manifold imposes many restrictions, for instance on the dimension of the manifold itself, but, at the same time, it produces a wilder scenario from the differential and curvature point of view. The very first evidence of this is the presence of many canonical connections preserving the complex and metric structure, at least in the non-Kähler case. Then, as for the Levi-Civita connection in Riemannian Geometry, it is natural to study the geometries of such connections. Especially in the case of the Chern and Bismut connection, the geometries arising highlight preferred types of Hermitian metrics, usually defined through cohomological conditions. The first section of this chapter has the objective to explain how these connections are defined and how they define special Hermitian metrics as preferred metrics for their study. We then describe the main properties of such metrics.

On the other hand, complex Geometry can be regarded as a special instance of a wider subject, called generalized Geometry, which was introduced with the precise aim of unifying the treatment of complex and symplectic Geometry. Among the most important facts defined and proved in generalized Geometry, an analogue of the classical Ricci flow, called generalized Ricci flow, was proposed to study generalized Geometry. The second section of this chapter discusses the main features in generalized Geometry towards the definition of such flow, describing its properties and its, both analytic and geometric, structure. As a direct connection with complex Geometry, the generalized Ricci flow has a strict relation with a geometric flow of Hermitian metrics, called pluriclosed flow. We will explain how these two flows are related.

Looking from another point of view, one may ask for even stricter geometries than complex one. A possibility one has is to require the given manifold to admit globally defined endomorphisms behaving like the quaternionic units. As a matter of fact, this requirement is equivalent to ask for a pair of anti-commuting complex structures on the manifold we are studying, giving rise to the so-called hypercomplex Geometry. As for the complex case, the imposition of a stricter geometry has the effect of generating, especially from the metric point of view, new features. The third section of this chapter serves the aim of introducing the basics concepts in hypercomplex and hyperHermitian Geometry and that of discussing the exclusive notions defined in this environment.

Finally, in the fourth and last section of this chapter, we define two types of convergences in the space of Riemannian manifolds, namely the Gromov-Hausdorff and the Chegeer-Gromov convergence. These two concepts turned out to be relevant in the study of geometric flows both from the topological and from the differential point of view, giving quantitative notions of collapsing and smooth convergence modulo diffeomorphisms.

1.1 Basics in complex Geometry

This first section has the aim of presenting all the basic knowledge in complex and Hermitian Geometry we will need throughout the thesis. The section is divided in three subsections. In Subsection 1.1.1 we quickly review some basic properties of complex manifolds, as, for instance, the splitting of the complefixied bundle of differential forms and the definition of first Chern class. Moreover, we present the classical blow-up procedure. Later, we recall general facts concerning the Hermitian geometry of a fixed manifold, discussing the definition of Gauduchon connections and of the Ricci 2-forms. In the last part, we present some definitions and results regarding complex orbifolds. In Subsection 1.1.2, we give the definition of balanced metrics and state their principal properties both from a geometric and analytical point of view. Finally, in Subsection 1.1.3, we give the definition of SKT metrics, describe the Fino-Vezzoni conjecture and, lastly, define the pluriclosed flow, ending the section discussing some results about the aforementioned flow.

1.1.1 Complex and Hermitian structures on manifolds and orbifolds

We start this subsection by recalling the main objects we want to study, i.e. complex manifolds.

Definition 1.1.1. Let M be a Hausdorff topological space. M is said to be a *complex manifold* of complex dimension n if, for any point $p \in M$, there exist a neighbourhood U_p of p and a homeomorphism $\varphi_p \colon U_p \to \varphi(U_p) \subset \mathbb{C}^n$ such that, whenever $U_p \cap U_q \neq \emptyset$, for some $p, q \in M$, the map

$$\varphi_p \circ \varphi_q^{-1} \colon \varphi_q(U_p \cap U_q) \to \varphi_p(U_p \cap U_q)$$

is a biholomorphism.

Example 1.1.2. Any open set of $\mathbb{C}^n$ is a complex manifold. Thus, $\mathbb{C}^n \backslash \{0\}$ is a complex manifold.

As in the smooth category, if we have a Lie group G acting holomorphically, i.e. for any $g \in \mathsf{G}$, the map $p \mapsto g \cdot p$ is holomorphic, freely and properly on a complex manifold M, then, the orbit space M/G inherits a unique structure of complex manifold such that the projection onto the quotient

$$\pi \colon M \to M/\mathsf{G}$$

is holomorphic. This gives rise to many examples. For instance, the complex projective space $\mathbb{CP}^n = \frac{\mathbb{C}^{n+1}\backslash\{0\}}{\mathbb{C}^*}$ is a complex manifold, where, of course, the action on $\mathbb{C}^*$ is defined by $\lambda \cdot z = \lambda z$, for all $\lambda \in \mathbb{C}^*$ and $z \in \mathbb{C}^{n+1}\backslash\{0\}$. The complex structure on $\mathbb{CP}^n$ defined in this way coincides with the usual complex structure defined using affine open sets.

Before discussing other examples, we want to explore in a better way what a complex manifold structure allows us to define. Let M^n be a complex manifold. For any $p \in M$, we can find holomorphic coordinates $\{z_1, \dots, z_n\}$ around the point p. On the other hand, we can write $z_i = x_i + \sqrt{-1}y_i$, for any $i = 1, \dots, n$, and infer that

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

Thus, for any point $p \in M$, we can define $J_p \in \mathrm{End}(T_pM)$ such that

$$J_p\frac{\partial}{\partial x_i} := \frac{\partial}{\partial y_i}, \quad J_p\frac{\partial}{\partial y_i} := -\frac{\partial}{\partial x_i}.$$

One can easily check that J_p does not depend on the choice of the coordinates we did and thus defines $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$. Moreover, the action of J can be extended by $\mathbb{C}$-linearity to the complexified tangent bundle

$$TM^{\mathbb{C}} := TM \otimes \mathbb{C}.$$

Now, since $J^2 = -\mathrm{Id}$, we obtain two eigenbundles of $TM^{\mathbb{C}}$

$$T^{1,0}M := \ker(J - \sqrt{-1}\mathrm{Id})\,, \quad T^{0,1}M := \ker(J + \sqrt{-1}\mathrm{Id})\,,$$

such that

$$TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M\,,$$

which are respectively called *holomorphic and anti-holomorphic tangent bundle* of M. One can easily check that

$$\Gamma(T^{1,0}M) \simeq \{X - \sqrt{-1}JX \mid X \in \Gamma(TM)\}\,, \quad \overline{T^{1,0}M} = T^{0,1}M\,. \tag{1.1}$$

It is easy to show that, if J is induced by a complex manifold structure on the manifold as above,

$$[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subseteq \Gamma(T^{1,0}M)\,,$$

i.e. $T^{1,0}M$ is a subbundle of $TM^{\mathbb{C}}$. This last condition is, using (1.1), equivalent to ask for

$$N_J(X,Y) := [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]\,, \quad X,Y \in \Gamma(TM)$$

to be identically vanishing. The tensor N_J is called *Nijenhuis tensor* of J. Usually, $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$ is called *almost-complex structure* on M. While if $N_J = 0$, J is called *integrable* almost-complex structure or just complex structure on M.
By Newlander-Niremberg Theorem, having an integrable almost-complex structure is equivalent to have a complex manifold structure on M which induces J.

Theorem 1.1.3 ([245])**.** *Let (M^{2n}, J) be a smooth manifold endowed with $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$. Then, J is induced by a complex manifold structure on M if and only if $N_J = 0$.*

Throughout this thesis, we will be concerned only on integrable almost-complex structures. For this, we will refer to J as the complex structure.
In view of Theorem 1.1.3, we can produce many more examples of complex manifolds.

Example 1.1.4. Let G be a Lie group endowed with an almost-complex structure J which is bi-invariant, i.e. both left and right-invariant. Then, J is integrable, thus, G is a complex manifold. Moreover, the multiplication rule and the inversion are holomorphic, giving G a structure of *complex Lie group.* Conversely, any complex Lie group can be endowed with a bi-invariant complex structure.

Let G be a Lie group endowed with a complex structure J which is left-invariant. This guarantees that (G, J) is a complex manifold. On the other hand, if we assume that Γ is a discrete subgroup of G acting freely and properly discontinuous by left multiplication, then, J descends to the quotient G/Γ, providing a complex structure on G/Γ.
An important example which will be central later is the Iwasawa manifold. We consider the Lie group

$$\mathrm{Heis}(3,\mathbb{C}) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z_1, z_2, z_3 \in \mathbb{C} \right\}\,.$$

As one may check easily, the multiplication rule on $\mathrm{Heis}(3,\mathbb{C})$ turns out to be holomorphic, giving $\mathrm{Heis}(3,\mathbb{C})$ a complex Lie group structure. This guarantees the existence of a complex structure J which is bi-invariant. Now, we can consider the subgroup $\mathrm{Heis}(3,\mathbb{Z}[\sqrt{-1}])$ of $\mathrm{Heis}(3,\mathbb{C})$ which is a lattice. Then, by the discussion above, we have a complex structure J on

$$M = \mathrm{Heis}(3,\mathbb{C})/\mathrm{Heis}(3,\mathbb{Z}[\sqrt{-1}])$$

which is called Iwasawa manifold. M is a threefold which is usual discussed as the first example of a non-Kähler complex manifold satisfying all the restrictions of Kähler manifolds on the Betti numbers. Indeed, M fails to be formal in the sense of Sullivan, see [309], which is a necessary condition to be Kähler, see [86].

Other examples of complex manifolds are the so-called holomorphic fiber bundles over a given complex manifold. These manifolds are, locally, biholomorphic to the product of an open set of the base manifold with another complex manifold, called the fibre.

Definition 1.1.5. Let M be a complex manifold. A *holomorphic fiber bundle with typical fibre F* is a complex manifold E, called total space, endowed with a holomorphic map

$$\pi\colon E \to M$$

such that, for any $p \in M$, there exist a neighbourhood U of p and a biholomorphism

$$\psi\colon U \times F \to \pi^{-1}(U)$$

such that $\pi \circ \psi = \mathrm{pr}_1$ where $\mathrm{pr}_1\colon U \times F \to U$ is the projection onto the first factor. If F is a complex vector space of dimension k, then, E is called *holomorphic vector bundle of rank k* over M. In the special case in which $k = 1$, E will be called *line bundle* over M.

Once the concept of line bundle is in hand, we can define the blow-up of a given complex manifold in a point. This procedure allows us to produce a different complex manifold by, heuristically, replacing a point of the starting manifold by a complex projective space. We start recalling the definition of the blow-up of $\mathbb{C}^n$ at the origin.

Definition 1.1.6. The blow-up $\mathrm{Bl}_0\mathbb{C}^n$ of $\mathbb{C}^n$ at the origin $0 \in \mathbb{C}^n$ is the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^{n-1}$.

The line bundle $\mathcal{O}(-1)$ is usually called *tautological line bundle* and its total space can be described as follows:

$$\mathcal{O}(-1) = \{(\ell, z) \in \mathbb{CP}^{n-1} \times \mathbb{C}^n \mid z \in \ell\}$$

with the projection map

$$p\colon \mathcal{O}(-1) \to \mathbb{CP}^{n-1}$$

such that $p(\ell, z) = \ell$, for all $(\ell, z) \in \mathcal{O}(-1)$. The zero section of p will be denoted with $E \simeq \mathbb{CP}^{n-1}$ and it will be called *exceptional divisor*. On the other hand, we can consider the map

$$\pi\colon \mathcal{O}(-1) \to \mathbb{C}^n$$

such that $\pi(\ell, z) = z$, for all $(\ell, z) \in \mathcal{O}(-1)$. Of course, π is holomorphic and restricts to a biholomorphism

$$\pi\colon \mathrm{Bl}_0\mathbb{C}^n \backslash E \to \mathbb{C}^n \backslash \{0\}\,.$$

The map π is usually called *blowdown map*.

Then, the blow-up of a given manifold in a certain point consists in a connected sum between the manifold and $\mathrm{Bl}_0\mathbb{C}^n$ via the blowdown map π.

Definition 1.1.7. Let M^n be a complex manifold and $p \in M$. We identify a neighbourhood centered at p with a ball $B \subset \mathbb{C}^n$. The blow-up Bl_pM of M at p is constructed by replacing B with $\pi^{-1}(B)$ using the biholomorphism

$$\pi\colon \pi^{-1}(B\backslash\{0\}) \to B\backslash\{0\}\,.$$

The blow-up of M at p is then a complex manifold endowed with a map

$$\pi\colon \mathrm{Bl}_pM \to M$$

which is a biholomorphism when restricted to

$$\pi\colon \mathrm{Bl}_pM\backslash E \to M\backslash\{p\}$$

where $E = \pi^{-1}(p) \simeq \mathbb{CP}^{n-1}$ is the exceptional divisor. The blow-up construction turned out to be fundamental, especially in Algebraic Geometry. To give a glimpse of this fact, we remark that, besides from $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, all the Fano surfaces, also known as Del Pezzo surfaces, are blow-ups of $\mathbb{CP}^2$ at $1 \leq k \leq 8$ points, see [85] or [129, Theorem 5.16]. Another important theorem stressing the importance of the blow-up is Theorem 1.1.41 that we will state later on in this section.

Let us now turn our attention to a more differential geometric point of view. From now on, let (M, J) be a complex manifold. The splitting of the complexified tangent bundle gives rise to a splitting of the complexified bundle of differential form in any degree:

$$\Lambda^k M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} M\,, \qquad \Lambda^{p,q} M := \Lambda^p (T^{1,0}M)^* \otimes \Lambda^q (T^{0,1}M)^*\,, \quad k \in \mathbb{N}\,, \tag{1.2}$$

where $\Lambda^{p,q}M$ is called the *bundle of* (p, q)*-forms*. Moreover, we will denote with $\Lambda^{p,p}_{\mathbb{R}}M$ the bundle of real (p, p)-forms, i.e. those $\alpha \in \Lambda^{p,p}M$ such that $\overline{\alpha} = \alpha$. Then, for any $k \in \mathbb{N}$, we can consider the projection

$$\pi^{p,q} \colon \Lambda^k M \otimes \mathbb{C} \to \Lambda^{p,q} M\,.$$

The presence of the above projections guarantees the splitting of the exterior differential: for any $k \in \mathbb{N}$,

$$d = \partial + \bar\partial\,, \quad \text{on } \Lambda^{p,q}M\,, \quad \partial := \pi^{p+1,q} \circ d\,, \quad \bar\partial := \pi^{p,q+1} \circ d\,.$$

It is easy to prove, using that $d^2 = 0$, that the following hold:

$$\partial^2 = 0\,, \quad \bar\partial^2 = 0\,, \quad \partial\bar\partial = -\bar\partial\partial\,. \tag{1.3}$$

Another interesting and useful differential operator on forms is the twisted exterior differential d^c. In order to define it, we extend the action of J on k-forms as follows:

$$(J\alpha)(X_1, \dots, X_k) = \alpha(JX_1, \dots, JX_k)\,, \quad X_1, \dots, X_k \in \Gamma(TM)\,. \tag{1.4}$$

We remark here that the action of J in (1.4) might differ by a sign from that used by many authors in literature.

Then, the twisted exterior differential is defined by

$$d^c := J^{-1} d J\,.$$

An easy computation shows that $d^c = \sqrt{-1}(\bar\partial - \partial)$. Using the latter expression and (1.3), we have that $dd^c = 2i\partial\bar\partial$.

The relations (1.3) give rise to different cohomology rings respectively defined by:

$$H_{\bar\partial}(M) := \frac{\ker \bar\partial}{\operatorname{Im} \bar\partial}\,, \quad H_{\mathrm{BC}}(M) := \frac{\ker \partial \cap \ker \bar\partial}{\operatorname{Im} \partial\bar\partial}\,, \quad H_{\mathrm{A}}(M) := \frac{\ker \partial\bar\partial}{\operatorname{Im} \partial + \operatorname{Im} \bar\partial}\,.$$

The cohomology ring $H_{\bar\partial}$ is called *Dolbeault* cohomology ring and it plays an important role in the study of Hodge theory in Kähler Geometry, see [191, Corollary 3.2.12], while H_{BC} and H_{A} are respectively called *Bott-Chern* and *Aeppli* cohomology rings. These two are characteristic of non-Kähler Geometry and, respectively, they play important roles in the study of balanced and SKT metrics.

Let us now focus on the study of line bundles over a given compact complex manifold M. It can easily be seen that the set of isomorphism classes of line bundles endowed with the tensor product is a group which is classically called *Picard group* and denoted with $\mathrm{Pic}(M)$. On the other hand, making use of the transition maps, one can infer that

$$\mathrm{Pic}(M) \simeq H^1(M, \mathcal{O}_M^*)\,,$$

where $\mathcal{O}_M^*$ is the sheaf of nowhere vanishing holomorphic functions on M. Then, we can consider the exponential exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \xrightarrow{e^{2\pi\sqrt{-1}}} \mathcal{O}_M^* \longrightarrow 0\ .$$

This of course will give rise to a long exact sequence in cohomology:

$$\dots \longrightarrow H^1(M,\mathbb{Z}) \longrightarrow H^1(M,\mathcal{O}_M) \longrightarrow \mathrm{Pic}(M) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \longrightarrow \dots \tag{1.5}$$

Definition 1.1.8. The *first Chern class* $c_1(L)$ of a line bundle L over M is the image of L via the boundary map c_1 in (1.5).

The splitting of differential forms gives rise to a special line bundle $K_M = \Lambda^{n,0}M$ over M^n which is called *canonical bundle* of M. The canonical bundle has a really important role both in Differential and Algebraic complex Geometry. Indeed, some of the most studied classes of complex manifolds are defined in terms of properties of the canonical bundle. For instance, Calabi-Yau manifolds are complex manifolds which have holomorphically trivial canonical bundle, while Fano manifolds are complex manifolds having $-K_M$ which is ample. Even though a general complex manifold might be neither Calabi-Yau nor Fano, classes of complex manifolds can be distinguished in terms of various invariants stemming from the canonical bundle. The very first one is the first Chern class of the manifold.

Definition 1.1.9. Let M be a complex manifold. The *first Chern class* $c_1(M)$ of M is the first Chern class of $-K_M$, i.e.

$$c_1(M) := c_1(-K_M) \in H^2_{\mathrm{dR}}(M,\mathbb{Z}) \cap H^{1,1}_{\bar\partial}(M)\,.$$

Even though the first Chern class is a holomorphic invariant, many compact complex manifolds can have the same first Chern class. For instance, for a compact complex manifold, it is sufficient to have topologically trivial canonical bundle to have vanishing first Chern class. Then, for example, any Lie group endowed with a left-invariant complex structure has vanishing first Chern class. However, the first Chern class of M is related to the curvature properties of the manifold itself, as we will see later.

Another invariant one can define using the canonical bundle is the Kodaira dimension.

Definition 1.1.10. Let M be a complex manifold. The *Kodaira dimension* of M is defined as

$$\kappa(M) := \limsup_{k\to\infty} \frac{\log \dim H^0(M, K_M^{\otimes k})}{\log k}\,,$$

where $\kappa(M) = -\infty$ if $K_M^{\otimes k}$ has no non-trivial holomorphic sections, for any $k \in \mathbb{N}$.

The Kodaira dimension turns out to be a birational invariant. Moreover, $\kappa(M^n) \in \{-\infty, 0, 1, \dots, n\}$. The Kodaira dimension was also found to be a useful tool to partially classify compact complex surfaces.

With all the basic definitions set, we now focus our attention on the metric properties of complex manifolds.

Definition 1.1.11. Let (M, J) be a complex manifold and g a Riemannian metric on M. The metric g will be called *Hermitian* if

$$g(JX, JY) = g(X, Y)\,, \quad X, Y \in \Gamma(TM)\,.$$

A Hermitian metric g on a complex manifold M defines the so-called fundamental form ω associated to g as follows:

$$\omega(X, Y) := g(JX, Y)\,, \quad X, Y \in \Gamma(TM)\,.$$

As one can check, $\omega \in \Lambda^2 M$ and $J\omega = \omega$, which, in turns, implies that $\omega \in \Lambda^{1,1}_{\mathbb{R}} M$. By definition, ω turns out to be positive definite, i.e.

$$\omega(X, JX) > 0\,, \quad X \in TM\,,\ X \neq 0\,.$$

Moreover, by knowing 2 elements among g, J, ω, one can recover the third one. Indeed,

$$g(X, Y) = \omega(X, JY)\,, \quad JX = (\iota_X \omega)^\sharp\,, \quad X, Y \in \Gamma(TM)\,.$$

Thanks to this, as it is customary in complex Geometry, we will refer to ω as the Hermitian metric. In what follows, we will be referring equivalently to the triple (M, J, ω) or to the pair (M, ω) as a *Hermitian manifold*, possibly omitting the complex structure when no confusion is made.

Let us fix a Hermitian metric ω on M. We know that ω is positive definite, hence, $\omega^n \in \Lambda^{n,n}_{\mathbb{R}} M$ is a volume form on M. It is easy to check that, up to an appropriate constant, it coincides with the volume form induced by the Riemannian metric:

$$\frac{\omega^n}{n!} = \mathrm{Vol}_g\,.$$

As in the Riemannian case, we can consider the Hodge star operator. We recall here its definition.

Definition 1.1.12. Let (M^n, ω) be a Hermitian manifold. We define the *Hodge star operator* $*$ as follows:

$$\alpha \wedge *\beta = g(\alpha, \beta)\frac{\omega^n}{n!}\,, \quad \alpha, \beta \in \Lambda^k M\,,$$

where the inner product g on k-forms is the naturally induced one by the Hermitian metric.

On the other hand, the Hodge star operator can be $\mathbb{C}$-linearily extended to

$$\Lambda^\bullet M \otimes \mathbb{C} := \bigoplus_{k \in \mathbb{N}} (\Lambda^k M \otimes \mathbb{C})\,.$$

The resulting operator satisfies the following:

$$\alpha \wedge *\bar{\beta} = g(\alpha, \beta)\frac{\omega^n}{n!}\,, \quad \alpha, \beta \in \Lambda^k M \otimes \mathbb{C}\,, \tag{1.6}$$

where here g is the Hermitian extension of g to $\Lambda^\bullet M \otimes \mathbb{C}$. In what follows, we will denote both the Hodge star operator and its $\mathbb{C}$-linear extension with the same notation, since no confusion will be possible.

The Hodge star operator satisfies many useful properties collected in the following proposition.

Proposition 1.1.13. *Let (M^n, ω) be a Hermitian manifold. Then, then following hold:*

1. $*\colon \Lambda^{p,q} M \to \Lambda^{n-q,n-p} M$ *;*
2. *for any $\alpha \in \Lambda^k M$ and $\beta \in \Lambda^{2n-k} M$, we have that*
$$g(\alpha, *\beta) = (-1)^{k(2n-k)} g(*\alpha, \beta)\,;$$
3. *$*$ is an isometry for g on $\Lambda^\bullet M$.*

We refer to [191, Proposition 1.2.20] for the proof of Proposition 1.1.13.

An important operator one can define when a non-degenerate 2-form ω is in hand is the *Lefschetz operator* $L_\omega\colon \Lambda^k M \to \Lambda^{k+2} M$ such that

$$L_\omega \alpha := \alpha \wedge \omega\,, \quad \alpha \in \Lambda^k M\,.$$

Starting from the Lefschetz operator, one can define its adjoint operator with respect the inner product on forms induced by the Hermitian metric.

Definition 1.1.14. Let (M, ω) be a Hermitian manifold. The adjoint Λ_ω of the Lefschetz operator is defined by:

$$g(\Lambda_\omega \alpha, \beta) = g(\alpha, L_\omega \beta)\,, \quad \alpha \in \Lambda^{k+2} M\,,\ \beta \in \Lambda^k M\,.$$

It is easy to verify that $\Lambda_\omega = *^{-1} L\, *$. A form $\alpha \in P^k := \ker \Lambda_\omega \cap \Lambda^k M$ is called *primitive* k-form.

Frequently, when no confusion can be made, we will omit the dependence on the Hermitian metric ω in both L and Λ.

In general, we do not have a precise expression for Λ_ω. However, one can obtain it in some special cases. The case we will use more frequently is that of $(1,1)$-forms. Indeed, one can easily see that, if $\alpha \in \Lambda^{1,1}M$, then,

$$\Lambda_\omega \alpha = n\frac{\alpha \wedge \omega^{n-1}}{\omega^n}\,. \tag{1.7}$$

In the next sections, from time to time, we will refer to $\Lambda_\omega\alpha$ as the trace of α with respect to ω or, when no confusion is possible, as the trace of α.

Proposition 1.1.15. *Let (M^n, ω) be a Hermitian manifold. The following hold:*

1. *If $H\colon \Lambda^k M \to \Lambda^k M$ is defined by $H\alpha := -(n-k)\alpha$, then the action of L, Λ and H determines a $\mathfrak{sl}(2,\mathbb{C})$-representation on $\Lambda^\bullet M$, i.e.*

$$[H, L] = 2L\,, \quad [H, \Lambda] = -2\Lambda\,, \quad [L, \Lambda] = H\,;$$

2. *We have the following orthogonal decomposition:*

$$\Lambda^k M = \bigoplus_{i\geq 0} L^i(P^{k-2i})\,;$$

3. *For any $k > n$, $P^k = 0$;*
4. *$L^{n-k}\colon \Lambda^k M \to \Lambda^{2n-k}M$ is bijective, for any $k \leq n$;*
5. *if $k \leq n$, $P^k = \ker L^{n-k+1} \cap \Lambda^k M$;*
6. *for any $\alpha \in P^k$, $k \leq n$,*

$$*L^j\alpha = (-1)^{\frac{k(k+1)}{2}}\frac{j!}{(n-j-k)!}J\alpha \wedge \omega^{n-j-k}\,.$$

We refer to [191, Proposition 1.2.26, Proposition 1.2.30, Proposition 1.2.31] for the proof of Proposition 1.1.15.

Let us now consider (M, ω) to be a compact Hermitian manifold. On M, we can define a inner product on the whole algebra of differential forms which is usually known as the L^2-inner product.

Definition 1.1.16. Let (M, ω) be a compact Hermitian manifold. The L^2*-inner product* is defined by:

$$\langle \alpha, \beta\rangle_{L^2} := \int_M g(\alpha, \beta)\frac{\omega^n}{n!} = \int_M \alpha \wedge *\bar{\beta}\,, \quad \alpha, \beta \in \Lambda^k M \otimes \mathbb{C}\,.$$

The L^2-inner product defines a Hilbert space structure on the whole algebra of differential forms. This allows us to define the formal adjoint operators of d, ∂ and $\bar\partial$.

Definition 1.1.17. Let (M, ω) be a compact Hermitian manifold. The operators d^*, ∂^* and $\bar\partial^*$ are defined, respectively, by the following:

$$\langle d^*\alpha, \beta\rangle_{L^2} = \langle \alpha, d\beta\rangle_{L^2}\,, \quad \langle \partial^*\alpha, \beta\rangle_{L^2} = \langle \alpha, \partial\beta\rangle_{L^2}\,, \quad \langle \bar\partial^*\alpha, \beta\rangle_{L^2} = \langle \alpha, \bar\partial\beta\rangle_{L^2}\,.$$

Moreover, we have that $d^* = - * d *$, $\partial^* = - * \partial *$, $\bar\partial^* = - * \bar\partial *$ and $d^* = \partial^* + \bar\partial^*$.

Then, with these operators, we can define various Laplacian operators.

Definition 1.1.18. Let (M, ω) be a compact Hermitian manifold. The *Hodge-Riemann Laplacian* and the *complex Laplacians* are defined, respectively, by:

$$\Delta_d := [d, d^*]\,, \quad \Delta_\partial := [\partial, \partial^*]\,, \quad \Delta_{\bar\partial} := [\bar\partial, \bar\partial^*]\,.$$

Here, we used the graded Lie bracket on the algebra of endomorphisms of $\Lambda^\bullet M$, i.e.

$$[A, B] := AB - (-1)^{\deg A \deg B} BA\,,$$

where $\deg A$ is defined as the integer such that

$$A\colon \Lambda^k M \to \Lambda^{k+\deg A} M\,.$$

Hence, with this in hand, for instance, we recover the classical expression of the Hodge-Riemann Laplacian as

$$\Delta_d = [d, d^*] = dd^* + d^* d\,.$$

Another Laplacian can be defined on fuctions on a Hermitian manifold and it is called Chern Laplacian.

Definition 1.1.19. Let (M^n, ω) be a Hermitian manifold. The *Chern Laplacian* Δ_ω is defined by:

$$\Delta_\omega f := \Lambda_\omega(\sqrt{-1}\partial\bar\partial f) = n\frac{\sqrt{-1}\partial\bar\partial f \wedge \omega^{n-1}}{\omega^n}\,, \quad f \in C^\infty(M, \mathbb{R})\,.$$

The relation between the Chern and the Hodge-Riemannan Laplacian was investigated by Gauduchon in [152].

Theorem 1.1.20 ([152], Formula (4))**.** *Let (M^n, ω) be a compact Hermitian manifold. Then, we have*

$$2\Delta_\omega f = -\Delta_d f + g(df, \theta)\,, \quad f \in C^\infty(M, \mathbb{R})\,, \tag{1.8}$$

where $\theta \in \Lambda^1 M$ is the Lee form of ω, defined by $d\omega^{n-1} = \theta \wedge \omega^{n-1}$.

Even though the Lee form is defined implictly, we can recover an explicit expression in terms of J and $d^*\omega$. First of all, we note that θ is well defined thanks to Item 4 of Proposition 1.1.15. On the other hand, using Item 6 of Proposition 1.1.15, one can infer that $\theta = -Jd^*\omega$.

Throughout all the thesis, we will be interested in study of Hermitian metrics satisfying some cohomological condition. Historically speaking, the first one appeared is the Kähler one.

Definition 1.1.21. Let (M, ω) be a Hermitian manifold. We say that ω is *Kähler* if $d\omega = 0$.

The Kähler condition might be thought as the intersection between Riemmanian, complex and symplectic Geometry and it plays an important role in the study of Differential and Algebraic Geometry. Indeed, for instance, classical examples of Kähler metrics are the Fubini-Study metric on $\mathbb{CP}^n$ and its pullback to any smooth projective variety.

Example 1.1.22. Another example of Kähler metric is the *Burn-Simanca metric* ω_{BS} on $\mathrm{Bl}_0\mathbb{C}^n$. This metric was defined in [227] and [285] and it is an asymptotically flat and scalar flat Kähler metric. The asymptotic flatness, in a more formal way, is equivalent to require that there exist a compact subset $K \subseteq \mathrm{Bl}_0\mathbb{C}^n$, $R > 1$ and a biholomorphism $\varphi\colon \mathbb{C}^n \backslash B(0, R) \to \mathrm{Bl}_0\mathbb{C}^n \backslash K$ such that there exist $\lambda > 1$ and $\tau > 0$ so that

$$\lambda^{-1}\omega_o \le \varphi^*\omega_{\mathrm{BS}} \le \lambda\omega_o\,,$$

where $\omega_o = \sqrt{-1}\partial\bar\partial|\zeta|^2$ is the flat Kähler metric on $\mathbb{C}^n\backslash B(0, R)$, and

$$\varphi^*\omega_{\mathrm{BS}} = \omega_o + \mathrm{O}(|\zeta|^{-\tau})\,, \quad |\zeta| \to \infty\,.$$

Indeed, assuming $n \ge 3$ and using the standard coordinates ζ on $\mathrm{Bl}_0\mathbb{C}^n \backslash E \simeq \mathbb{C}^n\backslash\{0\}$, ω_{BS} can be written as:

$$\omega_{\mathrm{BS}} = \sqrt{-1}\partial\bar\partial(|\zeta|^2 + \gamma(|\zeta|))\,, \quad \gamma(|\zeta|) = \mathrm{O}(|\zeta|^{4-2n})\,, \quad |\zeta| \to \infty\,, \tag{1.9}$$

where E is the exceptional divisor of $\mathrm{Bl}_0\mathbb{C}^n$.

The Kähler condition allows to compute several commutation rules between the differential operators d, ∂ and $\bar{\partial}$ and their adjoints with the operators L and Λ.

Theorem 1.1.23. *Let (M, ω) be a compact Kähler manifold. Then, the following hold:*

1. $[L, d] = 0$ *and* $[\Lambda, d^*] = 0$;
2. $[d^*, L] = -d^c$ *and* $[\Lambda, d] = -(d^c)^*$;
3. $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$.

For a proof of these identities, we refer to [191, Proposition 3.1.12] or to [155, Proposition 1.14.1]. Moreover, the Kähler condition is the right condition to impose on a Hermitian metric for the associated Levi-Civita connection to preserve the complex structure.

Proposition 1.1.24. *Let (M, J, ω) be a Hermitian manifold. Then, ω is Kähler if and only if*

$$DJ = 0 ,$$

where D is the Levi-Civita connection associated to ω.

On the other hand, since we will be concerned the most on non-Kähler geometry, we will be focusing on a special class of connections, called Hermitian.

Definition 1.1.25. Let (M, J, ω) be a Hermitian manifold. A connection ∇ is called *Hermitian* if

$$\nabla g = 0 , \quad \nabla J = 0 .$$

Gauduchon in [154] identified a line of special Hermitian connections, which are now called *canonical connections* or *Gauduchon connections.*

Theorem 1.1.26 ([154], Proposition 2(ii), Definition 2)**.** *Let (M, J, ω) be a Hermitian manifold. Then, for all $t \in \mathbb{R}$, there exists a Hermitian connection ∇^t characterized by the following:*

$$g(\nabla^t_X Y, Z) = g(D_X Y, Z) + \frac{t-1}{4}(d^c\omega)(X, Y, Z) + \frac{t+1}{4}(d^c\omega)(X, JY, JZ) , \quad X, Y, Z \in \Gamma(TM) . \tag{1.10}$$

In the case in which ω is Kähler, the Gauduchon connections all coincide with the Levi-Civita one. On the other hand, if ω is not Kähler, the Gauduchon line is a proper line and the choice of different connections in the line determines different geometries of the manifold. Among the Gauduchon connections, we can find two important Hermitian connections previously defined, respectively, by [78] and by [52, 308].

Definition 1.1.27. Let (M, J, ω) be a Hermitian manifold. The *Chern connection* ∇ is the unique Hermitian connection such that $T^{1,1} = 0$, where T is the torsion of ∇, i.e.

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] , \quad X, Y \in \Gamma(TM) .$$

The *Bismut connection* ∇^B is defined as the unique Hermitian connection such that

$$H := g(T^B(\cdot, \cdot), \cdot) \in \Lambda^3 M .$$

Respectively, the Chern and the Bismut connection correspond to the choice of the parameter $t = 1$ or $t = -1$ in the Gauduchon line. Then, using (1.10), one can infer that

$$H = d^c\omega . \tag{1.11}$$

The Chern and the Bismut connection play a natural role in the study of some special Hermitian metrics. Before discussing them, we recall the definition of the Ricci 2-form and the scalar curvature of the Gauduchon connections.

Definition 1.1.28. Let (M, J, ω) be a Hermitian manifold. For any $t \in \mathbb{R}$, we define the *Ricci form* of ∇^t as follows:

$$\mathrm{Ric}^t(\omega) := -\frac{1}{2}\mathrm{tr}(JR^t)\,, \tag{1.12}$$

where R^t is the curvature tensor associated to ∇^t. Moreover, the *scalar curvature* of ∇^t is defined as:

$$s^t(\omega) := \Lambda \mathrm{Ric}^t(\omega) = n\frac{\mathrm{Ric}^t(\omega)\wedge\omega^{n-1}}{\omega^n}\,. \tag{1.13}$$

In what follows, we will refer to $\mathrm{Ric}^{\mathrm{Ch}}(\omega) := \mathrm{Ric}^1(\omega)$ as the first Chern-Ricci form and to $\mathrm{Ric}^{\mathrm{B}}(\omega) := \mathrm{Ric}^{-1}(\omega)$ as the Bismut-Ricci form as well as $s^{\mathrm{Ch}}(\omega)$ and $s^{\mathrm{B}}(\omega)$ as, respectively, the Chern scalar curvature and the Bismut scalar curvature.

We have an explicit expression of the Ricci forms of ∇^t, for all $t \in \mathbb{R}$, in terms of the first Chern-Ricci form.

Lemma 1.1.29 ([170], Formula 8)**.** *Let (M, ω) be a Hermitian manifold. Then, for all $t \in \mathbb{R}$, we have*

$$\mathrm{Ric}^t(\omega) = \mathrm{Ric}^{\mathrm{Ch}}(\omega) + \frac{t-1}{2}dd^*\omega\,. \tag{1.14}$$

Let us focus our study in the particular cases in which $t = 1$. As we remarked before, choosing the parameter $t = 1$ is equivalent to study the Chern connection. Now, computing explicitly the components of the curvature tensor of the Chern connection in local holomorphic coordinates, we have that

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}\,.$$

Then, tracing as in (1.12), it is easy to check that locally

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\omega^n \in \Lambda^{1,1}_{\mathbb{R}}M\,,$$

giving that $\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ is d-closed and, hence, defining a De Rham cohomology class. We claim, that, up to a factor, this cohomology class coincide with the first Chern class of M.

First of all, we know that any Hermitian metric on M induces a Hermitian metric h on the fibres of $-K_M$. Then, one can consider the Chern connection ∇ associated to h on $-K_M$ and produce its curvature $F_\nabla := (\nabla)^2 \in \Gamma(M, \Lambda^{1,1}M \otimes \mathrm{End}(-K_M, h))$. Chern-Weil theory, see for instance [191, Section 4.4], then ensures that

$$c_1(M) = \left[\frac{\sqrt{-1}}{2\pi}\mathrm{tr}F_\nabla\right] \in H^2_{\mathrm{dR}}(M, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(M)\,.$$

On the other hand, in this specific case, $\sqrt{-1}\mathrm{tr}F_\nabla = \mathrm{Ric}^{\mathrm{Ch}}(\omega)$. Then, this allows us to infer that the Chern-Ricci form is a representative of the first Chern class of M, up to a factor 2π. On the other hand, a different Hermitian metric h' on $-K_M$ can be written as $h' = e^f h$, for some $f \in C^\infty(M, \mathbb{R})$. Then, the trace of the curvature of the Chern connection associated to h' will satisfy the following:

$$\sqrt{-1}\mathrm{tr}F_{\nabla_{h'}} = \sqrt{-1}\mathrm{tr}F_{\nabla_h} + \sqrt{-1}\partial\bar{\partial}f\,.$$

This guarantees that $c_1(M)$ does not depend on the choice of the metric on M. Moreover, it gives us that $\frac{1}{2\pi}\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ defines a class in $H^{1,1}_{\mathrm{BC}}(M, \mathbb{R})$ which does not depend on the choice of the metric on M.

Definition 1.1.30. Let (M, ω) be a Hermitian manifold. The *first Bott-Chern class* $c_1^{\mathrm{BC}}(M)$ of M is defined as

$$c_1^{\mathrm{BC}}(M) := \left[\frac{1}{2\pi}\mathrm{Ric}^{\mathrm{Ch}}(\omega)\right]_{\mathrm{BC}} \in H^{1,1}_{\mathrm{BC}}(M, \mathbb{R})\,.$$

More in general, if L is a line bundle over M, we define the *first Bott-Chern class* of L as follows:

$$c_1^{\mathrm{BC}}(L) := \left[\frac{\sqrt{-1}}{2\pi}\mathrm{tr}F_\nabla\right]_{\mathrm{BC}} \in H^{1,1}_{\mathrm{BC}}(M, \mathbb{R})\,,$$

where F_∇ is the curvature of the Chern connection associated to any Hermitian metric on L.

Example 1.1.31. Let M^n be a compact complex manifold and $p \in M$. Then,

$$K_{\mathrm{Bl}_p M} = \pi^* K_M + (n-1)E\,,$$

where π is the blowdown map and E is the exceptional divisor. This allows us to infer that

$$c_1^{\mathrm{BC}}(\mathrm{Bl}_p M) = \pi^* c_1^{\mathrm{BC}}(M) - (n-1)[E]_{\mathrm{BC}}\,, \tag{1.15}$$

where $[E]_{\mathrm{BC}}$ is the first Bott-Chern class of the line bundle associated to E in $\mathrm{Bl}_p M$.

Remark 1.1.32. Assuming ω to be Kähler, we know that the Guaduchon line will collapse to the Levi-Civita connection. This, in particular, gives us that, for all $t \in \mathbb{R}$,

$$\mathrm{Ric}^t(\omega) = \mathrm{Ric}(\omega)$$

the classical Ricci form of ω. Using the $\partial\bar{\partial}$-Lemma, we can deduce that $c_1(M) = c_1^{\mathrm{BC}}(M)$ when M is Kähler. Moreover, using Theorem 1.1.23, we can recover the classical *contracted second Bianchi identity*:

$$d^*\mathrm{Ric}(\omega) = -id^c s(\omega) \tag{1.16}$$

where $s(\omega)$ is the Riemannian scalar curvature, up to a factor $\frac{1}{2}$.

On the other hand, on non-Kähler manifolds, more precisely, on non $\partial\bar{\partial}$-manifolds, the first Chern and the first Bott-Chern classes might be different, see for instance [167].

In addiction, using the Chern-Ricci form, one can define a geometric flow of Hermitian metrics, called Chern-Ricci flow.

Definition 1.1.33 ([323])**.** Let (M, ω_0) be a Hermitian manifold. The *Chern-Ricci flow* is the following evolution equation:

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}^{\mathrm{Ch}}(\omega)\,, \quad \omega(0) = \omega_0\,.$$

As one may see directly from the equation, the Chern-Ricci flow is a generalization of the more classical Kähler-Ricci flow and, as that, it is equivalent to a parabolic complex Monge-Ampère equation for the potentials, see [164, 323]. The behaviour of the Chern-Ricci flow was studied in [27, 96, 100, 164, 225, 323, 322, 344]. We will study its behaviour on Oeljeklaus-Toma manifolds in Section 3.1 in more details.

Related to the scalar curvature of the Chern connection, we can define another invariant of a line bundle. Before doing that, we recall the definition of a Gauduchon metric.

Definition 1.1.34. Let (M^n, ω) be a Hermitian manifold. The metric ω is called *Gauduchon* if

$$\sqrt{-1}\partial\bar{\partial}\omega^{n-1} = 0\,,$$

or, equivalently, if $d^*\theta = 0$.

Gauduchon in [152] proved that Gauduchon metrics exist in any conformal class of a given Hermitian metric on a compact manifold.

Theorem 1.1.35. *Let (M^n, ω) be a compact Hermitian manifold. If $n \geq 2$, then, there exists a unique Gauduchon metric with unit volume in the conformal class $\{\omega\} = \{e^f\omega \mid f \in C^\infty(M, \mathbb{R})\}$ of ω.*

We can now define the Gauduchon degree with respect to a given Hermitian metric of a line bundle over a compact complex manifold as follows.

Definition 1.1.36. Let (M^n, ω) be a compact Hermitian manifold and L be a line bundle of M. The *Gauduchon degree* of L with respect to ω is

$$\Gamma(L, \{\omega\}) := \int_M \gamma \wedge \frac{\eta^{n-1}}{(n-1)!}\,,$$

where η is the unique Gauduchon metric in $\{\omega\}$ with unit volume while $[\gamma] = 2\pi c_1^{\mathrm{BC}}(L)$. If $L = -K_M$, we will denote with $\Gamma(\{\omega\}) := \Gamma(-K_M, \{\omega\})$ and we have that

$$\Gamma(\{\omega\}) = \int_M s^{\mathrm{Ch}}(\eta)\frac{\eta^n}{n!} = \int_M \mathrm{Ric}^{\mathrm{Ch}}(\eta) \wedge \frac{\eta^{n-1}}{(n-1)!}\,. \tag{1.17}$$

$\Gamma(\{\omega\})$ will be called *Gauduchon degree of the conformal class of* ω.

Let us conclude this section by discussing the basic concepts in the study of complex orbifolds.

Definition 1.1.37. Let X be a singular complex manifold of dimension n. We say that X is a *complex orbifold* if, for all $x \in X$, there exist a neighbourhood $U \subseteq X$ of x and a finite subgroup $\mathsf{G}_x \subset \mathsf{GL}(n, \mathbb{C})$ such that U is isomorphic to $\mathbb{C}^n/\mathsf{G}_x$. The points in which $\mathsf{G}_x \neq 1$ are called *orbifold points* and G_x is called *orbifold group.*

Examples of complex orbifolds can be constructed considering a finite group G acting holomorphically and faithfully on a complex manifold M. Then, $X := M/\mathsf{G}$ is a complex orbifold where orbifold points are precisely those in which $\mathrm{Stab}(x) \neq 1$ and $\mathrm{Stab}(x)$ is the orbifold group. On the other hand, not all complex orbifolds arise from finite quotients of complex manifolds, as the following example shows.

Example 1.1.38. Let $n \geq 1$ and $a_0, \ldots, a_n \in \mathbb{N}$ with no common factors. Then, we can define the following action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\backslash\{0\}$:

$$\lambda \cdot (z_0, \ldots, z_n) := (\lambda^{a_0} z_0, \ldots, \lambda^{a_n} z_n)\,, \quad \lambda \in \mathbb{C}^*\,, \quad (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}\backslash\{0\}\,.$$

Then, the quotient

$$\mathbb{CP}^n_{a_0,\ldots,a_n} := \frac{\mathbb{C}^{n+1}\backslash\{0\}}{\mathbb{C}^*}$$

is called weighted complex projective space. We refer to [202, p.135] for the proof that $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is a complex orbifold. On the other hand, $\mathbb{CP}^n_{a_0,\ldots,a_n}$ cannot be presented as a quotient of a complex manifold under the action of a finite group.

One can give a precise definition of a metric on a orbifold.

Definition 1.1.39. Let X^n be a complex orbifold. A metric g on X is a genuine metric on the regular part while, in a neighbourhood of a orbifold point $x \in X$ with orbifold group G_x, it can identified with a G_x-invariant metric near 0 in $\mathbb{C}^n$. We will say that g is Kähler if it is Kähler both on the regular part and near the orbifold points.

Especially in Chapter 2, we will be concerned on crepant resolutions of orbifolds. Crepant resolutions can be defined generally on complex algebraic varieties.

Definition 1.1.40. Let X be a singular complex algebraic variety. A *resolution* $(\hat{X}, \pi)$ of X is a pair consisting in a normal, see [175, p.177], smooth variety $\hat{X}$ and a proper birational map $\pi\colon \hat{X} \to X$. A resolution is called *crepant* if

$$K_{\hat{X}} \simeq \pi^* K_X\,.$$

We should remark that, given a complex algebraic variety, it always admits a resolution which is obtained by a finite sequence of blow-ups.

Theorem 1.1.41 ([186, 187])**.** *Let X be a complex algebraic variety. Then, there exists a resolution $\pi\colon \hat{X} \to X$ which is the result of a finite number of blow-ups.*

On the other hand, not all the resolutions are crepant. First of all, we turn our attention on quotient singularities. Let $\mathsf{G} \subset \mathsf{GL}(n, \mathbb{C})$ be a finite subgroup acting on $\mathbb{C}^n$. We further assume that G acts freely on $\mathbb{C}^n\backslash\{0\}$ so that 0 is the unique singular point of $\mathbb{C}^n/\mathsf{G}$. Since we want the canonical bundle of $\mathbb{C}^n/\mathsf{G}$ to be well-defined and since $g \in \mathsf{G}$ acts on $\Lambda^{n,0}\mathbb{C}^n$ via multiplication by $\det(g)$, we need to impose that $\mathsf{G} \subset \mathsf{SL}(n, \mathbb{C})$. Now, any finite subgroup of $\mathsf{SL}(n, \mathbb{C})$ is conjugated to a subgroup of $\mathsf{SU}(n)$. Then, we will suppose $\mathsf{G} \subset \mathsf{SU}(n)$.

Remark 1.1.42. In view of Definition 1.1.39, for any finite subgroup $\mathsf{G} \subset \mathsf{SU}(n)$, the flat Kähler metric on $\mathbb{C}^n$ descend to a Käher metric on $\mathbb{C}^n/\mathsf{G}$.

Theorem 1.1.43 ([287, 275])**.** *Let $\mathsf{G} \subset \mathsf{SL}(n, \mathbb{C})$ be a finite group and $n = 2, 3$. Then, $\mathbb{C}^n/\mathsf{G}$ admits a crepant resolution. If $n = 2$, the crepant resolution is unique.*

The scenario in higher dimensions is wilder. Given a finite subgroup G in $\mathsf{SL}(n, \mathbb{C})$, we can use a criterion to check if $\mathbb{C}^n/\mathsf{G}$ has no crepant resolutions, see [204, Theorem 7.3.3] and [244, Theorem 2.3]. Using this, we can actually prove that $\mathbb{C}^4/\{\pm \mathrm{Id}\}$ has no crepant resolutions.

Let us now turn our attention to general orbifolds. As we saw before, a necessary condition for an orbifold to admit crepant resolutions is that all the orbifold groups are cointaned in $\mathsf{SL}(n, \mathbb{C})$. Moreover, near the orbifold point $x \in X$, the crepant resolution will be isomorphic to a crepant resolution of $\mathbb{C}^n/\mathsf{G}_x$. Then, another necessary condition is that $\mathbb{C}^n/\mathsf{G}_x$ must have a crepant resolution, for all orbifold points $x \in X$. Using Theorem 1.1.43, one can prove the following.

Theorem 1.1.44. *Let X^3 be a complex orbifold with orbifold group in $\mathsf{SL}(3, \mathbb{C})$. Then, X admits a crepant resolution.*

To conclude this part, we will overview some basic facts about ALE metrics.

Definition 1.1.45. Let (M^n, g) be a Riemannian manifold. We say that (M, g) is *asymptotically locally euclidean*, or ALE, for short, to $\mathbb{R}^n/\mathsf{G}$, $\mathsf{G} \subset \mathsf{SO}(n)$ acting freely on $\mathbb{R}^n\backslash\{0\}$, if there exist a compact set $K \subseteq M$, $R > 0$ and $\pi\colon M\backslash K \to \mathbb{R}^n/\mathsf{G}$ such that

1. π is a diffeomorphism of $M\backslash K$ with $\{x \in \mathbb{R}^n/\mathsf{G} \mid r(x) > R\}$, where r is the distance induced by the flat metric g_o of x from the origin;
2. for all $k \geq 0$,
$$D^k(\pi_* g - g_o) = \mathrm{O}(r^{-n-k}), \quad \text{on } \{x \in \mathbb{R}^n/\mathsf{G} \mid r(x) > R\},$$
where D is the Levi-Civita connection of g_o.

We further say that a complex manifold (M^n, J, g) is *Kähler ALE* to $\mathbb{C}^n/\mathsf{G}$, $\mathsf{G} \subset \mathsf{U}(n)$ acting freely on $\mathbb{C}^n/\{0\}$, if g is Kähler, Item 1 and Item 2 hold with respect to the flat Kähler metric g_o and, for all $k \geq 0$,
$$D^k(\pi_* J - J_o) = \mathrm{O}(r^{-2n-k}), \quad \text{on } \{x \in \mathbb{C}^n/\mathsf{G} \mid r(x) > R\},$$
where J_o is the standard complex structure on $\mathbb{C}^n/\mathsf{G}$.

With this in mind, we can state the following theorem whose proof can be found in [202] or [203].

Theorem 1.1.46. *Let $\mathsf{G} \subset \mathsf{SU}(n)$ be a non-trivial finite group acting freely on $\mathbb{C}^n\backslash\{0\}$ and let (X, π) a crepant resolution of $\mathbb{C}^n/\mathsf{G}$. Then, any class of a ALE Kähler metric admits a Kähler Ricci-flat ALE metric. Moreover, such metric ω_{ALE} satisfies the following: there exist $R, A > 0$ and $\gamma \in (1-2n, 2-2n)$ such that*
$$\pi_*\omega_{\mathrm{ALE}} = \omega_o - 2A\sqrt{-1}\partial\bar{\partial}(r^{2-2n}) + \mathrm{O}(r^\gamma), \quad \textit{on } \{x \in \mathbb{C}^n/\mathsf{G} \mid r(x) > R\},$$
where ω_o is the flat Kähler metric on $\mathbb{C}^n/\mathsf{G}$.

With all this in mind, we can switch our attention to the study of some special cohomological conditions on Hermitian metrics, generalizing the Kähler one. We will be mostly interested in the so-called balanced and SKT condition. We firstly study balanced metrics.

1.1.2 Balanced metrics

First of all, we recall the definition of balanced metric on a given complex manifold.

Definition 1.1.47. Let (M^n, ω) be a Hermitian manifold. The Hermitian metric ω is called *balanced* if $d\omega^{n-1} = 0$.

Historically, the balanced condition was firstly studied by Gauduchon in [151, 153] under the name of *semi-Kähler* condition. Later, Michelsohn in [240] gave the name *balanced* to such metrics. A first remark one can do is the following. The balanced condition is the unique condition among those of type $d\omega^p = 0$, for some $p = 1, \ldots, n-1$, which does not force the metric to be Kähler. This can be easily viewed using Item 4 in Proposition 1.1.15. Another easy remark is that, on complex surfaces, the balanced condition coincides with the Kähler one. So, since we will be concerned in the study of balanced non-Kähler manifolds, we shall always assume that $n \geq 3$. The balanced condition has many equivalent conditions. We list all of them in the following proposition.

Proposition 1.1.48. *Let (M^n, ω) be a Hermitian manifold. The following are equivalent:*

1. *ω is balanced;*
2. *$d^*\omega = 0$;*
3. *$d\omega \in P^3$;*
4. *$\theta = 0$.*

Proof. We have that

$$d^*\omega = -*d*\omega = -\frac{1}{(n-1)!} * d\omega^{n-1},$$

which gives the equivalence between Item 1 and Item 2. Using Item 5 of Proposition 1.1.15, we conclude the equivalence between Item 1 and Item 3. Finally, recalling that θ is defined by $d\omega^{n-1} = \theta \wedge \omega^{n-1}$, we have the equivalence between Item 1 and Item 4. □

Examples of balanced non-Kähler manifolds were provided by many authors. Abbena and Grassi in [1] showed that a complex Lie group carries a left-invariant balanced metric if and only if it is unimodular. This, for instance, guarantees that on $\mathsf{SL}(2, \mathbb{C})$ or on $\mathrm{Heis}(3, \mathbb{C})$ all left-invariant Hermitian metrics are balanced. On the other hand, a complex Lie group is Kähler if and only if it is abelian. Then, $\mathsf{SL}(2, \mathbb{C})$ and $\mathrm{Heis}(3, \mathbb{C})$ are balanced non-Kähler manifolds. In particular, the Iwasawa manifold inherits a balanced metric, giving us a compact example of non-Kähler balanced manifold. The result by Abbena and Grassi combined with the classical result by Wang in [336] ensures that all complex parallelizable manifolds, i.e. such that $T^{1,0}M$ is holomorphically trivial, are balanced manifolds. Other examples were found on compact quotients of Lie groups, see for instance [18, 19, 43, 70, 73, 113, 118, 119, 120, 128, 167, 325, 326].

Regarding invariant examples, an important tool for the study of the existence and non-existence of balanced metrics is the symmetrization process, introduced firstly by Belgun in [49]. We recall here the main ideas.

Let G be a simply connected Lie group admitting a co-compact lattice Γ. In particular G is unimodular and admits a bi-invariant volume form ν, see e.g. [242, Lemma 6.2], which we may normalize so that on the compact quotient $M = \mathsf{G}/\Gamma$ we have $\int_M \nu = 1$.

Definition 1.1.49. The symmetrization map μ sends a k-form on M to a left-invariant one by averaging it against the bi-invariant volume:

$$\mu(\alpha)(X_1, \ldots, X_k) := \int_M \alpha_x(X_1|_x, \ldots, X_k|_x)\nu_x, \quad \alpha \in \Lambda^k M.$$

The symmetrization map has many properties, which are listed in the following proposition.

Proposition 1.1.50. *Let $M = \mathsf{G}/\Gamma$ be a compact quotient of a Lie group endowed with a left-invariant complex structure J. Then, the following hold:*

1. *if $\alpha \in \Lambda^k M$, $k \geq 0$, is left-invariant, then $\mu(\alpha) = \alpha$;*
2. *$\mu(\alpha \wedge \mu(\beta)) = \mu(\alpha) \wedge \mu(\beta)$, for all $\alpha, \beta \in \Lambda^\bullet M$;*
3. *$[d, \mu] = 0$;*
4. *$[J, \mu] = 0$.*

Using this process, one is led to prove the following.

Proposition 1.1.51. *Let $M = \mathsf{G}/\Gamma$ be a compact quotient of a Lie group endowed with a left-invariant complex structure J. Then, M admits a balanced metric if and only if it admits a left-invariant balanced metric.*

As for non-invariant examples, we should mention the work by Fu, Li and Yau [133] in which the authors produce compact non-Kähler balanced manifolds via conifold transitions of smooth Kähler Calabi-Yau threefolds. Giusti and Spotti in [168] proved that crepant resolutions of balanced orbifolds are balanced. Moreover, Goldstein and Prokushkin in [169] gave a general procedure to produce, starting form a given Hermitian manifold satisfying some mild assumptions, a T^2-principal bundle over the starting manifold. If we specify, this construction for Calabi-Yau surfaces, or more generally, for hyperKähler manifolds, we can produce Hermitian manifolds which carry balanced metrics. Finally, the twistor space of a hypercomplex manifold is balanced, see [205, 318].

On the other hand, many general properties of compact balanced manifolds are known, essentially, stemming from the characterization of the balanced condition by Michelsohn in terms of currents. We recall it, together with other simple properties, in the next theorem.

Theorem 1.1.52 ([240], Proposition 1.9, Theorem 4.7)**.** *Let M and N be two complex manifolds. Then,*

1. *if M and N are balanced, then $M \times N$ is balanced;*
2. *if M is balanced and $f \colon M \to N$ is a holomorphic submersion, then N is balanced;*
3. *M admits a balanced metric if and only if there are no non-trivial real, positive and d-closed $(1,1)$-currents.*

As previously anticipated, stemming from Item 3 of Theorem 1.1.52, Alessandrini and Bassanelli managed to prove that the class of balanced manifolds is closed under proper holomorphic modifications.

Theorem 1.1.53 ([10, 12, 13])**.** *Let M and N be two complex manifolds. Moreover, let $f \colon M \to N$ be a proper holomorphic modification, i.e there exists a complex submanifold Y of N such that*

$$f \colon M \backslash f^{-1}(Y) \to N \backslash Y$$

is a biholomorphism. Then, M is balanced if and only if N is balanced.

Theorem 1.1.53 guarantees that all Fujiki class $\mathcal{C}$ manifolds are balanced, since, by definition, they are bimeromorphic to a Kähler manifold. Moreover, Theorem 1.1.53 highlights a stronger closedness properties with respect to the Kähler case. Indeed, Hironaka in [185, 188] provided an example of a non-Kähler compact manifold obtained by a Kähler (projective) manifold via a finite sequence of blow-ups with smooth centers, proving the non-closedness of Kähler manifolds under proper modifications. In view of Theorem 1.1.53, the latter will be balanced.

On the other hand, balanced manifolds are not open under small deformations, unlike the Kähler ones, see [210]. Indeed, Alessandrini and Bassanelli in [11] provided a small deformation of the Iwasawa manifold which does not admit any balanced metric.

Turning our attention on more analytic and curvature aspects, balanced metrics can also be characterized by the coincidence of the Laplacians, as in the Kähler case, on smooth functions.

Theorem 1.1.54 ([151], Proposition 1)**.** *Let (M,ω) be a compact Hermitian manifold. Then, ω is balanced if and only if*

$$\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_d = -\Delta_\omega\,, \quad \text{on } C^\infty(M,\mathbb{R})\,.$$

Theorem 1.1.54 allows us to work equivalently in the balanced case, as in the Kähler one, with the Chern Laplacian instead of the Hodge-Riemannan one. This, referring also to Theorem 1.1.20, guarantees that $\Delta_\omega = \Delta_\omega^*$, i.e. Δ_ω is self-adjoint.

From the curvature point of view, balanced metrics are the unique Hermitian metrics whose Ricci 2-forms coincide, using (1.14).

Proposition 1.1.55. *Let (M,ω) be a compact Hermitian manifold. Then, ω is balanced if and only if*

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega) = \mathrm{Ric}^t(\omega)\,, \quad t\in\mathbb{R}\,.$$

Moreover, ω is balanced if and only if

$$s^{\mathrm{Ch}}(\omega) = s^t(\omega)\,, \quad t\in\mathbb{R}\,.$$

We shall also remark that the same condition holds even in the almost-Hermitian case, see [335, Corollary 3.3].

Every balanced metric is equipped with a Bott-Chern cohomology class which is usually called *balanced class*:

$$[\omega^{n-1}]_{\mathrm{BC}} \in H^{n-1,n-1}_{\mathrm{BC}}(M,\mathbb{R})\,.$$

With the balanced class in hand, one can easily observe that the Gauduchon degree of the conformal class of ω, defined in (1.17), is completely depending on the balanced class and the first Bott-Chern class of M. Indeed,

$$\Gamma(\{\omega\}) = \int_M \mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\frac{\omega^{n-1}}{(n-1)!} = \frac{2\pi}{(n-1)!}c_1^{\mathrm{BC}}(M)\cdot[\omega^{n-1}]_{\mathrm{BC}}\,. \tag{1.18}$$

As it is classically known, the Kähler class of a Kähler metric is $[\omega]\in H^{1,1}_{\mathrm{dR}}(M,\mathbb{R})$. Thanks to the $\partial\bar\partial$-Lemma, any other form cohomologous to ω will be of the form:

$$\omega + \sqrt{-1}\partial\bar\partial f\,, \quad f\in C^\infty(M,\mathbb{C})\,.$$

On the other hand, given a balanced metric ω, $\alpha\in[\omega^{n-1}]_{\mathrm{BC}}$ can be written as

$$\alpha = \omega^{n-1} + \sqrt{-1}\partial\bar\partial\varphi\,, \quad \varphi\in\Lambda^{n-2,n-2}M\,.$$

This fact is crucial when discussing the existence of balanced metrics with certain curvature properties in a fixed balanced class. Indeed, the great amount of possible deformations of the fixed balanced metric in its balanced class leads to the choice of ansatz in order to reduce the problem to an easier one. We will see how this problem arises in analysing the existence of constant Chern scalar curvature balanced metric on the blow-up of a fixed Chern-Ricci flat manifold, in Chapter 2.

1.1.3 SKT metrics

Let us now turn our attention on the SKT condition.

Definition 1.1.56. Let (M,ω) be a Hermitian manifold. The metric ω is called *Strong Kähler with torsion*, or *SKT*, for short, if

$$\sqrt{-1}\partial\bar\partial\omega = 0\,.$$

Equivalently, a Hermitian metric ω is SKT if the torsion of the Bismut connection H, see (1.11), is d-closed.

From a physical point of view, the SKT condition appeared to be relevant in type II string theory and 2-dimensional supersymmetric σ-model, see for instance [150, 196, 308]. From the mathematical point of view, besides being a generalization of the Kähler one, the SKT condition is of particular interest in generalized Kähler Geometry. Indeed, as proved by [28, 177], a generalized Kähler manifold is a quadruple (M, J_+, J_-, g) where $J_\pm$ are complex structures and g is $J_\pm$-Hermitian such that:

$$d^c_+\omega_+ = -d^c_-\omega_- \,, \quad dd^c_+\omega_+ = 0 \,.$$

In [52], Bismut proved a local index theory for non-Kähler manifolds carrying a SKT metric. Using Theorem 1.1.35 we know that any compact complex surface carries SKT metric in any conformal class of a given Hermitian metric. Examples of compact quotients of Lie groups admitting a left-invariant SKT structure were found in [18, 70, 81, 98, 116, 117, 118, 119, 120, 121, 124, 127, 174, 178, 238, 251, 253, 291]. As regards non-invariant ones, examples of SKT metrics are provided in [170] on some principal torus bundle over Kähler surfaces, such as on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$, for any $k \geq 1$.

Fino and Tomassini in [123] proved that the SKT condition is preserved for blow-ups along complex submanifolds. This, in particular, thanks to Hironaka's Theorem on resolution of singularities, see Theorem 1.1.41, implies that any complex orbifold endowed with a SKT metric admits a SKT resolution. Arroyo and Nicolini in [33] gives a procedure to produce SKT nilpotent Lie algebras starting from a fixed one. Barberis and Fino in [44] gives another procedure to construct strong HKT Lie algebras starting from a given one via quaternionic representations. We will analyse these procedures in Subsection 4.7.1 and in Subsection 4.7.2 in the hypercomplex setting. Via a twisting construction, Swann in [310] produced new examples of SKT metrics starting from a compact simply connected SKT manifold satisfying some mild assumptions. Other examples are constructed in [60, 61] via mapping tori of a product of a 3-dimensional torus or a 3-sphere and a (hyper)Kähler manifold.

SKT metrics are widely believed to be at the opposite pole with respect to the balanced ones, in the non-Kähler setting. This is mainly due to the Fino-Vezzoni conjecture.

Conjecture 1.1.57 ([125], Problem 3)**.** *Let M be a compact complex manifold. If M admits a balanced and a SKT metric, then M is Kähler.*

Moreover, as in contrast with the balanced setting, where flow approaches are way more difficult and not so well investigated, SKT metrics are preserved by a second-order parabolic flow which is called pluriclosed flow, introduced by Streets and Tian in [302].

Definition 1.1.58. Let (M, g_0) be a Hermitian manifold. The *pluriclosed flow* is the following evolution equation:

$$\frac{\partial}{\partial t} g = -S + Q \,, \quad g(0) = g_0 \,, \tag{1.19}$$

where $S_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}}$ is the *second Chern-Ricci tensor* while $Q_{i\bar{j}} = g^{k\bar{l}} g^{m\bar{n}} T_{ik\bar{n}} T_{\bar{j}\bar{l}m}$

The tensors R and T in Definition 1.1.58 are, respectively, the curvature tensor and the torsion tensor of the Chern connection of the evolving metric.

The pluriclosed flow is part of a larger class of geometric flows, called *Hermitian curvature flows*, introduced by Streets and Tian in [303], which share many common properties, such as short-time existence, the preservation of Kählerianity along the flow and the stability of Kähler-Einstein metrics. However, among the Hermitian curvature flows, the pluriclosed flow preserves the SKT condition.

Theorem 1.1.59 ([302], Theorem 3.4)**.** *Let (M, g_0) be a Hermitian manifold endowed with a SKT metric. Then, $(g(t))_{t\in[0,T)}$ is a solution of the pluriclosed flow starting at g_0 if and only if the family $(\omega(t))_{t\in[0,T)}$ of the fundamental forms associated to $g(t)$ is a solution of*

$$\frac{\partial}{\partial t}\omega = \partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega - \mathrm{Ric}^{\mathrm{Ch}}(\omega) \,, \quad \omega(0) = \omega_0 \,.$$

Using (1.14)*, we have that*

$$\frac{\partial}{\partial t}\omega = -(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1} \,, \quad \omega(0) = \omega_0 \,. \tag{1.20}$$

Moreover, the pluriclosed flow preserves the SKT condition.

Equation (1.20) highlights how the pluriclosed flow resembles the Kähler-Ricci flow, since it evolves a given Hermitian metric in the direction of a certain Ricci form. The pluriclosed flow has gained much importance in the last years, also due to its connection with generalized Kähler geometry. Moreover, it is believed that the pluriclosed flow, also in view of the fact that SKT metrics always exist on compact complex surfaces, recall Theorem 1.1.35, can be used as a tool to complete the classification of non-Kähler compact surfaces, see [299] for the precise conjectural picture. In Section 3.2, we will be concerned in the study of the generalized Ricci flow which is a geometric flow related to the pluriclosed flow. We refer to Section 1.2 for the precise relation. On the other hand, we here recall the following proposition due to Streets and Tian.

Proposition 1.1.60 ([305], Proposition 6.3, 6.4)**.** *Let (M,J,ω_0) be a SKT manifold and let $(\omega(t))_{t\in[0,T)}$ be a solution of* (1.20)*. Then,*

$$\begin{aligned}\frac{\partial}{\partial t}g &= -\operatorname{Ric}(g)+\frac{1}{4}H^2-\frac{1}{2}\mathcal{L}_{\theta^\sharp}g\,,\\ \frac{\partial}{\partial t}H &= \frac{1}{2}(\Delta_g H-\mathcal{L}_{\theta^\sharp}H)\,,\end{aligned}$$

where $\operatorname{Ric}(g)$ *is the classical Ricci tensor of g, Δ_g is the Laplace-Beltrami operator of g, $H=d^c\omega$, $\theta^\sharp$ is the Lee vector field and H^2 is the symmetric* 2*-tensor defined as follows:*

$$H^2(X,Y):=g(\iota_X H,\iota_Y H)\,,\quad X,Y\in\Gamma(TM)\,. \tag{1.21}$$

The importance of Proposition 1.1.60 is that it connects, via a gauge transformation generated by the Lee vector field of the time-varying metric, the pluriclosed flow with the coupled flow

$$\begin{aligned}\frac{\partial}{\partial t}g &= -\operatorname{Ric}(g)+\frac{1}{4}H^2\,,\\ \frac{\partial}{\partial t}H &= \frac{1}{2}\Delta_g H\,,\end{aligned}$$

which is closely related to the generalized Ricci flow. This, in particular, shows that the pluriclosed flow is gauge-equivalent to a gradient flow, see [250].

To conclude this section, let us briefly introduce and discuss pluriclosed solitons, which are particular solutions of the pluriclosed flow evolving self-similarly. A deep treatment of such topic in the general case can be found in [224].

Definition 1.1.61. Let (M,ω) be a SKT manifold. The metric ω is said to be a *pluriclosed soliton* if there exist $\lambda\in\mathbb{R}$ and a holomorphic vector field X such that

$$(\operatorname{Ric}^{\mathrm{B}}(\omega))^{1,1}=\lambda\omega+\mathcal{L}_X\omega\,. \tag{1.22}$$

Moreover, ω will be called *expanding, steady* or *shrinking* pluriclosed soliton if, respectively, $\lambda<0$, $\lambda=0$ or $\lambda>0$.

Definition 1.1.61 can be considered as the "static" definition of pluriclosed solitons, since no flow is involved in it. On the other hand, Definition 1.1.61 is equivalent to impose a precise behaviour on the solution of the pluriclosed flow starting from ω, as the next lemma shows.

Lemma 1.1.62. *Let (M,ω) be a SKT manifold. Then, ω is a pluriclosed soliton if and only if there exist $c(t)>0$, $c(0)=1$ and $\varphi_t\in\operatorname{Aut}(M,J)$ such that the solution of the pluriclosed flow $(\omega(t))_{t\in[0,T)}$ starting from ω can be written as:*

$$\omega(t)=c(t)\varphi_t^*\omega\,,\quad t\in[0,T)\,. \tag{1.23}$$

Proof. Clearly, if (1.23) holds, then it is sufficient to differentiate with respect to t and compute the result in $t=0$ to obtain (1.22). The converse is achieved by following the steps in [319, Proposition 1.2.1]. □

As (1.23) highlights, pluriclosed solitons are precisely those SKT metrics that evolve under the pluriclosed flow just by scalings and by the action of biholomorphisms. Despite having a quite simple evolution and being straightforward generalizations of *static* metrics, see [303], the importance of solitons also arises from the fact that they appear as models of the asymptotic behaviour of general solutions of the flow considered, as in the celebrated works by Perelman for the Ricci flow, see [258, 259, 260]. This behaviour was also observed in various works such as in [54] on some compact complex surfaces, in [32] on SKT nilmanifolds and SKT almost abelian solvmanifolds, and in [141] on SKT Oeljeklaus-Toma manifolds.

1.2 Basics in generalized Geometry

This section is dedicated to describe the basic notions of generalized Geometry. We mainly follow [149]. In particular, we give the definition of exact Courant algebroids and study their first properties. Then, we focus our attention on generalized metrics and on the generalized Ricci curvature on the generalized tangent bundle. Then, finally, we introduce the generalized Ricci flow, discuss its classical formulation and its gauge-fixed versions, ultimately showing that it is gauge equivalent to the pluriclosed flow, recall Definition 1.1.58.

For additional motivation and details on exact Courant algebroids we refer the reader to [84, 177, 236, 281, 282, 283] and the references therein.

1.2.1 Generalized metrics and generalized Ricci flow

We start this subsection with the definition of the fundamental objects in generalized Geometry, namely exact Courant algebroids.

Definition 1.2.1. Let M^n be a smooth manifold. An *exact Courant algebroid*, ECA, for short, is the datum of a vector bundle $E \to M^n$ of rank $2n$ endowed with a non-degenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$ of signature (n, n), called the *neutral inner product*, a bracket $[\cdot, \cdot]$ on $\Gamma(E)$, called the *Dorfman bracket* and a bundle map $\pi : E \to TM$, called the *anchor map*, such that, for all $a, b, c \in \Gamma(E)$ and $f \in C^\infty(M)$, the following axioms are satisfied:

1. $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$;
2. $\pi[a, b] = [\pi a, \pi b]$;
3. $[a, fb] = f[a, b] + \pi(a)fb$;
4. $\pi(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$;
5. $[a, b] + [b, a] = (\pi^* \circ d)\langle a, b\rangle$;
6. there is an exact sequence of vector bundles

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0\,. \tag{1.24}$$

In the last two axioms we have abused notation and considered $\pi^* : T^*M \to E^*$ as a map $\pi^* : T^*M \to E$, after composing with the isomorphism $E^* \simeq E$ given by $\langle \cdot, \cdot \rangle$. Notice also that d in Item 5 is the exterior differential in M, and the bracket in the right-hand-side of Item 2 is the Lie bracket of vector fields in M. The Dorfman bracket is not necessarily skew-symmetric, but it does satisfy the Jacobi identity, see [149, Lemma 2.5] for the proof. Although we will not be making use of it, it is worth mentioning that, in the literature, the skew-symmetrization of the Dorfman bracket is called *Courant bracket* and it is defined by:

$$[a, b]_c := \frac{1}{2}([a, b] - [b, a])\,, \quad a, b \in \Gamma(E)\,.$$

Despite being skew-symmetric, the Courant bracket does not satisfy the Jacobi identity. Indeed, one can show that the following

$$[a,[b,c]_c]_c + [c,[a,b]_c]_c + [b,[c,a]_c]_c = \frac{1}{3} d(\langle [a,b]_c, c\rangle + \langle [b,c]_c, a\rangle + \langle [c,a]_c, b\rangle)$$

holds, for all $a, b, c \in \Gamma(E)$.

The first example of exact Courant algebroid is the generalized tangent bundle. On this, we can consider different structures of exact Courant algebroids depending on the choice of a closed 3-form on M.

Example 1.2.2. The generalized tangent bundle $TM \oplus T^*M$ has a natural structure of exact Courant algebroid, with data given by:

$$\langle X+\xi, Y+\eta\rangle := \tfrac{1}{2}(\xi(Y)+\eta(X)), \qquad [X+\xi, Y+\eta] := [X,Y]+\mathcal{L}_X\eta - \iota_Y d\xi, \qquad \pi(X+\xi) := X, \tag{1.25}$$

for $X+\xi, Y+\eta \in \Gamma(TM \oplus T^*M)$.

Moreover, for any closed 3-form $H \in \Lambda^3 M$, we can consider another ECA structure on $T \oplus T^*$, denoted by

$$(T \oplus T^*)_H := (TM \oplus T^*M, \langle\,\cdot\,,\cdot\,\rangle, [\cdot,\cdot]_H, \pi).$$

Here, $\langle\,\cdot\,,\cdot\,\rangle$ and π are as in Equation (1.25), and $[\cdot,\cdot]_H$ denotes the *H-twisted Dorfman bracket* defined by:

$$[X+\xi, Y+\eta]_H := [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi + \iota_Y\iota_X H, \qquad X+\xi, Y+\eta \in \Gamma(TM \oplus T^*M). \tag{1.26}$$

Before we define the notion of isomorphism between two exact Courant algebroids, we recall the notion of isomorphism between vector bundles.

Definition 1.2.3. Let E_1 and E_2 be two vector bundles over, respectively, M_1 and M_2. A *vector bundle isomorphism* between E_1 and E_2 is a pair (f, F) consisting in a diffeomorphism $f\colon M_1 \to M_2$ and a map $F\colon E_1 \to E_2$ such that the following diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes and such that F restricts to a linear isomorphism on each fibre.

Now, we can specialize the notion of vector bundle isomorphism in the context of exact Courant algebroids.

Definition 1.2.4. Let $(E_i, \langle\,\cdot\,,\cdot\,\rangle_i, [\cdot,\cdot]_i, \pi_i)$ be two exact Courant algebroids over, respectively, M_i, $i = 1,2$. An isomorphism of vector bundles (f, F) is called *isomorphism of exact Courant algebroids* if, for any $a, b \in \Gamma(E_1)$, we have:

1. $\langle Fa, Fb\rangle_2 = \langle a, b\rangle_1$;
2. $[Fa, Fb]_2 = F[a,b]_1$.

It follows from the second condition above that $df \circ \pi_1 = \pi_2 \circ F$, see for instance [208, Lemma 2.5] for the detailed proof.

The next proposition guarantees that any exact Courant algebroid is isomorphic to one described in Example 1.2.2.

Proposition 1.2.5 ([149], Proposition 2.10). *Let E be an exact Courant algebroid. Any isotropic splitting σ of* (1.24) *induces an isomorphism of exact Courant algebroids*

$$E \simeq_\sigma (T \oplus T^*)_H \,.$$

Here $H \in \Lambda^3 M$ is the closed 3-form given by

$$H(X, Y, Z) = 2 \langle [\sigma X, \sigma Y], \sigma Z \rangle, \qquad X, Y, Z \in \Gamma(TM). \tag{1.27}$$

Moreover, given another isotropic splitting $\tilde{\sigma}$, the corresponding isomorphism satisfies

$$E \simeq_{\tilde{\sigma}} (T \oplus T^*)_{H+db} \,,$$

for some $b \in \Lambda^2 M$.

More generally, Ševera's classification of exact Courant algebroids is the content of the following theorem.

Theorem 1.2.6 ([281]). *The isomorphism classes of exact Courant algebroids over M are in one-to-one correspondence with $H^3(M, \mathbb{R})/\operatorname{Diff}(M) = H^3(M, \mathbb{R})/\Gamma_M$, where $\Gamma_M = \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is the mapping class group of M.*

Given the correspondence in Proposition 1.2.5, we can characterise *automorphisms* of any exact Courant algebroid. Before stating the result itself, let us introduce two classes of maps which play an important role in generalized Geometry. Any $b \in \Lambda^2 M$ may be viewed as a map $b : TM \to T^*M$ and thus it gives rise to an endomorphism of the generalized tangent bundle which in matrix form is represented by

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : T \oplus T^* \longrightarrow T \oplus T^*.$$

We then set

$$e^b := \begin{pmatrix} \mathrm{Id} & 0 \\ b & \mathrm{Id} \end{pmatrix} \in \operatorname{End}(T \oplus T^*), \qquad b \in \Lambda^2 M.$$

The latter transformations are classically known as *B-field transformations* which appear naturally as a generalization of the electromagnetic field in string theory.
Additionally, fixed $f \in \operatorname{Diff}(M)$, we can produce a bundle map of the generalized tangent bundle covering f, as follows:

$$\bar{f} := \begin{pmatrix} df & 0 \\ 0 & (f^{-1})^* \end{pmatrix} : T \oplus T^* \to T \oplus T^* \,.$$

Then, the group of automorphism of $(T \oplus T^*)_H$, and then of any ECA, consists of bundle maps which are composition of maps introduced above.

Theorem 1.2.7 ([149], Proposition 2.21). *For any closed 3-form $H \in \Lambda^3 M$, the group of automorphisms of the exact Courant algebroid $(T \oplus T^*)_H$ is given by:*

$$\operatorname{Aut}((T \oplus T^*)_H) = \left\{ \left(\bar{f} e^b, f \right) \mid f \in \operatorname{Diff}(M), \quad b \in \Lambda^2 M, \quad f^* H = H - db \right\}.$$

Moreover, the Lie algebra of $\operatorname{Aut}((T \oplus T^*)_H)$ *is*

$$\mathfrak{aut}((T \oplus T^*)_H) = \{ X + b \in \Gamma(TM) \oplus \Lambda^2 M \mid \mathcal{L}_X H = -db \}.$$

We refer the reader to [278] for a detailed study of the ILH Lie group structure on $\operatorname{Aut}((T \oplus T^*)_H)$.

In the context of generalized Geometry, there is a natural generalization of classical Riemannian metrics.

Definition 1.2.8. A *generalized metric* on an ECA E is an orthogonal endomorphism $\mathcal{G} \in \mathsf{O}(E, \langle\cdot,\cdot\rangle)$ such that the bilinear form

$$(a,b) \mapsto \langle \mathcal{G}a, b\rangle, \qquad a,b \in \Gamma(E),$$

is symmetric and positive definite. We call the pair $(E,\mathcal{G})$ a *metric Courant algebroid.* A Courant algebroid isomorphism $(F,f)\colon (E_1,\mathcal{G}_1) \to (E_2,\mathcal{G}_2)$ is a *metric Courant algebroid isometry* (or simply *isometry*) if $F \circ \mathcal{G}_1 = \mathcal{G}_2 \circ F$.

The presence of a generalized metric on E allows us to consider a preferred isotropic splitting. We quickly recall its construction since it will be useful in what follows. We refer the reader to [149, Section 2.3] for the detailed description. Since $\mathcal{G}^2 = \mathrm{Id}$, we can consider the eigenbundles $E_\pm$ of E associated to ± 1. It is not hard to see that $\pi\colon E_\pm \to TM$ is an isomorphism of vector bundles. Then, we can define

$$\sigma_\pm := (\pi_{|E_\pm})^{-1}\colon TM \to E_\pm\,, \quad \tau_\pm(X,Y) = \langle \sigma_\pm X, \sigma_\pm Y\rangle\,, \quad X,Y \in \Gamma(TM)\,.$$

With these ingredients, we can consider the following splitting:

$$\sigma = \sigma_\pm - \frac{1}{2}\pi^*\tau_\pm\,.$$

It is easy to see that σ is isotropic and it does not depend on the choices of $\sigma_\pm$, see [149, Lemma 2.29] for the proof. Moreover, we have that

$$\tau_+(X,X) = \langle \mathcal{G}\sigma_+ X, \sigma_+ X\rangle > 0\,, \quad X \in \Gamma(TM)\,,\, X \neq 0\,, \tag{1.28}$$

thus determining a Riemannian metric g on M.

Proposition 1.2.9 ([149], Proposition 2.38)**.** *Let E be an ECA with isotropic splitting σ. Under the isomorphism $E \simeq_\sigma (T\oplus T^*)_H$, any generalized metric on E corresponds to a generalized metric on $(T\oplus T^*)_H$ of the form*

$$\mathcal{G}(g,b) := e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b},$$

where $b \in \Lambda^2 M$ and g is a Riemannian metric on M. Moreover, the bundle map

$$\left(e^{-b}, \mathrm{Id}_M\right)\colon\ ((T\oplus T^*)_H, \mathcal{G}(g,b)) \to ((T\oplus T^*)_{H+db}, \mathcal{G}(g,0))\,, \tag{1.29}$$

is an isometry.

Remark 1.2.10. The isometry in (1.29) indicates that a generalized metric $\mathcal{G}$ on an ECA E with Ševera class $\alpha \in H^3(M,\mathbb{R})$ is equivalent to a choice of Riemannian metric g on M and a *preferred representative* $H \in \alpha$. This can be also be observed by considering the *preferred* isotropic splitting induced by $\mathcal{G}$ and the corresponding isomorphism $E \simeq (T\oplus T^*)_H$ under which $\mathcal{G}$ corresponds to $\mathcal{G}(g,0)$.

Furthermore, the existence of a preferred isotropic splitting induced by a generalized metric $\mathcal{G}$ allows us to characterize generalized isometries on a fixed exact Courant algebroid.

Proposition 1.2.11 ([149], Proposition 2.41)**.** *The group of generalized isometries of the metric Courant algebroid $((T\oplus T^*)_H, \mathcal{G}(g,0))$ is given by:*

$$\mathsf{Iso}((T\oplus T^*)_H, \mathcal{G}(g,0)) = \left\{ \left(\bar f, f\right) \mid f \in \mathsf{Iso}(M,g), \quad f^*H = H \right\} \subset \mathrm{Aut}((T\oplus T^*)_H).$$

Once a generalized metric is available, one can hope to define, as in the classical case, a tensor which is the equivalent version of the Ricci tensor in the context of generalized Geometry. We will just briefly discuss the explicit expression of the generalized Ricci curvature in the case of $((T\oplus T^*)_H, \mathcal{G}(g,0))$ which will be central in what follows. For a detailed investigation of the general definition, we refer to [82, 144, 149, 282, 283]. Before doing that, we recall that if (M, g) is a Riemannian manifold and $H \in \Lambda^3 M$, there exist unique connections $\nabla^{\pm}$, called *Bismut connections*, such that

$$\nabla g = 0\,, \quad gT^{\pm} = \pm H\,,$$

where the tensor $T^{\pm}$ is the torsion of $\nabla^{\pm}$. In what follows, we will abuse the notation denoting with

$$\mathrm{Ric}^{\mathrm{B}}_{g,H} := \mathrm{Ric}(g) - \frac{1}{4} H^2$$

the symmetric part of the Ricci tensor of the Bismut connections $\nabla^{\pm}$ associated to g and H. The symmetric 2-tensor H^2 is defined as in (1.21).
Then, following the notation in [149], the generalized metric $\mathcal{G}(g, 0)$ determines two eigenbundles as above, which, in this particular case, takes the following form:

$$E_{\pm} = \{X \pm g(X) : X \in \Gamma(TM)\}\,.$$

Now, using [149, Definition 3.31] and [149, Proposition 3.30], it is easy to see that, for any $X \in \Gamma(TM)$,

$$\begin{aligned} \mathcal{R}c^{+}(X - g(X)) &= g^{-1}\mathrm{Ric}^{\mathrm{B}}_{g,H}X - \frac{1}{2}g^{-1}d^*_g HX + \mathrm{Ric}^{\mathrm{B}}_{g,H}X - \frac{1}{2}d^*_g HX\,, \\ \mathcal{R}c^{-}(X + g(X)) &= -\,g^{-1}\mathrm{Ric}^{\mathrm{B}}_{g,H}X - \frac{1}{2}g^{-1}d^*_g HX + \mathrm{Ric}^{\mathrm{B}}_{g,H}X + \frac{1}{2}d^*_g HX\,. \end{aligned} \tag{1.30}$$

$$\mathcal{R}c(\mathcal{G}) = \mathcal{R}c^{+} - \mathcal{R}c^{-} = \begin{pmatrix} g^{-1}\mathrm{Ric}^{\mathrm{B}}_{g,H} & \frac{1}{2}\,g^{-1}\,d^*_g H g^{-1} \\ -\frac{1}{2}d^*_g H & -\mathrm{Ric}^{\mathrm{B}}_{g,H}g^{-1} \end{pmatrix}.$$

In [282, Theorem 3], the authors prove the $\mathrm{Aut}(E)$-equivariance of the generalized Ricci curvature which will be extensively used in Section 3.2.

Now, since a generalized Ricci tensor is available, we can mimick the definition of the classical Ricci flow to define a flow of generalized metrics which is called generalized Ricci flow.

Definition 1.2.12. A one-parameter family $(\mathcal{G}(t))_{t\in[0,T)}$ of generalized metrics on an ECA E is a solution to the *generalized Ricci flow* if it satisfies

$$\mathcal{G}^{-1}\frac{\partial}{\partial t}\mathcal{G} = -2\mathcal{R}c(\mathcal{G}). \tag{1.31}$$

If we use the isotropic splitting σ_0 associated to the initial metric $\mathcal{G}(0)$ to identify $E \simeq_{\sigma_0} (T \oplus T^*)_{H_0}$, then $\mathcal{G}(t)$ corresponds to $\mathcal{G}(g(t), b(t))$, for some one-parameter families of Riemannian metrics $g(t)$ and two-forms $b(t)$ on M. Then, Equation (1.31) is equivalent to the following coupled system:

$$\begin{aligned} \frac{\partial}{\partial t}g &= -2\,\mathrm{Ric}^{\mathrm{B}}_{g,H}, \\ \frac{\partial}{\partial t}b &= -d^*_g H, \end{aligned} \tag{1.32}$$

where $H(t) = H_0 + db(t)$. Since H_0 is closed, it follows that $H(t)$ evolves accordingly to the classical heat equation:

$$\frac{\partial}{\partial t}H = \Delta_g H.$$

We should emphasize that Equation (1.32) firstly appeared in [69] as the *renormalization group flow*. We refer to [296] for a detailed explanation of the relation between the renormalization group flow and generalized Geometry.

Since the generalized Ricci curvature is $\mathrm{Aut}(E)$-equivariant, the PDE system (1.31) defining the generalized Ricci flow is only weakly parabolic. A suitable application of the DeTurck trick can be applied to obtain short-time existence and uniqueness, provided M is compact, see [149, Theorem 5.6] for details. The presence of the gauge group $\mathrm{Aut}(E)$ allows us to produce different but equivalent flows starting from the generalized Ricci flow.

Definition 1.2.13. A one-parameter family $(\mathcal{G}(t))_{t\in[0,T)}$ of generalized metrics on an ECA E solves the *gauge-fixed generalized Ricci flow* if there exists a one-parameter family $(F_t)_{t\in[0,T)} \subseteq \mathrm{Aut}(E)$ so that $F_t\mathcal{G}(t)F_t^{-1}$ is a solution to the generalized Ricci flow (1.31).

Using again the isotropic splitting σ_0 induced by $\mathcal{G}(0)$ to obtain $(E, \mathcal{G}(t)) \simeq_{\sigma_0} ((T\oplus T^*)_{H_0}, \mathcal{G}(g(t), b(t)))$ as above, it can be seen that $\mathcal{G}(t)$ is a solution of the gauge-fixed generalized Ricci flow if only if there exists a one-parameter family of generalized vector fields $X_t + B_t \in \mathfrak{aut}((T\oplus T^*)_{H_0})$ such that $(g(t), b(t))_{t\in[0,T)}$ is a solution of the following coupled system:

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\,\mathrm{Ric}^{\mathrm{B}}_{g,H} + \mathcal{L}_X g\,, \\ \frac{\partial}{\partial t} b &= -d_g^* H - B + \mathcal{L}_X b\,. \end{aligned} \tag{1.33}$$

It follows that $H(t)$ will evolve by:

$$\frac{\partial}{\partial t} H = \Delta_g H + \mathcal{L}_X H\,. \tag{1.34}$$

A first remark we can do is that, using Proposition 1.1.60, up to a time reparametrization, a solution of the pluriclosed flow is a solution of a gauged fixed generalized Ricci flow where the gauge is generated by the generalized vector field $-\theta^\sharp - d\theta + \iota_{\theta^\sharp} H_0 + d\iota_{\theta^\sharp} b \in \mathfrak{aut}((T\oplus T^*)_{H_0})$.

Remark 1.2.14. Streets and Tian in [304, Corollary 3.3] proved that, given a SKT Hermitian manifold (M, J, g), the generalized Ricci flow coupled with an evolution equation for J:

$$\frac{\partial}{\partial t} J = \Delta J + [J, g^{-1}\mathrm{Ric}(g)] + \mathcal{Q}(DJ)\,, \tag{1.35}$$

where $\mathcal{Q}(DJ)$ is an appropriate quadratic term in the covariant derivative of J with respect to the Levi-Civita connection, see [304, (3.24)] for the precise expression of $\mathcal{Q}(DJ)$, preserves the SKT condition, namely $g(t)$ is $J(t)$-Hermitian and SKT for all times of existence. Then, gauging with the family of diffeomorphisms generated by $(-J(t)d^*_{g(t)}\omega(t))^\sharp$ the generalized Ricci flow coupled with (1.35) and using [304, Proposition 3.1], we obtain a solution for the pluriclosed flow, up to a time reparametrization.

The generalized Ricci flow recently has gained much importance and many properties of the flow are now known. For instance, we have the following.

Theorem 1.2.15 ([149], Theorem 5.23)**.** *Let E be a ECA over M. Assume that the solution of the generalized Ricci flow starting from a given generalized metric $\mathcal{G}_0$ exists on the maximal time interval $[0, T)$, $T < \infty$. Then,*

$$\limsup_{t\to T} \sup_{M\times\{t\}} |\mathrm{Rm}| = \infty\,.$$

A better version of the above result was obtained by Streets and Tian in [305, Theorem 1.2] for the pluriclosed flow. Following Perelman's path, in [250] it was showed that the generalized Ricci flow is the gradient flow of

$$\lambda_1(g, H) := \inf_{\{f\,|\,\|e^{-f}\|_{L^1}=1\}} \mathcal{F}(g, H, f)$$

where

$$\mathcal{F}(g, H, f) := \int_M \left(\mathrm{R}_g - \frac{1}{12}|H|^2 + |\nabla f|^2 \right) e^{-f}\mathrm{Vol}_g\,.$$

Other results can be found in [29, 30, 165, 166, 179, 228, 229, 233, 254, 269, 273, 296, 300, 301, 304, 306]. The Bismut flat spaces - which are particular fixed points of the generalized Ricci flow - were classified by [71, 72, 337]. They are covered by compact Lie groups with bi-invariant metric, and the form H given by the contraction of the Lie bracket with said metric. The very first and striking difference with the classical Ricci flow is the presence of non-flat homogeneous fixed point of the generalized Ricci flow.

Indeed, in view of [3], we know that any homogeneous Ricci flat manifold has to be flat. On the other hand, examples of non-flat homogeneous Bismut-Ricci flat manifolds were found in [226, 267, 268]. In [267] the authors construct an example of $\mathsf{SO}(3)$-invariant steady, gradient generalized soliton on $\mathbb{R}^3$, resembling the structure of the classical Bryant soliton for the Ricci flow, see [62].

1.3 Basics in hyperHermitian Geometry

This section is devoted to illustrate the basic features of hyperHermitian manifolds. The section is divided in two subsections. In Subsection 1.3.1, we define what a hypercomplex structure is and how it affects the geometry of the manifold. A twisted operator, formally replacing $\bar{\partial}$, is taken into account allowing us to produce new cohomology rings in the hypercomplex setting. Then, we define the notion of $\mathsf{SL}(n,\mathbb{H})$-manifold and study its relation with the holomorphic triviality of the canonical bundles of the given manifold. In Subsection 1.3.2, we study hyperHermitian metrics and their basic properties. The presence of such metrics allows to define a preferred $(2,0)$-form on which is possible to impose cohomological condition for the metric to be "special". The HyperKähler and the HKT conditions are defined together with other weaker ones, such as the quaternionic Gauduchon and quaternionic balanced condition.

1.3.1 Hypercomplex structures

Let us start recalling the definition of hypercomplex structure on a smooth manifold.

Definition 1.3.1. Let M^{4n} be a smooth manifold. A *hypercomplex structure* (I,J) on M is a pair of complex structure which anti-commutes, i.e.

$$IJ = -JI\,.$$

We will refer to n as the quaternionic dimension of M. We will then say that the triple (M,I,J) is a *hypercomplex manifold*.

As in Section 1.1, we will always assume that both I and J are integrable.

Example 1.3.2. The very first example of hypercomplex manifold is $\mathbb{H}^n$ with the left multiplication by i,j and k, the quaternionic units. Consequently, any open set $A\subseteq\mathbb{H}^n$ is again hypercomplex.
Thanks to a result by Boyer in [56], the unique 4-manifolds admitting hypercomplex structures are the 4-torus, K3 surfaces, i.e. the unique compact simply connected complex surfaces with holomorphically trivial canonical bundle, and the quaternionic Hopf surfaces. We describe in details the latter.
Let $q\in\mathbb{H}$, $|q|>1$ and consider the compact quotient

$$M=\frac{\mathbb{H}\backslash\{0\}}{\langle q\rangle}\,,$$

where $\langle q\rangle$ is the group of right multiplications generated by q. Clearly, the left multiplication by the quaternionic units i and j commutes with the right action of q. Thus, it induces a hypercomplex structure on M. M endowed with the latter hypercomplex structure is called quaternionic Hopf surface. Equivalently, we can regard M as $(\mathbb{C}^2\setminus\{0\})/\sim$ where the equivalence relation is given by:

$$(z,w)\sim(az-\bar{b}w, bz+\bar{a}w)\,,$$

where $a+bj=q$. The standard hypercomplex structure on the universal cover $\mathbb{C}^2\setminus\{0\}$ is given by $Jdz^1=-d\bar{z}^2$ and it descends to M. Clearly, this construction can be generalized in any quaternionic dimension.
Many other examples can be found in the literature. A hypercomplex structure was defined by Joyce in [201] on suitable products of a torus and a compact semisimple Lie group, we refer to Subsection 4.7.3 for a precise description of these examples. Other examples and constructions can be found in [16, 40, 41, 42, 43, 44, 57, 58, 59, 92, 93, 94, 110, 171, 172, 200, 232, 255, 256, 257, 289]

On a hypercomplex manifold (M, I, J), we can define a new complex structure:

$$K := IJ$$

so that (I, J, K) behaves as the quaternionic units, i.e.

$$I^2 = J^2 = K^2 = -\mathrm{Id}\,, \quad IJ = -JI = K\,.$$

The triple (I, J, K) induces a whole 2-sphere of complex structures

$$\mathsf{H} := \{aI + bJ + cK \mid (a, b, c) \in S^2\}\,.$$

Note that $aI + bJ + cK$ and $a'I + b'J + c'K$ anti-commute if and only if $(a, b, c), (a', b', c')$ are orthogonal vectors in $\mathbb{R}^3$. We emphasize that the role of (I, J) is not preferential and we can replace it with any pair of anti-commuting complex structures in H. For this, we will often refer to (M, H) as a hypercomplex manifold, rather than (M, I, J). On the other hand, it will often be useful to think in terms of a fixed basis (I, J, K) for H.

Obviously, for any complex structure in H, the graded algebra of differential forms on M will split according to (1.2). Thus, in what follows, we will denote with $\Lambda^{p,q}_L M$ the space of (p, q)-forms on M with respect to $L \in \mathsf{H}$, emphasizing the complex structure we choose. Since I and J anti-commute, an easy observation is the following:

$$J\Lambda^{p,q}_I M \subseteq \Lambda^{q,p}_I M\,,$$

i.e. J interchanges the bi-degrees with respect to I, where the action of J on $\Lambda^\bullet M$ is defined as in (1.4).

Moreover, the presence of multiple complex structures allows us to define a positivity and realness condition adapted to the hypercomplex setting.

Definition 1.3.3. Let (M, I, J) be a hypercomplex manifold. A differential form $\gamma \in \Lambda^{2p,2q}_I M$ is called *q-real* if $J\bar{\gamma} = \gamma$ and *q-semipositive* (resp. *q-positive*) if additionally

$$\gamma(Z_1, J\bar{Z}_1, \cdots, Z_p, J\bar{Z}_p, \bar{Z}_{p+1}, JZ_{p+1}, \ldots, \bar{Z}_{p+q}, JZ_{p+q}) \geq 0\,, \quad (\text{resp. } > 0)$$

for every non-vanishing $Z_1, \ldots, Z_{p+q} \in \Gamma(T^{1,0}_I M)$. Equivalently,

$$\gamma(X_1, JX_1, \ldots, X_{p+q}, JX_{p+q}) \geq 0\,, \quad (\text{resp. } > 0)$$

for any non-zero $X_1, \ldots, X_{p+q} \in \Gamma(TM)$.

We should emphasize that the conjugation in the definition of q-realness is the one induced by the complex structure I.

Moreover, it will be useful to observe the following well-known fact whose proof is essentially the same as in [240, Proof of Theorem 4.7] adapted to the hypercomplex case.

Lemma 1.3.4. *Let (M^n, H) be a hypercomplex manifold. Then, the $(n-1)$-th wedge power is a bijective correspondence between the cone of q-positive $(2, 0)$-forms and the cone of q-positive $(2n-2, 0)$-forms with respect to I.*

Furthermore, having available two anti-commuting complex structures allows us to define a twisted operator, firstly introduced by Verbitsky in [329].

Definition 1.3.5. Let (M, I, J) be a hypercomplex manifold and let $d = \partial + \bar{\partial}$ be the splitting of the exterior differential induced by I. We define the *twisted operator* ∂_J as follows:

$$\partial_J := J^{-1}\bar{\partial}J \colon \Lambda^{p,q}_I M \to \Lambda^{p+1,q}_I M\,.$$

As in the complex case, we have that

$$\partial_J^2 = 0\,, \quad \partial_J \partial = -\partial \partial_J\,, \quad \bar{\partial}\partial_J = -\partial_J \bar{\partial}\,. \tag{1.36}$$

We refer to [89, Lemma 2.12] for the detailed proof. As a first difference from the complex setting, the operator ∂_J raises the same degree as ∂. On the other hand, it is the natural analogue of $\bar{\partial}$ in the hypercomplex setting, from both a cohomological and analytical point of view. Indeed, for instance, we can define the *quaternionic Bott-Chern* and *Aeppli* cohomology as follows:

$$H_{\mathrm{qBC}}(M) := \frac{\ker \partial \cap \ker \partial_J}{\operatorname{Im} \partial \partial_J}\,, \quad H_{\mathrm{qA}}(M) := \frac{\ker \partial \partial_J}{\operatorname{Im} \partial + \operatorname{Im} \partial_J}\,.$$

We will see in Section 4.2 how one can use the quaternionic Bott-Chern cohomology to define, in analogy with the complex case, the first quaternionic Bott-Chern class of a hypercomplex manifold.

An important tool to study the metric properties of a hypercomplex manifold is the following lemma describing the correspondence between $(1,1)$-forms and $(2,0)$-forms with respect to I.

Lemma 1.3.6 ([8]). *Let (M,I,J) be a hypercomplex manifold. Then, the map*

$$\Phi\colon \Lambda^{1,1}_I M \to \Lambda^{2,0}_I M$$

given by

$$\Phi(\gamma)(X,Y) := \frac{\sqrt{-1}\gamma(JX,Y) - \gamma(KX,Y)}{2}\,, \quad \gamma \in \Lambda^{1,1}_I M\,, \quad X,Y \in \Gamma(TM) \tag{1.37}$$

is bijective. Furthermore, $\Phi(\gamma)$ is q-real, respectively q-positive, if and only if γ is real, respectively positive.

In Chapter 4, it will be useful to look at the J-anti-invariant parts of $(1,1)$-forms with respect to I. Thus, let $\gamma \in \Lambda^{1,1}_I M$, then, using (1.37), we have:

$$\Phi\left(\frac{\gamma - J\gamma}{2}\right)(X,Y) = \frac{1}{4}\left(\sqrt{-1}\gamma(JX,Y) + \sqrt{-1}\gamma(X,JY) - \gamma(KX,Y) - \gamma(X,KY)\right)\,, \quad X,Y \in \Gamma(TM)\,.$$

Furthermore, $\Phi(\frac{\gamma - J\gamma}{2})$ is q-real if and only if $\gamma - J\gamma$ is real, i.e. $\gamma - \bar{\gamma}$ is J-invariant. When $\gamma = \sqrt{-1}\bar{\partial}\psi$, we actually have $\Phi(\frac{\gamma - J\gamma}{2}) = \frac{1}{2}\partial_J\psi$ as it is shown in the next lemma, whose proof can be adapted from that of [47, Lemma 2.1] (cf. also [292, Remark 4.1]).

Lemma 1.3.7. *Let (M,I,J) be a hypercomplex manifold. Then, for any $\psi \in \Lambda^{1,0}_I M$, we have*

$$\partial_J\psi(X,Y) = -\frac{1}{2}\left(\bar{\partial}\psi(JX,Y) + \bar{\partial}\psi(X,JY) + \sqrt{-1}\bar{\partial}\psi(KX,Y) + \sqrt{-1}\bar{\partial}\psi(X,KY)\right)\,,$$

for all $X,Y \in \Gamma(TM)$. In particular, $\partial_J\psi$ is q-real if and only if $\bar{\partial}\psi + \partial\bar{\psi}$ is J-invariant.

Before going into the discussion of the metric properties of a hypercomplex manifold, we shall recall here the definition of a current on a hypercomplex manifold, which will be useful in Chapter 4.

Definition 1.3.8. Let (M,I,J) a hypercomplex manifold. The space $\mathcal{D}^{p,q}_I(M)$ of currents of bi-degree (p,q) with respect to I on M is by definition the topological dual to the space of $(2n-p,2n-q)$-forms with respect to I with compact support endowed with the usual family of seminorms, see [87, (2.1) p. 13].

Example 1.3.9. As in the complex case, any

$$\psi \overset{\mathrm{loc}}{=} \sum_{|P|=p\,,|Q|=q} \psi_{P\bar{Q}} dz^P \wedge dz^{\bar{Q}}\,, \quad \psi_{P\bar{Q}} \in L^1_{\mathrm{loc}}\,,$$

where $P = (i_1, \ldots, i_p)$, $Q = (j_1, \ldots, j_q)$ with $i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$ and $dz^P := dz^{i_1} \wedge \ldots \wedge dz^{i_p}$, defines a (p,q)-current T_ψ given by integration:

$$T_\psi(\gamma) := \int_M \psi \wedge \gamma \,,$$

for any compactly supported $\gamma \in \Lambda_I^{2n-p,2n-q} M$.

The action of the hypercomplex structure H naturally extends to currents: $L \in \mathsf{H}$ acts on $T \in \mathcal{D}_I^{p,q}(M)$ in the following way:

$$(LT)(\gamma) := T(L\gamma) \,,$$

for any compactly supported form $\gamma \in \Lambda_I^{2n-p,2n-q} M$. Similarly, the differential operators $\partial, \partial_J \colon \mathcal{D}_I^{p,q}(M) \to \mathcal{D}_I^{p+1,q}(M)$ are extended to (p,q)-currents by duality:

$$(\partial T)(\gamma) := (-1)^{p+q+1} T(\partial \gamma) \,, \quad (\partial_J T)(\gamma) := (-1)^{p+q+1} T(\partial_J \gamma) \,,$$

for any compactly supported $\gamma \in \Lambda_I^{2n-p-1,2n-q} M$. Finally, if $T \in \mathcal{D}_I^{p,q}(M)$, we define the conjugate $\bar{T} \in \mathcal{D}_I^{q,p}(M)$ as $\bar{T}(\gamma) := \overline{T(\bar{\gamma})}$, for any $\gamma \in \Lambda_I^{2n-q,2n-p} M$ with compact support.

Definition 1.3.10. A $(2p,2q)$-current T is called *q-real* if $J\bar{T} = T$. If further $T(\gamma) \geq 0$, for any q-positive $\gamma \in \Lambda_I^{2n-2p,2n-2q} M$, we say that T is *q-positive*.

We conclude this section by discussing the so-called $\mathsf{SL}(n,\mathbb{H})$ condition. In order to define it, we need to introduce the Obata connection, the unique torsion-free connection which preserves the hypercomplex structure.

Theorem 1.3.11 ([246])**.** *Let (M, H) be a hypercomplex manifold. Then, there exists a unique torsion-free connection ∇^{Ob}, called* Obata connection*, such that*

$$\nabla^{\mathrm{Ob}} L = 0 \,, \quad L \in \mathsf{H} \,.$$

As a difference with the Levi-Civita connection, the Obata connection is intrisically related to the hypercomplex structure rather than to the metric one. The explicit formula for the Obata connection can be found, even in the non integrable case, in [4]. In the integrable case, the expression can be simplified resulting in

$$\nabla^{\mathrm{Ob}}_X Y = \frac{1}{2}\left([X,Y] + I[IX,Y] - J[X,JY] + K[IX,JY]\right) \,, \quad X, Y \in \Gamma(TM) \,,$$

we refer to [288] for the proof of the above. The curvature of the Obata connection can be viewed as a measure of how much the hypercomplex manifold fails to be locally isomorphic to open sets of $\mathbb{H}^n$.

Theorem 1.3.12 ([290])**.** *Let (M, H) be a hypercomplex manifold. Then, the Obata connection is flat if and only if M has affine quaternionic transition maps.*

By the Holonomy principle, we immediately have that

$$\mathrm{Hol}(\nabla^{\mathrm{Ob}}) \subseteq \mathsf{GL}(n,\mathbb{H}) \,.$$

On the other hand, inside $\mathsf{GL}(n,\mathbb{H})$, we can define the quaternionic special linear group as follows.

Definition 1.3.13. The quaternionic special linear group $\mathsf{SL}(n,\mathbb{H})$ is defined as the commutator group of $\mathsf{GL}(n,\mathbb{H})$, i.e.

$$\mathsf{SL}(n,\mathbb{H}) := [\mathsf{GL}(n,\mathbb{H}), \mathsf{GL}(n,\mathbb{H})] \,.$$

We should remark that the quaternionic special linear group cannot be defined as the set of matrices with determinant 1 as in the real case, due to an ambiguity on the definition of determinant in the quaternionic case.

Definition 1.3.14. Let (M, H) be a hypercomplex manifold. Then, (M, H) is called $\mathsf{SL}(n, \mathbb{H})$-manifold if

$$\mathrm{Hol}(\nabla^{\mathrm{Ob}}) \subseteq \mathsf{SL}(n, \mathbb{H})\,.$$

The $\mathsf{SL}(n, \mathbb{H})$ assumption is very helpful and often desirable. $\mathsf{SL}(n, \mathbb{H})$-manifolds have been studied extensively, see [9, 172, 197, 230, 231, 232, 333]. One of the main properties of $\mathsf{SL}(n, \mathbb{H})$-manifolds is the following. Since we can identify $\mathsf{SL}(n, \mathbb{H}) = \mathsf{GL}(n, \mathbb{H}) \cap \mathsf{SL}(2n, \mathbb{C})$, one can see that if $(M, \mathbb{H})$ is a $\mathsf{SL}(n, \mathbb{H})$-manifold, $K_{(M,L)}$ is holomorphically trivial, for all $L \in \mathsf{H}$, see for instance [331].

Consequently, Verbitsky in [332] conjectures that, in the compact case, the $\mathsf{SL}(n, \mathbb{H})$ condition is equivalent to $K_{(M,I)}$ being holomorphically trivial. Recently, Andrada e Tolcachier in [17, Example 6.3] provided a counterexample to the above conjecture, which is a compact solvmanifold M admitting a hypercomplex structure (I, J) such that $K_{(M,I)}$ is holomorphically trivial while $K_{(M,J)}$ is not. In view of this, Andrada e Tolcachier posed the following question.

Question 1.3.15 ([17], Remark 6.4)**.** Let (M, H) a hypercomplex manifold such that $K_{(M,L)}$ is holomorphically trivial, for all $L \in \mathsf{H}$. Then, (M, H) is $\mathsf{SL}(n, \mathbb{H})$.

In order to address this question, we will need to consider hyperHermitian metrics, even though the $\mathsf{SL}(n, \mathbb{H})$ condition is not related to those.

1.3.2 HyperHermitian metrics

To start this subsection, we immediately give the definition of hyperHermitian metric.

Definition 1.3.16. Let (M, H) be a hypercomplex manifold. A Riemannian metric g is called *hyperHermitian* if it is Hermitian with respect to any complex structure in H. Then, the triple (M, H, g) will be called *hyperHermitian manifold.*

Example 1.3.17. The flat metric on $\mathbb{H}^n$ is hyperHermitian with respect to the standard hypercomplex structure on $\mathbb{H}^n$. Moreover, as in the real case, it is invariant under traslations, descending to a hyperHermitian metric on T^{4n}.
Let now consider M to be the quaternionic Hopf surface described in Example 1.3.2. The Hermitian metric induced by

$$\omega_I = \frac{\sqrt{-1}}{|z|^2}\left(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2\right) \in \Lambda^{1,1}_I(\mathbb{C}^2 \backslash \{0\}) \tag{1.38}$$

is hyperHermitian with respect to the standard hypercomplex structure on $\mathbb{C}^2 \backslash \{0\}$ and it is invariant under the action of $\langle q \rangle$, $q \in \mathbb{H}$, $|q| > 1$. Thus, it descends to a hyperHermitian metric on M.
Any K3 surface is known to admit a hyperHermitian metric which is additionally hyperKähler, see Definition 1.3.22.

As in the Hermitian case, for any $L \in \mathsf{H}$, we can consider the fundamental form $\omega_L \in \Lambda^{1,1}_L M$ associated to the Hermitian structure (g, L). A straightforward equivalent condition for a I-Hermitian metric on a hypercomplex manifold to be hyperHermitian is the following.

Lemma 1.3.18. *Let (M, I, J) be a hypercomplex manifold and let g be a I-Hermitian metric. Then, g is hyperHermitian if and only if ω_I is J-anti-invariant, i.e. $J\omega_I = -\omega_I$.*

Furthermore, the hyperHermitian structure can also be completely described in terms of a distinguished $(2, 0)$-form with respect to I. Indeed, one can apply the bijection Φ in Lemma 1.3.6 to ω_I to obtain

$$\Omega_I := \Phi(\omega_I) = \frac{\omega_J + \sqrt{-1}\omega_K}{2}\,, \tag{1.39}$$

which is a q-real and q-positive $(2, 0)$-form with respect to I. Conversely, any $(2, 0)$-form with respect to I which is q-real and q-positive induces a hyperHermitian metric. Rewriting (1.39) in terms of ω_I, we have, for every $X, Y \in \Gamma(TM)$,

$$\Omega_I(X,Y) = \frac{\sqrt{-1}\omega_I(JX,Y) - \omega_I(KX,Y)}{2} .$$

As it is customary, in view of this correspondence, we shall often say that Ω_I is a hyperHermitian metric, by a slight abuse of language. Furthermore, for the sake of notation, if no confusion is possible we shall omit the reference to the complex structure I and simply write Ω.

A straightforward observation is that the volume form induced by Ω coincides with the Riemannian one. Indeed, one can easily check that:

$$\frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2} = \frac{\omega_I^{2n}}{(2n)!} , \tag{1.40}$$

where n is the quaternionic dimension of M. Another observation which is in order is that the n-th power of Ω, since it is non-degenerate, will give rise to a nowhere vanishing section of the canonical bundle $K_{(M,I)}$, implying its topological triviality. Then, in particular, any hyperHermitian manifold has vanishing first Chern class. We will see, however, in Section 4.2 that there is a natural generalization of the first Chern and Bott-Chern class in this setting which may distinguish classes of hypercomplex manifolds behaving differently, in some precise sense.
We can however deduce a more precise expression for the n-th power of Ω as follows. We fix I-holomorphic coordinates $\{z_1, \dots, z_{2n}\}$ we can write

$$\Omega = \Omega_{ij} dz^i \wedge dz^j ,$$

where the complex matrix (Ω_{ij}) is skew-symmetric. The Pfaffian $\mathrm{pf}(\Omega_{ij})$ of (Ω_{ij}) is defined via the relation:

$$\frac{\Omega^n}{n!} = \mathrm{pf}(\Omega_{ij})\, dz^1 \wedge \cdots \wedge dz^{2n} . \tag{1.41}$$

From (1.40) we deduce

$$|\mathrm{pf}(\Omega_{ij})|^2 = \det(g_{r\bar{s}}) , \tag{1.42}$$

where $(g_{r\bar{s}})$ is the Hermitian matrix that describes the hyperHermitian metric in the given coordinates. We will see how the Pfaffian of Ω is related to the first Chern-Ricci form of ω_I in Section 4.1. A similar theory as that described in Section 1.1 can be conducted in the hyperHermitian setting. The presence of Ω makes, however, possible to define and study properties, for instance, of the Lefschetz operator associated to Ω.

Definition 1.3.19. Let (M, H, Ω) be a hyperHermitian manifold. We define the *Lefschetz operator* associated with Ω as

$$L_\Omega \colon \Lambda^k M \to \Lambda^{k+2} M , \quad L_\Omega \alpha := \alpha \wedge \Omega , \quad \alpha \in \Lambda^k M .$$

Then, the dual of L_Ω with respect to the standard inner product induced by the hyperHermitian metric on differential forms will be denoted with Λ_Ω, namely

$$g(\Lambda_\Omega \alpha, \beta) = g(\alpha, L_\Omega \beta) , \quad \alpha \in \Lambda^k M , \beta \in \Lambda^{k-2} M .$$

An important property of the Leftschetz operator associated with Ω is the following.

Proposition 1.3.20. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then, for any $p \leq n$,*

$$L_\Omega^{n-p} \colon \Lambda_I^{p,0} M \to \Lambda_I^{2n-p} M$$

is an isomorphism.

The full theory of these operators, mimicking the complex case, can be studied, see for instance [156]. We will not be interested in that, so it will be omitted. However, we will frequently make use of the explicit expression for Λ_Ω on $(2,0)$-forms with respect to I. Let $\xi \in \Lambda^{2,0}_I M$, an easy computation shows that

$$\Lambda_\Omega \xi = n \frac{\xi \wedge \Omega^{n-1}}{\Omega^n} \, .$$

Thanks to this expression, one can easily observe that, if ξ is q-real, then $\Lambda_\Omega \xi$ is a real-valued function on M. Furthermore, at any given point, chosen I-holomorphic coordinates $(z^1, \dots, z^{2n})$ such that $\Omega = \sum_{i=1}^n dz^{2i-1} \wedge dz^{2i}$ we have

$$\Lambda_\Omega \xi = \sum_{i=1}^n \xi \left(\frac{\partial}{\partial z^{2i-1}}, \frac{\partial}{\partial z^{2i}} \right) . \tag{1.43}$$

Recalling (1.7) and choosing $\xi = \Phi(\frac{\gamma - J\gamma}{2})$ in (1.43), by straightforward calculations, we have

$$\Lambda_\Omega \left(\Phi \left(\frac{\gamma - J\gamma}{2} \right) \right) = \frac{1}{2} \Lambda_{\omega_I} \gamma \, , \tag{1.44}$$

where we used that, pointwise in the chosen coordinates, $J \frac{\partial}{\partial z^{2i-1}} = \frac{\partial}{\partial \bar{z}^{2i}}$, for all $i = 1, \dots, n$. In particular, for a function $\varphi \in C^\infty(M, \mathbb{R})$, we recover, from Lemma 1.3.7 and (1.44), the well-known fact that the operator

$$\Delta_\Omega \varphi := \Lambda_\Omega(\partial \partial_J \varphi) = \Lambda_{\omega_I}(\sqrt{-1}\partial\bar{\partial}\varphi)$$

is the Chern Laplacian. Again recalling the definition of Hodge $*$-star operator in Definition 1.1.12, we can deduce the following identities:

$$*\Omega = \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{n!(n-1)!} \, , \quad *\psi = -J\bar{\psi} \wedge \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{n!(n-1)!} \, , \quad \psi \in \Lambda^{1,0}_I M \, .$$

Moreover, one can easily prove that:

$$*\zeta = -J\bar{\zeta} \wedge \frac{\Omega^{n-2} \wedge \bar{\Omega}^n}{n!(n-2)!} + \Lambda_\Omega(J\bar{\zeta}) \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{n!(n-1)!} \, , \quad \zeta \in \Lambda^{2,0}_I M \, . \tag{1.45}$$

Before going into the discussion of special hyperHermitian metrics, we will prove the following general lemma which will be useful in Section 4.5.

Lemma 1.3.21. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then, for every $\psi, \zeta \in \Lambda^{2,0}_I M$, we have:*

$$\psi \wedge \zeta \wedge \frac{\Omega^{n-2}}{(n-2)!} = \left(\Lambda_\Omega(\psi)\Lambda_\Omega(\zeta) - g(\psi, J\bar{\zeta}) \right) \frac{\Omega^n}{n!} \, . \tag{1.46}$$

Proof. Fixed $\psi, \zeta \in \Lambda^{2,0}_I M$, we have that, using (1.45),

$$\begin{aligned} g(\psi, J\bar{\zeta}) \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2} &= -\psi \wedge \zeta \wedge \frac{\Omega^{n-2} \wedge \bar{\Omega}^n}{n!(n-2)!} + \Lambda_\Omega(\zeta)\psi \wedge \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{n!(n-1)!} \\ &= -\psi \wedge \zeta \wedge \frac{\Omega^{n-2} \wedge \bar{\Omega}^n}{n!(n-2)!} + \Lambda_\Omega(\zeta)\Lambda_\Omega(\psi) \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2} \, . \end{aligned}$$

Using that wedging with $\frac{\bar{\Omega}^n}{n!}$ is an isomorphism, we conclude. □

As in the Hermitian case, we can impose cohomological constraints on the form Ω to obtain classes of hyperHermitian metrics which can be considered special. The first one we consider is the hyperKähler condition.

Definition 1.3.22. Let (M, H, Ω) be a hyperHermitian manifold. We say that the hyperHermitian metric is *hyperKähler* if

$$d\Omega = 0\,. \tag{1.47}$$

The hyperKähler condition can be equivalently expressed by requiring that $d\omega_L = 0$, for all $L \in \mathsf{H}$. A straightforward consequence of Proposition 1.1.24 is that (1.47) is equivalent to require that the Obata connection is preserving the hyperHermitian metric and, thus, it coincides with the Levi-Civita connection of the metric. (1.47) readily implies that the trivialization Ω^n of $K_{(M,I)}$ is holomorphic implying that hyperKähler manifolds are Kähler Calabi-Yau manifolds. Moreover, one can show that hyperKähler metric are Ricci-flat, even in the non compact case.

Example 1.3.23. The flat metric on T^{4n} is hyperKähler. Moreover, any K3 surface is hyperKähler. Furthermore, one can produce hyperKähler metrics on the Hilbert scheme $M^{[n]}$, $n \in \mathbb{N}$ when M is either a 4-torus or a K3 surface, see [45]. Other two exceptional and compact examples were found by O'Grady in [248, 249].

Non-compact 4-dimensional ALE, recall Definition 1.1.45, examples of hyperKähler metrics are particular instances of the so-called gravitational instantons. For instance, the Eguchi-Hanson metric, see [97], on T^*S^2 is an example of complete, asymptotically flat metric which is hyperKähler. Calabi in [68] showed that the cotangent bundle of $\mathbb{CP}^n$, $n \in \mathbb{N}$, admits always a complete hyperKähler metric. Examples with S^1-symmetries of the above can be constructed using the Gibbons-Hawking ansatz, see [161]. Kronheimer in [211, 212] gave the full classification of 4-dimensional ALE hyperKähler manifolds.

However, the hyperKähler condition, as we observed above, forces the geometry of the manifold to satisfy some strict properties. Moreover, many examples of hypercomplex manifolds do not admit any hyperKähler metric. The very first example is the quaternionic Hopf surface which is, of course, not HyperKähler since it is not even Kähler.

Thus, we are led to consider weaker cohomological conditions generalizing the hyperKähler one. For sure, the most studied is the hyperKähler with torsion (HKT, for short) condition, firstly considered in [189].

Definition 1.3.24. Let (M, I, J, Ω) be a hyperHermitian manifold. The metric Ω is called *HKT* if

$$\partial\Omega = 0\,,$$

where ∂ is the operator induced by the splitting of d with respect to I.

Originally, the definition of HKT metrics involved the Bismut connections associated to ω_L, for all $L \in \mathsf{H}$.

Theorem 1.3.25 ([174], Proposition 2)**.** *Let (M, H, Ω) be a hyperHermitian manifold. Then, the following are equivalent:*

1. *Ω is HKT;*
2. *the Bismut connections of ω_L coincide, for all $L \in \mathsf{H}$;*
3. $$Id\omega_I = Jd\omega_J = Kd\omega_K\,.$$

Example 1.3.26. For dimensional reasons, any hyperHermitian metric on a quaternionic Hopf surface is HKT. Thanks to a result by Dotti and Fino in [93], any nilmanifold with abelian hypercomplex structure admits a left-invariant HKT metric. Barberis, Dotti and Verbitsky, see [43, Theorem 4.6], proved also the converse, i.e. that the existence of a left-invariant HKT metric forces the hypercomplex structure to be abelian. Dotti and Fino in [92] classified 8-dimensional 2-step nilpotent Lie algebras admitting an abelian hypercomplex structure. Other examples were found by Verbitsky in [330] as a by-product of holomorphic bundles. Joyce's examples, see [201], admits a HKT metric. Inhomogeneous examples of HKT manifolds can be found in [174]. On the other hand, the existence of HKT metrics on a given hypercomplex manifold is not always guaranteed. The first example of hypercomplex manifold not admitting HKT metrics was produced by Fino and Grantcharov in [110]. Then, obstructions to the existence of such metrics were proved, see [172, 289].

Other examples of HKT metrics can be found in [16, 44].

Despite being defined using the Bismut connection, which, as we saw in Section 1.1, is a non-Kähler feature, HKT manifolds are believed to be the analogue of Kähler manifolds in hypercomplex Geometry. The evidences towards this assertion can be found in the work by many authors. First of all, an analogue of the Buchdahl-Lamari criterion [63, 215] for complex surfaces was proved by Grantcharov, Lejmi and Verbistky in [172] for $\mathsf{SL}(2,\mathbb{H})$-manifolds. Cohomologically speaking, the HKT condition allows to obtain similar results as in Theorem 1.1.23, see for instance [329]. Moreover, Banos and Swann in [35, 270], proved that, for HKT metrics, local potentials always exist. Other general properties of HKT metrics can be found in [8, 173, 195, 197, 329, 332]

However, the most important fact that highlights the analogy of HKT manifolds with Kähler ones is the so-called *quaternionic Calabi conjecture*. In order to give the precise statement of the conjecture we need some other preliminaries.

Let (M, I, J, Ω) be a HKT manifold. The HKT condition and the q-realness of Ω allow us to define a cohomological class

$$[\Omega]_{\mathrm{qBC}} \in H^{2,0}_{\mathrm{qBC}}(M)\,,$$

which is also called *HKT class*. Then, any other q-real $(2,0)$-form $\alpha \in [\Omega]_{\mathrm{qBC}}$ will be of the following form:

$$\alpha = \Omega + \partial\partial_J\varphi\,, \quad \varphi \in C^\infty(M,\mathbb{R})\,.$$

The quaternionic Calabi conjecture was firstly stated by Alesker and Verbitsky in [9] and it mimicks the statement of the Calabi conjecture in Kähler Geometry.

Conjecture 1.3.27. *Let (M^n, I, J, Ω) be a compact HKT manifold. Let Θ be a q-real, q-positive, $(2n,0)$-form. Then, there exists a unique $\varphi \in C^\infty(M,\mathbb{R})$ such that*

$$(\Omega + \partial\partial_J\varphi)^n = \Theta\,, \quad \Omega + \partial\partial_J\varphi > 0\,, \quad \sup_M \varphi = 0\,.$$

As in the Kähler case, the validity of Conjecture 1.3.27 will give rise, in some special cases, to canonical hyperHermitian metrics. Indeed, for instance, we know that on $\mathsf{SL}(n,\mathbb{H})$-manifolds we always have a holomorphic volume form Θ which is q-real. Then, with this particular choice, Conjecture 1.3.27 will produce a balanced HKT metric on M which, in particular, is Chern-Ricci flat.

As for the Calabi conjecture, Conjecture 1.3.27 has a more analytic and equivalent formulation involving the so-called *quaternionic Monge-Ampère equation*.

Conjecture 1.3.28. *Let (M^n, I, J, Ω) be a compact HKT manifold and $F \in C^\infty(M,\mathbb{R})$. Then, there exists a unique $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}$ such that*

$$(\Omega + \partial\partial_J\varphi)^n = e^{F+b}\Omega^n\,, \quad \Omega + \partial\partial_J\varphi > 0\,, \quad \sup_M \varphi = 0\,. \tag{1.48}$$

Partial results are known for Conjecture 1.3.27 to hold. As far as the author knows, the most general result concerning the validity of the conjecture can be found in [89], where Conjecture 1.3.27 is proved under the assumption of hyperKählerianity of the manifold. The proof of the latter result involves the standard continuity method. Of course, the hardest part is the proof of a priori estimates. Recently, Sroka in [292] proved the C^0 estimate for (1.48) in the general hyperHermitian setting.

As we saw, we can combine the HKT condition with another condition coming from complex Geometry without forcing the metric to be hyperKähler. Indeed, we saw that Conjecture 1.3.27 is essentially formutaled to find balanced HKT metrics on any fixed $\mathsf{SL}(n,\mathbb{H})$-manifold. On the other hand, mainly motivated by the coincidence of the Bismut connections, one can ask d-closedness of its torsion, resulting in the so-called strong HKT metrics.

Definition 1.3.29. Let (M, I, J, Ω) be a HKT manifold. Ω is called *strong HKT* if the torsion of the Bismut connection of ω_I is closed, i.e.

$$dd^c_I\omega_I = 0.$$

As for the all the conditions we will impose on the metric, the strong HKT condition can be rephrased in the terms of some cohomological condition.

Theorem 1.3.30 ([332], Proposition 5.4)**.** *Let (M, I, J, Ω) be a HKT manifold. Ω is strong HKT if and only if*

$$\partial\partial_J\overline{\Omega} = 0\,.$$

Few examples of strong HKT metrics are known. The largest class of such examples are Joyce's examples, we refer to Subsection 4.7.3 for the construction of such examples. Indeed, in [251, 174], the authors proved that a suitable extension of the Cartan-Killing form of the compact semisimple factor gives rise to a strong HKT left-invariant metric. Barberis and Fino in [44] provided a construction for producing new examples of left-invariant strong HKT metrics starting from a fixed one, using quaternionic representations. Surprisingly, the moduli space of Hermitian-Einstein connections on a given holomorphic vector bundle over a quaternionic Hopf surface inherits a strong HKT structure, see [243].

On the other hand, other conditions on the hyperHermitian metric were imposed throughout the past years. As it is customary, they are all conditions on the $(2,0)$-form Ω. We shall recall them in the following definition.

Definition 1.3.31. Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. The metric Ω is called *quaternionic Gauduchon* if

$$\partial\partial_J\Omega^{n-1} = 0\,.$$

Moreover, the metric Ω is called *quaternionic strongly Gauduchon* if $\partial\Omega^{n-1}$ is ∂_J-exact. Finally, Ω is called *quaternionic balanced* if

$$\partial\Omega^{n-1} = 0\,.$$

One can easily see that all the definitions above are mimicking the analogous definitions of, respectively, Gauduchon, recall Definition 1.1.34, strongly Gauduchon, see [271], and balanced metrics, see Definition 1.1.47, defined in complex Geometry.
As in the complex case, we have that a HKT metric is, in particular, a quaternionic balanced metric. Moreover, any quaternionic balanced metric is quaternionic strongly Gauduchon. Finally, any quaternionic strongly Gauduchon metric is of course quaternionic Gauduchon. On the other hand, the converses are not true in general. We shall collect in Section 4.7 examples showing that the inclusions among the classes of manifolds admitting these metrics might be strict. For instance, as far as the author knows, Examples 4.7.4, 4.7.5 and 4.7.6 are the first examples of compact hypercomplex manifolds admitting quaternionic strongly Gauduchon metrics but no quaternionic balanced metrics.
The quaternionic Gauduchon condition was firstly considered in [172]. In particular, the authors noticed that on a compact $\mathsf{SL}(n, \mathbb{H})$-manifold quaternionic Gauduchon metrics really behave as Gauduchon metrics. Namely, on a compact $\mathsf{SL}(n, \mathbb{H})$-manifold, we can always find a quaternionic Gauduchon metric in each conformal class of a given hyperHermitian metric. On the other hand, on non $\mathsf{SL}(n, \mathbb{H})$-manifold, we cannot expect, in general, the existence of quaternionic Gauduchon metric. An example of such manifolds was firstly provided by Andrada and Tolcachier in [17]. We will address that example in details in Example 4.7.9. The main aim of Section 4.3 is that of giving sufficient and necessary conditions for the existence of such metrics.

As regards the quaternionic balanced condition, it was firstly introduced in [231] and the first example of a hypercomplex manifold admitting a quaternionic balanced metric but no HKT ones was found in [112]. We should also mention that these kinds of metric are also the object of an interesting form-type Calabi-Yau problem [131, 132, 159, 160]. On the other hand, properties of quaternionic balanced manifolds are not perfectly known, as in the complex case. We will study some of them in Section 4.4.

Finally, quaternionic strongly Gauduchon metrics were firstly defined in [232]. Unfortunately, such metrics were not studied much in the literature. As we will see in Example 4.7.11, this condition, as a difference with the quaternionic balanced and quaternionic Gauduchon one, relies on the preferred choice of the pair of anti-commuting complex structures, ultimately indicating the not well-suited behaviour of such condition in the hypercomplex setting.

1.4 Gromov-Hausdorff and Cheeger-Gromov convergence

In this short section, we quickly recall the definition of Gromov-Hausdorff and Cheeger-Gromov convergence and their main properties.

Definition 1.4.1 ([64], Definition 7.3.1, Definition 7.3.10)**.** Let (X, d) be a metric space and $A, B \subseteq X$. The *Hausdorff distance* between A and B is defined as:

$$d_{\mathrm{H}}(A, B) := \inf\{r > 0 \,|\, A \subseteq B_r(B) \text{ and } B \subseteq B_r(A)\}$$

where $B_r(A) = \{x \in X \,|\, d(x, A) < r\}$.

Given X, Y two metric spaces, the *Gromov-Hausdorff distance* $d_{\mathrm{GH}}(X, Y)$ between X and Y is defined as the infimum of the Hausdorff distances between $A', B' \subseteq Z$, where Z is a metric space and A' and B' are, respectively, isometric to X and Y.

Finally, let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of metric spaces and X be a metric space. We say that X_n converges to X *in the Gromov-Hausdorff sense* if $d_{\mathrm{GH}}(X_n, X) \to 0$, as $n \to \infty$.

We collect here the main properties of the Gromov-Hausdorff distance and convergence.

Theorem 1.4.2. *1. Let X and Y be two isometric metric spaces, then $d_{\mathrm{GH}}(X, Y) = 0$;*

2. d_{GH} defines a metric on the space of isometry classes of compact metric spaces;

3. if there exist maps $F \colon X \to Y$ and $G \colon Y \to X$, not necessary continuous, such that

$$\begin{aligned} |d_Y(F(x_1), F(x_2)) - d_X(x_1, x_2)| < \varepsilon\,, \quad d_X(x, GF(x)) < \varepsilon\,, \quad x_1, x_2, x \in X\,, \\ |d_X(G(y_1), G(y_2)) - d_Y(y_1, y_2)| < \varepsilon\,, \quad d_Y(y, FG(y)) < \varepsilon\,, \quad y_1, y_2, y \in Y\,, \end{aligned}$$

then, $d_{\mathrm{GH}}(X, Y) < \frac{3}{2}\varepsilon$.

The proof of the first statement is straightforward, the second one is proved in [64, Theorem 7.3.30] while the proof of the third can be found in [277, Lemma 1.3.3].

Besides being purely topological, Gromov-Hausdorff convergence gives a quantitative picture of collapsing in Differential Geometry. As we will see in Section 3.1, the Gromov-Hausdorff convergence of the solution of certain geometric flows highlights how the flow transforms the starting manifold, ultimately collapsing into a lower dimensional manifold.

On the other hand, the collapsing behaviour might be caused by the presence of some gauge group, especially in the study of geometric flows, acting on our objects and producing degenerating directions. Moreover, one may be interested not only on the topology but rather on the differential structure of the limit. To do this, we introduce the notion of Cheeger-Gromov convergence.

Definition 1.4.3 ([221], Definition 6.2)**.** Let $\{(M_k, g_k, p_k)\}_{k\in\mathbb{N}}$ be a sequence of pointed Riemannian manifolds and (M, g, p) be a pointed Riemannian manifold. We say that (M_k, g_k, p_k) converges to (M, g, p) in the *Cheeger-Gromov sense* if there exist an exhaustion $\{\Omega_k\}_{k\in\mathbb{N}}$ of open sets of M containing p and $\varphi_k \colon \Omega_k \to M_k$ embeddings such that $\varphi_k(p) = p_k$, for any $k \in \mathbb{N}$, and $\varphi_k^* g_k \to g$ smoothly on compact sets.

We recall here some of the properties of the Cheeger-Gromov convergence.

Theorem 1.4.4. *Let $\{(M_k, g_k, p_k)\}_{k\in\mathbb{N}}$ be a sequence of pointed Riemannian manifolds converging to (M, g, p) in the Cheeger-Gromov sense.*

1. *If M is compact, then $\varphi_k \colon M \to M_k$ are diffeomorphisms. In particular, (M_k, g_k) converges smoothly to (M, g), up to diffeomorphisms;*

2. *(M, g, p) is unique up to isometries;*

3. *If (M_k, g_k, p_k) is homogeneous, for all $k \in \mathbb{N}$, then (M, g, p) is homogeneous;*

4. *If (M_k, g_k, p_k) is homogeneous, for all $k \in \mathbb{N}$, the choice of the base points p_k is not influent on the limit.*

In contrast with the first Item of Theorem 1.4.4, we can find examples of compact Riemannian manifolds converging to a non compact one, see [22, 9.2.2]. Moreover, the choice of the base points, in the general case, is crucial, see [221]. We will see both in Section 3.1 and Section 3.2 the study of the Cheeger-Gromov convergence of the solutions of pluriclosed and generalized Ricci flow highlights soliton solutions.

Chapter 2

Constructing constant Chern scalar curvature balanced metrics

In the Kähler realm, cscK metrics are nowadays believed to be canonical representatives of a fixed Kähler class, mainly due to the Yau-Tian-Donaldson conjecture, see [91, 316, 339]. A great source of examples of cscK metrics was provided by a result by Arezzo and Pacard in [31]. This result ensures the existence of cscK metrics on the blow-up in a finite number of points of a cscK orbifold with no non-trivial holomorphic vector field vanishing somewhere. The aim of this chapter is to extend the result by Arezzo and Pacard to the balanced case, thus proving Theorem B.

The strategy we follow to prove Theorem B is a classical gluing procedure. The chapter is divided as follows.

In Section 2.1, we discuss the general set up within we wish to perform the deformation argument.

Section 2.2 is fully devoted to the proof of Theorem B. After comparing our problem with the Kähler case, we prove the invertibility of the linearized operator and solve the equation by means of Banach's fixed point Theorem.

Section 2.3 is dedicated to the proof of Theorem 2.3.1 which requires the existence of a suitable $(n-2, n-2)$-form on the starting manifold.

Finally, in Section 2.4, we discuss some classes of examples in which Theorem 2.3.1 can be applied.

The present chapter is an account of a joint work in progress with Federico Giusti.

2.1 Set up of the problem

In this section, we set up the problem in the general setting. This section will be divided in two subsections. In Subsection 2.1.1, we construct the approximate solution and prove some estimates on it to ensure we can perform the deformation argument in Section 2.2. In Subsection 2.1.2, we set up the equation we want to solve and compute the linearized operator of it.

2.1.1 The approximate solution

Let $(M, \tilde{\omega})$ be a compact Chern-Ricci flat manifold of dimension $n \geq 3$ and let $\hat{M}$ be the blow-up at a point $x \in M$. For the sake of simplicity, we will focus in the case of the blow-up of a point, but the argument applies in the same way when blowing up a finite family of points. Following the usual strategy of gluing constructions (see [31] or [312]), the first step will be to construct an *approximate solution* to the problem on $\hat{M}$, which in our case will consist of an *approximately constant Chern scalar curvature balanced metric*. In order to do this, we shall implement the cut-off argument for balanced metrics introduced in [168]. With this in hand, we can glue together the background metric $\tilde{\omega}$ to the

Burns-Simanca metric ω_{BS} on the blow-up of $\mathbb{C}^n$ at 0, using a flat region as bridge, making sure that it also satisfies suitable properties for the following deformation argument.

Let us then start to describe this gluing process by seeing how the balanced property intervenes. This happens with the following lemma, whose proof can be found in [168].

Lemma 2.1.1. *Let $(Y^n, \tilde{\omega})$ be a balanced manifold. Then, for every $y \in Y$ and $p > 0$, there exist a sufficiently small $\varepsilon > 0$, coordinates z centered at y and a balanced metric $\tilde{\omega}_\varepsilon$ such that*

$$\tilde{\omega}_\varepsilon = \begin{cases} \omega_o & \text{if } |z| < \varepsilon^p \,, \\ \tilde{\omega} & \text{if } |z| > 2\varepsilon^p \,, \end{cases}$$

where ω_o is the flat metric around y, and such that $|\tilde{\omega}_\varepsilon - \omega_o|_{\omega_o} < c\varepsilon^p$ on $\{\varepsilon^p \le |z| \le 2\varepsilon^p\}$.

Thus, starting from $\tilde{\omega}$, we can obtain the corresponding $\tilde{\omega}_\varepsilon$, which is exactly flat in a neighbourhood of x.
On the other hand, we can consider the standard coordinates ζ on $\hat{X} := \mathrm{Bl}_0\mathbb{C}^n \backslash E \simeq \mathbb{C}^n \backslash \{0\} =: X \setminus \{0\}$. Recalling that ω_{BS} admits the expansion (1.9) away from the exceptional divisor, we can introduce a cut-off function

$$\psi(y) := \begin{cases} 1 & \text{if } y \le \frac{1}{4}, \\ \text{non increasing} & \text{if } \frac{1}{4} < y < \frac{1}{2}, \\ 0 & \text{if } y \ge \frac{1}{2}, \end{cases}$$

which, for all $q > 0$, can be rescaled to

$$\psi_\varepsilon(y) := \psi(\varepsilon^q y),$$

making the cut-off happen *far away* from the exceptional divisor, i.e. in the asymptotically flat part. This allows us to introduce the family of closed $(1,1)$-forms:

$$\omega_{\mathrm{BS},\varepsilon} := \sqrt{-1}\partial\overline{\partial}(|\zeta|^2 + \psi_\varepsilon(|\zeta|)\gamma(|\zeta|)) \,.$$

From here, it is easily seen that, on the cut-off region $\{\frac{1}{4}\varepsilon^{-q} \le |\zeta| \le \frac{1}{2}\varepsilon^{-q}\}$, it holds

$$\omega_{\mathrm{BS},\varepsilon} = \omega_o + \mathrm{O}(|\zeta|^{2-2n}), \tag{2.1}$$

where now ω_o denotes the flat metric on $\mathbb{C}^n \backslash \{0\}$ induced by the coordinates ζ. This ensures that for sufficiently small ε, $\omega_{\mathrm{BS},\varepsilon}$ is an asymptotically exactly flat Kähler metric on $\hat{X}$.
If we then consider the biholomorphism

$$z = \varepsilon^{p+q}\zeta,$$

we get the identification

$$\left\{\frac{1}{4}\varepsilon^{-q} \le |\zeta| \le 2\varepsilon^{-q}\right\} \equiv \left\{\frac{1}{4}\varepsilon^{p} \le |z| \le 2\varepsilon^{p}\right\}$$

with which we can topologically realize $\hat{M}$. Moreover, it also allows us to obtain that

$$|z|^2 = \varepsilon^{2(p+q)}|\zeta|^2,$$

telling that on $\hat{M}$, the metrics $\varepsilon^{2(p+q)}\omega_{\mathrm{BS},\varepsilon}$ and $\tilde{\omega}_\varepsilon$ coincide with the flat metric on the region

$$\left\{\frac{1}{2}\varepsilon^{-q} \le |\zeta| \le \varepsilon^{-q}\right\} \equiv \left\{\frac{1}{2}\varepsilon^{p} \le |z| \le \varepsilon^{p}\right\} \,. \tag{2.2}$$

This last fact gives us the possibility to glue $\tilde{\omega}_\varepsilon$ and $\varepsilon^{2(p+q)}\omega_{\mathrm{BS},\varepsilon}$ to a global balanced metric ω_ε on $\hat{M}$. In the following, as it will not create any confusion, we will avoid writing explicitly the dependence on ε, thus we will always write just $\omega = \omega_\varepsilon$.

Remark 2.1.2. The metric ω is a suitable approximate solution. Indeed, it is clear that the metric is unaltered on $\{\varepsilon^p \leq |z| \leq 2\varepsilon^p\}$, on which we still have

$$|\nabla^k_{\omega_o}(\omega - \omega_o)|_{\omega_o} \leq c|z|^{1-k},$$

for all $k \geq 0$.
On the other hand, since to obtain ω we had to rescale the metric $\omega_{\mathrm{BS},\varepsilon}$ on $\hat{X}$, we have to check how it has affected the distance from the flat metric. To have clearer estimates, we will express also this one in terms of the "small"coordinates z. The main thing to observe, is that on $\{\frac{1}{4}\varepsilon^{-q} \leq |\zeta| \leq \frac{1}{2}\varepsilon^{-q}\}$ it holds

$$\begin{aligned}\langle \omega - \omega_o, \omega - \omega_o\rangle_{\omega_o}(z) &= \varepsilon^{-4(p+q)}\langle \varepsilon^{2(p+q)}(\omega_{\mathrm{BS},\varepsilon} - \omega_o), \varepsilon^{2(p+q)}(\omega_{\mathrm{BS},\varepsilon} - \omega_o)\rangle_{\omega_o}(\zeta)\\ &= \langle \omega_{\mathrm{BS},\varepsilon} - \omega_o, \omega_{\mathrm{BS},\varepsilon} - \omega_o\rangle_{\omega_o}(\zeta)\,,\end{aligned}$$

implying that $|\omega - \omega_o|_{\omega_o}(z) = |\omega_{\mathrm{BS},\varepsilon} - \omega_o|_{\omega_o}(\zeta)$. From here, we can recall the expansion (2.1) and obtain

$$|\omega - \omega_o|_{\omega_o}(z) \leq |\omega_{\mathrm{BS},\varepsilon} - \omega_o|_{\omega_o}(\zeta) \leq c|\zeta|^{2-2n} \leq c\varepsilon^{(2n-2)q} \leq c|z|^{(2n-2)q/p}\,,$$

which implies, on the whole gluing region, that, for all $k \geq 0$, holds

$$|\nabla^k_{\omega_o}(\omega - \omega_o)|_{\omega_o} \leq c|z|^{m-k},$$

where $m = \min\{1, (2n-2)q/p\}$, showing again that ω is indeed a metric on $\hat{M}$. Moreover, the closeness between the metric ω and the flat metric ω_o shows us that ω is suitable to perform analysis with, and hence we can try to search for a constant Chern scalar curvature balanced metric through a deformation argument.

Remark 2.1.3. As we will see, the deformation argument we will introduce in the following subsection will allow us to work within the balanced class of ω. Hence, in light of [23, Proposition 2.6], we can predict the sign of the Chern scalar curvature of a genuine solution (which we will obtain in the next sections) by describing the cohomology class of ω. Indeed, we have that

$$[\omega^{n-1}]_{\mathrm{BC}} = \pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} + [\varepsilon^{2(p+q)}\omega_{\mathrm{BS}}]^{n-1}_{\mathrm{BC}},$$

and, since $[\omega_{\mathrm{BS}}]_{\mathrm{BC}} = -[E]_{\mathrm{BC}}$, we get

$$[\omega^{n-1}]_{\mathrm{BC}} = \pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} + (-1)^{n-1}\varepsilon^{(2n-2)(p+q)}[E]^{n-1}_{\mathrm{BC}}.$$

Now, recalling (1.18), (1.15) and that $[E]^n_{\mathrm{BC}} = (-1)^{n-1}$, it follows that

$$\begin{aligned}\Gamma(\{\omega\}) &= \frac{2\pi}{(n-1)!}(\pi^* c_1^{\mathrm{BC}}(M) - (n-1)[E]_{\mathrm{BC}}) \cdot (\pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} + (-1)^{n-1}\varepsilon^{(2n-2)(p+q)}[E]^{n-1}_{\mathrm{BC}})\\ &= \Gamma(\{\tilde{\omega}\}) - \frac{2\pi}{(n-2)!}\varepsilon^{(2n-2)(p+q)}.\end{aligned} \tag{2.3}$$

2.1.2 Setting up the equation

We now wish to obtain a constant Chern scalar curvature balanced metric starting from the approximate solution, and as done, for instance, in [31, 51, 312]. We plan to do it through a deformation argument. Since we wish to work inside the balanced class of ω, we will consider the general deformation, considered firstly in [130] :

$$\omega^{n-1}_\varphi := \omega^{n-1} + \sqrt{-1}\partial\overline{\partial}\varphi, \quad \varphi \in \Lambda^{n-2,n-2}_{\mathbb{R}}\hat{M} \text{ such that } \omega^{n-1}_\varphi > 0. \tag{2.4}$$

Thus, the problem we are interested in solving, following what was done in [312], is the equation

$$s^{\mathrm{Ch}}(\omega_\varphi) = const. \tag{2.5}$$

for $\varphi \in \Lambda^{n-2,n-2}_{\mathbb{R}}\hat{M}$ such that $\omega_\varphi^{n-1} > 0$. Now, as showed by (2.3), we can expect the solution to have the Chern scalar curvature near to the one of $\tilde{\omega}$, thus we can rephrase equation (2.5) as

$$\mathcal{S}(\varphi) := s^{\mathrm{Ch}}(\omega_\varphi) - s^{\mathrm{Ch}}(\tilde{\omega}) = c \tag{2.6}$$

for $\varphi \in \Lambda^{n-2,n-2}_{\mathbb{R}}\hat{M}$ and $c \in \mathbb{R}$. Moreover, we can get rid of the unknown constant by rewriting the equation as

$$\tilde{\mathcal{S}}(\varphi) := s^{\mathrm{Ch}}(\omega_\varphi) - s^{\mathrm{Ch}}(\tilde{\omega}) - \int_{\hat{M}} f(\varphi)\frac{\omega^n}{n!} = 0, \tag{2.7}$$

where $f\colon \Lambda^{n-2,n-2}_{\mathbb{R}}\hat{M} \to C^\infty(M,\mathbb{R})$ is a suitable operator to be evaluated on φ which will be chosen later to help us to get rid of the kernel of some operator. This will help us in obtaining the invertibility of the linearization of $\tilde{\mathcal{S}}$, which is a key ingredient to allow us to turn the problem of solving equation (2.7) into a fixed point problem to be solved with Banach's fixed-point Theorem in a suitably chosen neighbourhood of zero. Hence, our next step will be to compute the linearization at 0 of $\tilde{\mathcal{S}}$.

First of all, we observe that, directly from (2.7), we have that the linearized operator

$$\tilde{\mathcal{L}}(\varphi) := d_0\tilde{\mathcal{S}}(\varphi) = \mathcal{L}(\varphi) - \int_{\hat{M}} d_0 f(\varphi)\frac{\omega^n}{n!}\,, \quad \varphi \in \Lambda^{n-2,n-2}_{\mathbb{R}}\hat{M}\,,$$

where $\mathcal{L}(\varphi) := d_0 s^{\mathrm{Ch}}(\varphi)$. We thus need to obtain an explicit expression for the operator $\mathcal{L}(\varphi) = \frac{d}{dt}\big|_{t=0} s^{\mathrm{Ch}}(\omega_{t,\varphi})$, where $\omega_{t,\varphi}$ is an arbitrary curve of Hermitian metrics lying in $[\omega^{n-1}]_{\mathrm{BC}}$ and such that $\omega_{0,\varphi}^{n-1} = \omega^{n-1}$ and $(\omega_{t,\varphi}^{n-1})'(0) = \varphi$. Thus, we consider the curve of Hermitian metrics defined by $\omega_{t,u}^{n-1} = \omega^{n-1} + t\sqrt{-1}\partial\bar{\partial}\varphi$ and we observe that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\omega^n_{t,\varphi} &= n\frac{d}{dt}\Big|_{t=0}\omega_{t,\varphi}\wedge\omega^{n-1}\,,\\ \frac{d}{dt}\Big|_{t=0}\omega^n_{t,\varphi} &= \frac{d}{dt}\Big|_{t=0}\omega_{t,\varphi}\wedge\omega^{n-1} + \omega\wedge\sqrt{-1}\partial\bar{\partial}\varphi\,. \end{aligned} \tag{2.8}$$

Then, from (2.8), we obtain

$$\frac{d}{dt}\Big|_{t=0}\omega^n_{t,u} = \frac{n}{n-1}\omega\wedge\sqrt{-1}\partial\bar{\partial}\varphi\,. \tag{2.9}$$

For the sake of simplicity, we will denote with

$$F_\omega(\varphi) = \frac{d}{dt}\Big|_{t=0}\log\omega^n_{t,\varphi} = \frac{n}{n-1}\frac{\omega\wedge\sqrt{-1}\partial\bar{\partial}\varphi}{\omega^n}\,.$$

From this, we easily obtain that

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Ric}^{\mathrm{Ch}}(\omega_{t,\varphi}) = -\sqrt{-1}\partial\bar{\partial}F_\omega(\varphi)\,. \tag{2.10}$$

Now, differentiating (1.13) in the case of the Chern connection and using (2.9) and (2.10), we obtain that

$$\mathcal{L}(\varphi) = -\Delta_\omega F_\omega(\varphi) + n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}\varphi}{\omega^n} - s^{\mathrm{Ch}}(\omega)F_\omega(\varphi)\,. \tag{2.11}$$

Clearly such an operator will not stand the possibility to have no kernel, as the input space is much larger then the target space, on top of being extremely complicate to understand. For this reason, in the next two sections, we will introduce two different ansatz with the objective to turn the operator into one between two function spaces, allowing us to perform the usual analysis involved in gluing constructions.

2.2 Balanced deformation

In this section we consider the balanced ansatz for the deformation argument. In particular, we analyse the equation in (2.4) assuming $\varphi = u\omega^{n-2}$, for $u \in C^\infty(M, \mathbb{R})$, as previously done in [168], from which we get, as from [168, Lemma 3.2], that the operator F_ω takes the form

$$F_\omega(u) = \frac{1}{n-1}\left(\Delta_\omega u + \frac{1}{n-1}|\partial\omega|^2 u\right). \tag{2.12}$$

We will then choose $f(u\omega^{n-2}) = u|\partial\omega|^2$. With these choices, we are able to turn operator $\tilde{\mathcal{S}}$ into an operator taking smooth functions in input defined as

$$\tilde{\mathcal{S}}(u) = s^{\mathrm{Ch}}(\omega_u) - s^{\mathrm{Ch}}(\tilde{\omega}) - \int_{\hat{M}} u|\partial\omega|^2 \frac{\omega^n}{n!}, \tag{2.13}$$

whose linearization is now suitable to be inverted, up to working in the correct functional spaces. Let us conclude by writing again the linearized operator $\tilde{\mathcal{L}}$ implementing this ansatz:

$$\tilde{\mathcal{L}}(u) := \tilde{\mathcal{L}}(u\omega^{n-2}) = -\Delta_\omega F_\omega(u) + n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \sqrt{-1}\partial\bar{\partial}(u\omega^{n-2})}{\omega^n} - s^{\mathrm{Ch}}(\omega)F_\omega(u) - \int_{\hat{M}} u|\partial\omega|^2\frac{\omega^n}{n!}. \tag{2.14}$$

Before proving Theorem B, we want to highlight and discuss the differences and similarities of this setting with the Kähler one.

2.2.1 Comparison with the Kähler case

As highlighted before, this deformation makes sense on every balanced manifold, hence it is worth analyzing the linearized operator in a more general setting, aiming to understand something more about it in the case of constant Chern scalar curvature balanced metrics. As we will see, this linearization will show up as a perfectly fitting generalization of the Kähler case, as it will reduce to the Lichnerowicz operator whenever the metric on the base manifold is chosen to be cscK. In order to obtain this, we will first recall some ingredients from the Kähler setting, then obtain some significant formulas for the balanced setting, and finally put all the pieces together.

First of all, we recall that from the case of [31] for cscK metrics on blow-ups, a key role is played by the Lichnerowicz operator:

$$\mathcal{D}^*\mathcal{D}\colon C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C}) \quad \text{defined locally by } \mathcal{D}^*\mathcal{D}\psi := g^{i\bar{j}}g^{k\bar{l}}D_iD_kD_{\bar{j}}D_{\bar{l}}\psi$$

where here D is the Levi-Civita connection of a given Kähler metric g. More globally, using (1.16), we can write, see for instance [312, Definition 4.3],

$$\mathcal{D}^*\mathcal{D}\psi = \Delta^2_\omega\psi + g(i\partial\bar{\partial}\psi, \mathrm{Ric}(\omega)) + g(\partial s(\omega), \partial\bar{\psi}), \quad \psi \in C^\infty(M,\mathbb{C}),$$

where $\mathrm{Ric}(\omega)$ is the Ricci form of the metric g while $s(\omega) = \mathrm{tr}_\omega\mathrm{Ric}(\omega)$ is the Riemannian scalar curvature of g. The kernel of the Lichnerowicz operator is well-known and it consists on those function ψ such that $(D\psi)^{1,0}$ is a holomorphic vector field.

Now, let ω be a balanced manifold. Following the Kähler case, we would like to consider the operator

$$\mathcal{D} := \nabla_{\bar{k}}\nabla_{\bar{q}}\colon C^\infty(M,\mathbb{R}) \to \Gamma(\Lambda^{0,1}M \otimes \Lambda^{0,1}M),$$

where ∇ is the Chern connection of ω (which is now different from the Levi-Civita connection), and compute, given $\psi \in C^\infty(M,\mathbb{C})$, the Chern-Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$. As the contracted Bianchi identity (1.16) is only satisfied by the Levi-Civita connection, in order to obtain a reasonable expression for the operator, we are going to develop some formulas for the codifferentials of the Chern-Ricci forms of a given balanced metric ω. We will then see some interesting applications of them, not only in direct relation to our problem, but in the wider framework of constant Chern scalar curvature balanced metrics.

To make things clear, let us quickly establish some local coordinates conventions. We will locally write any Hermitian metric ω as

$$\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j ,$$

where, as usual, $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$ are the components of the Hermitian metric g associated to ω. As we know, the presence of a Riemannian metric determines a natural metric on all the tensor bundles. In what follows, we will make use of that on differential forms. Encoding directly the complex structure in the discussion, we will use the following convention: given $\alpha, \beta \in \Lambda^{p,q}M$, then, in local holomorphic coordinates, we have that

$$g(\alpha, \beta) = \frac{1}{p!q!} g^{i_1\bar{j}_1} \cdots g^{i_p\bar{j}_p} g^{k_1\bar{l}_1} \cdots g^{k_q\bar{l}_q} \alpha_{i_1\cdots i_p\bar{l}_1\cdots\bar{l}_q} \overline{\beta_{j_1\cdots j_p\bar{k}_1\cdots\bar{k}_q}} .$$

Recalling Definition 1.1.12 and 1.6, we have:

$$\alpha \wedge *\bar{\beta} = g(\alpha, \beta) \frac{\omega^n}{n!} .$$

With these conventions, we can recall some well-known Riemannian relations we will use later, specified in the Hermitian case. The first one is the fact that that the interior product and the exterior one are formal adjoints with respect to the inner product above. More precisely, given $Z \in T^{1,0}M$, $\alpha \in \Lambda^{p,q}M$ and $\beta \in \Lambda^{p-1,q}M$ we have that

$$g(\iota_Z \alpha, \beta) = g(\alpha, \bar{Z}^b \wedge \beta) , \tag{2.15}$$

where $Z^b(W) = g(Z, W)$, for all $W \in T^{0,1}M$. From this equality, we derive the following:

$$\iota_{\bar{Z}} \beta = *(\bar{Z}^b \wedge *\beta) \quad \text{and} \quad \iota_Z \beta = *(Z^b \wedge *\beta) . \tag{2.16}$$

We then have the following formulas, which can be interpreted as the contracted Bianchi identities for the Chern connection.

Lemma 2.2.1. *Let (M, ω) be a balanced manifold. We have,*

$$\bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \sqrt{-1} \partial s^{\mathrm{Ch}}(\omega) - \frac{\sqrt{-1}}{2} \Lambda^2 (\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial \omega) ,$$
$$\bar{\partial}^* \mathrm{Ric}^{(2)}(\omega) = \sqrt{-1} \partial s^{\mathrm{Ch}}(\omega) - \frac{\sqrt{-1}}{2} \Lambda^2 (\mathrm{Ric}^{(2)}(\omega) \wedge \partial \omega) - \sqrt{-1} \Lambda (\partial Ric^{(2)}(\omega)) .$$

As a consequence, we have that

$$i\partial^* \bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \Delta_\omega s^{\mathrm{Ch}}(\omega) - g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \partial^* \partial \omega) ,$$
$$i\partial^* \bar{\partial}^* \mathrm{Ric}^{(2)}(\omega) = \Delta_\omega s^{\mathrm{Ch}}(\omega) - \frac{1}{2} \Lambda^2 (\sqrt{-1} \partial \bar{\partial} \mathrm{Ric}^{(2)}(\omega)) + 2\mathrm{Re}(g(\partial \mathrm{Ric}^{(2)}(\omega), \partial \omega)) - g(\mathrm{Ric}^{(2)}(\omega), \partial^* \partial \omega) .$$

Proof. First of all, using [48, Lemma 2.1], we know that

$$*\mathrm{Ric}^{\mathrm{Ch}}(\omega) = \frac{1}{(n-1)!} s^{\mathrm{Ch}}(\omega) \omega^{n-1} - \frac{1}{(n-2)!} \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \omega^{n-2} . \tag{2.17}$$

Now, using the balanced condition, we have that

$$\partial * \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \partial s^{\mathrm{Ch}}(\omega) \wedge *\omega - \frac{1}{(n-2)!} \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial \omega^{n-2} .$$

Applying (2.16), for any $Z \in T^{1,0}M$, we can infer that

$$\iota_Z \bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = *\left(Z^b \wedge \partial * \mathrm{Ric}^{\mathrm{Ch}}(\omega)\right) = *(Z^b \wedge \partial s^{\mathrm{Ch}}(\omega) \wedge *\omega) - \frac{1}{(n-2)!} * \left(Z^b \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial \omega^{n-2}\right).$$

On the other hand, applying again Item 6 of Proposition 1.1.15, we obtain

$$*\partial s^{\mathrm{Ch}}(\omega) = -\frac{\sqrt{-1}}{(n-1)!} \partial s^{\mathrm{Ch}}(\omega) \wedge \omega^{n-1} = -\sqrt{-1} \partial s^{\mathrm{Ch}}(\omega) \wedge *\omega,$$

which gives us that

$$*(Z^b \wedge \partial s^{\mathrm{Ch}}(\omega) \wedge *\omega) = \sqrt{-1} \iota_Z \partial s^{\mathrm{Ch}}(\omega).$$

Now, thanks to the balanced condition, we have that $\partial\omega$ is primitive, recall Item 3 of Proposition 1.1.48. So, using again Item 6 of Proposition 1.1.15, we can easily see that

$$*\partial \omega = \frac{\sqrt{-1}}{(n-2)!} \partial \omega^{n-2}. \tag{2.18}$$

Then,

$$\frac{1}{(n-2)!} Z^b \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial \omega^{n-2} = -\sqrt{-1} Z^b \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge *\partial\omega = -\sqrt{-1} g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \iota_{\bar{Z}} \bar{\partial}\omega) \frac{\omega^n}{n!}.$$

Hence, we obtain that

$$\iota_Z \bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \sqrt{-1} \iota_Z \partial s^{\mathrm{Ch}}(\omega) + \sqrt{-1} g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \iota_{\bar{Z}} \bar{\partial}\omega). \tag{2.19}$$

In order to achieve the claim, using again the balanced condition, we observe that, for any $Z \in T^{1,0}M$, we have that

$$\begin{aligned} \frac{1}{2} \iota_Z \Lambda^2(\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial\omega) &= \frac{1}{2} \Lambda^2(\iota_Z(\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial\omega)) = \frac{1}{2} g(\iota_Z(\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \partial\omega), \omega^2) \\ &= * \left(\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \iota_Z \partial\omega \wedge \frac{\omega^{n-2}}{(n-2)!} \right) = -g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \iota_{\bar{Z}} \bar{\partial}\omega), \end{aligned} \tag{2.20}$$

where we used that

$$*\iota_Z \partial\omega = -\frac{1}{(n-2)!} \iota_Z \partial\omega \wedge \omega^{n-2}.$$

Now, we can conclude by using (2.20) in (2.19).
As regards the formula for the second Chern Ricci form, we just need to analyze the term involving $\partial \mathrm{Ric}^{(2)}(\omega) \wedge \omega^{n-2}$. Applying again Item 6 of Proposition 1.1.15, we have that

$$*LZ^b = \sqrt{-1} Z^b \wedge \frac{\omega^{n-2}}{(n-2)!},$$

which implies that

$$Z^b \wedge \partial \mathrm{Ric}^{(2)}(\omega) \wedge \frac{\omega^{n-2}}{(n-2)!} = \sqrt{-1} g(\mathrm{Ric}^{(2)}(\omega), L\bar{Z}^b) = \sqrt{-1} \iota_Z \Lambda \partial \mathrm{Ric}^{(2)}(\omega),$$

giving the desired formula.

As regards the second part of the statement, we know that

$$\sqrt{-1} \partial^* \bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \sqrt{-1} * \partial \bar{\partial} * \mathrm{Ric}^{\mathrm{Ch}}(\omega).$$

Now, recalling (2.17), using the balanced condition and the fact that $\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ is both ∂ and $\bar{\partial}$-closed, we have that

$$\begin{aligned}\sqrt{-1}\partial^*\bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega) &= *\left(\frac{1}{(n-1)!}\sqrt{-1}\partial\bar{\partial}s^{\mathrm{Ch}}(\omega)\wedge\omega^{n-1} - \frac{1}{(n-2)!}\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}\omega^{n-2}\right)\\ &= \Delta_\omega s^{\mathrm{Ch}}(\omega) - \frac{1}{(n-2)!} * (\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}\omega^{n-2})\,.\end{aligned} \tag{2.21}$$

Hence, using (2.18), we obtain

$$-\frac{1}{(n-2)!}\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}\omega^{n-2} = -\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge *\bar{\partial}^*\bar{\partial}\omega = -g(\mathrm{Ric}^{\mathrm{Ch}}(\omega),\partial^*\partial\omega)\frac{\omega^n}{n!}\,, \tag{2.22}$$

which used in (2.21) gives the claim.
Finally, using that

$$*\mathrm{Ric}^{(2)}(\omega) = s^{\mathrm{Ch}}(\omega)\wedge\frac{\omega^{n-1}}{(n-1)!} - \mathrm{Ric}^{(2)}(\omega)\wedge\frac{\omega^{n-2}}{(n-2)!}\,,$$

we obtain

$$\begin{aligned}\sqrt{-1}\partial^*\bar{\partial}^*\mathrm{Ric}^{(2)}(\omega) &= *\left(\sqrt{-1}\partial\bar{\partial}s^{\mathrm{Ch}}(\omega)\wedge\frac{\omega^{n-1}}{(n-1)!} - \frac{1}{(n-2)!}\sqrt{-1}\partial\bar{\partial}(\mathrm{Ric}^{(2)}(\omega)\wedge\omega^{n-2})\right)\\ &= \Delta_\omega s^{\mathrm{Ch}}(\omega) - *\left(\frac{1}{(n-2)!}\sqrt{-1}\partial\bar{\partial}(\mathrm{Ric}^{(2)}(\omega)\wedge\omega^{n-2})\right)\,.\end{aligned} \tag{2.23}$$

Moreover, we have that

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}(\mathrm{Ric}^{(2)}(\omega)\wedge\omega^{n-2}) &= \sqrt{-1}\partial\bar{\partial}\mathrm{Ric}^{(2)}(\omega)\wedge\omega^{n-2} + 2\mathrm{Re}(\sqrt{-1}\partial\mathrm{Ric}^{(2)}(\omega)\wedge\bar{\partial}\omega^{n-2})\\ &\quad + \mathrm{Ric}^{(2)}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}\omega^{n-2}\,.\end{aligned}$$

Now, in the same fashion as in (2.22), we can conclude that

$$-\frac{1}{(n-2)!}\mathrm{Ric}^{(2)}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}\omega^{n-2} = -g(\mathrm{Ric}^{(2)}(\omega),\partial^*\partial\omega)\,.$$

Furthermore, one can check that

$$\sqrt{-1}\partial\mathrm{Ric}^{(2)}(\omega)\wedge\frac{\bar{\partial}\omega^{n-2}}{(n-2)!} = -\partial\mathrm{Ric}^{(2)}(\omega)\wedge *\bar{\partial}\omega = -g(\partial\mathrm{Ric}^{(2)}(\omega),\partial\omega)\,.$$

Then,

$$-\frac{1}{(n-2)!}\sqrt{-1}\partial\bar{\partial}(\mathrm{Ric}^{(2)}(\omega)\wedge\omega^{n-2}) = -\frac{1}{2}\Lambda^2(\sqrt{-1}\partial\bar{\partial}\mathrm{Ric}^{(2)}(\omega)) + 2\mathrm{Re}(g(\partial\mathrm{Ric}^{(2)}(\omega),\partial\omega)) - g(\mathrm{Ric}^{(2)}(\omega),\partial^*\partial\omega)\,,$$

which inserted in (2.23) concludes the proof. □

This result has several very interesting consequences which we shall now discuss (up to obtaining a nice expression for the Chern-Lichnerowicz operator). The very first formula in Lemma 2.2.1, for example, guarantees a new way of proving a well-known fact, i.e. that balanced first Chern-Ricci Einstein metrics are either flat or Kähler-Einstein, see [24].

Corollary 2.2.2. *Let (M,ω) be a first Chern-Einstein balanced manifold. Then, ω is either first Chern-Ricci flat or Kähler-Einstein.*

Proof. The first Chern-Einstein condition together with the balanced condition ensures that

$$\bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \sqrt{-1}\partial s^{\mathrm{Ch}}(\omega)\,.$$

On the other hand, if $\mathrm{Ric}^{\mathrm{Ch}}(\omega) = \lambda\omega$, for some $\lambda \in C^\infty(M, \mathbb{R})$, we have that

$$\bar{\partial}^* \mathrm{Ric}^{\mathrm{Ch}}(\omega) = \bar{\partial}^*(\lambda\omega) = - * \frac{1}{(n-1)!}(\partial\lambda \wedge \omega^{n-1}) = \sqrt{-1}\partial\lambda\,.$$

Then, we have that

$$\sqrt{-1}\partial\lambda = \sqrt{-1}\partial s^{\mathrm{Ch}}(\omega) = n\sqrt{-1}\partial\lambda$$

which implies that λ is constant. So, in the case in which $\lambda \neq 0$ we conclude the proof, using that $\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ is d-closed. □

Another consequence of Lemma 2.2.1 is that, on a compact balanced manifold (M, ω), the term $g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \partial^*\partial\omega)$ appearing in the expression of $\sqrt{-1}\partial^*\bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ is such that

$$\int_M g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \partial^*\partial\omega)\frac{\omega^n}{n!} = 0\,.$$

The above condition directly implies that if (M, ω) is a compact quotient of a Lie group endowed with a left-invariant balanced metric, then $\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ is Bott-Chern harmonic, i.e.

$$\partial\mathrm{Ric}^{\mathrm{Ch}}(\omega) = 0\,, \quad \bar{\partial}\mathrm{Ric}^{\mathrm{Ch}}(\omega) = 0 \ \text{ and } \sqrt{-1}\partial^*\bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega) = 0\,.$$

On the other hand, a well know result, stemming from 1.16, states that a Kähler metric has harmonic Ricci form if and only if the metric is cscK. Then, one can formulate the following problem:

Question 2.2.3. Let (M, ω) be a compact balanced manifold. The first Chern-Ricci form of ω is Bott-Chern harmonic if and only if ω has constant Chern scalar curvature.

Remark 2.2.4. In general, we cannot expect the first Chern-Ricci form to be d-harmonic. Indeed, in [167], the authors constructed balanced compact quotient of Lie groups with vanishing first Chern class but non-zero Chern-Ricci form. Then, if the Chern-Ricci form was d-harmonic, it would readily imply that it is zero, which is not possible.

One more consequence of the proof of Lemma 2.2.1 is the following identity.

Remark 2.2.5. For ω balanced metric, we can easily use (2.18) and [168] to infer that

$$\Lambda\partial^*\partial\omega = \frac{1}{(n-2)!} * (\sqrt{-1}\partial\bar{\partial}\omega^{n-2} \wedge \omega) = |\partial\omega|^2_\omega\,.$$

With the same strategy, we can easily re-prove a result from Liu and Yang (see [235, Corollary 5.3]) relating balanced metrics with k-Gauduchon metrics, i.e. metrics such that

$$\sqrt{-1}\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0\,,$$

for some $k = 1, \ldots, n-1$. In particular, we can actually extend the result to the non-compact case and obtain:

Proposition 2.2.6. *Let (M, ω) be a Hermitian manifold with ω both balanced and k-Gauduchon, for some $k = 1, \ldots, n-2$. Then, ω is Kähler.*

Proof. As in (2.18), we have that

$$*\partial\omega^{j+1} = \sqrt{-1}\frac{(j+1)!}{(n-2-j)!}\partial\omega^{n-2-j}\,, \quad j=0,\dots,n-3\,.$$

Then,

$$\partial^*\partial\omega^{j+1} = \frac{(j+1)!}{(n-2-j)!} * \sqrt{-1}\partial\bar\partial\omega^{n-2-j}\,, \quad j=0,\dots,n-3\,. \tag{2.24}$$

Then, combining (2.24) for $j=n-2+k$ with the k-Gauduchon condition we get

$$\Lambda^{n-1-k}\partial^*\partial\omega^{n-1-k} = \frac{(n-1-k)!}{k!} * (\sqrt{-1}\partial\bar\partial\omega^k\wedge\omega^{n-1-k}) = 0$$

On the other hand, for $k\geq 2$, it holds

$$\frac{1}{k!}\sqrt{-1}\partial\bar\partial\omega^k\wedge\omega^{n-1-k} = \frac{1}{(k-1)!}\sqrt{-1}\partial\bar\partial\omega\wedge\omega^{n-2} - \frac{1}{(k-2)!}\sqrt{-1}\bar\partial\omega\wedge\partial\omega\wedge\omega^{n-3}\,,$$

while the balanced condition gives us that

$$\sqrt{-1}\partial\bar\partial\omega\wedge\omega^{n-2} = (n-2)\bar\partial\omega\wedge\partial\omega\wedge\omega^{n-3}\,, \tag{2.25}$$

from which follows

$$0 = \Lambda^{n-1-k}\partial^*\partial\omega^{n-1-k} = \frac{(n-1-k)(n-1-k)!}{(k-1)!}|\partial\omega|^2\,,$$

which gives us the claim.
The 1-Gauduchon case follows directly by combining (2.25) and (2.18). □

Let us now discuss the main consequence of Lemma 2.2.1 for our purposes, that is a formula for the Chern-Lichnerowicz operator.

Lemma 2.2.7. *Let (M^n,ω) be a compact balanced manifold. Then, for any $u\in C^\infty(M,\mathbb{R})$, we have that*

$$\mathcal{D}^*\mathcal{D}u = \Delta^2_\omega u + g\left(\mathrm{Ric}^{\mathrm{Ch}}(\omega) - \frac{1}{2}\Xi, i\partial\bar\partial u\right) + g\left(\bar\partial^*\left(\mathrm{Ric}^{\mathrm{Ch}}(\omega) - \frac{1}{2}\Xi\right), i\partial u\right)\,,$$

where the $(1,1)$*-form* Ξ *is defined as*

$$\Xi(X,Y) := Q^2(JX,Y)\,, \quad X,Y\in TM\,,$$

with $Q^2_{i\bar j} = g^{k\bar l}g^{p\bar q}T_{kp\bar j}T_{\bar l\bar p i}$.

Proof. In order to have the explicit expression of $\mathcal{D}^*$ we observe that, if $\beta\in\Lambda^{0,1}(M)\otimes\Lambda^{0,1}(M)$,

$$g^{i\bar j}g^{l\bar k}\nabla_{\bar j}\nabla_{\bar k}u\bar\beta_{il} = -\,\mathrm{tr}_\omega(\bar\partial\nabla^*(u\bar\beta)) + 2\mathrm{tr}_\omega(\bar\partial(u\nabla^*\bar\beta)) + ug^{i\bar j}g^{l\bar k}\nabla_{\bar j}\nabla_{\bar k}\bar\beta_{il}\,, \tag{2.26}$$

where $(\nabla^*\bar\beta)_m = -g^{p\bar q}\nabla_{\bar q}\bar\beta_{pm}$. Then, using the balanced condition, we can infer that

$$\langle\mathcal{D}u,\beta\rangle = \int_M ug^{i\bar j}g^{l\bar k}\nabla_{\bar j}\nabla_{\bar k}\bar\beta_{il}\frac{\omega^n}{n!}\,,$$

obtaining that $\mathcal{D}^*\beta = g^{j\bar i}g^{k\bar l}\nabla_j\nabla_k\beta_{\bar i\bar l}$. This guarantees that

$$\mathcal{D}^*\mathcal{D}u = g^{j\bar k}g^{p\bar q}\nabla_p\nabla_j\nabla_{\bar k}\nabla_{\bar q}u\,,$$

as in the Kähler case. On the other hand, we have that, for any $\beta\in\Lambda^{1,0}(M)\otimes\Lambda^{1,0}(M)$,

$$\nabla_p\nabla_j\beta_{\bar k\bar q} = \nabla_j\nabla_p\beta_{\bar k\bar q} + T^s_{jp}\nabla_s\beta_{\bar k\bar q}\,. \tag{2.27}$$

We can then use (2.27) to infer that

$$\mathcal{D}^*\mathcal{D}u = g^{p\bar{q}}g^{j\bar{k}}\nabla_j\nabla_p\nabla_{\bar{k}}\nabla_{\bar{q}}u + g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_s\nabla_{\bar{k}}\nabla_{\bar{q}}u\,.$$

Furthermore, we can recall that, for any $\alpha \in \Lambda^{0,1}(M)$,

$$\nabla_p\nabla_{\bar{k}}\alpha_{\bar{q}} = \nabla_{\bar{k}}\nabla_p\alpha_{\bar{q}} + R_{\bar{k}p\bar{q}}^{\bar{t}}\alpha_{\bar{t}}\,. \tag{2.28}$$

Hence, by making use of (2.28), we obtain that

$$\mathcal{D}^*\mathcal{D}u = g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_{\bar{k}}\nabla_p\nabla_{\bar{q}}u + g^{j\bar{k}}g^{p\bar{q}}\nabla_j(R_{\bar{k}p\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u) + g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_s\nabla_{\bar{k}}\nabla_{\bar{q}}u\,.$$

Now, recalling that, thanks to the second Bianchi identity, see for instance [327, Proposition 1.6], we have

$$\nabla_j R_{\bar{k}p\bar{q}}^{\bar{t}} = \nabla_p R_{\bar{k}j\bar{q}}^{\bar{t}} + T^s_{pj}R_{\bar{k}s\bar{q}}^{\bar{t}}\,,$$

we can conclude that

$$\begin{aligned} g^{j\bar{k}}g^{p\bar{q}}\nabla_j(R_{\bar{k}p\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u) &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j R_{\bar{k}p\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u + g^{j\bar{k}}g^{p\bar{q}}R_{\bar{k}p\bar{q}}^{\bar{t}}\nabla_j\nabla_{\bar{t}}u \\ &= g(\mathrm{Ric}^{(3)}(\omega), i\partial\bar{\partial}u) + g^{p\bar{q}}\nabla_p\mathrm{Ric}^{(2)}(g)_{\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u + g^{j\bar{k}}g^{p\bar{q}}T^s_{pj}R_{\bar{k}s\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u\,, \end{aligned}$$

where $\mathrm{Ric}^{(3)}(\omega)$ is the third Chern-Ricci form, see, for instance, [235] for its definition. On the other hand, using [235, Theorem 4.1], we know that, in the balanced case, $\mathrm{Ric}^{(3)}(\omega) = \mathrm{Ric}^{\mathrm{Ch}}(\omega)$. This gives us that

$$\mathcal{D}^*\mathcal{D}u = \Delta^2_\omega u + g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), i\partial\bar{\partial}u) + g^{p\bar{q}}\nabla_p\mathrm{Ric}^{(2)}(g)_{\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u + g^{j\bar{k}}g^{p\bar{q}}T^s_{pj}R_{\bar{k}s\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u + g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_s\nabla_{\bar{k}}\nabla_{\bar{q}}u\,.$$

Now, we can use (2.28) to infer that

$$g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_s\nabla_{\bar{k}}\nabla_{\bar{q}}u - g^{j\bar{k}}g^{p\bar{q}}T^s_{jp}R_{\bar{k}s\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u = g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_{\bar{k}}\nabla_s\nabla_{\bar{q}}u\,.$$

On the other hand, one can easily check that

$$g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_{\bar{k}}\nabla_s\nabla_{\bar{q}}u = \frac{1}{2}g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}(\nabla_{\bar{k}}\nabla_s\nabla_{\bar{q}}u - \nabla_{\bar{q}}\nabla_s\nabla_{\bar{k}}u)\,,$$

moreover, using that, for any $\gamma \in \Lambda^{1,1}(M)$, we have that

$$\nabla_{\bar{k}}\gamma_{s\bar{q}} - \nabla_{\bar{q}}\gamma_{s\bar{k}} = (\bar{\partial}\gamma)_{\bar{k}s\bar{q}} + T^{\bar{t}}_{\bar{q}\bar{k}}\gamma_{s\bar{t}}\,,$$

concluding that

$$g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}\nabla_{\bar{k}}\nabla_s\nabla_{\bar{q}}u = \frac{1}{2}g^{p\bar{q}}g^{j\bar{k}}T^s_{jp}T^{\bar{t}}_{\bar{q}\bar{k}}\nabla_s\nabla_{\bar{t}}u = -\frac{1}{2}g(\Xi, i\partial\bar{\partial}u)\,.$$

Then, putting all the pieces together we obtain that

$$\begin{aligned} \mathcal{D}^*\mathcal{D}u &= \Delta^2_\omega u + g\left(\mathrm{Ric}^{\mathrm{Ch}}(\omega) - \frac{1}{2}\Xi, i\partial\bar{\partial}u\right) + g^{p\bar{q}}\nabla_p\mathrm{Ric}^{(2)}(g)_{\bar{q}}^{\bar{t}}\nabla_{\bar{t}}u \\ &= \Delta^2_\omega u + g\left(\mathrm{Ric}^{\mathrm{Ch}}(\omega) - \frac{1}{2}\Xi, i\partial\bar{\partial}u\right) + g(\bar{\partial}^*\mathrm{Ric}^{(2)}(\omega), i\partial u)\,, \end{aligned}$$

where the last equality is due to the fact that

$$\bar{\partial}^*\mathrm{Ric}^{(2)}(\omega)_s = -g^{p\bar{q}}\nabla_p\mathrm{Ric}^{(2)}(\omega)_{s\bar{q}}\,,$$

see, for instance [105, Appendix D] for a more general statement. The claim is obtained making use of [303, Proposition 4.3]. □

This formula highlights the fundamental differences between the problem in our case and in the Kähler one. Indeed, formally, the role which in the Kähler case belongs to the Ricci form now is played by the first Chern-Ricci form, and the last torsion term appears. As one can see from the proof, the balanced condition is playing a crucial role both in computing $\mathcal{D}^*$ and in identifying the third and the first Chern-Ricci tensors. This last property is really exclusive of the balanced setting, see [235, Theorem 4.1].

If we now go back to the operator $\mathcal{L}$, which in our current setting is given by

$$\mathcal{L}(u) = -\Delta_\omega F_\omega(u) + n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \sqrt{-1}\partial\bar{\partial}(u\omega^{n-2})}{\omega^n} - s^{\mathrm{Ch}}(\omega)F_\omega(u),$$

we can obtain a clearer alternative expression, which will allow us to compare it to the Kähler case.

Theorem 2.2.8. *Let (M^n, ω) be a compact balanced manifold. Then, for all $u \in C^\infty(M, \mathbb{R})$,*

$$\begin{aligned}\mathcal{L}(u) = &-\frac{1}{n-1}\left(\Delta_\omega^2 u + g(\sqrt{-1}\partial\bar{\partial}u, \mathrm{Ric}^{\mathrm{Ch}}(\omega)) + \frac{1}{n-1}(\Delta_\omega + s^{\mathrm{Ch}}(\omega)\mathrm{Id})(|\partial\omega|^2 u)\right)\\ &-\frac{1}{n-1}\left(-\mathrm{Re}(g(\sqrt{-1}\partial u, \sqrt{-1}\Lambda^2(\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\partial\omega)) - ug(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \partial^*\partial\omega)\right).\end{aligned} \tag{2.29}$$

In particular, if ω is cscK,

$$\mathcal{L}(u) = -\frac{1}{n-1}\mathcal{D}^*\mathcal{D}u\,,$$

while if ω is a constant Chern scalar curvature balanced metric, we have

$$\mathcal{L}(u) = -\frac{1}{n-1}\left(\mathcal{D}^*\mathcal{D}u + g(\sqrt{-1}\partial u, \bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega)) + \frac{1}{n-1}(\Delta_\omega + s^{\mathrm{Ch}}(\omega)\mathrm{Id})(|\partial\omega|^2 u) + u\sqrt{-1}\partial^*\bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega)\right).$$

Proof. In order to obtain (2.29), we will compute the L^2-formal adjoint of $\mathcal{L}$ and then deduce the claim. First of all, recalling (2.12), we observe that F_ω is L^2-self-adjoint. This implies that, for any $u, \varphi \in C^\infty(M, \mathbb{R})$,

$$\langle\varphi, \Delta_\omega F_\omega(u)\rangle_{L^2} = \langle u, F_\omega\Delta_\omega\varphi\rangle_{L^2}\,, \quad \langle\varphi, s^{\mathrm{Ch}}(\omega)F_\omega(u)\rangle_{L^2} = \langle u, F_\omega(s^{\mathrm{Ch}}(\omega)\varphi)\rangle_{L^2}\,. \tag{2.30}$$

On the other hand, it is straightforward to check that

$$F_\omega(s^{\mathrm{Ch}}(\omega)\varphi) = s^{\mathrm{Ch}}(\omega)F_\omega(\varphi) + \frac{1}{n-1}(\varphi\Delta_\omega s^{\mathrm{Ch}}(\omega) + 2\mathrm{Re}(g(\partial s^{\mathrm{Ch}}(\omega), \partial\varphi))) \tag{2.31}$$

So, using (2.31) in the second equation of (2.30), we have that

$$\langle\varphi, s^{\mathrm{Ch}}(\omega)F_\omega(u)\rangle_{L^2} = \langle u, s^{\mathrm{Ch}}(\omega)F_\omega\varphi\rangle_{L^2} + \frac{1}{n-1}\langle u, \varphi\Delta_\omega s^{\mathrm{Ch}}(\omega) + 2\mathrm{Re}(g(\partial s^{\mathrm{Ch}}(\omega), \partial\varphi))\rangle_{L^2}\,. \tag{2.32}$$

Moreover, using the fact that $\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ is both ∂ and $\bar{\partial}$-closed, we observe that

$$\begin{aligned}\sqrt{-1}\varphi\partial\bar{\partial}(u\omega^{n-2}) \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) = {}& d((\varphi\sqrt{-1}\bar{\partial}(u\omega^{n-2}) + \sqrt{-1}\partial\varphi \wedge u\omega^{n-2}) \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega))\\ &+ u\sqrt{-1}\partial\bar{\partial}\varphi \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \omega^{n-2}\,.\end{aligned} \tag{2.33}$$

Thus, using (2.33) and Stokes' Theorem, we infer that

$$n\left\langle\varphi, \frac{\sqrt{-1}\partial\bar{\partial}(u\omega^{n-2}) \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega)}{\omega^n}\right\rangle_{L^2} = \frac{1}{(n-1)!}\int_M u\sqrt{-1}\partial\bar{\partial}\varphi \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \omega^{n-2}\,. \tag{2.34}$$

On the other hand, using [312, Lemma 4.7], we obtain that

$$\sqrt{-1}\partial\bar{\partial}\varphi \wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \frac{\omega^{n-2}}{(n-2)!} = (s^{\mathrm{Ch}}(\omega)\Delta_\omega\varphi - g(\sqrt{-1}\partial\bar{\partial}\varphi, \mathrm{Ric}^{\mathrm{Ch}}(\omega)))\frac{\omega^n}{n!}\,,$$

which can be used in (2.34) to deduce

$$n\left\langle \varphi, \frac{\sqrt{-1}\partial\bar\partial(u\omega^{n-2})\wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega)}{\omega^n}\right\rangle_{L^2} = \frac{1}{n-1}\langle u, s^{\mathrm{Ch}}(\omega)\Delta_\omega\varphi - g(\sqrt{-1}\partial\bar\partial\varphi, \mathrm{Ric}^{\mathrm{Ch}}(\omega))\rangle_{L^2}\,. \tag{2.35}$$

Now, using (2.30), (2.32) and (2.35), we have

$$\begin{aligned}\langle\varphi,\mathcal{L}(u)\rangle_{L^2} =& -\langle u, F_\omega\Delta_\omega\varphi\rangle_{L^2} - \langle u, s^{\mathrm{Ch}}(\omega)F_\omega(\varphi)\rangle_{L^2} - \frac{1}{n-1}\langle u,\varphi\Delta_\omega s^{\mathrm{Ch}}(\omega)\rangle_{L^2}\\ &-\frac{2}{n-1}\langle u, \mathrm{Re}(g(\sqrt{-1}\partial s^{\mathrm{Ch}}(\omega),\sqrt{-1}\partial\varphi))\rangle_{L^2} + \frac{1}{n-1}\langle u, s^{\mathrm{Ch}}(\omega)\Delta_\omega\varphi\rangle_{L^2}\\ &-\frac{1}{n-1}\langle u, g(\sqrt{-1}\partial\bar\partial\varphi,\mathrm{Ric}^{\mathrm{Ch}}(\omega))\rangle_{L^2}\,.\end{aligned} \tag{2.36}$$

Now, recalling again (2.12), we obtain

$$\begin{aligned}\left\langle u, F_\omega\Delta_\omega\varphi + \frac{1}{n-1}g(\sqrt{-1}\partial\bar\partial\varphi,\mathrm{Ric}^{\mathrm{Ch}}(\omega))\right\rangle_{L^2} =& \frac{1}{n-1}\left\langle u, \Delta^2_\omega\varphi + g(\sqrt{-1}\partial\bar\partial\varphi,\mathrm{Ric}^{\mathrm{Ch}}(\omega))\right\rangle_{L^2}\\ &+\frac{1}{n-1}\left\langle u, \frac{|\partial\omega|^2\Delta_\omega\varphi}{n-1}\right\rangle_{L^2}\,.\end{aligned} \tag{2.37}$$

Moreover,

$$\left\langle u, -s^{\mathrm{Ch}}(\omega)F_\omega(\varphi) + \frac{1}{n-1}s^{\mathrm{Ch}}(\omega)\Delta_\omega\varphi\right\rangle_{L^2} = -\frac{1}{(n-1)^2}\langle u, s^{\mathrm{Ch}}(\omega)|\partial\omega|^2\varphi\rangle_{L^2}\,. \tag{2.38}$$

Finally, using (2.37) and (2.38) in (2.36), we conclude that

$$\begin{aligned}\mathcal{L}^*(\varphi) =& -\frac{1}{n-1}\left(\Delta^2_\omega\varphi + g(\sqrt{-1}\partial\bar\partial\varphi,\mathrm{Ric}^{\mathrm{Ch}}(\omega)) + \frac{|\partial\omega|^2}{n-1}(\Delta_\omega\varphi + s^{\mathrm{Ch}}(\omega)\varphi)\right)\\ &-\frac{1}{n-1}\left(\varphi\Delta_\omega s^{\mathrm{Ch}}(\omega) + 2\mathrm{Re}(g(\sqrt{-1}\partial s^{\mathrm{Ch}}(\omega),\sqrt{-1}\partial\varphi))\right)\,.\end{aligned}$$

In order to conclude, it sufficient to understand what are the L^2-formal adjoint of the operators $g(\sqrt{-1}\partial\bar\partial\varphi,\mathrm{Ric}^{\mathrm{Ch}}(\omega))$ and $g(\partial s^{\mathrm{Ch}}(\omega),\partial\varphi)$. So, we consider $u\in C^\infty(M,\mathbb{R})$, we have that

$$\langle g(\partial s^{\mathrm{Ch}}(\omega),\partial\varphi),u\rangle_{L^2} = \langle u\partial s^{\mathrm{Ch}}(\omega),\partial\varphi\rangle_{L^2} = \langle\partial^*(u\partial s^{\mathrm{Ch}}(\omega)),\varphi\rangle_{L^2}\,.$$

Now, using (1.8), one checks that

$$\begin{aligned}2\mathrm{Re}(\partial^*(u\partial s^{\mathrm{Ch}}(\omega))) =& -2\mathrm{Re}(g(\sqrt{-1}\partial u,\sqrt{-1}\partial s^{\mathrm{Ch}}(\omega))) + 2u\partial^*\partial s^{\mathrm{Ch}}(\omega)\\ =& -2\mathrm{Re}(g(\sqrt{-1}\bar\partial u,\sqrt{-1}\bar\partial s^{\mathrm{Ch}}(\omega))) - 2u\Delta_\omega s^{\mathrm{Ch}}(\omega)\,.\end{aligned} \tag{2.39}$$

Moreover

$$\langle g(\sqrt{-1}\partial\bar\partial\varphi,\mathrm{Ric}^{\mathrm{Ch}}(\omega)),u\rangle_{L^2} = \langle\sqrt{-1}\partial\bar\partial\varphi, u\mathrm{Ric}^{\mathrm{Ch}}(\omega)\rangle_{L^2} = \langle\varphi,\sqrt{-1}\partial^*\bar\partial^*(u\mathrm{Ric}^{\mathrm{Ch}}(\omega))\rangle_{L^2}\,.$$

On the other hand, one can easily verify that

$$\sqrt{-1}\partial^*\bar\partial^*(u\mathrm{Ric}^{\mathrm{Ch}}(\omega)) = g(\sqrt{-1}\partial\bar\partial u,\mathrm{Ric}^{\mathrm{Ch}}(\omega)) + 2\mathrm{Re}(g(\sqrt{-1}\partial u,\bar\partial^*\mathrm{Ric}^{\mathrm{Ch}}(\omega))) + u\sqrt{-1}\partial^*\bar\partial^*\mathrm{Ric}^{\mathrm{Ch}}(\omega)\,. \tag{2.40}$$

Furthermore, from (2.39) and (2.40), using Lemma 2.2.1, we have that

$$-2\mathrm{Re}(g(\bar\partial u,\bar\partial s^{\mathrm{Ch}}(\omega))) + 2\mathrm{Re}(g(\sqrt{-1}\partial u,\bar\partial^*\mathrm{Ric}^{\mathrm{Ch}}(\omega))) = -\mathrm{Re}(g(\partial u,\Lambda^2(\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\partial\omega))$$

and

$$-\Delta_\omega s^{\mathrm{Ch}}(\omega) + \sqrt{-1}\partial^*\bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega) = -g(\mathrm{Ric}^{\mathrm{Ch}}(\omega), \partial^*\partial\omega)\,.$$

From these we obtain the first claim. Assume now ω to be cscK, then

$$\mathcal{L}(u) = -\frac{1}{n-1}\left(\Delta_\omega^2 u + g(\sqrt{-1}\partial\bar{\partial}u, \mathrm{Ric}^{\mathrm{Ch}}(\omega))\right) = -\frac{1}{n-1}\mathcal{D}^*\mathcal{D}u\,,$$

see [312, Definition 4.3].
Finally, let ω be a balanced metric with constant Chern scalar curvature. Using the fact that

$$2\mathrm{Re}(g(\sqrt{-1}\partial u, \bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega))) = g(\sqrt{-1}\partial u, \bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega)) + g(\bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega), \sqrt{-1}\partial u)\,,$$

we can easily infer that

$$\Delta_\omega^2 u + g(\sqrt{-1}\partial\bar{\partial}u, \mathrm{Ric}^{\mathrm{Ch}}(\omega)) + 2\mathrm{Re}(g(\sqrt{-1}\partial u, \bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega))) = \mathcal{D}^*\mathcal{D}u + g(\sqrt{-1}\partial u, \bar{\partial}^*\mathrm{Ric}^{\mathrm{Ch}}(\omega))\,.$$

This last relation inserted in (2.29) gives us the claim. □

Thus, we see that the operator $\mathcal{L}$ arising from the balanced deformation generalizes to the balanced case, with the choice of the Chern connection, the Lichnerowicz operator, giving further motivation to widen the understanding of the associated equation, and making it a good candidate to obtain in the future the result in full generality.

Remark 2.2.9. It is very interesting to notice that, when starting with ω a Kähler metric, despite the deformation is not corresponding to a deformation in the Kähler class (indeed the balanced deformation does not preserve the Kähler condition in general), it produces an operator whose linearization is again the Lichnerowicz operator, up to a constant factor. This in particular shows that, along this ansatz, holomorphic vector fields appear again as an obstruction to successfully deform the pregluing metric to a genuine constant Chern scalar curvature balanced metric.

2.2.2 Construction of constant Chern scalar curvature balanced metrics

Having now convinced ourselves of the significance of this deformation, we can proceed with the proof of Theorem B.

In order to prove the theorem, we shall now introduce suitable weighted spaces, as done in [51], as they will turn out to be the right spaces on which we are able to invert (uniformly) the operator $\tilde{\mathcal{L}}$. Since we can always assume, up to rescaling, that the neighbourhood of x on which the z coordinates are defined contains the region $\{|z| \le 1\}$, we define

$$\rho = \rho_\varepsilon(z) := \begin{cases} \varepsilon^{p+q} & \text{on } |z| \le \varepsilon^{p+q}, \\ \text{non decreasing} & \text{on } \varepsilon^{p+q} \le |z| \le 2\varepsilon^{p+q}, \\ |z| & \text{on } 2\varepsilon^{p+q} \le |z| \le 1/2, \\ \text{non decreasing} & \text{on } 1/2 \le |z| \le 1, \\ 1 & \text{on } |z| \ge 1\,. \end{cases}$$

We then introduce, for all $b \in \mathbb{R}$, the *weighted Hölder norm* as

$$\begin{aligned} \|u\|_{C^{k,\alpha}_{b,\varepsilon}(\hat{M})} := &\sum_{i=0}^{k} \sup_{\hat{M}} |\rho^{b+i}\nabla^i_\varepsilon u|_\omega \\ &+ \sup_{d_\varepsilon(x,y)<inj_\varepsilon} \left| \min\left(\rho^{b+k+\alpha}(x), \rho^{b+k+\alpha}(y)\right) \frac{\nabla^k_\varepsilon u(x) - \nabla^k_\varepsilon u(y)}{d_\varepsilon(x,y)^\alpha} \right|_\omega, \end{aligned}$$

where inj_ε is the injectivity radius of the metric ω. Consequently, we define the corresponding *weighted Hölder spaces* $C^{k,\alpha}_{b,\varepsilon}(\hat{M}) := \{u \in C^k(\hat{M}) \mid \|u\|_{C^{k,\alpha}_{b,\varepsilon}(\hat{M})} < \infty\}$, where $k \ge 0$, $\alpha \in (0,1)$ is the Hölder constant, and ε indicates the dependence on the pre-gluing metric ω obtained in Subsection 2.1.1. Hence, we can interpret $\tilde{\mathcal{S}}$ as

$$\tilde{\mathcal{S}} : C^{4,\alpha}_{b,\varepsilon}(\hat{M}) \to C^{0,\alpha}_{b+4,\varepsilon}(\hat{M}) .$$

As in [168, Theorem 1], there is an obstacle given by the kernel of the operator F_ω (actually its limit - with respect to ε - on M_x) in (2.12). Nevertheless, we will see that it is possible to work orthogonally to this kernel, in order to ensure the invertibility of the operator. Indeed, we can introduce the functional space

$$\mathcal{V}_\varepsilon := (\ker F_\omega)^{\perp_{L^2}} \subseteq C^{4,\alpha}_{b,\varepsilon}(\hat{M}),$$

which inherits the Banach structure thanks to the fact that the topology induced by the weighted Hölder norm is finer than the L^2-topology, which guarantees that $\mathcal{V}_\varepsilon$ is a closed subspace of $C^{4,\alpha}_{b,\varepsilon}(\hat{M})$. We are then able to obtain the uniform invertibility of the linearized operator, following the strategy in [51]. Before proving this result, however, we need a preliminary lemma, which will be central later.

Lemma 2.2.10. *For all $v \in \ker F_{\tilde{\omega}}$, it exists $v_\varepsilon \in C^{4,\alpha}_{b,\varepsilon}(\hat{M})$ such that*

$$v_\varepsilon \in \ker F_{\omega_\varepsilon} \quad \textit{and} \quad v_\varepsilon \equiv v\,, \quad \textit{on} \;\; \{|z| \geq 2\varepsilon^p\}\,, \tag{2.41}$$

for all $\varepsilon > 0$.

Proof. For all $v \in \ker F_{\tilde{\omega}}$, we consider the same cut-off function χ_ε of Lemma 2.1.1 and call $R_\varepsilon := \{\varepsilon^p < |z| < 2\varepsilon^p\}$ the cut-off region, over which we consider the boundary value problem:

$$\begin{cases} F_{\tilde{\omega}_\varepsilon}(u_\varepsilon) = -F_{\tilde{\omega}_\varepsilon}(\chi_\varepsilon v) & \text{on } R_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial R_\varepsilon, \end{cases} \tag{2.42}$$

for u_ε smooth on R_ε.

In order to solve problem (2.42), we consider its weak formulation:

$$\begin{cases} B(\varphi, u_\varepsilon) := \langle \partial\varphi, \partial u_\varepsilon \rangle_{L^2_\varepsilon} - \frac{1}{n-1}\langle \varphi, |\partial\tilde{\omega}_\varepsilon|^2_{\tilde{\omega}_\varepsilon} u_\varepsilon \rangle_{L^2_\varepsilon} = \langle \varphi, F_{\tilde{\omega}_\varepsilon}(\chi_\varepsilon v) \rangle_{L^2_\varepsilon}\,, & \varphi \in C^\infty_c(R_\varepsilon) \\ u_\varepsilon \in W^{1,2}_{0,\varepsilon}(R_\varepsilon), \end{cases} \tag{2.43}$$

where L^2_ε identifies the L^2-product induced by $\tilde{\omega}_\varepsilon$, and same for the Sobolev space $W^{1,2}_{0,\varepsilon}(R_\varepsilon)$.

In order to obtain a solution to problem (2.43), we notice that the bilinear form B satisfies the Gårding inequality

$$B(\varphi, \varphi) \geq \frac{1}{2}||\varphi||^2_{W^{1,2}_\varepsilon} - \left(\frac{\max|\partial\tilde{\omega}_\varepsilon|^2_{\tilde{\omega}_\varepsilon}}{n-1} + \frac{1}{2} \right) ||\varphi||^2_{L^2_\varepsilon}\,, \qquad \varphi \in W^{1,2}_{0,\varepsilon}. \tag{2.44}$$

This ensures us that we can apply [2, Theorem 8.5], to obtain that problem (2.43) has solution if and only if $F_{\tilde{\omega}_\varepsilon}(\chi_\varepsilon v)$ is orthogonal to $\ker F_{\tilde{\omega}_\varepsilon}$, where here $F_{\tilde{\omega}_\varepsilon} : W^{2,2}_{0,\varepsilon}(R_\varepsilon) \to L^2_\varepsilon(R_\varepsilon)$, which is clearly self-adjoint. On the other hand, it is straightforward to notice that this last condition is indeed verified, ensuring us a solution u_ε.

Now, if we extend u_ε to the function $\tilde{u}_\varepsilon$, defined on the whole M as identically zero outside of R_ε, it is clear that $\tilde{u}_\varepsilon$ solves (weakly) on M the problem

$$F_{\tilde{\omega}_\varepsilon}(\tilde{u}_\varepsilon) = -F_{\tilde{\omega}_\varepsilon}(\chi_\varepsilon v),$$

ensuring that $\tilde{u}_\varepsilon$ is actually smooth on M, and hence also a classical solution of the latter equation.

Finally, it is straightforward to see that $\tilde{u}_\varepsilon$ extends smoothly to $\hat{u}_\varepsilon$ on $\hat{M}$ (by setting $\hat{u}_\varepsilon \equiv 0$ on the exceptional divisor), yielding a classical solution on $\hat{M}$ of

$$F_{\omega_\varepsilon}(u) = -F_{\omega_\varepsilon}(\chi_\varepsilon v).$$

Thus, the function

$$v_\varepsilon := \chi_\varepsilon v + \hat{u}_\varepsilon$$

is exactly the function we wanted. ☐

We are now ready to prove the main theorem of this subsection.

Theorem 2.2.11. *For any $b \in (0, 2n-4)$, there exists $C > 0$ such that, for all $u \in \mathcal{V}_\varepsilon$, we have*

$$\|u\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq C \|\tilde{\mathcal{L}} u\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})} .$$

Proof. Suppose by contradiction that statement does not hold. Hence, we can find sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ and $\{u_k\}_{k\in\mathbb{N}}$ such that $u_k \in \mathcal{V}_k := \mathcal{V}_{\varepsilon_k}$, for all $k \in \mathbb{N}$, and

$$\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0 , \quad ||u_k||_{C^{4,\alpha}_{b,\varepsilon_k}(\hat{M})} = 1 , \quad k \in \mathbb{N} , \tag{2.45}$$

and

$$||\tilde{\mathcal{L}} u_k||_{C^{0,\alpha}_{b+4,\varepsilon_k}(\hat{M})} < \frac{1}{k} , \quad k \in \mathbb{N}. \tag{2.46}$$

We will focus firstly on $M_x := M \backslash \{x\}$. By applying Ascoli-Arzelà's Theorem, we have that $u_k \to u_\infty$ uniformly on compact subsets of M_x in the sense of $C^{4,\alpha}_b$, with respect to the background metric $\tilde{\omega}$. This implies, in particular, that on any compact subset of M_x, thanks to the fact that $\tilde{\omega}$ is a Chern-Ricci flat balanced metric, it holds

$$\mathcal{L} u_k \to -\Delta_{\tilde{\omega}} F_{\tilde{\omega}}(u_\infty) , \quad \text{as } k \to \infty , \tag{2.47}$$

i.e. $\mathcal{L} u_k$ converges uniformly on compact sets to a continuous function on M_x. If we then fix a point $y \in M_x$, in the region where $\rho \equiv 1$, condition (2.46) implies that

$$\tilde{\mathcal{L}} u_k(y) \to 0,$$

which, combined with equation (2.47), implies that the real sequence $\int_{\hat{M}} u_k |\partial \omega|^2 \frac{\omega^n}{n!}$ has finite limit. Hence, by Lebesgue's Theorem, we get

$$\int_{\hat{M}} u_k |\partial \omega|^2 \frac{\omega^n}{n!} \to \int_{M_x} u_\infty |\partial \tilde{\omega}|^2 \frac{\tilde{\omega}^n}{n!} , \quad \text{as } k \to \infty . \tag{2.48}$$

If we now integrate $\tilde{\mathcal{L}} u_k$ on M_x, using equations (2.47) and (2.48) and assuming $b < 2n - 4$, we obtain

$$0 = \int_{M_x} \tilde{\mathcal{L}} u_\infty \tilde{\omega}^n = -\int_{M_x} \Delta_{\tilde{\omega}} F_{\tilde{\omega}}(u_\infty) \frac{\tilde{\omega}^n}{n!} + \mathrm{Vol}(M, \tilde{\omega}) \int_{M_x} u_\infty |\partial \tilde{\omega}|^2 \frac{\tilde{\omega}^n}{n!} = \mathrm{Vol}(M, \tilde{\omega}) \int_{M_x} u_\infty |\partial \tilde{\omega}|^2 \frac{\tilde{\omega}^n}{n!}, \tag{2.49}$$

hence $\int_{M_x} u_\infty |\partial \tilde{\omega}|^2 \frac{\tilde{\omega}^n}{n!} = 0$. From this, we can infer that

$$\Delta_{\tilde{\omega}} F_{\tilde{\omega}}(u_\infty) = 0 ,$$

from which follows that u_∞ is such that $F_{\tilde{\omega}}(u_\infty) \equiv c \in \mathbb{R}$. Now, recalling (2.12), we again integrate the equation $F_{\tilde{\omega}}(u_\infty) \equiv c$ over the whole M_x yielding that, since again $b < 2n - 4$,

$$0 = \int_{M_x} u_\infty |\partial \tilde{\omega}|^2 \frac{\tilde{\omega}^n}{n!} = c \mathrm{Vol}(M, \omega) ,$$

which implies that $c = 0$ and then $F_{\tilde{\omega}}(u_\infty) = 0$ on M_x.

Now, using that $u_\infty \in C^{4,\alpha}_{b,\varepsilon}(M_x)$ and (2.12) again, we can conclude that

$$\Delta_d u_\infty \in C^{4,\alpha}_{b,\varepsilon}(M_x) .$$

Thanks to this, following the argument in [312, Proposition 8.10], we have that $u_\infty \in C^{6,\alpha}_{b-2,\varepsilon}(M_x)$. Iterating this process, we can infer that $u_\infty \in C^{4+2j,\alpha}_{b-2j,\varepsilon}(M_x)$ where $j \in \mathbb{N}$ is the first integer such that $b - 2j + 1 < 0$. Now, we can extend u_∞ to a function in $C^{4+2j,\alpha}(M)$ such that

$$u_\infty(x) = 0 , \quad \Delta_\omega u_\infty(x) = 0$$

so that $u \in \ker F_{\tilde{\omega}}$ on the whole M. Now, elliptic regularity allows us to conclude that $u_\infty \in C^\infty(M,\mathbb{R})$ such that $\int_M u_\infty |\partial\tilde{\omega}|^2 \frac{\tilde{\omega}^n}{n!} = 0$. Now, we want to use that $u_k \in \mathcal{V}_k$ in order to conclude that u_∞ is L^2-orthogonal to $\ker F_{\tilde{\omega}}$. If we show this, we will have that $u_\infty \in \ker F_{\tilde{\omega}} \cap (\ker F_{\tilde{\omega}})^\perp$ concluding that $u_\infty = 0$.

For all $v \in \ker F_{\tilde{\omega}}$, we can apply Lemma 2.2.10 and obtain a corresponding $v_\varepsilon \in C^{4,\alpha}_{b,\varepsilon}(\hat{M})$. Then, denoting with $v_k := v_{\varepsilon_k}$, for all $k \in \mathbb{N}$, using that $u_k \in \mathcal{V}_k$, it holds

$$\langle u_k, v_k \rangle_{\omega_{\varepsilon_k}} = 0\,, \quad \text{for all } k \in \mathbb{N}\,.$$

This implies that, on any compact subset on M_x, we have

$$\langle u_\infty, v \rangle_{\tilde{\omega}} = 0.$$

Hence, considering an exhaustion of compact subsets of M_x and using the fact that u_∞ and v are actually functions on M, we obtain

$$\langle u_\infty, v \rangle_{\tilde{\omega}} = 0$$

on M, which means exactly that

$$u_\infty \perp_{L^2} \ker F_{\tilde{\omega}}.$$

This allows to conclude that $u_\infty = 0$, as explained above.

We thus fix the compact set $M_c := M \setminus \{|z| < 1/2\}$, and focus on $A := \{|z| < 1/2\}$, on which we wish to obtain uniform convergence to zero. For convenience, we shall shift to the "large"coordinates ζ, i.e. the coordinates on the blow-up $\hat{X}$ defined outside the exceptional divisor. Recalling then that

$$\zeta = \varepsilon^{-(p+q)} z \quad \text{and} \quad |z| = \varepsilon^{p+q}|\zeta|,$$

we have the identification

$$A \simeq \tilde{A} = \tilde{A}_\varepsilon := \left\{ |\zeta| < \frac{1}{2}\varepsilon^{-(p+q)} \right\} \subseteq \hat{X},$$

and the last description will be the one we will use.
First of all, we shall rewrite ρ with respect to ζ on $\tilde{A}$, giving

$$\rho = \begin{cases} \varepsilon^{p+q} & \text{on } |\zeta| \le 1, \\ \text{non decreasing} & \text{on } 1 \le |\zeta| \le 2, \\ \varepsilon^{p+q}|\zeta| & \text{on } 2 \le |\zeta| \le 1/2\varepsilon^{-(p+q)}. \end{cases}$$

It follows that, going back to $\{u_k\}_{k\in\mathbb{N}}$ and recalling (2.45), we have, in particular, that on all $\tilde{A}_k := \tilde{A}_{\varepsilon_k}$ it holds

$$|\rho^b u_k| \le C.$$

This suggests us to introduce the new sequence

$$U_k := \varepsilon_k^{b(p+q)} u_k,$$

and using again (2.45), we obtain

$$\begin{cases} |U_k| \le C & \text{on } |\zeta| \le 1, \\ |U_k| \le C & \text{on } 1 \le |\zeta| \le 2, \\ |U_k| \le C|z|^{-b}(\zeta) & \text{on } 2 \le |\zeta| \le 1/2\varepsilon_k^{-(p+q)}, \end{cases}$$

and the same for its derivatives up to the fourth order. These estimates for U_k bring us to consider a new weight function $\tilde{\rho} = \tilde{\rho}_k$ on $\tilde{A}_k$ given by

$$\tilde{\rho}(\zeta) = \begin{cases} 1 & \text{on } |\zeta| \le 1, \\ \text{non decreasing} & \text{on } 1 \le |\zeta| \le 2, \\ |\zeta| & \text{on } 2 \le |\zeta| \le 1/2\varepsilon_k^{-(p+q)}, \end{cases}$$

which gives that

$$|\tilde{\rho}^b U_k| \leq C, \tag{2.50}$$

and estimates also for $\nabla^m U_k$, for all $m = 1, ..., 4$. Hence, again by Ascoli-Arzelà's Theorem, we have that $U_k \to U_\infty$, as $k \to \infty$, uniformly on compact sets of $\hat{X}$ (since $\tilde{A}_k \to \hat{X}$, as $k \to \infty$) in the sense of $\tilde{C}^{4,\alpha}_b := C^{4,\alpha}_b(\tilde{\rho})$, where this last space is the weighted Hölder space on $\hat{X}$ given by the weight $\tilde{\rho}$ and the metric ω_{BS}.
On the other hand, on any compact subset of $\hat{X}$, for sufficiently large k, it holds

$$\rho^{b+4}\mathcal{L}u_k = -\frac{1}{n-1}\tilde{\rho}^{b+4}\mathcal{D}^*\mathcal{D}U_k, \tag{2.51}$$

where $\mathcal{D}^*\mathcal{D}$ is the Lichnerowicz operator corresponding to ω_{BS}. Thus, since (2.46) holds, taking the limit in (2.51), we obtain that U_∞ is in the kernel of $\mathcal{D}^*\mathcal{D}$ with respect to the Burns-Simanca metric ω_{BS}. Thus, applying [312, Proposition 8.10], we get that U_∞ is necessarily constant, which needs to be zero as U_∞ decays at infinity (from inequality (2.50)). Hence, $U_k \to 0$ uniformly on compact sets of $\hat{X}$ in $\tilde{C}^{4,\alpha}_b$.
In order to conclude, we will show that U_k admits a subsequence uniformly convergent to zero on the whole $\hat{X}$ in the sense $\tilde{C}^0_b$. This, combined with the scaled Schauder estimates, see for instance [51, formula (6)], will imply that also $U_k \to 0$ uniformly in $\tilde{C}^{4,\alpha}_b$. On the other hand, this is equivalent to $u_k \to 0$ uniformly on $\{|z| < 1/2\}$ in $C^{4,\alpha}_{b,\varepsilon}$. Together with the fact that u_k converges uniformly to zero on M_c, it gives a contradiction with the fact that $||u_k||_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} = 1$, for all $k \in \mathbb{N}$.
The final step of the proof will be to show that such subsequence necessarily exists. Indeed, if we assume by contradiction that such subsequence does not exist, then we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \hat{X}$ and a $\delta > 0$ such that

$$R_k := |\zeta(x_k)| \to +\infty \tag{2.52}$$

and

$$\tilde{\rho}^b(x_k)|U_k(x_k)| \geq \delta\,, \quad k \geq 0\,. \tag{2.53}$$

This last condition can be rewritten (up to choosing sufficiently large k) as

$$R_k^b|U_k(x_k)| \geq \delta\,, \quad k \geq 0\,. \tag{2.54}$$

If we then define $r_k := |z(x_k)|$, we have that $r_k = \varepsilon_k^{p+q}R_k$, for all $k \in \mathbb{N}$, from which, up to subsequences, we see that we can only fall into two cases:

- if $\lim_{k\to+\infty} r_k = r > 0$, then it means that we can assume $x_k \to x_\infty$, which combined with the uniform convergence to zero on compact sets (of M_x) of the sequence $\{u_k\}_{k\in\mathbb{N}}$ gives

$$0 < \delta \leq R_k^b|U_k(x_k)| = r_k^b u_k(x_k) \to 0,$$

  i.e. a contradiction;

- if instead $\lim_{k\to+\infty} r_k = 0$, we take X' a copy of $\hat{X}$, and for all $k \geq 0$ we introduce the holomorphic maps

$$\sigma_k : B_k \to A^* := A \setminus \{0\},$$

  given by $\sigma_k(z') = r_k z'$, where $B_k := \{0 < |z'| < r_k^{-1}/2\} \subseteq X'$. Using these, we can define the metrics and the forms

$$\theta_k := r_k^{-2}\sigma_k^*\omega\,,$$

  and easily observe that $(B_k, \theta_k) \to (X', \omega_o)$, as $k \to \infty$, where ω_o here denotes the flat metric induced by the coordinates z'. Then, it is natural to consider the functions on each B_k given by

$$W_k := r_k^b\sigma_k^* u_k\,, \quad k \in \mathbb{N},$$

and the pullback weight function

$$\rho'(z') = \sigma_k^*\rho(z') = \begin{cases} \varepsilon_k^{p+q} & \text{on } |z'| \leq R_k^{-1}\,, \\ \text{non increasing} & \text{on } R_k^{-1} < |z'| < 2R_k^{-1}\,, \\ r_k|z'| & \text{on } 2R_k^{-1} \leq |z'| < \frac{1}{2}r_k^{-1}. \end{cases} \tag{2.55}$$

Now, if we pullback (2.45) using σ_k, we immediately obtain that the sequence $\{W_k\}_{k\in\mathbb{N}}$ is uniformly bounded on compact sets in the $C_b^{4,\alpha}$ sense. Thus, by Ascoli-Arzela's Theorem, we can assume that $W_k \to W_\infty$, and, again, from pulling back (2.45), we obtain that W_∞ is a $C^{4,\alpha}$-function on X' decaying to infinity. Moreover, analyzing the pieces of the pullback

$$\sigma_k^*(\mathcal{L}u_k) = \sigma_k^*\left(\Delta_\omega F_\omega(u_k) + n\frac{\sqrt{-1}\partial\overline{\partial}(u_k\omega^{n-2})\wedge \mathrm{Ric}^{\mathrm{Ch}}(\omega)}{\omega^n} - s^{\mathrm{Ch}}(\omega)F_\omega(u_k)\right)$$

we can see that:

- $\sigma_k^*\Delta_\omega(F_\omega u_k) = r_k^{-(b+4)}\Delta_{\theta_k}F_k(W_k)$, where $F_k := F_{\theta_k}$;
- $\sigma_k^*\left(\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega)\wedge\sqrt{-1}\partial\bar{\partial}(u_k\omega^{n-2})}{\omega^n}\right) = r_k^{-(b+4)}\left(\frac{\mathrm{Ric}^{\mathrm{Ch}}(\theta_k)\wedge\sqrt{-1}\partial\bar{\partial}(W_k\theta_k^{n-2})}{\theta_k^n}\right)$;
- $\sigma_k^*(s^{\mathrm{Ch}}(\omega)F_\omega(u_k)) = r_k^{-(b+4)}s^{\mathrm{Ch}}(\theta_k)F_k(W_k)$.

Moreover, it is easy to show that $F_k \to \frac{1}{n-1}\Delta_{\omega_o}$ and that, of course, $s^{\mathrm{Ch}}(\theta_k) \to s^{\mathrm{Ch}}(\omega_o) = 0$, as $k \to \infty$. Hence, pulling back with σ_k (2.46) and taking the limit in k, we obtain that W_∞ is biharmonic on X'. Pulling back (2.45) and recalling (2.55), we obtain that W_∞ decays at infinity, implying necessarily that $W_\infty \equiv 0$ on X'. On the other hand, if we define the sequence $y_k := \sigma_k^{-1}(x_k) \in X'$, it is straightforward to see that $|y_k| = 1$, for all $k \in \mathbb{N}$. Hence, it can be assumed to be convergent to some y_∞, which combined with the limit of pullback via σ_k of (2.54), implies $W_\infty(y_\infty) > 0$, i.e. a contradicition with the fact that $W_\infty \equiv 0$.

Hence the thesis is proven. □

Remark 2.2.12. We stress how in the proof, the assumption of Chern-Ricci flatness of the metric $\tilde{\omega}$ has allowed to make the problem significantly more approachable by erasing the second degree terms involving the Chern-Ricci tensor and the Chern scalar curvature. We however expect that it is possible to overcome this technical assumption and obtain the result in the more general constant Chern scalar curvature balanced case, as a (not straightforward) consequence of the fact that both the above mentioned terms are only second order terms.

The above estimate allows us to easily obtain the uniform invertibility.

Lemma 2.2.13. *The operator*

$$\tilde{\mathcal{L}}\colon \mathcal{V}_\varepsilon \to C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})$$

is an isomorphism.

Proof. Thanks to Theorem 2.2.11, we have that $\tilde{\mathcal{L}}$ is injective. Moreover, thanks to Remark 2.3.2, it is clear that $\tilde{\mathcal{L}}$ is elliptic and with the same index of Δ_ω^2, which is 0. This automatically guarantees the claim. □

2.2.3 Setting up the fixed point problem

We can now reformulate $\tilde{\mathcal{S}}(u) = 0$ by considering the expansion

$$s^{\mathrm{Ch}}(\omega_u) = s^{\mathrm{Ch}}(\omega) + \mathcal{L}u + Q(u)\,,$$

where Q is the quadratic part of $s^{\mathrm{Ch}}(\omega_u)$. Then, (2.7) can be rewritten as:

$$s^{\mathrm{Ch}}(\omega) + \tilde{\mathcal{L}}u + Q(u) = 0\,,$$

and hence, using Lemma 2.2.13, we obtain that

$$\mathcal{N}(u) := -\tilde{\mathcal{L}}^{-1}(s^{\mathrm{Ch}}(\omega) + Q(u)) = u\,. \tag{2.56}$$

So, thanks to Banach's fixed-point Theorem, in order to find a solution to $\tilde{\mathcal{S}}(u) = 0$, it is sufficient to show that

$$\mathcal{N}\colon \mathcal{V}_\varepsilon \to \mathcal{V}_\varepsilon$$

is a contraction on a suitable open neighbourhood of zero in $\mathcal{V}_\varepsilon$.

In order to determine the open set we are looking for, we note that, if $\|\psi\|_{C^{4,\alpha}_{-2,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau$, for some $C, \tau > 0$, then,

$$\|\sqrt{-1}\partial\bar{\partial}(\psi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \leq C\|\psi\|_{C^{4,\alpha}_{-2,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau\,, \tag{2.57}$$

where the first inequality is due to the fact that $\|\omega^{n-2}\|_{C^{4,\alpha}_{0,\varepsilon}(\hat{M})} \leq C$. Up to choosing ε sufficiently small, this guarantees that $\omega_\psi^{n-1} > 0$, hence provides a balanced metric, thanks to [240], as well as, using (2.57),

$$\|\omega_\psi^{n-1} - \omega^{n-1}\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} = \|\sqrt{-1}\partial\bar{\partial}(\psi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau\,. \tag{2.58}$$

Moreover, arguing as in [168, Remark 2.8], we can fix a point $y \in \hat{M}$ and consider holomorphic coordinates so that, in y, ω is the identity and ω_ψ is diagonal with eigenvalues λ_i. On the other hand, ω^{n-1} will be again the "identity "and ω_ψ^{n-1} will have eigenvalues Λ_i. But, thanks to (2.58), we know that

$$\Lambda_i = 1 + \mathrm{O}(\varepsilon^\tau)\,,$$

which implies that $\lambda_i = \left(\prod_{j\neq i}\Lambda_j\right)^{\frac{1}{n-1}} = 1 + O(\varepsilon^\tau)$. This last fact readily guarantees that

$$\|\omega_\psi - \omega\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau\,, \tag{2.59}$$

which, in particular, gives that $\omega_\psi \to \omega$, as $\varepsilon \to 0$. As in [312] and [168], we then consider the open set

$$U_\tau := \{\psi \in \mathcal{V}_\varepsilon \quad | \quad \|\psi\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq C\varepsilon^{(p+q)(b+2)+\tau}\}\,.$$

We can readily note that, if $\psi \in U_\tau$, then

$$\|\psi\|_{C^{4,\alpha}_{-2,\varepsilon}(\hat{M})} \leq \varepsilon^{-(p+q)(b+2)}\|\psi\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau\,, \tag{2.60}$$

where the first inequality is due to the fact that $\|\psi\|_{C^{k,\alpha}_{a,\varepsilon}(\hat{M})} \leq \varepsilon^{(p+q)(-b+a)}\|\psi\|_{C^{k,\alpha}_{b,\varepsilon}(\hat{M})}$, for any $k \geq 0$, $a \leq b$, thanks to the definition of our weight. This inequality guarantees also that every $\psi \in U_\tau$, is not only small in the weighted sense, but it is so also in the standard sense, ensuring that our setting for the problem makes sense in this set.
We are thus left with the estimates to obtain that $\mathcal{N}$ preserves U_τ and it is a contraction on it.

2.2.4 Weighted estimates

We first show that $\mathcal{N}$ contracts distances on U_τ, which thanks to (2.56) and Theorem 2.2.13 reduces to showing that Q contracts distances. Thus, fixed $\varphi_1, \varphi_2 \in U_\tau$, the Mean value Theorem guarantees that there exists $t \in [0,1]$ such that, defined $\chi := t\varphi_1 + (1-t)\varphi_2 \in U_\tau$, we have

$$Q(\varphi_1) - Q(\varphi_2) = d_\chi Q(\varphi_1 - \varphi_2) = (\mathcal{L}_\chi - \mathcal{L})(\varphi_1 - \varphi_2)\,.$$

We now need to compute $\mathcal{L}_\chi := d_\chi \mathcal{S}$. As done in Subsection 2.1.2, we consider the curve of Hermitian metrics in $[\omega^{n-1}]_{\mathrm{BC}} = [\omega_\chi^{n-1}]_{\mathrm{BC}}$ defined by $\omega_{\chi,v}(s)^{n-1} := \omega_\chi^{n-1} + s\sqrt{-1}\partial\bar\partial(v\omega^{n-2})$, then,

$$\mathcal{L}_\chi(v) = \frac{d}{ds}\Big|_{s=0} s^{\mathrm{Ch}}(\omega_{\chi,v}(s))\,.$$

But, differentiating again (1.13), we obtain that

$$\mathcal{L}_\chi(v)\omega_\chi^n = n\frac{d}{ds}\Big|_{s=0}\mathrm{Ric}^{\mathrm{Ch}}(\omega_{\chi,v}(s))\wedge\omega_\chi^{n-1} + n\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\wedge\frac{d}{ds}\Big|_{s=0}\omega_\chi^{n-1}(s) - s^{\mathrm{Ch}}(\omega_\chi)\frac{d}{ds}\Big|_{s=0}\omega_{\chi,v}(s)^n\,.$$

As done in (2.8), we conclude that

$$\frac{d}{ds}\Big|_{s=0}\omega_{\chi,v}(s)^n = \frac{n}{n-1}\sqrt{-1}\partial\bar\partial(v\omega^{n-2})\wedge\omega_\chi\,,\quad \frac{d}{ds}\Big|_{s=0}\mathrm{Ric}^{\mathrm{Ch}}(\omega_{\chi,v}(s)) = -\sqrt{-1}\partial\bar\partial F_\chi(v)\,, \tag{2.61}$$

where $F_\chi := F_{\omega_\chi}$. Then, we have that

$$\mathcal{L}_\chi(v) = -\Delta_{\omega_\chi}F_\chi(v) + n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\wedge\sqrt{-1}\partial\bar\partial(v\omega^{n-2})}{\omega_\chi^n} - s^{\mathrm{Ch}}(\omega_\chi)F_\chi(v)\,.$$

Before going through the estimates, we need to explore the relation between the differential operators we are working with. First of all, we define the function

$$g(\chi) := \frac{\omega^n}{\omega_\chi^n}\,. \tag{2.62}$$

Then, for any $v \in C^2(\hat M)$, we have

$$\Delta_{\omega_\chi}v = n\frac{\sqrt{-1}\partial\bar\partial v\wedge\omega_\chi^{n-1}}{\omega_\chi^n} = g(\chi)\left(\Delta_\omega v + n\frac{\sqrt{-1}\partial\bar\partial v\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})}{\omega^n}\right)\,, \tag{2.63}$$

which gives us that

$$\Delta_{\omega_\chi}v - \Delta_\omega v = (g(\chi)-1)\Delta_\omega v + ng(\chi)\left(\frac{\sqrt{-1}\partial\bar\partial v\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})}{\omega^n}\right)\,.$$

For the sake of simplicity, we will denote

$$E(v) := ng(\chi)\left(\frac{\sqrt{-1}\partial\bar\partial v\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})}{\omega^n}\right) \tag{2.64}$$

so that (2.63) can be rewritten as

$$\Delta_{\omega_\chi}v = g(\chi)\Delta_\omega v + E(v)\,. \tag{2.65}$$

Moreover, we define

$$\begin{aligned}G(v) :=& n\frac{\sqrt{-1}\partial\bar\partial(v\omega^{n-2})\wedge\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)}{\omega_\chi^n} - n\frac{\sqrt{-1}\partial\bar\partial(v\omega^{n-2})\wedge\mathrm{Ric}^{\mathrm{Ch}}(\omega)}{\omega^n} + s^{\mathrm{Ch}}(\omega)F(v) - s^{\mathrm{Ch}}(\omega_\chi)F_\chi(v)\\ =& n\frac{\sqrt{-1}\partial\bar\partial(v\omega^{n-2})\wedge\left(g(\chi)\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi) - \mathrm{Ric}^{\mathrm{Ch}}(\omega) - \frac{1}{n-1}(g(\chi)s^{\mathrm{Ch}}(\omega_\chi)\omega_\chi - s^{\mathrm{Ch}}(\omega)\omega)\right)}{\omega^n}\,.\end{aligned}$$

Then, using (2.65) and these new notations, we have

$$\begin{aligned}(\mathcal{L}_\chi - \mathcal{L})(v) =& -(g(\chi)\Delta_\omega F_\chi(v) - \Delta_\omega F(u)) + E(F_\chi(v)) + G(v)\\ =& -g(\chi)\Delta_\omega(F_\chi - F)(v) - (g(\chi)-1)\Delta_\omega F(v) + E(F_\chi(v)) + G(v)\,.\end{aligned} \tag{2.66}$$

We will then breakdown the estimates in a series of smaller lemmas which will be used to conclude. The first one gives estimates on the function g defined in (2.62).

Lemma 2.2.14. *Let $\chi \in U_\tau$, then*

$$\|g(\chi)-1\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \le C\varepsilon^\tau\,, \quad \|g(\chi)\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \le 1 + C\varepsilon^\tau\,. \tag{2.67}$$

Proof. Obviously, it is sufficient to prove the first inequality, since the second one can be recovered by that one using the triangle inequality and the fact that $\|1\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} = 1$. In order to prove the first inequality in (2.67), we observe that

$$\omega^n - \omega_\chi^n = \omega^{n-1}\wedge(\omega-\omega_\chi) + \sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})\wedge(\omega-\omega_\chi) - \omega\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})\,.$$

Now, from this, we have

$$\begin{aligned}\|\omega^n-\omega_\chi^n\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \le & C(\|\omega-\omega_\chi\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} + \|\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})}\|\omega-\omega_\chi\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})}\\ &+\|\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})})\,.\end{aligned}$$

Thus, using (2.57), (2.58) and (2.59), we have

$$\|\omega^n-\omega_\chi^n\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})} \le C\varepsilon^\tau\,. \tag{2.68}$$

Now, (2.68) readily implies that

$$\omega^n = \omega_\chi^n + \mathrm{O}(\varepsilon^\tau)\,,$$

giving us the claim. □

The next lemma shows the continuity of $E\colon C^{2,\alpha}_{b,\varepsilon}(\hat{M}) \to C^{0,\alpha}_{b+2,\varepsilon}(\hat{M})$ and that its operator norm is bounded by ε^τ, and it follows immediately from (2.57) and (2.67).

Lemma 2.2.15. *For ε sufficiently small, we have that, for any $v \in C^{2,\alpha}_{b,\varepsilon}(\hat{M})$,*

$$\|E(v)\|_{C^{0,\alpha}_{b+2,\varepsilon}(\hat{M})} \le C\varepsilon^\tau\|v\|_{C^{2,\alpha}_{b,\varepsilon}(\hat{M})}\,.$$

Before showing the next lemma, we notice that, for any $v \in C^2(\hat{M})$, it holds

$$(F_\chi - F)(v) = \frac{n}{n-1}\left(g(\chi)\left(\frac{\sqrt{-1}\partial\bar\partial(v\omega^{n-2})\wedge(\omega_\chi-\omega)}{\omega^n}\right) + (g(\chi)-1)F(v))\right)\,. \tag{2.69}$$

Again, the next lemma shows that $F_\chi - F\colon C^{4,\alpha}_{b,\varepsilon}(\hat{M}) \to C^{2,\alpha}_{b+2,\varepsilon}(\hat{M})$ is a continuous operator with operator norm bounded by ε^τ.

Lemma 2.2.16. *For ε sufficiently small, for any $v \in C^{4,\alpha}_{b,\varepsilon}(\hat{M})$, we have that*

$$\|(F_\chi-F)(v)\|_{C^{2,\alpha}_{b+2,\varepsilon}(\hat{M})} \le C\varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\,.$$

Proof. Thanks to (2.69) and Lemma 2.2.14, we can obtain that

$$\begin{aligned}\|(F_\chi-F)(v)\|_{C^{2,\alpha}_{b+2,\varepsilon}(\hat{M})} \le & C\left((1+\varepsilon^\tau)\|\omega_\chi-\omega\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})}\|\sqrt{-1}\partial\bar\partial(v\omega^{n-2})\|_{C^{2,\alpha}_{b+2,\varepsilon}(\hat{M})} + \varepsilon^\tau\|F(v)\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})}\right)\\ \le & C\left((1+\varepsilon^\tau)\|\omega_\chi-\omega\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})}\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} + \varepsilon^\tau\|F(v)\|_{C^{2,\alpha}_{0,\varepsilon}(\hat{M})}\right)\,.\end{aligned}$$

Now, we can use (2.59), (2.67) and the continuity of $F\colon C^{4,\alpha}_{b,\varepsilon}(\hat{M}) \to C^{2,\alpha}_{b+2,\varepsilon}(\hat{M})$ to obtain that

$$\|(F_\chi-F)(v)\|_{C^{2,\alpha}_{b+2,\varepsilon}(\hat{M})} \le C((1+\varepsilon^\tau)\varepsilon^\tau)\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} + \varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}) \le C\varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\,,$$

concluding the proof. □

It remains to analize the operator G. In order to do so, we need two more estimates.

Lemma 2.2.17. *For ε sufficiently small, we have:*

$$\|g(\chi)\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)-\mathrm{Ric}^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\le C\varepsilon^\tau\,,\quad \|g(\chi)s^{\mathrm{Ch}}(\omega_\chi)\omega_\chi-s^{\mathrm{Ch}}(\omega)\omega\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\le C\varepsilon^\tau\,.$$

Proof. We have that

$$\begin{aligned} g(\chi)\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)-\mathrm{Ric}^{\mathrm{Ch}}(\omega) &= g(\chi)(\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)-\mathrm{Ric}^{\mathrm{Ch}}(\omega))+(g(\chi)-1)\mathrm{Ric}^{\mathrm{Ch}}(\omega)\\ &= g(\chi)\sqrt{-1}\partial\bar\partial\log g(\chi)+(g(\chi)-1)\mathrm{Ric}^{\mathrm{Ch}}(\omega)\end{aligned}$$

On the other hand, we have

$$\|\mathrm{Ric}^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\le C\,,\quad \|s^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\le C\,. \tag{2.70}$$

Indeed, we know that $\omega=\omega_o+O(|z|^m)$, implying that $\omega^n=\omega_o^n+O(|z|^m)$. This allows us to infer that

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)=\mathrm{O}(|z|^{m-2})\,,\quad s^{\mathrm{Ch}}(\omega)=\mathrm{O}(|z|^{m-2})\,.$$

Trivially, this gives that

$$\rho^2\mathrm{Ric}^{\mathrm{Ch}}(\omega)=\mathrm{O}(|z|^m)\,,\quad \rho^2 s^{\mathrm{Ch}}(\omega)=\mathrm{O}(|z|^m),$$

obtaining the claim. Now, using (2.67) and (2.70), we have that

$$\|g(\chi)\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)-\mathrm{Ric}^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\le C(1+\varepsilon^\tau)\|\log g(\chi)\|_{C^{2,\alpha}_{0,\varepsilon}(\hat M)}+C\varepsilon^\tau\,. \tag{2.71}$$

But if we now recall again inequalities (2.67), we can use the Taylor expansion and obtain from (2.71) the first claim. As for the second one, we observe that

$$g(\chi)s^{\mathrm{Ch}}(\omega_\chi)\omega_\chi-s^{\mathrm{Ch}}(\omega)\omega=g(\chi)(s^{\mathrm{Ch}}(\omega_\chi)-s^{\mathrm{Ch}}(\omega))\omega_\chi+g(\chi)s^{\mathrm{Ch}}(\omega)(\omega_\chi-\omega)+(g(\chi)-1)s^{\mathrm{Ch}}(\omega)\omega\,.$$

Moreover, using (2.59), (2.67) and (2.70), we have that

$$\begin{aligned}\|g(\chi)s^{\mathrm{Ch}}(\omega_\chi)\omega_\chi-s^{\mathrm{Ch}}(\omega)\omega\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)} &\le C(1+\varepsilon^\tau)\|(s^{\mathrm{Ch}}(\omega_\chi)-s^{\mathrm{Ch}}(\omega))\omega_\chi\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}+C(1+\varepsilon^\tau)\varepsilon^\tau+C\varepsilon^\tau\\ &\le C\varepsilon^\tau+C(1+\varepsilon^\tau)\|s^{\mathrm{Ch}}(\omega_\chi)-s^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\|\omega_\chi\|_{C^{0,\alpha}_{0,\varepsilon}(\hat M)}\,.\end{aligned} \tag{2.72}$$

Now, (2.59) yields

$$\|\omega_\chi\|_{C^{0,\alpha}_{0,\varepsilon}(\hat M)}\le\|\omega\|_{C^{0,\alpha}_{0,\varepsilon}(\hat M)}+\|\omega_\chi-\omega\|_{C^{0,\alpha}_{0,\varepsilon}(\hat M)}\le C(1+\varepsilon^\tau)\,,$$

which put into (2.72) gives that

$$\|g(\chi)s^{\mathrm{Ch}}(\omega_\chi)\omega_\chi-s^{\mathrm{Ch}}(\omega)\omega\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\le C\varepsilon^\tau+C(1+\varepsilon^\tau)^2\|s^{\mathrm{Ch}}(\omega_\chi)-s^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\,. \tag{2.73}$$

On the other hand, we have

$$\begin{aligned} s^{\mathrm{Ch}}(\omega_\chi)-s^{\mathrm{Ch}}(\omega) &= n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\wedge\omega^{n-1}}{\omega_\chi^n}+n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})}{\omega_\chi^n}-s^{\mathrm{Ch}}(\omega)\\ &= (g(\chi)-1)s^{\mathrm{Ch}}(\omega)+g(\chi)\Delta_\omega\log(g(\chi))+g(\chi)\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})}{\omega^n}\,.\end{aligned}$$

Then, using again (2.67) and (2.70),

$$\begin{aligned}\|s^{\mathrm{Ch}}(\omega_\chi)-s^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)} &\le C\varepsilon^\tau+C(1+\varepsilon^\tau)\varepsilon^\tau+C(1+\varepsilon^\tau)\|\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\wedge\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\\ &\le C\varepsilon^\tau+C(1+\varepsilon^\tau)\|\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat M)}\|\sqrt{-1}\partial\bar\partial(\chi\omega^{n-2})\|_{C^{0,\alpha}_{0,\varepsilon}(\hat M)}\,.\end{aligned}$$

But, we have

$$\begin{aligned}\|\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat{M})} &\leq \|\mathrm{Ric}^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat{M})} + \|\sqrt{-1}\partial\bar{\partial}\log g(\chi)\|_{C^{0,\alpha}_{2,\varepsilon}(\hat{M})} \leq C(1+\varepsilon^\tau)\,,\\ \|\sqrt{-1}\partial\bar{\partial}(\chi\omega^{n-2})\|_{C^{0,\alpha}_{0,\varepsilon}(\hat{M})} &\leq \|\chi\omega^{n-2}\|_{C^{2,\alpha}_{-2,\varepsilon}(\hat{M})} \leq C\|\chi\|_{C^{4,\alpha}_{-2,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau\,,\end{aligned} \tag{2.74}$$

where the last inequality is due to (2.60). Putting (2.74) into (2.73), we have the claim. □

Thus, using Lemma 2.2.17, we can conclude that

$$\begin{aligned}\|G(v)\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})} &\leq C\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\left\|g(\chi)\mathrm{Ric}^{\mathrm{Ch}}(\omega_\chi) - \mathrm{Ric}^{\mathrm{Ch}}(\omega) - \frac{g(\chi)s^{\mathrm{Ch}}(\omega_\chi)\omega_\chi - s^{\mathrm{Ch}}(\omega)\omega}{n-1}\right\|_{C^{0,\alpha}_{2,\alpha}(\hat{M})}\\ &\leq C\varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\,.\end{aligned} \tag{2.75}$$

We are finally ready to prove that $\mathcal{N}$ is a contraction operator on U_τ.

Proposition 2.2.18. *For ε sufficiently small and $b < 2n-4$, the operator $\mathcal{N}$ is a contraction and $\mathcal{N}(U_\tau) \subseteq U_\tau$.*

Proof. Consider $v = \varphi_1 - \varphi_2$ as above,

$$\|\mathcal{N}(\varphi_1) - \mathcal{N}(\varphi_2)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq C\|(\mathcal{L}_\chi - \mathcal{L})(v)\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})}\,.$$

Using (2.66), (2.67), Lemma 2.2.16, Lemma 2.2.15 and (2.75) and the continuity of $\Delta_\omega\colon C^{4,\alpha}_{b,\varepsilon}(\hat{M}) \to C^{2,\alpha}_{b+2}(\hat{M})$ and that of $F\colon C^{2,\alpha}_{b+2,\varepsilon}(\hat{M}) \to C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})$, we have

$$\|(\mathcal{L}_\chi - \mathcal{L})(v)\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})} \leq C\varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}$$

which, after choosing ε sufficiently small, guarantees that $\mathcal{N}$ is a contraction. Now fix $\varphi \in U_\tau$, we have that

$$\begin{aligned}\|\mathcal{N}(\varphi)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} &\leq \|\mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} + \|\mathcal{N}(\varphi) - \mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq \|\mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} + C\varepsilon^\tau\|\varphi\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\\ &\leq \|\mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} + C\varepsilon^{2\tau+(p+q)(b+2)} \leq \|\tilde{\mathcal{L}}^{-1}(s^{\mathrm{Ch}}(\omega))\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} + C\varepsilon^{2\tau+(p+q)(b+2)}\\ &\leq C\|s^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})} + C\varepsilon^{2\tau+(p+q)(b+2)}\,.\end{aligned}$$

On the other hand,

$$\|s^{\mathrm{Ch}}(\omega)\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})} \leq C\varepsilon^{p(m+b+2)}\,,$$

from which it follows

$$\|\mathcal{N}(\varphi)\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq C\varepsilon^{p(m+b+2)} + C\varepsilon^{2\tau+(p+q)(b+2)} \leq C\varepsilon^{\min\{\tau, pm-q(b+2)-\tau\}}\varepsilon^{\tau+(p+q)(b+2)}\,.$$

It is then sufficient to notice that τ can be chosen such that $pm - q(b+2) > \tau > 0$, giving us the claim. □

Hence, Theorem B is proven.

The construction done to prove Theorem B can also be used in the case in which the chosen points are not smooth, provided the resolutions of the singularity model at such points satisfy some extra conditions. More precisely, we need to impose the following:

1. for any $x \in M$, let $\mathsf{G}_x \subseteq \mathrm{SU}(n)$ acting freely on $\mathbb{C}^n\backslash\{0\}$ so that a suitable neighbourhood of p is biholomorphic to a neighbourhood of $\mathbb{C}^n/\mathsf{G}_x$. $\mathbb{C}^n/\mathsf{G}_x$ has a ALE resolution $(X, \omega_{\mathrm{ALE}})$, where ω_{ALE} is a scalar flat ALE Kähler metric on X such that, away from the exceptional divisors, takes the following form:
$$\omega_{\mathrm{ALE}} = \omega_o + \sqrt{-1}\partial\bar{\partial}\gamma\,, \quad \gamma = \mathrm{O}(r^{4-2n}) \tag{2.76}$$

Once (1) is satisfied, we can repeat all the step substituting the Burns-Simanca metric with ω_{ALE} and conclude.

It is also clear that this setting can be considered in the case of an orbifold admitting crepant resolutions, since, as we recalled in Theorem 1.1.46, the singularity resolution model carries Kähler Ricci-flat ALE metrics with fast decay. In particular, one can easily adapt the proof of Theorem B (where the key fact in repeating the proof is Lemma 2.2.10) to prove Theorem 2.2.19 which can be considered an extension of [168, Theorem 1] to the general case of balanced Chern-Ricci flat orbifolds admitting crepant resolutions.

Theorem 2.2.19. *Let $(M^n, \tilde{\omega})$ be a compact Chern-Ricci flat balanced orbifold with isolated singularities. Furthermore, assume that M admits a crepant resolution $\hat{M}$. Then, $\hat{M}$ carries a Chern-Ricci flat balanced metric $\hat{\omega}_\varepsilon$ such that*

$$\hat{\omega}_\varepsilon^{n-1} \in \pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} - \varepsilon^{2n-2} \sum_{i=1}^{k} a_i^{n-1} [E_i]_{\mathrm{BC}}^{n-1} ,$$

where $[E_i]_{\mathrm{BC}}$ is the first Bott-Chern class of the line bundle associated to the exceptional divisor E_i of exceptional set of the resolution and $\varepsilon \in (0, \varepsilon_0)$.

2.3 Non-positive trace deformation

The second ansatz we will be considering is based on assuming the existence of a suitable $(n-2, n-2)$-form to restrict the deformation argument to an easier subspace of the cohomology class. More specifically, we will assume that M is endowed with $\tilde{\Omega} \in \Lambda_{\mathbb{R}}^{n-2,n-2} M$ and such that

$$\tilde{\omega} \wedge \tilde{\Omega} > 0 \quad \text{ and } \quad \Lambda_{\tilde{\omega}}^{n-1}(\sqrt{-1}\partial\bar{\partial}\tilde{\Omega}) \leq 0 .$$

With this assumption, our main objective is to prove the following theorem.

Theorem 2.3.1. *Let $(M^n, \tilde{\omega})$ be a compact balanced Chern-Ricci flat manifold or orbifold with isolated singularities endowed with $\tilde{\Omega} \in \Lambda_{\mathbb{R}}^{n-2,n-2} M$ such that*

$$\tilde{\omega} \wedge \tilde{\Omega} > 0 \quad \textit{ and } \quad \Lambda_{\tilde{\omega}}^{n-1}(\sqrt{-1}\partial\bar{\partial}\tilde{\Omega}) \leq 0 . \tag{2.77}$$

Then, given $p_1, \ldots, p_k \in M$ and $a_1, \ldots, a_k > 0$ there exists $\varepsilon_0 > 0$ such that the blow-up of M at $p_1, \ldots, p_k$ admits a balanced negative constant Chern scalar curvature metric

$$\omega_\varepsilon^{n-1} \in \pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} - \varepsilon^{2n-2} \sum_{i=1}^{k} a_i^{n-1} [E_i]_{\mathrm{BC}}^{n-1} ,$$

where $[E_i]_{\mathrm{BC}}$ is the first Bott-Chern class of the line bundle associated to the exceptional divisor E_i of the blow-up at p_i and $\varepsilon \in (0, \varepsilon_0)$.

First of all, we can consider $\varepsilon > 0$ sufficiently small and a small neighbourhood of x, which will be identified with $B(0,1) \subset \mathbb{C}^n$, with holomorphic coordinates z on M. As before, we can consider the cut-off function $\chi \colon [0,1] \to [0,1]$ from Lemma 2.1.1 and consider

$$\tilde{\Omega}_\varepsilon = (1 - \chi_\varepsilon(|z|^2))\tilde{\Omega} + \chi_\varepsilon(|z|^2)\omega_o^{n-2} \in \Lambda_{\mathbb{R}}^{n-2,n-2} M$$

where again ω_o is the flat metric induced by z on $B(0,1)$. As for $\tilde{\omega}_\varepsilon$, $\tilde{\Omega}_\varepsilon$ coincides with the $(n-2)$-th power of the flat metric in a small neighbourhood of x, hence we can repeat the strategy to construct ω, and glue together $\tilde{\Omega}_\varepsilon$ with $\varepsilon^{2(n-2)(p+q)}\omega_{\mathrm{BS},\varepsilon}^{n-2}$ on the flat region (2.2), obtaining $\Omega \in \Lambda_{\mathbb{R}}^{n-2,n-2} \hat{M}$.

Remark 2.3.2. It is easy to check that $\omega \wedge \Omega > 0$. Indeed, it is straightforward from the construction that the only region in which we have to check this is the cut-off region $\{\varepsilon^p \leq |z| \leq 2\varepsilon^p\}$, in which we have

$$\omega \wedge \Omega = (\omega_o + \mathrm{O}(|z|)) \wedge ((1-\chi)\tilde{\Omega} + \chi\omega_o^{n-2}) = (1-\chi)\omega_o \wedge \tilde{\Omega} + \chi\omega_o^{n-1} + \mathrm{O}(|z|),$$

which is positive, for sufficiently small ε. Indeed, up to the decaying term, it is a pointwise convex combination of positive forms. This, thanks to the work of Michelsohn ([240]), also implies that $\omega \wedge \Omega = (\omega')^{n-1}$, where ω' is an Hermitian metric on $\hat{M}$.

On the other hand, the condition $\Lambda_{\tilde{\omega}}(\sqrt{-1}\partial\bar{\partial}\tilde{\Omega}) \leq 0$ might not be preserved, but, as we will see, we will just need it on the base manifold.

Thanks to this construction, we can thus choose $\varphi = u\Omega$ with $u \in C^\infty(\hat{M}, \mathbb{R})$ such that $\omega_u^{n-1} := \omega_{u\Omega}^{n-1} > 0$, and along with assuming $f(u\Omega) = u$, we are able to turn again the operator $\tilde{\mathcal{S}}$ into an operator taking smooth functions in input defined as

$$\tilde{\mathcal{S}}(u) = s^{\mathrm{Ch}}(\omega_u) - s^{\mathrm{Ch}}(\tilde{\omega}) - \int_{\hat{M}} u\frac{\omega^n}{n!}\,. \tag{2.78}$$

Let us start by writing again the linearized operator $\tilde{\mathcal{L}}$ implementing this new ansatz:

$$\tilde{\mathcal{L}}(u) := \tilde{\mathcal{L}}(u\Omega) = -\Delta_\omega F_{\omega,\Omega}(u) + n\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega) \wedge \sqrt{-1}\partial\bar{\partial}(u\Omega)}{\omega^n} - s^{\mathrm{Ch}}(\omega)F_{\omega,\Omega}(u), \tag{2.79}$$

where

$$F_{\omega,\Omega}(u) := F_\omega(u\Omega) = \frac{n}{n-1}\frac{\omega \wedge \sqrt{-1}\partial\bar{\partial}(u\Omega)}{\omega^n}. \tag{2.80}$$

Now, the proof of Theorem 2.3.1 is considerably simpler than that of Theorem B. This is mainly due to the fact that there is no need to restrict our attention on suitably chosen subspaces. Indeed, the following lemma asserts that the operator $F_{\omega,\Omega}$ has trivial kernel, when restricted to zero-mean value functions, on compact manifolds.

Lemma 2.3.3. *Let $(M^n, \tilde{\omega})$ be a compact balanced manifold and let $\tilde{\Omega} \in \Lambda_{\mathbb{R}}^{n-2,n-2}M$ satisfying (2.77). If*

$$F_{\tilde{\omega},\tilde{\Omega}} \colon C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})\,, \quad F_{\tilde{\omega},\tilde{\Omega}}(u) := \frac{n}{n-1}\frac{\tilde{\omega} \wedge \sqrt{-1}\partial\bar{\partial}(u\tilde{\Omega})}{\tilde{\omega}^n}\,,$$

then, there are no non-trivial functions $u \in C^\infty(M,\mathbb{R})$ such that

$$F_{\tilde{\omega},\tilde{\Omega}}(u) = c\,, \quad c \in \mathbb{R} \quad \text{and} \quad \int_M u\frac{\tilde{\omega}^n}{n!} = 0\,.$$

Then, the restriction of $F_{\tilde{\omega},\tilde{\Omega}}$ to smooth functions with zero $\tilde{\omega}$-mean value is injective.

Proof. Expanding the definition of $F_{\tilde{\omega},\tilde{\Omega}}$, we have

$$F_{\tilde{\omega},\tilde{\Omega}}(u) = \frac{n}{n-1}\left(\frac{\sqrt{-1}\partial\bar{\partial}u \wedge \tilde{\Omega} \wedge \tilde{\omega}}{\tilde{\omega}^n} + 2\mathrm{Re}\left(\frac{\sqrt{-1}\partial u \wedge \bar{\partial}\tilde{\Omega} \wedge \tilde{\omega}}{\tilde{\omega}^n}\right) + u\frac{\sqrt{-1}\partial\bar{\partial}\tilde{\Omega} \wedge \tilde{\omega}}{\tilde{\omega}^n}\right).$$

On the other hand, we know that $\tilde{\Omega} \wedge \tilde{\omega}$ is a positive $(n-1, n-1)$-form. Then, thanks to a result in [240], there exists a Hermitian metric $\tilde{\omega}'$ such that $\tilde{\omega}'^{n-1} = \tilde{\Omega} \wedge \tilde{\omega}$. This implies that

$$n\frac{\sqrt{-1}\partial\bar{\partial}u \wedge \tilde{\Omega} \wedge \tilde{\omega}}{\tilde{\omega}^n} = n\frac{\sqrt{-1}\partial\bar{\partial}u \wedge \tilde{\omega}'^{n-1}}{\tilde{\omega}^n} = \frac{\tilde{\omega}'^n}{\tilde{\omega}^n}\Delta_{\tilde{\omega}'}u\,.$$

Moreover,

$$\frac{\sqrt{-1}\partial\bar{\partial}\tilde{\Omega} \wedge \tilde{\omega}}{\tilde{\omega}^n} = \frac{1}{n!(n-1)!}\tilde{g}(\sqrt{-1}\partial\bar{\partial}\tilde{\Omega}, \tilde{\omega}^{n-1}) = \frac{1}{n!(n-1)!}\Lambda_{\tilde{\omega}}^{n-1}(\sqrt{-1}\partial\bar{\partial}\tilde{\Omega})\,.$$

Then, we can conclude that

$$F_{\tilde{\omega},\tilde{\Omega}}(u)=\frac{1}{n-1}\frac{\tilde{\omega}'^n}{\tilde{\omega}^n}\Delta_{\tilde{\omega}'}u+\frac{2n}{n-1}\mathrm{Re}\left(\frac{\sqrt{-1}\partial u\wedge\bar{\partial}\tilde{\Omega}\wedge\tilde{\omega}}{\tilde{\omega}^n}\right)+\frac{u}{(n-1)((n-1)!)^2}\Lambda^{n-1}_{\tilde{\omega}}(\sqrt{-1}\partial\bar{\partial}\tilde{\Omega})\,, \quad (2.81)$$

which is a second order elliptic operator with non-positive zero order coefficient. Now, if $u\in C^\infty(M,\mathbb{R})\backslash\{0\}$ with $\int_M u\frac{\tilde{\omega}^n}{n!}=0$, then $\max_M u>0$. On the other hand, thanks to the maximum principle, being a solution of

$$F_{\tilde{\omega},\tilde{\Omega}}(u)=c,\quad c>0\,,$$

allows us to conclude that u is constant. Using again the condition $\int_M u\frac{\tilde{\omega}^n}{n!}=0$, we conclude $u=0$. The case in which $c<0$ can be deduced by the above, applying the same argument to $v=-u$ satisfying

$$F_{\tilde{\omega},\tilde{\Omega}}(v)=-c>0\,,$$

obtaining the claim. □

Theorem 2.3.4. *For any $b\in(0,2n-4)$, there exists $C>0$ such that, for all $u\in C^{4,\alpha}_{b,\varepsilon}(\hat{M})$, we have*

$$\|u\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\leq C\|\tilde{\mathcal{L}}u\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})}\,.$$

Then,

$$\tilde{\mathcal{L}}\colon C^{4,\alpha}_{b,\varepsilon}(\hat{M})\to C^{,\alpha}_{b+4,\varepsilon}(\hat{M})$$

is a isomorphism.

Proof. The proof of the first part follows the ideas of the proof of Lemma 2.2.13. In this case, we have that $u_\infty\in C^{4,\alpha}_b(M_x)$ is such that

$$F_{\tilde{\omega},\tilde{\Omega}}(u_\infty)=c\,,\quad c\in\mathbb{R}\,,\quad \int_{M_x}u_\infty\frac{\tilde{\omega}^n}{n!}=0\,.$$

Now, using (2.81), we can use a bootstrap argument to infer that $u_\infty\in\ker F_{\tilde{\omega},\tilde{\Omega}}$ on M, then, in particular, it is smooth. Now, we can conclude that $u_\infty=0$, using Lemma 2.3.3 . As regards the second part, one may notice that the index of $\tilde{\mathcal{L}}$ is equal to that of $\Delta_\omega\circ\Delta_{\omega'}$, where ω' is the Hermitian metric such that $\Omega\wedge\omega=(\omega')^{n-1}$, as in Remark 2.3.2, which is 0. On the other hand, the first part of the statement is telling that $\tilde{\mathcal{L}}$ is injective and thus, since of index 0, surjective. □

Now, the rest of the proof of Theorem 2.3.1 goes as the one of Theorem B. Indeed, We can now reformulate $\tilde{\mathcal{S}}(u)=0$ as the following fixed point problem:

$$\mathcal{N}(u):=-\tilde{\mathcal{L}}^{-1}(s^{\mathrm{Ch}}(\omega)+Q(u))=u$$

where

$$\mathcal{N}\colon C^{4,\alpha}_{b,\varepsilon}(\hat{M})\to C^{4,\alpha}_{b,\varepsilon}(\hat{M})\,.$$

Proposition 2.3.5. *For ε sufficiently small and $b<2n-4$, the operator $\mathcal{N}$ is a contraction and $\mathcal{N}(U_\tau)\subseteq U_\tau$, where*

$$U_\tau:=\{\psi\in C^{4,\alpha}_{b,\varepsilon}(\hat{M})\quad|\quad \|\psi\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})}\leq C\varepsilon^{(p+q)(b+2)+\tau}\}\,.$$

Proof. First of all, we notice that

$$\|\Omega\|_{C^{4,\alpha}_{0,\varepsilon}(\hat{M})}\leq C\,.$$

Indeed, we recall that

$$\Omega = \begin{cases} \varepsilon^{2(n-2)(p+q)}\omega_{\mathrm{BS},\varepsilon}^{n-2} & |z| \le \varepsilon^p\,, \\ \tilde{\Omega}' & \varepsilon^p < |z| < 2\varepsilon^p\,, \\ \tilde{\Omega} & |z| \ge 2\varepsilon^p\,, \end{cases} \qquad \omega = \begin{cases} \varepsilon^{2(p+q)}\omega_{\mathrm{BS},\varepsilon} & |z| \le \varepsilon^p\,, \\ \omega'_\varepsilon & \varepsilon^p < |z| < 2\varepsilon^p\,, \\ \tilde{\omega} & |z| \ge 2\varepsilon^p\,. \end{cases}$$

Of course, $|\nabla^k\Omega|_\omega \le C$, if $|z| \ge 2\varepsilon^p$ and $0 \le |z| \le \varepsilon^p$, for all $k = 0, \dots, 4$. On the other hand, both $\tilde{\Omega}'$ and ω'_ε depend on ε just for their domain of definition and so we can infer that $|\nabla^k\Omega|_\omega \le C$ on the whole $\hat{M}$, giving the claim. Once, we have this, the proof is analogue to that of Proposition 2.2.18, using an easily adapted version of the estimates found in Subsection 2.2.4. □

Hence Theorem 2.3.1 is proven.

In the same fashion as discussed in the last part of section 2.2, we can prove the same result even for singular points provided Condition 1 is satisfied. In particular, again one can repeat the proof of Theorem 2.3.1 (where the key fact in repeating the proof is Theorem 2.3.4) to prove the Theorem, which can be considered as a variation of [168, Theorem 1] and Theorem 2.2.19 with this new ansatz.

Theorem 2.3.6. *Let $(M^n, \tilde{\omega})$ be a compact Chern-Ricci flat balanced orbifold with isolated singularities endowed with $\tilde{\Omega} \in \Lambda_{\mathbb{R}}^{n-2,n-2}M$ and satisfying (2.77) on the smooth part. Furthermore, assume that M admits a crepant resolution $\hat{M}$. Then, $\hat{M}$ carries a Chern-Ricci flat balanced metric $\hat{\omega}_\varepsilon$ such that*

$$\hat{\omega}_\varepsilon^{n-1} \in \pi^*[\tilde{\omega}^{n-1}]_{\mathrm{BC}} - \varepsilon^{2n-2}\sum_{i=1}^{k} a_i^{n-1}[E_i]_{\mathrm{BC}}^{n-1}\,,$$

where $[E_i]_{\mathrm{BC}}$ is the first Bott-Chern class of the line bundle associated to the exceptional divisor E_i of exceptional set of the resolution and $\varepsilon \in (0, \varepsilon_0)$.

In Section 2.4 we will discuss some examples in which Theorems 2.3.1 and Theorem 2.2.19 can be applied.

2.4 Examples

In this section, we will describe families of compact balanced manifolds satisfying the hypothesis of Theorem 2.3.1. In [168, Theorem 1], the authors require the base manifold to satisfy the following

$$\ker F_\omega = \{0\} \tag{2.82}$$

Condition (2.82) arises naturally, as we saw in Section 2.2, in the study of invertibility of the linearized operator with the balanced ansatz. Motivated by this, we study examples in which (2.82) is verified.

2.4.1 Non-positive trace examples

A very special case in which we can apply Theorem 2.3.1 is when $\sqrt{-1}\partial\bar{\partial}\Omega = 0$. This setting can arise in multiple scenarios, one of which is given by the case in which the balanced manifold $(M, \tilde{\omega})$ carries also *astheno-Kähler* metrics, introduced by Jost and Yau in [199], i.e. Hermitian metrics with fundamental form η such that

$$\sqrt{-1}\partial\bar{\partial}\eta^{n-2} = 0\,,$$

whose coexistence with the balanced structure was shown [113] and in [216]) not to force the existence of a Kähler metric. In this case, the $(n-2)$-th power of η is trivially satisfying conditions (2.77), thus they are very natural to be considered.

Once compact balanced manifolds admitting also an astheno-Kähler structure are found, it is not hard to find they carry Chern-Ricci flat balanced metrics, as we can see in the following remark.

Remark 2.4.1. The existence of Chern-Ricci flat balanced metrics in a given balanced class when the manifold carries astheno-Kähler metrics depends only on the first Bott-Chern class. Indeed, in [314], generalizing the same result on Kähler manifolds in [324], the authors proved, as a direct consequence of the solvability of suitable complex Monge-Ampère equations, that on a compact balanced manifold (M, ω) admitting an astheno-Kähler metric with $c_1^{\mathrm{BC}}(M) = 0$, we can always find a Chern-Ricci flat balanced metric in $[\omega^{n-1}]_{\mathrm{BC}}$.

A first explicit example in complex dimension 4 for this setting is the following.

Example 2.4.2 ([113], Example 4.1)**.** Consider the T^2-principal bundle $\pi : M \to T^6$, where T^6 has the standard complex structure with holomorphic coordinates (z_1, z_2, z_3), and with characteristic classes

$$a_1 := dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 - 2dz_3 \wedge d\bar{z}_3 \quad \text{and} \quad a_2 := dz_2 \wedge d\bar{z}_2 - dz_3 \wedge d\bar{z}_3.$$

We then consider the Kähler metrics

$$\eta_1 := dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 \quad \text{and} \quad \eta_2 := dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + 5dz_3 \wedge d\bar{z}_3$$

on T^6, the connection 1-forms θ^j, $j = 1, 2$, such that $d\theta^j = \pi^* a_j$, and define

$$\omega_j := \pi^* \eta_j + \theta^1 \wedge \theta^2, \quad j = 1, 2,$$

which correspond respectively to a balanced and an astheno-Kähler metric on M. Moreover, denoting with z_0 the holomorphic coordinate on T^2, it is straightforward to notice that

$$\Theta := dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3,$$

defines a global holomorphic volume form, from which we also see that ω_1 (and actually also ω_2) is Chern-Ricci flat, hence satisfying all the hypothesis are into place to apply Theorem 2.3.1.

A further family of examples was given in any complex dimension $n \geq 4$ as follows.

Example 2.4.3 ([216],Theorem 2.4, Remark 2.6)**.** For $n \geq 4$, consider the nilpotent Lie group and the left-invariant complex structure identified by a $(1, 0)$-coframe satisfying the following structure equations:

$$d\omega^i = 0\,, \quad i = 1, \ldots, n-1\,, \quad d\omega^n = \sum_{i=1}^{n-1} a_i \omega^{i\bar{i}}\,,$$

where $a_1, \ldots, a_{n-1} \in \mathbb{R}$ such that $a_i \neq 0$, for all $i = 1, \ldots, n-1$, and $\sum_{i=1}^{n-1} a_i = 0$. Choosing $a_i \in \mathbb{Q}$, for all $i = 1, \ldots, n$ guarantees the existence of a co-compact lattice, giving a compact nilmanifold. On top of this, we know that nilmanifolds have vanishing first Bott-Chern class, as any left-invariant metric is Chern-Ricci flat (see [225, Proposition 2.1]). Thus we are in condition to apply Theorem 2.3.1.
Moreover, in the same paper the authors produced another suitable family of 4-dimensional nilmanifolds depending on three parameters in $\mathbb{Q}(i)$, where one can find Chern-Ricci flat balanced and astheno-Kähler metrics, identified by the following structure equation:

$$d\omega^1 = d\omega^2 = d\omega^3 = 0\,, \quad d\omega^4 = A\omega^{12} + B\omega^{13} + C\omega^{23} + \omega^{1\bar{1}} + \omega^{2\bar{2}} - 2\omega^{3\bar{3}}\,, \quad A, B, C \in \mathbb{Q}(i).$$

To cover also the case of complex dimension 3, we need to follow a different path. Indeed, in complex dimension 3, the astheno-Kähler condition coincide with the SKT one, recall Definition 1.1.56. Then, we cannot hope for the co-existence of balanced and astheno-Kähler metrics on threefolds, due to Conjecture 1.1.57. Not having astheno-Kähler metrics is not, however, a big problem, mainly because the positivity of the form Ω for the ansatz is not a necessary condition, as we only need its positivity when wedged with the metric on the base. With this in mind, we are able to identify an interesting class of examples in dimension $n \geq 3$, still satisfying the condition $\sqrt{-1}\partial\bar{\partial}\Omega = 0$, without necessarily having Ω to be the $(n-2)$-th power of an astheno-Kähler metric. This class is given by compact balanced manifolds admitting a holomorphic submersion with 1-dimensional fibres.

Proposition 2.4.4. *Let $\pi\colon M^n \to X^{n-1}$ be a holomorphic submersion and assume (M,ω) is a compact balanced manifold. Then, there exists $\Omega \in \Lambda^{n-2,n-2}_{\mathbb{R}} M$ satisfying (2.77).*

Proof. Thanks to [240, Proposition 1.9], we know that X admits balanced metrics. Let us fix ω_X a balanced metric on X. So, we can consider $\Omega = \pi^*\omega_X^{n-2}$. Thus, we have

$$d\Omega = 0\,.$$

Then, we just need to check if $\omega \wedge \Omega > 0$. We then fix a point $p \in M$ and choose holomorphic coordinates $\{z_1, \dots, z_n\}$ on a neighbourhood of p such that $\{z_1, \dots, z_{n-1}\}$ are holomorphic coordinates on a neighbourhood of $\pi(p)$ in X and z_n is the holomorphic coordinate of the fibre over p, see for instance [321, Lemma 5.6], such that

$$\omega_{n\bar{n}} = 1\,, \quad \pi^*\omega_X = \sqrt{-1}\sum_{i=1}^{n-1} dz^{i\bar{i}} \quad \text{and } \omega\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_k}\right) = \lambda_i \delta_{ik}\,, \quad i,k = 1,\dots,n-1\,.$$

From this, we immediately have that

$$\Omega = \pi^*\omega_X^{n-2} = (\sqrt{-1})^{n-2}(n-2)!\sum_{i=1}^{n-1} dz^{1\bar{1}\dots\hat{i}\hat{\bar{i}}\dots n-1\overline{n-1}}\,.$$

Then, denoting with $\Lambda = \sum_{i=1}^{n-1}\lambda_i$, it is easy to see that

$$\omega \wedge \Omega = (\sqrt{-1})^{n-1}(n-2)!\left(\Lambda dz^{1\bar{1}\dots\hat{n}\hat{\bar{n}}} + \sum_{i=1}^{n-1} dz^{1\bar{1}\dots\hat{i}\hat{\bar{i}}\dots n\bar{n}} + \omega_{i\bar{n}} dz^{1\bar{1}\dots i\hat{\bar{i}}\dots\hat{n}\bar{n}} - \omega_{n\bar{i}} dz^{1\bar{1}\dots\hat{i}\bar{i}\dots j\hat{\bar{j}}\dots n\hat{\bar{n}}}\right)\,.$$

Thus, $\omega \wedge \Omega$ is represented in p by the matrix

$$B = \frac{1}{n-1}\begin{pmatrix} \mathrm{Id} & \omega_{i\bar{n}} \\ \omega_{n\bar{i}} & \Lambda \end{pmatrix}$$

which is positive definite if and only if

$$\det(B) = \frac{1}{(n-1)^n}\sum_{i=1}^{n-1}\left(\lambda_i - |\omega_{i\bar{n}}|^2\right) > 0\,.$$

On the other hand, using that ω is a metric, we have that $\lambda_i - |\omega_{i\bar{n}}|^2 > 0$, for all $i = 1,\dots,n-1$, concluding the proof. □

This proposition is thus very interesting, as in most explicit examples gives us an explicit choice of Ω, since we frequently have an explicit balanced metric on the base.

One explicit example in the homogeneous case in which we can follow this approach is given by the Iwasawa manifold, which we shall discuss in detail in Subsection 2.4.2. Another very interesting one is given by the construction by Goldstein and Prokushkin, see [169], which Fu and Yau in [136] showed to be highly relevant in the study of the Hull-Strominger system.

Example 2.4.5 ([169])**.** Let M be the total space of a T^2 bundle on a Calabi-Yau surface (S,ω_S) with holomorphic volume $\Theta \in \Lambda^{2,0}S$. Choose S such that it admits closed 2-forms ω_P and ω_Q such that $\omega_P + \sqrt{-1}\omega_Q \in \Lambda^{2,0}S$, and $[\omega_P/2\pi], [\omega_Q/2\pi] \in H^2(S,\mathbb{Z})$. Then we can find a $(1,0)$ form θ on M such that $d\theta = \omega_P + \sqrt{-1}\omega_Q$ and $\sqrt{-1}\theta\wedge\bar{\theta} > 0$. Now, $\theta\wedge\Theta$ defines a holomorphic volume form on M, along with $\pi^*\omega_S + \sqrt{-1}\theta\wedge\bar{\theta}$, which corresponds to a Chern-Ricci flat balanced metric. Finally, choosing ω_P and ω_Q such that either one identifies a non-zero cohomology class guarantees that M does not carry Kähler metrics. Hence, $\pi^*\omega_S$ satisfies conditions (2.77), and thus is a natural choice of Ω in order to apply Theorem 2.3.1.

Finally, we can also recover an example on threefolds in a case where $\sqrt{-1}\partial\bar{\partial}\Omega \neq 0$

Example 2.4.6. Consider the six-dimensional nilmanifolds (corresponding to the case [26, Example (Ni)]) identified by the structure equations

$$d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = \rho\varphi^{12} + \varphi^{1\bar{1}} + \lambda\varphi^{1\bar{2}} + D\varphi^{2\bar{2}},$$

for $\{\varphi^1, \varphi^2, \varphi^3\}$ a coframe of invariant $(1,0)$-forms, $\rho \in \{0,1\}$, $\lambda \in \mathbb{R}_{\geq 0}$ and $D \in \mathbb{C}$ such that $\mathrm{Im}(D) \geq 0$. When the corresponding Lie algebra is $\mathfrak{h}_2$, $\mathfrak{h}_3$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$ (in the notation of [279, 325, 326]), these manifolds carry both a balanced metric ω (which is automatically Chern-Ricci flat thanks to the nilmanifold structure) and a plurinegative metric ω', i.e. $\sqrt{-1}\partial\bar{\partial}\omega' \leq 0$, making $\Omega = \omega'$ a natural choice to apply Theorem 2.3.1.

Let us conclude by focusing on two specific locally homogeneous examples whose explicit structure will allow us to construct examples suitable to apply Theorems 2.3.1, 2.2.19 which satisfy (2.82).

2.4.2 The Iwasawa manifold

Recall that the Iwasawa manifold $M = \mathrm{Heis}(3,\mathbb{C})/\mathrm{Heis}(3,\mathbb{Z}[\sqrt{-1}])$ is the unique complex parallelizable nilmanifold of complex dimension 3, recall Example 1.1.4.

Example 2.4.7. The center of $\mathrm{Heis}(3,\mathbb{C})$ is given by $\mathbb{C}$, whose natural action descends to a T^2 action on M, which gives rise to (see for a more general assertion [276]) a holomorphic principal T^2-bundle structure

$$\pi\colon M \to T^4. \tag{2.83}$$

Moreover, the nilmanifold structure once again guarantees that any left-invariant Hermitian metric is balanced and Chern-Ricci flat. Thus, we are in the position to apply Proposition 2.4.4 to find Ω satisfying (2.77) and then apply Theorem 2.3.1. We also have an explicit choice of Ω, as we have the flat torus metric ω_{T^4} that naturally yields the choice $\Omega = \pi^*\omega_{T^4}$.

Let us now recall the standard coframe of invariant (with respect to the Heisenberg group operation) 1-forms:

$$\varphi^1 := dz^1, \quad \varphi^2 := dz^2, \quad \varphi^3 := dz^3 - z^2 dz^1,$$

which satisfy the following structure equations:

$$d\varphi^1 = d\varphi^2 = 0\,, \quad d\varphi^3 = \varphi^1 \wedge \varphi^2\,.$$

Using [326, Lemma 2.5], we are led to infer that, with such a frame, any left-invariant balanced metric on $\mathrm{Heis}(3,\mathbb{C})$ is biholomorphically isometric to

$$\omega_t := \frac{\sqrt{-1}}{2}(\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2 + t^2\varphi^3 \wedge \bar{\varphi}^3)\,, \quad t^2 > 0\,,$$

which descends to a Chern-Ricci flat balanced metric on the Iwasawa manifold M, for every $t \neq 0$. We can thus just focus on the family ω_t, making use of the invariance of (2.82) under biholomorphic isometries.

Example 2.4.8. For the metrics ω_t, the kernel of the operator F_{ω_t} (see (2.12)) is described by the equation

$$\Delta_{\omega_t} u + \frac{1}{2}|\partial\omega_t|^2_{\omega_t} u = 0\,. \tag{2.84}$$

Now, an easy computation shows that $\partial\omega_t = \sqrt{-1}\frac{t^2}{2}\varphi_1 \wedge \varphi_2 \wedge \varphi_3$, implying $|\partial\omega_t|^2_{\omega_t}\frac{\omega_t^3}{3!} = \sqrt{-1}\,\overline{\partial}\omega_t \wedge \partial\omega_t = t^2\frac{\omega_t^3}{3!}$, from which we get $|\partial\omega_t|^2_{\omega_t} = t^2$. Hence, equation (2.84) becomes

$$\Delta_{\omega_t} u + \frac{t^2}{2} u = 0\,. \tag{2.85}$$

We claim that $u \equiv 0$ is the unique solution of (2.85), for any $t > 0$ outside a countable set. Indeed, first of all we notice that the bundle map

$$\pi\colon (M, \omega_t) \to (T^4, \omega_{T^4}), \quad \omega_{T^4} = \frac{\sqrt{-1}}{2}(\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2)$$

corresponding to the projection on the first two coordinates, is a Riemannian submersion, for any $t > 0$. Moreover, we observe that the fibres are minimal, as the mean curvature vector of the fibres has to be left-invariant. Then, the claim is equivalent to prove that $Z(\mathrm{Heis}(3, \mathbb{C}))$ is minimal in $\mathrm{Heis}(3, \mathbb{C})$. This is trivially true since the Levi-Civita connection computed on central vector fields is identically zero, see also [95, Example 3.4]. Now, we can integrate equation (2.85) along the fibres of π, and obtain

$$\Delta_{\omega_{T^4}} \hat{u} + \frac{t^2}{2} \hat{u} = 0\,, \tag{2.86}$$

where $\hat{u}(x) := \int_{T_x} u\omega_{T^2,t}$ is the average of u along the fibres, $\omega_{T^2,t}$ is the metric obtained by rescaling the flat metric on T^2_x by a factor t^2, which coincides with the metric induced on the fibres by ω_t).
Here, equation (2.86) gives us two possibilities: either $\frac{t^2}{2} \notin \mathrm{Spec}(T^4, \omega_{T^4})$ or $\frac{t^2}{2} \in \mathrm{Spec}(T^4, \omega_{T^4})$. Choosing then t in order to land in the first case, puts us in the position to conclude that $\hat{u} \equiv 0$. Indeed, this allows us to apply [55, Corollary 1.7], telling us that, in our setting, the k-th eigenvalue corresponding to eigenfunctions with vanishing average along the fibres is bounded from below by the k-th eigenvalue of $\Delta_{\omega_{T^2,t}}$, which is equal to $4\pi^2 t^2 k^2 > t^2/2$. Hence, for these values of t, u is necessarily vanishing, and thus equation (2.85) has only the trivial solution. Hence, condition (2.82) is satisfied.

Remark 2.4.9. The case in which $\frac{t^2}{2} \in \mathrm{Spec}(T^4, \omega_{T^4})$ does not satisfy condition (2.82), since $u = f \circ \pi$, with $f\colon T^4 \to \mathbb{R}$ an eigenfunction of $\Delta_{\omega_{T^4}}$ with eigenvalue $\frac{t^2}{2}$ is clearly contained in $\ker F_{\omega_t}$.

Remark 2.4.10. Since the metrics ω_t considered in Example 2.4.8 all descend to Chern-Ricci flat balanced metrics on the orbifold in [168, Example 2.6], constructed in [284]. Hence, for the same values of t, F_{ω_t} has vanishing kernel also as an operator on weighted Hölder spaces on the orbifold. Indeed, through a bootstrap argument, the kernel reduces to the kernel on the smooth cover, i.e. the Iwasawa manifold, giving also an example in which [168, Theorem 1] can be applied to construct Chern-Ricci flat balanced metrics on the crepant resolution.

Example 2.4.11. The orbifold quotient constructed by Sferruzza and Tomassini considered in the previous remark provides also an example on which we can apply Theorem 2.2.19. Indeed, the action the authors consider preserves the flat torus metric ω_{T^4} on the base, making $\pi^*\omega^2_{T^4}$ a suitable choice for Ω also on the orbifold quotient and hence produces again Chern-Ricci flat balanced metrics on the crepant resolution.

Remark 2.4.12. The fact that for the same metric (and corresponding cohomology class) we are able to achieve our constructions with both ansatz, suggests the expected fact that Chern-Ricci flat balanced metrics, as well as constant Chern scalar balanced metrics, might have very large moduli space inside the balanced class. Hence, in order to be able to geometrize such classes, we forsee the need to introduce additional constraint, possibly on the torsion.

2.4.3 Nakamura manifolds

The final example we want to discuss is the one of Nakamura manifolds, as constructed by Cattaneo and Tomassini in [73].

To briefly recall the construction of Nakamura manifolds, fix $M \in \mathsf{SL}(n, \mathbb{Z})$ to be diagonalizable and let $P \in \mathsf{GL}(n, \mathbb{R})$ such that $P^{-1}MP = \mathrm{diag}(e^{\lambda_1}, ..., e^{\lambda_n})$, where $\lambda_i \in \mathbb{R}$ are such that

$$\sum_{i=1}^{n} \lambda_i = 0. \tag{2.87}$$

From here, we can consider the group action $\rho : \mathbb{C} \to \mathsf{GL}(n,\mathbb{C})$ given by

$$\rho(w) := \operatorname{diag}\left(e^{\frac{\lambda_1}{2}(w+\bar{w})}, \ldots, e^{\frac{\lambda_n}{2}(w+\bar{w})}\right),$$

which allows to consider the semidirect product $\mathsf{G}_M := \mathbb{C} \ltimes_\rho \mathbb{C}^n$. Then, we consider the lattices

$$\Gamma'_\tau := \mathbb{Z} \oplus \sqrt{-1}\tau \cdot \mathbb{Z} \quad \text{and} \quad \Gamma''_P := \mathbb{Z}^n \oplus \sqrt{-1}P \cdot \mathbb{Z}^n,$$

where $\tau \in \mathbb{R} \setminus \{0\}$. After noticing that $\Gamma_{P,\tau} := \Gamma'_\tau \ltimes_\rho \Gamma''_P \leq \mathsf{G}_M$, we define Nakamura manifolds as the quotients

$$N = N_{M,P,\tau} := \mathsf{G}_M / \Gamma_{P,\tau}.$$

These $(n+1)$-dimensional compact manifolds inherit the left-invariant coframe

$$\varphi^0 := dw\,, \quad \varphi^j := e^{\frac{\lambda_j}{2}(w+\bar{w})} dz_j\,, \quad j = 1, \ldots, n,$$

of $(1,0)$-forms from G_M (as they are invariant under $\Gamma_{P,\tau}$'s action) satisfying the following structure equations:

$$d\varphi^0 = 0, \quad d\varphi^j = -\frac{\lambda_j}{2}(\varphi^0 + \bar{\varphi}^0) \wedge \varphi^j\,, \quad j = 1, \ldots, n\,.$$

It can then be easily checked that

$$\omega_t := \frac{\sqrt{-1}}{2}\left(t^2 \varphi^0 \wedge \bar{\varphi}^0 + \sum_{j=1}^n \varphi^j \wedge \bar{\varphi}^j\right) \quad \text{and} \quad \Theta := \varphi^0 \wedge \ldots \wedge \varphi^n$$

define respectively a family of balanced metrics and a holomorphic volume form of constant (on N) ω_t-norm, making ω_t also Chern-Ricci flat, and N a Calabi-Yau manifold.

Example 2.4.13. Following the discussion in [73, Section 5], any Nakamura manifold inherits a holomorphic T^{2n}-bundle over a 2 dimensional torus. A fundamental difference with the previous example is that now the metrics induced by ω_t on the fibres are not equal, but they vary depending on the base parameter w. Nevertheless, it is straightforward to notice (by fixing a fiber, rescaling the coordinates and using (2.87)) that the Laplacians induced on each fibre all share the same eigenvalues, which are also the ones of the flat metric on T^{2n}. Hence, we can repeat the argument used in Example 2.4.8 to obtain once again that the operator F_{ω_t} has vanishing kernel up to a countable set of values for t, giving thus another family of examples on which (2.82) is satisfied.

On the other hand, Theorem 2.3.1 cannot be applied to this manifolds, as we explain in the following remark.

Remark 2.4.14. Using Proposition 1.1.50, we can reduce ourselves to work with invariant forms. We will prove the claim in complex dimension 3. In this case, the structure equations are:

$$d\varphi^0 = 0\,, \quad d\varphi^1 = -\frac{\lambda}{2}(\varphi^0 + \bar{\varphi}^0) \wedge \varphi^1\,, \quad d\varphi^1 = \frac{\lambda}{2}(\varphi^0 + \bar{\varphi}^0) \wedge \varphi^1\,, \quad \lambda \in \mathbb{R}\,.$$

Using these, one can easily prove that

$$\sqrt{-1}\partial\bar{\partial}(\varphi^i \wedge \bar{\varphi}^j) = 0\,, \quad i < j\,, \tag{2.88}$$

while

$$\sqrt{-1}\partial\bar{\partial}(\varphi^0 \wedge \bar{\varphi}^0) = 0\,, \quad \sqrt{-1}\partial\bar{\partial}(\varphi^i \wedge \bar{\varphi}^i) = \sqrt{-1}\lambda^2 \varphi^0 \wedge \bar{\varphi}^0 \wedge \varphi^i \wedge \bar{\varphi}^i\,, \quad i = 1, 2\,. \tag{2.89}$$

Then, given a general left-invariant $\Omega \in \Lambda^{1,1}_{\mathbb{R}} N$, we can write it as:

$$\Omega = \sqrt{-1} \sum_{i,j=0}^{2} a_{i\bar{j}} \varphi^i \wedge \bar{\varphi}^j\,.$$

Then, using (2.88) and (2.89), we have that

$$\sqrt{-1}\partial\bar{\partial}\Omega = (\sqrt{-1})^2\lambda^2(a_{1\bar{1}}\varphi^0 \wedge \bar{\varphi}^0 \wedge \varphi^1 \wedge \bar{\varphi}^1 + a_{2\bar{2}}\varphi^0 \wedge \bar{\varphi}^0 \wedge \varphi^2 \wedge \bar{\varphi}^2)\,.$$

Now,

$$\Lambda^2_{\omega_t}(\sqrt{-1}\partial\bar{\partial}\Omega)\frac{\omega_t^3}{3!} = 2\sqrt{-1}\partial\bar{\partial}\Omega \wedge \omega_t\,.$$

On the other hand, we easily see that

$$\sqrt{-1}\partial\bar{\partial}\Omega \wedge \omega_t = (\sqrt{-1})^3\lambda^2(a_{1\bar{1}} + a_{2\bar{2}})\varphi^0 \wedge \bar{\varphi}^0 \wedge \varphi^1 \wedge \bar{\varphi}^1 \wedge \varphi^2 \wedge \bar{\varphi}^2 = \lambda^2(a_{1\bar{1}} + a_{2\bar{2}})\frac{\omega_t^3}{3!}\,.$$

Then, we conclude that $\Lambda^2_{\omega_t}(\sqrt{-1}\partial\bar{\partial}\Omega) = \lambda^2(a_{1\bar{1}}+a_{2\bar{2}})$. Then, $\Lambda^2_{\omega_t}(\sqrt{-1}\partial\bar{\partial}\Omega) \leq 0$ if and only if $a_{1\bar{1}}+a_{2\bar{2}} \leq 0$. But, computing $\omega \wedge \Omega$ and imposing its positivity, we need to have $a_{1\bar{1}} + a_{2\bar{2}} > 0$ which is absurd, giving the claim.

Remark 2.4.15. What was shown in Remark 2.4.14 and Example 2.4.8 suggests that the existence of metrics satisfying condition (2.82) might be strictly weaker then the assumption of having a form satisfying (2.77), fueling the interest towards the study of operator (2.12).

Chapter 3

The Pluriclosed flow on solvmanifolds

After the proof of Thurston's geometrization conjecture by Perelman using the Ricci flow, the study of geometric flows has gained much importance and these are frequently defined ad hoc to address many problems in Geometry. A large class of manifolds on which the behaviour of the Ricci flow is largely understood even in higher dimensions is the class of homogeneous spaces. Analogues of the Ricci flow in the complex setting were defined by many authors. For sure, as we saw in Subsection 1.1.3 and Section 1.2, among all of them, the pluriclosed flow is one of the most studied. In addiction, it is the object of many open conjectures, for instance involving the classification of compact complex surfaces, see [299, 305]. This chapter is focused on the study of the behaviour of the pluriclosed in the locally homogeneous setting, i.e. on compact quotients of Lie groups. This study is conducted in two different ways.

In Section 3.1, we study the pluriclosed flow on Oeljeklaus-Toma manifolds, an explicit class of compact complex manifolds. These manifolds are, in particular, compact quotients of a solvable Lie group, endowed with a left-invariant complex structure, by a discrete subgroup. The complete behaviour of the left-invariant pluriclosed flow on such manifolds is deducted by the study of the Bismut-Ricci form of a general left-invariant SKT metric. This first section is an account of a joint work with Luigi Vezzoni, see [141].

In Section 3.2, we make use of the equivalence between the pluriclosed flow and the generalized Ricci flow saw in Subsection 1.2.1. In view of this equivalence, we study the behaviour of the homogeneous generalized Ricci flow by introducing a flow of Dorfman brackets which allows us to understand the behaviour the generalized Ricci flow. The second section is a collection of the results of a joint work with Ramiro A. Lafuente and James Stanfield, see [140].

3.1 Pluriclosed flow on Oeljeklaus-Toma manifolds

In this section, we collect the results regarding the behaviour of the pluriclosed flow on Oeljeklaus-Toma manifolds. The section is divided as follows. Subsection 3.1.1 is dedicated to the description of Oeljeklaus-Toma manifolds and their structure of solvmanifold. Moreover, we prove a general result concerning the Gromov-Hausdorff convergence of Hermitian metrics on Oeljeklaus-Toma manifolds.

In Subsection 3.1.2, we study the left-invariant Chern-Ricci flow and discuss its long-time behaviour on Oeljeklaus-Toma manifolds.

The statement and the proof of the main result of this section can be found in Subsection 3.1.3. The long-time behaviour of the pluriclosed flow will be obtained through an explicit characterization of left-invariant SKT metrics and the direct computation of their Bismut-Ricci form.

Finally, in Subsection 3.1.4 part of the main results will be generalized to a wider class of Lie algebras.

3.1.1 Gromov-Hausdorff convergence on Oeljeklaus-Toma manifolds

Oeljeklaus-Toma manifolds were defined firstly in [247] using algebraic number theory. Such manifolds are compact complex manifolds generalizing the classical Inoue surfaces, see [194]. We briefly present their construction in details.

Let $\mathbb{Q} \subseteq \mathbb{K}$ be an algebraic number field with $[\mathbb{K} : \mathbb{Q}] = r + 2s$ and $r, s \geq 1$. Let $\sigma_1, \dots, \sigma_r \colon \mathbb{K} \to \mathbb{R}$ be the real embeddings of $\mathbb{K}$ and $\sigma_{r+1}, \dots, \sigma_{r+2s} \colon \mathbb{K} \to \mathbb{C}$ be the complex embeddings of $\mathbb{K}$ satisfying $\sigma_{r+s+i} = \bar{\sigma}_{r+i}$, for every $i = 1, \dots, s$. We denote by $\mathcal{O}_{\mathbb{K}}$ the ring of algebraic integers of $\mathbb{K}$, i.e.

$$\mathcal{O}_{\mathbb{K}} := \{a \in \mathbb{K} \mid \exists f \in \mathbb{Z}[x]\,, f(a) = 0\}\,,$$

and by $\mathcal{O}_{\mathbb{K}}^*$ the group of units of $\mathcal{O}_{\mathbb{K}}$. Let

$$\mathcal{O}_{\mathbb{K}}^{*,+} := \{u \in \mathcal{O}_{\mathbb{K}}^* \quad | \quad \sigma_i(u) > 0\,, \quad i = 1, \dots, r\}$$

be the group of totally positive units of $\mathcal{O}_{\mathbb{K}}$. The groups $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{K}}^{*,+}$ act on $\mathbf{H}^r \times \mathbb{C}^s$ as:

$$a \cdot (z_1, \dots, z_r, w_1, \dots, w_s) = (z_1 + \sigma_1(a), \dots, z_r + \sigma_r(a), w_1 + \sigma_{r+1}(a), \dots, w_s + \sigma_{r+s}(a))\,, \quad a \in \mathcal{O}_{\mathbb{K}}$$

and

$$u \cdot (z_1, \dots, z_r, w_1, \dots, w_s) = (\sigma_1(u)z_1, \dots, \sigma_r(u)z_r, \sigma_{r+1}(u)w_1, \dots, \sigma_{r+s}(u)w_s)\,, \quad u \in \mathcal{O}_{\mathbb{K}}^{*,+}\,,$$

where $\mathbf{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ is the upper-half plane in $\mathbb{C}$.

There always exists a free subgroup U of rank r of $\mathcal{O}_{\mathbb{K}}^{*,+}$ such that $\mathrm{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in $\mathbb{R}^r$, see [247], where $l \colon \mathcal{O}_{\mathbb{K}}^{*,+} \to \mathbb{R}^{r+s}$ is the logarithmic representation of units

$$l(u) = (\log \sigma_1(u), \dots, \log \sigma_r(u), 2\log|\sigma_{r+1}(u)|, \dots, 2\log|\sigma_{r+s}(u)|)$$

and $\mathrm{pr}_{\mathbb{R}^r} \colon \mathbb{R}^{r+s} \to \mathbb{R}^r$ is the projection on the first r coordinates. The action of $U \ltimes \mathcal{O}_{\mathbb{K}}$ on $\mathbf{H}^r \times \mathbb{C}^s$ is free, properly discontinuous and co-compact. An *Oeljeklaus-Toma manifold* is then defined as the quotient

$$M := \frac{\mathbf{H}^r \times \mathbb{C}^s}{U \ltimes \mathcal{O}_{\mathbb{K}}}$$

and it is a compact complex manifold having complex dimension $r + s$. Directly from their construction, we derive a structure of torus bundle for Oeljeklaus-Toma manifolds. This structure can be seen as follows: we have

$$\frac{\mathbf{H}^r \times \mathbb{C}^s}{\mathcal{O}_{\mathbb{K}}} = \mathbb{R}^r_+ \times T^{r+2s}$$

and that the action of U on $\mathbf{H}^r \times \mathbb{C}^s$ induces an action on $\mathbb{R}^r_+ \times T^{r+2s}$ such that, for every $x \in \mathbb{R}^r_+$ and $u \in U$, the induced map

$$u \colon (x, T^{r+2s}) \mapsto (\sigma_1(u)x_1, \dots, \sigma_r(u)x_r, T^{r+2s})$$

is a diffeomorphism. Hence

$$M \simeq \frac{\mathbb{R}^r_+ \times T^{r+2s}}{U}$$

inherits the structure of a T^{r+2s}-bundle over T^r. We denote by π and F the projections

$$\pi \colon \mathbf{H}^r \times \mathbb{C}^s \to M\,, \quad F \colon M \to T^r\,.$$

Finally, we observe that the Poincaré metric $\omega_{\mathbf{H}^r} = \sqrt{-1}\sum_{a=1}^r \frac{dz_a \wedge d\bar{z}_a}{4(\mathrm{Im} z_a)^2}$ on $\mathbf{H}^r$ can be pulled back to $\mathbf{H}^r \times \mathbb{C}^s$, obtaining a degenerate $(1,1)$-form ω_∞. One can easily see that ω_∞ is invariant under the action of $U \ltimes \mathcal{O}_{\mathbb{K}}$. Thus, it descends to a $(1,1)$-form on M, which will be denoted again with ω_∞.

From the viewpoint of Lie groups, the universal covering of an Oeljeklaus-Toma manifold M has a natural structure of solvable Lie group G and the complex structure on M lifts to a left-invariant complex structure, see [206] for the detailed proof of this fact. Therefore, Oeljeklaus-Toma manifolds can be seen as compact solvmanifolds with a left-invariant complex structure. The solvable structure on the universal covering of M can be described in terms of the existence of a left-invariant $(1,0)$-coframe $\{\omega^1,\dots,\omega^r,\gamma^1,\dots,\gamma^s\}$ such that

$$\begin{cases} d\omega^k = \frac{\sqrt{-1}}{2}\omega^k\wedge\bar\omega^k & k=1,\dots,r\,,\\ d\gamma^i = \sum_{k=1}^r \lambda_{ki}\,\omega^k\wedge\gamma^i - \sum_{k=1}^r \lambda_{ki}\bar\omega^k\wedge\gamma^i & i=1,\dots,s\,,\end{cases} \tag{3.1}$$

where

$$\lambda_{ki} = \frac{\sqrt{-1}}{4}b_{ki} - \frac{1}{2}c_{ki}$$

and $b_{ki}, c_{ki}\in\mathbb{R}$ depend on the embeddings σ_j as

$$\sigma_{r+i}(u) = \left(\prod_{k=1}^r (\sigma_k(u))^{\frac{b_{ki}}{2}}\right) e^{\sqrt{-1}\sum_{k=1}^r c_{ki}\log\sigma_k(u)}\,, \tag{3.2}$$

for any $u\in U$, $k=1,\dots,r$ and $i=1,\dots,s$. Since $U\subseteq\mathcal{O}^*_{\mathbb{K}}$, it is easy to see that

$$l(U)\subseteq\left\{x\in\mathbb{R}^{r+s}\;\middle|\;\sum_{i=1}^{r+s}x_i=0\right\}.$$

This fact together with (3.2) implies that, for every $u\in U$,

$$\sum_{i=1}^r \log\sigma_i(u)\left(1+\sum_{k=1}^s b_{ik}\right)=0\,,$$

which, since $\mathrm{pr}_{\mathbb{R}^r}\circ l(U)$ is a lattice of rank r in $\mathbb{R}^r$, is equivalent to

$$\sum_{k=1}^s b_{ik} = -1\,,\quad i=1,\dots,r\,. \tag{3.3}$$

The dual frame $\{Z_1,\dots,Z_r,W_1,\dots,W_s\}$ to $\{\omega^1,\dots,\omega^r,\gamma^1,\dots,\gamma^s\}$ satisfies the following structure equations:

$$[Z_k,\bar Z_k] = -\frac{\sqrt{-1}}{2}(Z_k+\bar Z_k)\,,\quad [Z_k,W_i]=-\lambda_{ki}W_i\,,\quad [Z_k,\bar W_i]=\bar\lambda_{ki}\bar W_i\,,$$

for $k=1,\dots,r$, $i=1,\dots,s$. Consequently, the Lie algebra $\mathfrak{g}$ of the universal covering of M splits as vector space as

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{I}$$

where $\mathfrak{I}$ is an abelian ideal and $\mathfrak{h}$ is a subalgebra isomorphic to $\underbrace{\mathfrak{f}\oplus\cdots\oplus\mathfrak{f}}_{r\text{-times}}$, where $\mathfrak{f}$ is the *filiform* Lie algebra $\mathfrak{f}=\langle e_1,e_2\rangle$, $[e_1,e_2]=-\frac{1}{2}e_1$. The complex structure J induced on $\mathfrak{g}$ preserves both $\mathfrak{h}$ and $\mathfrak{I}$ and its restriction $J_{\mathfrak{h}}$ on $\mathfrak{h}$ satisfies

$$J_{\mathfrak{h}} = \underbrace{J_{\mathfrak{f}}\oplus\cdots\oplus J_{\mathfrak{f}}}_{r\text{-times}}\,,$$

where $J_{\mathfrak{f}}$ is the complex structure on $\mathfrak{f}$ defined by $J_{\mathfrak{f}}(e_1)=e_2$. Moreover

$$[\mathfrak{h}^{1,0},\mathfrak{I}^{0,1}]\subseteq\mathfrak{I}^{0,1}\,.$$

Now that the construction and the geometric structure of Oeljeklaus-Toma manifolds is well-understood, we turn our focus on the study of Gromov-Hausdorff convergence of such manifolds.

Let $\{\omega_t\}_{t\in[0,\infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus-Toma manifold M and let d_t be the induced distance on M. For a smooth curve γ on M, let $L_t(\gamma)$ be the length of γ with respect to ω_t. We further denote by $\mathcal{H}$ the foliation induced by $\mathfrak{h}$ on M.

Proposition 3.1.1. *Let M be an Oeljeklaus-Toma manifold and $\{\omega_t\}_{t\in[0,\infty)}$ be a smooth curve of Hermitian metrics on M such that $\omega_t \to \omega_\infty$ pointwise, as $t \to \infty$. Assume that there exist $T \in (0,\infty)$ and $C > 0$ such that*

1. *$L_t(\gamma) \leq CL_0(\gamma)$, for every smooth curve γ in M;*
2. *$L_t(\gamma) \leq (C/\sqrt{t})L_0(\gamma)$, for every smooth curve γ in M such that $\dot\gamma \in \ker\omega_\infty$.*
3. *for every $\varepsilon, \ell > 0$, there exists $T > 0$ such that $|L_t(\gamma) - L_\infty(\gamma)| < \varepsilon$, for every $t > T$ and every curve γ in M tangent to $\mathcal{H}$ and such that $L_\infty(\gamma) < \ell$.*

Then, (M, d_t) converges to (T^r, d) in the Gromov-Hausdorff sense, where d is the distance induced by ω_∞ onto T^r.

Proof. To prove the claim, we will construct $F\colon M \to T^r$ and $G\colon T^r \to M$ satisfying the properties in Item 3 of Theorem 1.4.2. In order to do this, we follow the approach in [322, Section 5] and in [344, Proof of Theorem 1.1].

We consider the structure of M as T^{r+2s}-bundle over a T^r and let $F\colon M \to T^r$ be the projection onto the base and let $G\colon T^r \to M$ be an arbitrary map such that $F \circ G = \mathrm{Id}_{T^r}$. We show that, for every $\varepsilon > 0$, there exists $T > 0$ such that

$$|d_t(p,q) - d(F(p),F(q))| \leq \varepsilon\,, \tag{3.4}$$
$$|d(a,b) - d_t(G(a),G(b))| \leq \varepsilon\,, \tag{3.5}$$
$$d_t(p, G(F(p))) \leq \varepsilon\,, \tag{3.6}$$
$$d(a, F(G(a))) \leq \varepsilon\,, \tag{3.7}$$

for every $t \geq T$, $p,q \in M$, $a,b \in T^r$ which implies the statement.

First of all, we note that (3.7) is trivial since

$$d(a, F(G(a))) = 0\,,$$

for every $a \in T^r$.

Then, we show that (3.6) is satisfied. Let $p,q \in M$ be two points in the same fiber over T^r. Assume $p = \pi(z,w)$. We denote with $\mathcal{L}_{(z,w)}$ the leaf of the foliation $\ker\omega_\infty$ on the universal covering of M passing through (z,w). We easily see that, for all $(z,w) \in \mathbf{H}^r \times \mathbb{C}^s$, $\mathcal{L}_{(z,w)} = \{z\} \times \mathbb{C}^s$. In view of [334, Section 2], for every $z \in \mathbf{H}^r$, $\pi(\{z\} \times \mathbb{C}^s)$ is the leaf of the foliation $\ker\omega_\infty$ on M passing through p and it is dense in the fiber $F^{-1}(F(p))$. Let B_R be the standard ball in $\mathbb{C}^s$ around the origin having radius R. We can choose R so that every point in $F^{-1}(F(p))$ has distance with respect to d_0 less than $\varepsilon/2C$ to $\pi(\{z\} \times \bar B_R)$. On the other hand, given two points in $\pi(\{z\} \times \bar B_R)$, they can be joined with a curve γ in $F^{-1}(F(p))$ which is tangent to $\ker\omega_\infty$. Hence, for any such curve, Item 2 implies

$$L_t(\gamma) \leq \frac{C'}{\sqrt{t}}\,,$$

for a uniform constant C' depending only on R. Let $p_0 = \pi(z,0)$, γ_1 be a curve in $F^{-1}(F(p))$ connecting p with p_0 tangent to $\ker\omega_\infty$ and γ_2 be a curve connecting p_0 with q having minimal length with respect to d_0. Hence, by using Item 1, for t sufficiently large, we have

$$d_t(p,q) \leq L_t(\gamma_1) + L_t(\gamma_2) \leq \frac{C'}{\sqrt{t}} + CL_0(\gamma_2) \leq \frac{C'}{\sqrt{t}} + \frac{\varepsilon}{2} \leq \varepsilon\,,$$

from which (3.6) follows.

Next we show (3.4) and (3.5). First of all, we denote with g the riemannian metric on T^r induced by ω_∞, for an explicit expression of g see [344, Section 2], and we observe that

$$L_g(F(\gamma)) \leq L_\infty(\gamma)\,, \tag{3.8}$$

for every curve γ in M, and the equality holds if and only if

$$\dot{\gamma} \in \mathcal{Y} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{1}{2\sqrt{-1}} \left(Z_i - \bar{Z}_i \right) \;\middle|\; i = 1, \dots, r \right\}.$$

Let $p, q \in M$. We can find a curve γ in M connecting p with a point $\tilde{q}$ in the T^{r+2s}-fiber containing q which is tangent to $\mathcal{Y}$ and such that $F(\gamma)$ is a minimal geodesic on (T^r, g), see for instance [322, Proof of Theorem 5.1] or [344, Proof of Theorem 1.1]. By applying Item 3 we have

$$d_t(p,q) \leq d_t(p,\tilde{q}) + d_t(\tilde{q},q) \leq d_t(p,\tilde{q}) + \varepsilon \leq L_t(\gamma) + \varepsilon \leq L_\infty(\gamma) + 2\varepsilon = L_g(F(\gamma)) + 2\varepsilon = d(F(p),F(q)) + 2\varepsilon\,,$$

for t big enough, i.e.

$$d_t(p,q) - d(F(p),F(q)) \leq 2\varepsilon\,, \tag{3.9}$$

for t sufficiently large.

Next, using again (3.8), we obtain, for $p, q \in M$,

$$d(F(p),F(q)) \leq L_g(F(\gamma)) \leq L_\infty(\gamma) \leq L_t(\gamma) + \varepsilon = d_t(p,q) + \varepsilon\,,$$

for t big enough, where γ is curve which realizes the distance $d_t(p,q)$. Hence we obtain

$$d(F(p),F(q)) - d_t(p,q) \leq \varepsilon\,. \tag{3.10}$$

By substituting $p = G(a)$ and $q = G(b)$ in (3.9) and (3.10) we infer

$$-\varepsilon \leq d_t(G(a),G(b)) - d(a,b) \leq 2\varepsilon$$

and (3.4) and (3.5) follow. □

3.1.2 The left-invariant Chern-Ricci flow on Oeljeklaus-Toma manifolds

Before moving to prove the main results about the left-invariant pluriclosed flow on Oeljeklaus-Toma manifolds, we study the behaviour of the Chern-Ricci flow, recall Definition 1.1.33, on such manifolds.

In this section we will prove the following theorem.

Proposition 3.1.2. *Let ω be a left-invariant Hermitian metric on an Oeljeklaus-Toma manifold M. Then, ω lifts to an expanding algebraic soliton for the Chern-Ricci flow on the universal covering of M if and only if it takes the following expression with respect to the coframe $\{\omega^1, \dots, \omega^r, \gamma^1, \dots, \gamma^s\}$ satisfying (3.1):*

$$\omega = \sqrt{-1} \left(A \sum_{i=1}^{r} \omega^i \wedge \bar{\omega}^i + \sum_{i,j=1}^{s} g_{r+i\overline{r+j}} \gamma^i \wedge \bar{\gamma}^j \right) . \tag{3.11}$$

Moreover, the Chern-Ricci flow starting from ω has a long-time solution $\{\omega_t\}$ such that $(M, \frac{\omega_t}{1+t})$ converges as $t \to \infty$ in the Gromov-Hausdorff sense to (T^r, d), where d is the distance induced by ω_∞ onto T^r. Finally, $(\mathbf{H}^r \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbf{H}^r \times \mathbb{C}^s, \tilde{\omega}_\infty)$ where $\tilde{\omega}_\infty$ is an algebraic soliton.

Proof. Let M be an Oeljeklaus-Toma manifold. Since the Chern-Ricci form does not depend on the choice of the left-invariant Hermitian metric, see, for instance, [225, 335], it is enough to compute $\mathrm{Ric}^{\mathrm{Ch}}(\omega)$ for the following metric:

$$\omega = \sqrt{-1} \left(\sum_{i=1}^{r} \omega^i \wedge \bar{\omega}^i + \sum_{j=1}^{s} \gamma^j \wedge \bar{\gamma}^j \right) . \tag{3.12}$$

We recall that the Chern-Ricci form of a left-invariant Hermitian metric $\omega = \sqrt{-1}\sum_{a=1}^{n}\alpha^a\wedge\bar\alpha^a$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)(X,Y) = -\sum_{a=1}^{n}(\omega([[X,Y]^{0,1},X_a],\bar X_a)+\omega([[X,Y]^{1,0},\bar X_a],X_a))\,, \tag{3.13}$$

for every left-invariant vector fields X, Y on G, where $\{\alpha^i\}$ is a left-invariant unitary $(1,0)$-coframe with dual frame $\{X_a\}$ (see e.g. [335]). By applying (3.13) to the metric (3.12), we have

$$\begin{aligned}\mathrm{Ric}^{\mathrm{Ch}}(\omega)(X,Y) = &-\sum_{a=1}^{r}\{\omega([[X,Y]^{0,1},Z_a],\bar Z_a)+\omega([[X,Y]^{1,0},\bar Z_a],Z_a)\}\\ &-\sum_{b=1}^{s}\{\omega([[X,Y]^{0,1},W_b],\bar W_b)+\omega([[X,Y]^{1,0},\bar W_b],W_b)\}\,.\end{aligned}$$

Clearly,

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)(Z_i,\bar Z_j)=0\,,\quad i\neq j\,,\quad \mathrm{Ric}^{\mathrm{Ch}}(\omega)(W_i,\bar W_j)=0\,,\quad i,j=1,\dots,s\,.$$

Moreover, since $\mathfrak{J}$ is an abelian ideal and ω makes $\mathfrak{J}$ and $\mathfrak{h}$ orthogonal, we have:

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)(Z_i,\bar W_j)=0\,,\quad i=1,\dots,r\,,\quad j=1,\dots,s\,.$$

Moreover we have

$$\omega([[Z_i,\bar Z_i]^{0,1},Z_a],\bar Z_a)=\frac{\sqrt{-1}}{4}\delta_{ia}\,,\quad \omega([[Z_i,\bar Z_i]^{1,0},\bar Z_a],Z_a)=\frac{\sqrt{-1}}{4}\delta_{ia}$$

and

$$\omega([[Z_i,\bar Z_i]^{0,1},W_b],\bar W_b)=\frac{1}{2}\lambda_{ib}\,,\quad \omega([[Z_i,\bar Z_i]^{1,0},\bar W_b],W_b)=-\frac{1}{2}\bar\lambda_{ib}$$

which imply

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)(Z_i,\bar Z_i)=-\sqrt{-1}\left(\frac{1}{2}+\sum_{b=1}^{s}\mathrm{Im}(\lambda_{ib})\right)=-\frac{\sqrt{-1}}{4}\,.$$

Consequently,

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)=-\omega_\infty\,,$$

where ω_∞ is the degenerate metric induced on M by the Poincaré metric on $\mathbf{H}^r$, namely,

$$\omega_\infty=\frac{\sqrt{-1}}{4}\sum_{i=1}^{r}\omega^i\wedge\bar\omega^i\,.$$

Now, consider P as the endomoprhism associated to the Chern-Ricci form, i.e. $\mathrm{Ric}^{\mathrm{Ch}}(\omega)(\cdot,\cdot)=\omega(P\cdot,\cdot)$. In general, we have that

$$P_i^j=(\mathrm{Ric}^{\mathrm{Ch}}(\omega))_{i\bar k}g^{\bar kj}=\begin{cases}-\frac{1}{4}g^{\bar ij} & \text{if } i\in\{1,\dots,r\}\,,\\ 0 & \text{otherwise}\ .\end{cases}$$

Then, using [225, Part (iii) Proposition 4.2], we can infer that any left-invariant Hermitian metrics of the form (3.11) lifts to an expanding algebraic soliton on the universal covering of M with cosmological constant $c=\frac{1}{4A}$. Conversely, let ω be an algebraic soliton for the Chern-Ricci flow. Then, thanks to [225, Part (ii) Proposition 4.2], we have that

$$P-cI\in\mathrm{Der}(\mathfrak{g})\,.$$

On the other hand, we can easily see that, if $D \in \mathrm{Der}(\mathfrak{g})$, then $\mathfrak{h} \subseteq \ker D$, see proof of Corollary 3.1.6 for the details. This readily implies that

$$-\frac{1}{4}g^{i\bar{i}} = -\frac{1}{4}g^{\bar{j}j} = c\,, \quad i,j = 1,\ldots,r\,, \quad g^{\bar{i}j} = 0\,, \quad i \in \{1,\ldots,r\}\,, j \neq i\,,$$

from which the claim follows.

Moreover, the Chern-Ricci flow evolves an arbitrary left-invariant Hermitian metric ω as $\omega_t = \omega + t\omega_\infty$ and $\frac{\omega_t}{1+t} \to \omega_\infty$ as $t \to \infty$. In order to obtain the claim regarding the Gromov-Hausdorff convergence, we show that $\frac{\omega_t}{1+t}$ satisfies Items 1 to 3 of Proposition 3.1.1. Here we denote by $|\cdot|_t$ the norm induced by ω_t. On the other hand, Item 2 of Proposition 3.1.1 is trivially satisfied since $\omega_t|_{\mathfrak{J}\oplus\mathfrak{J}} = \omega_0$, for every $t \geq 0$, and

$$L_t(\gamma) = \frac{1}{\sqrt{1+t}}L_0(\gamma)\,,$$

for every curve γ in M tangent to $\ker \omega_\infty$.

On the other hand, for a vector $v \in \mathfrak{h}$, we have

$$\frac{1}{\sqrt{1+t}}|v|_t \leq C|v|_0\,,$$

for a constant $C > 0$ independent on v. This, together with Item 2, guarantees Item 1.

In order to prove Item 3, let $\varepsilon, \ell > 0$ and $T > 0$ be such that

$$\left|\frac{|v|_t}{\sqrt{1+t}} - |v|_\infty\right| \leq \frac{\varepsilon}{\ell}\,,$$

for every $v \in \mathfrak{h}$ and $t \geq T$. Let γ be a curve in M tangent to $\mathcal{H}$ which is parametrized by arclength with respect to ω_∞ and such that $L_\infty(\gamma) < \ell$. Then

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left|\frac{1}{\sqrt{1+t}}|\dot\gamma|_t - |\dot\gamma|_\infty\right| da \leq \frac{\varepsilon}{\ell}b \leq \varepsilon\,,$$

since $b \leq \ell$.

For the last statement, we identify ω_t with its pull-back onto $\mathbf{H}^r \times \mathbb{C}^s$ and we fix as base point the identity element of $\mathbf{H}^r \times \mathbb{C}^s$. Firstly, we observe that the endomorphism D represented with respect to the frame $\{Z_1,\ldots,Z_r,W_1,\ldots,W_s\}$ by the following matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\mathfrak{J}} \end{pmatrix}$$

is a derivation of $\mathfrak{g}$. Moreover, we can construct

$$\exp(s(t)D) = \begin{pmatrix} \mathrm{I}_{\mathfrak{h}} & 0 \\ 0 & e^{s(t)}\mathrm{I}_{\mathfrak{J}} \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}, J)\,, \quad t \geq 0\,,$$

where $s(t) = \log(\sqrt{1+t})$ and define the 1-parameter family $\{\varphi_t\} \subseteq \mathrm{Aut}(\mathbf{H}^r \times \mathbb{C}^s, J)$ such that

$$d\varphi_t = \exp(s(t)D)\,, \quad t \geq 0\,.$$

Trivially, we see that

$$\varphi_t^*\frac{\omega_t}{1+t}(Z_i, \bar{Z}_j) = \sqrt{-1}\frac{1}{1+t}\left(g_{i\bar{j}} + \frac{t}{4}\delta_{ij}\right) \to \frac{\sqrt{-1}}{4}\delta_{ij}\,, \quad t \to \infty\,,$$

$$\varphi_t^*\frac{\omega_t}{1+t}(Z_i, \bar{W}_j) = \sqrt{-1}\frac{e^{s(t)}}{1+t}g_{i\overline{r+j}} \to 0\,, \quad t \to \infty\,,$$

$$\varphi_t^*\frac{\omega_t}{1+t}(W_i, \bar{W}_j) = \sqrt{-1}\frac{e^{2s(t)}}{1+t}g_{r+i\overline{r+j}} \to \sqrt{-1}g_{r+i\overline{r+j}}\,, \quad t \to \infty\,.$$

These facts guarantee that

$$\varphi_t^*\frac{\omega_t}{1+t} \to \omega_\infty + \omega|_{\mathfrak{J}\oplus\mathfrak{J}}\,, \quad t \to \infty\,,$$

hence, the assertion follows. ☐

3.1.3 Behaviour of the pluriclosed flow on Oeljeklaus-Toma manifolds

In this section, we prove long-time existence and convergence in both Gromov-Haudorff and Cheeger-Gromov sense of a solution of the pluriclosed flow on a given Oeljeklaus-Toma manifold starting from a left-invariant SKT metric. The result will be proved in steps. The first one consists in characterizing all the SKT metrics on a Oeljeklaus-Toma manifold. Then, we will compute the $(1,1)$-part of the Bismut-Ricci form and, from that, we will deduce long-time existence and the characterization of algebraic solitons. Finally, we will conclude the proof applying Proposition 3.1.1 and obtain Cheeger-Gromov convergence as in the proof of Proposition 3.1.2.

The existence of SKT metrics on Oeljeklaus-Toma manifolds was studied in [25], [115] and [253].

Theorem 3.1.3 ([25], Corollary 3)**.** *An Oeljeklaus-Toma manifold of type (r,s) admits a SKT metric if and only if $r=s$ and*

$$\sigma_j(u)|\sigma_{r+j}(u)|^2 = 1\,, \quad j=1,\dots,s\,,\ u\in U\,. \tag{3.14}$$

Condition (3.14) in the previous theorem can be rewritten in terms of the structure constants appearing in (3.1). Indeed, (3.1) together with (3.14) forces $b_{ki}\in\{0,-1\}$ and $b_{ki}b_{li}=0$, for every $i,k,l=1,\dots,s$ with $k\neq l$. In particular, using (3.3), for every fixed index $k\in\{1,\dots,s\}$, there exists a unique $i_k\in\{1,\dots,s\}$ such that

$$b_{ki_k}=-1\,,\quad b_{ki}=0\,,$$

for all $i\neq i_k$ and, if $k\neq l$, then $i_k\neq i_l$. Hence, up to a reorder of the γ_j's, we may and do assume, without loss of generality, $i_k=k$, for every $k\in\{1,\dots,s\}$, i.e.

$$\lambda_{ki}=\begin{cases}-\frac{1}{2}c_{ki} & \text{if } i\neq k\,,\\ -\frac{1}{2}c_{kk}-\frac{\sqrt{-1}}{4} & \text{if } i=k\,.\end{cases} \tag{3.15}$$

The next proposition gives a full characterization of left-invariant SKT metric on a Oeljeklaus-Toma manifold in terms of the standard $(1,0)$-coframe.

Proposition 3.1.4. *A left-invariant metric ω on an Oeljeklaus-Toma manifold admitting SKT metrics is SKT if and only if it takes the following expression with respect to a coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ satisfying* (3.1) *and* (3.15)*:*

$$\omega=\sqrt{-1}\sum_{i=1}^{s}A_i\omega^i\wedge\bar\omega^i+B_i\gamma^i\wedge\bar\gamma^i+\sqrt{-1}\sum_{r=1}^{k}\left(C_r\omega^{p_r}\wedge\bar\gamma^{p_r}+\bar C_r\gamma^{p_r}\wedge\bar\omega^{p_r}\right) \tag{3.16}$$

for some $A_1,\dots,A_s,B_1,\dots,B_s\in\mathbb{R}_+$, $C_1,\dots,C_k\in\mathbb{C}$, where $\{p_1,\dots,p_k\}\subseteq\{1,\dots,s\}$ are such that

$$\lambda_{jp_i}=0\,,\quad j\neq p_i\,,\ i=1,\dots,k\,.$$

Proof. We assume $s>1$ since the case $s=1$ is trivial. Let

$$\omega=\sqrt{-1}\sum_{p,q=1}^{s}A_{p\bar q}\omega^p\wedge\bar\omega^q+B_{p\bar q}\gamma^p\wedge\bar\gamma^q+C_{p\bar q}\omega^p\wedge\bar\gamma^q+\bar C_{p\bar q}\gamma^q\wedge\bar\omega^p$$

be an arbitrary real left-invariant $(1,1)$-form on M, with $A_{p\bar p},B_{p\bar p}\in\mathbb{R}$, for every $p=1,\dots,s$, $A_{p\bar q},B_{p\bar q}\in\mathbb{C}$, for all $p,q=1,\dots,s$ with $p\neq q$, and $C_{p\bar q}\in\mathbb{C}$, for every $p,q=1,\dots,s$.

From the structure equations (3.1), it easily follows

$$\begin{cases}\partial\bar\partial(\omega^p\wedge\bar\omega^q)\in\langle\omega^p\wedge\omega^q\wedge\bar\omega^p\wedge\bar\omega^q\rangle\,,\\ \partial\bar\partial(\omega^p\wedge\bar\gamma^q)\in\langle\omega^i\wedge\omega^j\wedge\bar\omega^l\wedge\bar\gamma^m\rangle\,,\\ \partial\bar\partial(\gamma^p\wedge\bar\gamma^q)\in\langle\omega^i\wedge\bar\omega^j\wedge\gamma^l\wedge\bar\gamma^m\rangle\,,\end{cases} \tag{3.17}$$

and that ω is SKT if and only if the following three conditions are satisfied

$$\sum_{p,q=1}^{s} A_{p\bar q}\partial\bar\partial(\omega^p\wedge\bar\omega^q)=0\,; \tag{3.18}$$

$$\sum_{p,q=1}^{s} B_{p\bar q}\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=0\,; \tag{3.19}$$

$$\sum_{p,q=1}^{s} C_{p\bar q}\partial\bar\partial(\omega^p\wedge\bar\gamma^q)=0\,. \tag{3.20}$$

The first relation in (3.17) yields that (3.18) is satisfied if and only if

$$A_{p\bar q}=0\,,\quad p\neq q\,.$$

Next we focus on (3.19). We have

$$\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=\partial\left(-\sum_{\delta=1}^{s}\lambda_{\delta p}\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q-\gamma^p\wedge\sum_{\delta=1}^{s}\bar\lambda_{\delta q}\bar\omega^\delta\wedge\bar\gamma^q\right)$$

and

$$\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=\sum_{\delta=1}^{s}(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\partial\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q-\bar\omega^\delta\wedge\partial\gamma^p\wedge\bar\gamma^q+\bar\omega^\delta\wedge\gamma^p\wedge\partial\bar\gamma^q\right)\,,$$

which implies

$$\begin{aligned}\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=&\sum_{\delta=1}^{s}\frac{\sqrt{-1}}{2}(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^\delta\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q-\sum_{\delta=1}^{s}(\bar\lambda_{\delta q}-\lambda_{\delta p})\bar\omega^\delta\wedge\left(\sum_{a=1}^{s}\lambda_{ap}\omega^a\wedge\gamma^p\right)\wedge\bar\gamma^q\\&+\sum_{\delta=1}^{s}(\bar\lambda_{\delta q}-\lambda_{\delta p})\bar\omega^\delta\wedge\gamma^p\wedge\left(-\sum_{a=1}^{s}\bar\lambda_{aq}\omega^a\wedge\bar\gamma^q\right)\\=&\sum_{\delta=1}^{s}\frac{\sqrt{-1}}{2}(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^\delta\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q+\sum_{\delta,a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q\,.\end{aligned}$$

Finally, we get

$$\begin{aligned}\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=&\sum_{\delta=1}^{s}(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q\\&+\sum_{\delta\neq a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q\end{aligned}$$

and that condition (3.19) is equivalent to

$$B_{p\bar q}\left(\sum_{\delta=1}^{s}(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\delta\neq a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta\right)=0\,,$$

for every $p,q=1,\dots,s$.

By using our conditions on the b_{ki}'s, it is easy to show that the quantity

$$\sum_{\delta=1}^{s}(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\delta\neq a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta$$

is vanishing for $p = q$ and, consequently, there are no restrictions on the $B_{q\bar{q}}$'s. Now we observe that the real part of

$$(\bar{\lambda}_{pq} - \lambda_{pp}) \left(\frac{\sqrt{-1}}{2} + \lambda_{pp} - \bar{\lambda}_{pq} \right)$$

is different from 0, for every p, q with $p \neq q$, which forces $B_{p\bar{q}} = 0$, for $p \neq q$. Indeed, we have

$$\begin{aligned}
\bar{\lambda}_{\delta q} - \lambda_{\delta p} &= \frac{1}{2}(c_{\delta p} - c_{\delta q}) - \frac{\sqrt{-1}}{4}(b_{\delta p} + b_{\delta q})\,, \\
\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar{\lambda}_{\delta q} &= -\frac{1}{2}(c_{\delta p} - c_{\delta q}) + \frac{\sqrt{-1}}{2}\left(1 + \frac{b_{\delta p} + b_{\delta q}}{2}\right)\,,
\end{aligned}$$

which implies

$$\operatorname{Re}\left((\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \left(\frac{\sqrt{-1}}{2} + \lambda_{\delta q} - \bar{\lambda}_{\delta p} \right) \right) = -\frac{(c_{\delta p} - c_{\delta q})^2}{4} + \frac{1}{4}\left(\frac{b_{\delta p} + b_{\delta q}}{2} \right)\left(1 + \frac{b_{\delta p} + b_{\delta q}}{2} \right)\,. \tag{3.21}$$

Since $p \neq q$, we have

$$b_{pp} = -1\,, \quad b_{pq} = 0\,,$$

and so (3.21) computed for $\delta = q$ gives

$$\operatorname{Re}\left((\bar{\lambda}_{pq} - \lambda_{pp}) \left(\frac{\sqrt{-1}}{2} + \lambda_{pq} - \bar{\lambda}_{pp} \right) \right) = \frac{1}{4}\left(-(c_{pp} - c_{pq})^2 - \frac{1}{4} \right) \neq 0\,,$$

as required. Therefore equation (3.19) is satisfied if and only if

$$B_{p\bar{q}} = 0\,, \quad p \neq q\,.$$

Next we focus on (3.20). We have

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \partial \left(\frac{\sqrt{-1}}{2} \omega^p \wedge \bar{\omega}^p \wedge \bar{\gamma}^q - \omega^p \wedge \left(\sum_{\delta=1}^{s} \bar{\lambda}_{\delta q} \bar{\omega}^\delta \wedge \bar{\gamma}^q \right) \right)$$

and

$$\begin{aligned}
\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = {} & \frac{\sqrt{-1}}{2} \left(-\frac{\sqrt{-1}}{2} \omega^p \wedge \omega^p \wedge \bar{\omega}^p \wedge \bar{\gamma}^q + \omega^p \wedge \bar{\omega}^p \wedge \left(-\sum_{\delta=1}^{s} \bar{\lambda}_{\delta q} \omega^\delta \wedge \bar{\gamma}^q \right) \right) \\
& + \sum_{\delta=1}^{s} \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\delta=1}^{s} \bar{\lambda}_{\delta q} \omega^p \wedge \bar{\omega}^\delta \wedge \left(\sum_{a=1}^{s} \bar{\lambda}_{aq} \omega^a \wedge \bar{\gamma}^q \right)\,.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = {} & \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q \\
& + \sum_{\substack{\delta, a \\ a\neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{aq} \omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q
\end{aligned}$$

and

$$\begin{aligned}
\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = {} & \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \sum_{\substack{a=1 \\ a\neq p}}^{s} \bar{\lambda}_{pq} \bar{\lambda}_{aq} \omega^p \wedge \bar{\omega}^p \wedge \omega^a \wedge \bar{\gamma}^q \\
& + \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta, a \\ \delta\neq p \\ a\neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{aq} \omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q\,.
\end{aligned}$$

Therefore

$$\partial\bar\partial(\omega^p\wedge\bar\gamma^q)=\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}+\bar\lambda_{pq}\right)\omega^p\wedge\bar\omega^p\wedge\omega^\delta\wedge\bar\gamma^q+\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{\delta q}\right)\omega^p\wedge\omega^\delta\wedge\bar\omega^\delta\wedge\bar\gamma^q$$
$$+\sum_{\substack{\delta\neq a\\ \delta\neq p\\ a\neq p}}\bar\lambda_{\delta q}\bar\lambda_{aq}\omega^p\wedge\bar\omega^\delta\wedge\omega^a\wedge\bar\gamma^q$$

and (3.20) is equivalent to

$$C_{p\bar q}\left(\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}+\bar\lambda_{pq}\right)\bar\omega^p\wedge\omega^\delta+\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\substack{\delta\neq a\\ \delta\neq p\\ a\neq p}}\bar\lambda_{\delta q}\bar\lambda_{aq}\bar\omega^\delta\wedge\omega^a\right)=0\,,$$

for every $p,q=1,\dots,s$. Since

$$\lambda_{pq}\neq\pm\frac{\sqrt{-1}}{2}\,,\quad p,q=1,\dots,s\,,$$

the quantity

$$E_{p\bar q}:=\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}+\bar\lambda_{pq}\right)\bar\omega^p\wedge\omega^\delta+\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\substack{\delta\neq a\\ \delta\neq p\\ a\neq p}}\bar\lambda_{\delta q}\bar\lambda_{aq}\bar\omega^\delta\wedge\omega^a$$

is vanishing if and only if

$$\lambda_{\delta q}=0\,,\quad \delta\neq p\,.$$

Since $\lambda_{qq}\neq 0$, it follows

$$E_{p\bar q}\neq 0\,,\quad p\neq q$$

and

$$E_{p\bar p}=0 \text{ if and only if } c_{\delta p}=0\,,\ \delta\neq p\,.$$

Hence the claim follows. □

Proposition 3.1.5. *Let*

$$\omega=\sqrt{-1}\sum_{i=1}^{s}A_i\omega^i\wedge\bar\omega^i+B_i\gamma^i\wedge\bar\gamma^i+\sqrt{-1}\sum_{r=1}^{k}\left(C_r\omega^{p_r}\wedge\bar\gamma^{p_r}+\bar C_r\gamma^{p_r}\wedge\bar\omega^{p_r}\right)\tag{3.22}$$

be a left-invariant SKT Hermitian metric on an Oeljeklaus-Toma manifold, where the components are with respect to a coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ *satisfying* (3.1) *and* (3.15) *and* $\{p_1,\dots,p_k\}\subseteq\{1,\dots,s\}$ *are such that*

$$\lambda_{jp_i}=0\,,\quad j\neq p_i\,,\ i=1,\dots,k\,.$$

Then, the $(1,1)$*-part of the Bismut-Ricci form of* ω *takes the following expression:*

$$(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}=-\sqrt{-1}\sum_{r=1}^{k}\frac34\left(1+\frac{|C_r|^2}{A_{p_r}B_{p_r}-|C_r|^2}\right)\omega^{p_r}\wedge\bar\omega^{p_r}-\sqrt{-1}\sum_{i\notin\{p_1,\dots,p_k\}}\frac34\omega^i\wedge\bar\omega^i$$
$$-\sqrt{-1}\sum_{r=1}^{k}\left(-\frac{3}{16}-\frac{c_{p_rp_r}^2}{4}-\frac{\sqrt{-1}c_{p_rp_r}}{4}\right)\frac{B_{p_r}C_r}{A_{p_r}B_{p_r}-|C_r|^2}\omega^{p_r}\wedge\bar\gamma^{p_r}+\quad\text{conjugates}\,.$$

Proof. We recall that the Bismut-Ricci form of a left-invariant Hermitian metric $\omega = \sqrt{-1}g_{a\bar{b}}\alpha^a \wedge \bar{\alpha}^b$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(X,Y) = -g^{a\bar{b}}\omega([[X,Y]^{1,0},X_a],\bar{X}_b) + g^{\bar{a}b}\omega([[X,Y]^{0,1},\bar{X}_a],X_b) + \sqrt{-1}g^{a\bar{b}}\omega([X,Y],J[X_a,\bar{X}_b])\,, \tag{3.23}$$

for every left-invariant vector fields X, Y on G, where $\{\alpha^i\}$ is a left-invariant $(1,0)$-coframe with dual frame $\{X_a\}$ and $(g^{\bar{b}a})$ is the inverse matrix to $(g_{i\bar{j}})$ (see e.g. [335]). We apply (3.23) to a left-invariant SKT metric on an Oeljeklaus-Toma manifold of the form (3.22).

We have

$$g^{\bar{i}s+i} = \begin{cases} 0 & \text{if } i \notin \{p_1,\dots,p_k\}\,, \\ -\frac{C_i}{A_iB_i-|C_i|^2} & \text{otherwise}\,, \end{cases} \qquad g^{\bar{i}i} = \frac{B_i}{A_iB_i-|C_i|^2}\,, \quad g^{\overline{s+i}s+i} = \frac{A_i}{A_iB_i-|C_i|^2}$$

and taking into account that the ideal $\mathfrak{I}$ is abelian, we have

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(X,Y) = -\sum_{i=1}^{4}\mathrm{Ric}^{\mathrm{B}}_i(\omega)(X,Y)\,,$$

where

$$\begin{aligned}
\mathrm{Ric}^{\mathrm{B}}_1(\omega)(X,Y) &= \sum_{a=1}^{s} g^{a\bar{a}}(\omega([[X,Y]^{1,0},Z_a],\bar{Z}_a) - \frac{\sqrt{-1}}{2}\omega([X,Y],Z_a-\bar{Z}_a) + \omega([[X,Y]^{0,1},\bar{Z}_a],Z_a))\,,\\
\mathrm{Ric}^{\mathrm{B}}_2(\omega)(X,Y) &= \sum_{a=1}^{s} g^{s+a\overline{s+a}}(\omega([[X,Y]^{1,0},W_a],\bar{W}_a) + \omega([[X,Y]^{0,1},\bar{W}_a],W_a))\,,\\
\mathrm{Ric}^{\mathrm{B}}_3(\omega)(X,Y) &= \sum_{r=1}^{k} g^{p_r\overline{s+p_r}}\left(\omega([[X,Y]^{1,0},Z_{p_r}],\bar{W}_{p_r}) - \omega([X,Y],[Z_{p_r},\bar{W}_{p_r}])\right)\\
&\quad + \sum_{r=1}^{k} g^{\overline{p_r}s+p_r}\omega([[X,Y]^{0,1},\bar{Z}_{p_r}],W_{p_r})\,,\\
\mathrm{Ric}^{\mathrm{B}}_4(\omega)(X,Y) &= \sum_{r=1}^{k} g^{s+p_r\bar{p}_r}\left(\omega([[X,Y]^{1,0},W_{p_r}],\bar{Z}_{p_r}) + \omega([X,Y],[W_{p_r},\bar{Z}_{p_r}])\right)\\
&\quad + \sum_{r=1}^{k} g^{\overline{s+p_r}p_r}\omega([[X,Y]^{0,1},\bar{W}_{p_r}],Z_{p_r})\,.
\end{aligned}$$

Next we focus on the computation of $\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_i,\bar{Z}_j)$. Thanks to (3.1), we easily obtain that

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_i,\bar{Z}_j) = 0\,,\quad i,j=1,\dots,s\,,\ i\neq j\,.$$

On the other hand,

$$\mathrm{Ric}^{\mathrm{B}}_1(\omega)(Z_i,\bar{Z}_i) = -\frac{\sqrt{-1}}{2}\sum_{a=1}^{s} g^{a\bar{a}}\left(-\frac{\sqrt{-1}}{2}\omega(Z_i+\bar{Z}_i,Z_a-\bar{Z}_a)\right) = \frac{\sqrt{-1}}{2}g^{i\bar{i}}A_i = \frac{\sqrt{-1}}{2}\left(\frac{A_iB_i}{A_iB_i-|C_i|^2}\right).$$

Moreover, we have

$$\begin{aligned}
\mathrm{Ric}^{\mathrm{B}}_2(\omega)(Z_i,\bar{Z}_i) &= -\frac{\sqrt{-1}}{2}\sum_{a=1}^{s} g^{s+a\overline{s+a}}(\omega([Z_i,W_a],\bar{W}_a) + \omega([\bar{Z}_i,\bar{W}_a],W_a)\\
&= -\sqrt{-1}\sum_{a=1}^{s} g^{s+a\overline{s+a}}\mathrm{Re}\,\omega([Z_i,W_a],\bar{W}_a)\,.
\end{aligned}$$

Using (3.1), we have

$$\omega([Z_i, W_a], \bar{W}_a) = -\sqrt{-1}\lambda_{ia} B_a\,,$$
$$\mathrm{Re}\,\omega([Z_i, W_a], \bar{W}_a) = \frac{B_a b_{ia}}{4} = -\frac{B_a}{4}\delta_{ia}\,.$$

Then,

$$\mathrm{Ric}_3^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = \sqrt{-1}\frac{g^{s+i\overline{s+i}}B_i}{4} = \frac{\sqrt{-1}}{4}\frac{A_i B_i}{A_i B_i - |C_i|^2}\,.$$

Next we observe that

$$\mathrm{Ric}_3^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) + \mathrm{Ric}_4^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = 0$$

which implies

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = \begin{cases} -\sqrt{-1}\frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2}\right) & \text{if there exists } r = 1, \dots, k \text{ such that } i = p_r\,, \\ -\sqrt{-1}\frac{3}{4} & \text{if } i \notin \{p_1, \dots, p_k\}\,. \end{cases} \tag{3.24}$$

We have

$$\begin{aligned}\mathrm{Ric}_3^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) &= \sum_{j=1}^{k} g^{p_j\overline{s+p_j}}\omega([Z_i, \bar{Z}_i], [Z_{p_j}, \bar{W}_{p_j}]) = -\frac{\sqrt{-1}}{2}\sum_{j=1}^{k} g^{p_j\overline{s+p_j}}\bar{\lambda}_{p_j p_j}\omega(Z_i + \bar{Z}_i, \bar{W}_{p_j}) \\ &= \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\}\,, \\ \frac{1}{2}g^{i\overline{s+i}}\bar{\lambda}_{ii}C_i & \text{otherwise}\,. \end{cases}\end{aligned}$$

We compute the three addends in the expression of $\mathrm{Ric}_4^{\mathrm{B}}(\omega)$ separately:

$$\omega([[Z_i, \bar{Z}_i]^{1,0}, W_{p_j}], \bar{Z}_{p_j}) = -\frac{1}{2}\lambda_{ip_j}\bar{C}_{p_j} = \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\} \quad \text{or } i \neq p_j\,, \\ -\frac{1}{2}\lambda_{ii}\bar{C}_i & \text{otherwise}\,; \end{cases}$$
$$\omega([Z_i, \bar{Z}_i], [W_{p_j}, \bar{Z}_{p_j}]) = \frac{1}{2}\lambda_{p_j p_j} g_{\bar{i}s+p_j} = \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\} \quad \text{or} \quad i \neq p_j\,, \\ \frac{1}{2}\lambda_{ii}\bar{C}_i & \text{otherwise}\,; \end{cases}$$
$$\omega([[Z_i, \bar{Z}_i]^{0,1}, \bar{W}_{p_j}], Z_{p_j}) = \frac{1}{2}\bar{\lambda}_{ip_j} g_{\overline{s+p_j}p_j} = \begin{cases} 0 & \text{if } i \neq p_j\,, \\ \frac{1}{2}\bar{\lambda}_{ii}C_i & \text{otherwise}\,. \end{cases}$$

It follows

$$\mathrm{Ric}_3^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = \mathrm{Ric}_4^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = 0\,, \quad i \notin \{p_1, \dots, p_k\}\,,$$

and, for $i \in \{p_1, \dots, p_k\}$,

$$\mathrm{Ric}_3^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) + \mathrm{Ric}_4^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = \frac{1}{2}\left(-g^{i\overline{s+i}}\bar{\lambda}_{ii}C_i - g^{s+i\bar{i}}\lambda_{ii}\bar{C}_i + g^{s+i\bar{i}}\lambda_{ii}\bar{C}_i + g^{\overline{s+i}i}\bar{\lambda}_{ii}C_i\right) = 0\,.$$

Now, we focus on the calculation of $\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_i, \bar{W}_j)$. We have

$$\begin{aligned}\mathrm{Ric}_1^{\mathrm{B}}(\omega)(Z_i, \bar{W}_j) &= \sum_{a=1}^{s} g^{a\bar{a}}\bar{\lambda}_{ij}\left(-\frac{\sqrt{-1}}{2}\omega(\bar{W}_j, Z_a - \bar{Z}_a) + \omega([\bar{W}_j, \bar{Z}_a], Z_a)\right) \\ &= \begin{cases} \sqrt{-1}g^{i\bar{i}}C_i\bar{\lambda}_{ii}\left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{ii}\right) & \text{if } i = j \in \{p_1, \dots, p_k\}\,, \\ 0 & \text{otherwise}\,, \end{cases}\end{aligned}$$

and, since $\mathfrak{I}$ is abelian,

$$\mathrm{Ric}_2^{\mathrm{B}}(\omega)(Z_i, \bar{W}_j) = 0\,.$$

Furthermore

$$\mathrm{Ric}_3^{\mathrm{B}}(\omega)(Z_i,\bar W_j)=\sum_{j=1}^k g^{\overline{p_j}s+p_j}\omega([[Z_i,\bar W_j]^{0,1},\bar Z_{p_j}],W_{p_j})=-\sqrt{-1}\sum_{j=1}^k g^{\overline{p_j}s+p_j}\bar\lambda_{ij}\bar\lambda_{p_jp_j}g_{\overline{s+j}s+p_j}$$
$$=\begin{cases}-\sqrt{-1}\bar\lambda_{jj}^2 g^{\bar j s+j}B_j & \text{if } i=j\in\{p_1,\dots,p_k\}\,,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathrm{Ric}_4^{\mathrm{B}}(\omega)(Z_i,\bar W_j)=\sum_{j=1}^k g^{s+p_j\bar p_j}\omega([Z_i,\bar W_j],[W_{p_j},\bar Z_{p_j}])=\sqrt{-1}\sum_{j=1}^k g^{s+p_j\bar p_j}\bar\lambda_{ij}\lambda_{p_jp_j}g_{\overline{s+j}s+p_j}$$
$$=\begin{cases}\sqrt{-1}g^{s+j\bar j}\bar\lambda_{jj}\lambda_{jj}B_j & \text{if } i=j\in\{p_1,\dots,p_k\}\\ 0 & \text{otherwise}\,.\end{cases}$$

It follows that $\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_i,\bar W_j)\neq 0$ if and only if $i=j\in\{p_1,\dots,p_k\}$. In such a case, we have

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_j,\bar W_j)=-\sqrt{-1}\left(g^{s+j\bar j}B_j\left(|\lambda_{jj}|^2-\bar\lambda_{jj}^2\right)+g^{j\bar j}C_j\bar\lambda_{jj}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{jj}\right)\right)\,.$$

Since

$$g^{s+j\bar j}B_j=-\frac{B_jC_j}{A_jB_j-|C_j|^2}\quad\text{and}\quad g^{j\bar j}C_j=\frac{B_jC_j}{A_jB_j-|C_j|^2}\,,$$

we infer

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_j,\bar W_j)=-\sqrt{-1}\left(\bar\lambda_{jj}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{jj}\right)-\left(|\lambda_{jj}|^2-\bar\lambda_{jj}^2\right)\right)\frac{B_jC_j}{A_jB_j-|C_j|^2}\,.$$

Taking into account that $\lambda_{jj}=-\frac{\sqrt{-1}}{4}-\frac{c_{jj}}{2}$, we obtain

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_j,\bar W_j)=-\sqrt{-1}\left(-\frac{3}{16}-\frac{c_{jj}^2}{4}-\frac{\sqrt{-1}c_{jj}}{4}\right)\frac{B_jC_j}{A_jB_j-|C_j|^2}$$

and the claim follows. □

The next result gives the classification of algebraic solitons for the pluriclosed flow.

Corollary 3.1.6. *Let ω be a left-invariant SKT Hermitian metric on an Oeljeklaus-Toma manifold M. Then ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if it takes the following diagonal expression with respect to a coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ satisfying* (3.1) *and* (3.15)*:*

$$\omega=\sqrt{-1}\sum_{i=1}^s A\omega^i\wedge\bar\omega^i+B_i\gamma^i\wedge\bar\gamma^i\,. \tag{3.25}$$

Proof. Let ω be a SKT left-invariant metric on an Oeljeklaus-Toma manifold M. In view of [223, Section 7], ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if

$$(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}(\cdot,\cdot)=c\omega(\cdot,\cdot)+\frac{1}{2}\left(\omega(D\cdot,\cdot)+\omega(\cdot,D\cdot)\right)\,,$$

for some $c\in\mathbb{R}_-$ and $D\in\mathrm{Der}(\mathfrak{g})\cap\mathfrak{gl}(\mathfrak{g},J)$.

Assume that ω takes the expression in formula (3.25). Proposition 3.1.5 implies that $\mathrm{Ric}^{\mathrm{B}}(\omega)$ is represented with respect to the basis $\{Z_1,\dots,Z_s,W_1,\dots,W_s\}$ by the matrix

$$P=-\frac{3}{4A}\begin{pmatrix}\mathrm{I}_{\mathfrak{h}} & 0\\ 0 & 0\end{pmatrix}\,.$$

Since

$$\frac{3}{4A}\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\mathfrak{J}} \end{pmatrix}$$

induces a symmetric derivation on $\mathfrak{g}$, ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M and the first part of the claim follows.

In order to prove the second part of the statement, we need some preliminary observations on derivations D of $\mathfrak{g}$ that commute with J, i.e. such that

$$D(\mathfrak{g}^{1,0}) \subseteq \mathfrak{g}^{1,0}\,, \quad D(\mathfrak{g}^{0,1}) \subseteq \mathfrak{g}^{0,1}\,.$$

We can write

$$DZ_i = \sum_{j=1}^{s} k^i_j Z_j + m^i_j W_j \quad \text{and} \quad D\bar{Z}_i = \sum_{j=1}^{s} l^i_j \bar{Z}_j + r^i_j \bar{W}_j\,.$$

Since D is a derivation, we have, for all $i = 1, \dots, s$,

$$D[Z_i, \bar{Z}_i] = [DZ_i, \bar{Z}_i] + [Z_i, D\bar{Z}_i]\,. \tag{3.26}$$

On the other hand

$$\begin{aligned} D[Z_i, \bar{Z}_i] &= -\frac{\sqrt{-1}}{2}\left(\sum_{j=1}^{s} k^i_j Z_j + l^i_j \bar{Z}_j + m^i_j W_j + r^i_j \bar{W}_j\right)\,, \\ [DZ_i, \bar{Z}_i] &= -\frac{\sqrt{-1}}{2} k^i_i (Z_i + \bar{Z}_i) - \sum_{j=1}^{s} m^i_j \lambda_{ij} W_j\,, \\ [Z_i, D\bar{Z}_i] &= -\frac{\sqrt{-1}}{2} l^i_j (Z_i + \bar{Z}_i) + \sum_{j=1}^{s} r^i_j \bar{\lambda}_{ij} \bar{W}_j\,. \end{aligned}$$

Then, (3.26) rewrites as:

$$0 = -\frac{\sqrt{-1}}{2}\sum_{j\neq i} k^i_j Z_j + l^i_j \bar{Z}_j + \frac{\sqrt{-1}}{2} l^i_i Z_i + \frac{\sqrt{-1}}{2} k^i_i \bar{Z}_i + \sum_{j=1}^{s} m^i_j \left(\lambda_{ij} - \frac{\sqrt{-1}}{2}\right) W_j - r^i_j \left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{ij}\right) \bar{W}_j\,,$$

which forces $DZ_i, D\bar{Z}_i = 0$, for all $i = 1, \dots, s$. It follows that $D|_{\mathfrak{h}} = 0$. Moreover, for all $I, I' \in \mathfrak{J}$, we have

$$0 = D[I, I'] = [DI, I'] + [I, DI']\,,$$

which implies

$$[DI, I'] = -[I, DI']\,.$$

Assume

$$DW_i = \sum_{j=1}^{s} k^{s+i}_j Z_j + m^{s+i}_j W_j \quad \text{and} \quad D\bar{W}_i = \sum_{j=1}^{s} l^{s+i}_j \bar{Z}_j + r^{s+i}_j \bar{W}_j\,,$$

then

$$[DW_i, \bar{W}_i] = \sum_{j=1}^{s} k^{s+i}_j [Z_j, \bar{W}_i] \in \mathfrak{J}^{0,1} \quad \text{and} \quad [W_i, D\bar{W}_i] = \sum_{j=1}^{s} l^{s+i}_j [W_i, \bar{Z}_j] \in \mathfrak{J}^{1,0}\,.$$

This implies

$$DW_i = \sum_{j=1}^{s} m^{s+i}_j W_j\,, \quad D\bar{W}_i = \sum_{j=1}^{s} r^{s+i}_j \bar{W}_j\,,$$

i.e. $D(\mathfrak{J}) \subseteq \mathfrak{J}$. Moreover, for all $i = 1, \ldots, s$, we have that

$$D[Z_i, W_i] = -\lambda_{ii} DW_i = -\sum_{j=1}^{s} \lambda_{ii} m_j^{s+i} W_j \,,$$

while $[DZ_i, W_i] = 0$ and

$$[Z_i, DW_i] = -\sum_{j=1}^{s} m_j^{s+i} \lambda_{ij} W_j \,.$$

Using again the fact that D is a derivation, we have

$$DW_i = \sum_{j \in J_i} m_j W_j$$

where

$$J_i = \{j \in \{1, \ldots, s\} \quad | \quad \lambda_{ii} = \lambda_{ij}\} \,.$$

With analogous computations, we infer

$$D\bar{W}_i = \sum_{j \in J_i} r_j^{s+i} \bar{W}_j \,.$$

Clearly, $i \in J_i$. On the other hand, for all $i = 1, \ldots, s$, we know that $\mathrm{Im}(\lambda_{ii}) \neq 0$, while, for all $i \neq j$, $\lambda_{ij} \in \mathbb{R}$. This guarantees that, for all $i = 1, \ldots, s$,

$$J_i = \{i\} \,.$$

This allows us to write

$$DW_i = m_i^{s+i} W_i \,, \quad D\bar{W}_i = r_i^{s+i} \bar{W}_i \,.$$

From the relations above, we obtain that

$$\mathrm{Der}(\mathfrak{g})^{1,0} = \{E \in \mathrm{End}(\mathfrak{g})^{1,0} \quad | \quad \mathfrak{h} \subseteq \ker(E) \,,\ E(\langle W_i \rangle) \subseteq \langle W_i \rangle \,, \quad i = 1, \ldots, s\} \,.$$

First of all, we suppose that ω is a SKT Hermitian metric which takes the following diagonal expression with respect to a coframe $\{\omega^1, \ldots, \omega^s, \gamma^1, \ldots, \gamma^s\}$ satisfying (3.1) and (3.15):

$$\omega = \sqrt{-1} \sum_{i=1}^{s} A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i \,.$$

such that there exist $i, j \in \{1, \ldots, s\}$ such that $A_i \neq A_j$ and we suppose that ω is an algebraic soliton. Thanks to the facts regarding derivations proved before, we have that

$$-\sqrt{-1}\frac{3}{4} = \mathrm{Ric}^{\mathrm{B}}(\omega)(Z_i, \bar{Z}_i) = c\omega(Z_i, \bar{Z}_i) + \frac{1}{2}\left(\omega(DZ_i, \bar{Z}_i) + \omega(Z_i, D\bar{Z}_i)\right) = \sqrt{-1} c A_i \,,$$
$$-\sqrt{-1}\frac{3}{4} = \mathrm{Ric}^{\mathrm{B}}(\omega)(Z_j, \bar{Z}_j) = c\omega(Z_j, \bar{Z}_j) + \frac{1}{2}\left(\omega(DZ_j, \bar{Z}_j) + \omega(Z_j, D\bar{Z}_j)\right) = \sqrt{-1} c A_j \,,$$

which is impossible, since $A_i \neq A_j$.

Now suppose that ω is a SKT metric on M which is not diagonal. So, we suppose that there exists $\tilde{j} = 1, \ldots, s$ such that $C_{\tilde{j}} \neq 0$. Then, assume that there exist a constant $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g}) \cap \mathfrak{gl}(\mathfrak{g}, J)$ such that

$$(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}(\cdot, \cdot) = c\omega(\cdot, \cdot) + \frac{1}{2}\left(\omega(D\cdot, \cdot) + \omega(\cdot, D\cdot)\right) \,.$$

On the other hand

$$0 = \mathrm{Ric}^{\mathrm{B}}(\omega)(W_{\tilde{j}}, \bar{W}_{\tilde{j}}) = c\omega(W_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \frac{1}{2}\left(\omega(DW_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \omega(W_{\tilde{j}}, D\bar{W}_{\tilde{j}})\right) = \sqrt{-1}cB_{\tilde{j}} + \frac{\sqrt{-1}}{2}(r_{\tilde{j}}^{s+\tilde{j}} + m_{\tilde{j}}^{s+\tilde{j}})B_{\tilde{j}}\,,$$

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_{\tilde{j}}, \bar{W}_{\tilde{j}}) = c\omega(Z_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \frac{1}{2}\left(\omega(DZ_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \omega(Z_{\tilde{j}}, D\bar{W}_{\tilde{j}})\right) = \sqrt{-1}cC_{\tilde{j}} + \frac{\sqrt{-1}}{2}r_{\tilde{j}}^{s+\tilde{j}}C_{\tilde{j}}\,,$$

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(\bar{Z}_{\tilde{j}}, W_{\tilde{j}}) = c\omega(\bar{Z}_{\tilde{j}}, W_{\tilde{j}}) + \frac{1}{2}\left(\omega(D\bar{Z}_{\tilde{j}}, W_{\tilde{j}}) + \omega(\bar{Z}_{\tilde{j}}, DW_{\tilde{j}})\right) = -\sqrt{-1}c\bar{C}_{\tilde{j}} - \frac{\sqrt{-1}}{2}m_{\tilde{j}}^{s+\tilde{j}}\bar{C}_{\tilde{j}}\,,$$

which implies that

$$c = -\frac{1}{2}(r_{\tilde{j}}^{s+\tilde{j}} + m_{\tilde{j}}^{s+\tilde{j}})\,.$$

On the other hand,

$$\mathrm{Ric}^{\mathrm{B}}(\omega)(Z_{\tilde{j}}, \bar{W}_{\tilde{j}}) = \sqrt{-1}KC_{\tilde{j}}\,,$$

where

$$K = \left(\frac{3}{16} + \frac{c_{\tilde{j}\tilde{j}}^2}{4} + \frac{\sqrt{-1}c_{\tilde{j}\tilde{j}}}{4}\right)\frac{B_{\tilde{j}}}{A_{\tilde{j}}B_{\tilde{j}} - |C_{\tilde{j}}|^2}\,.$$

Then,

$$K = c + \frac{1}{2}r_{\tilde{j}}^{s+\tilde{j}} = -\frac{1}{2}m_{\tilde{j}}^{s+\tilde{j}}$$

and

$$\bar{K} = c + \frac{1}{2}m_{\tilde{j}}^{s+\tilde{j}} = -\frac{1}{2}r_{\tilde{j}}^{s+\tilde{j}}\,.$$

From this, we obtain that

$$c = K + \bar{K} = 2\mathrm{Re}(K) > 0\,.$$

On the other hand, we have

$$-\sqrt{-1}\frac{3}{4}\left(1 + \frac{|C_{\tilde{j}}|^2}{A_{\tilde{j}}B_{\tilde{j}} - |C_{\tilde{j}}|^2}\right) = \mathrm{Ric}^{\mathrm{B}}(\omega)(Z_{\tilde{j}}, \bar{Z}_{\tilde{j}}) = c\omega(Z_{\tilde{j}}, \bar{Z}_{\tilde{j}}) + \frac{1}{2}\left(\omega(DZ_{\tilde{j}}, \bar{Z}_{\tilde{j}}) + \omega(Z_{\tilde{j}}, D\bar{Z}_{\tilde{j}})\right) = \sqrt{-1}cA_{\tilde{j}}\,,$$

which implies that c must be negative. From this the claim follows. □

Using Proposition 3.1.5, we deduce the long-time existence of the pluriclosed flow starting from any left-invariant SKT metric.

Corollary 3.1.7. *Let ω be a SKT Hermitian metric on an Oeljeklaus-Toma manifold which takes the form* (3.16)*. Then the pluriclosed flow starting from ω is equivalent to the following system of ODEs:*

$$\begin{cases} A_i' = \frac{3}{4} & \text{if } i \notin \{p_1, \ldots, p_k\}\,, \\ A_{p_r}' = \frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2}\right) & \text{for all } r = 1, \ldots, k\,, \\ B_j' = 0 & \text{for all } j = 1, \ldots, s\,, \\ C_r' = -\left(\frac{3}{16} + \frac{c_{p_rp_r}^2}{4} + \frac{\sqrt{-1}c_{p_rp_r}}{4}\right)\frac{B_{p_r}C_r}{A_{p_r}B_{p_r} - |C_r|^2} & \text{for all } r = 1, \ldots, k\,. \end{cases} \tag{3.27}$$

Moreover, $|C_r|$ is bounded, for all $r = 1, \ldots, k$, the solution exists for all $t \in [0, +\infty)$ and $A_i \sim \frac{3}{4}t$, as $t \to +\infty$, for all $i = 1, \ldots, s$.

In particular,

$$\frac{\omega_t}{1+t} \to 3\omega_\infty$$

as $t \to \infty$.

Proof. Observe that, for every $r \in \{1, \dots, k\}$,

$$(|C_r|^2)' = -\left(\frac{3}{8} + \frac{c^2_{p_r p_r}}{2}\right) \frac{B_{p_r}|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \leq 0\,,$$

which guarantees that $|C_r|^2$ is bounded. On the other hand, denote, for all $r = 1, \dots, k$,

$$u_r = A_{p_r}B_{p_r} - |C_r|^2.$$

We have that

$$u_r' = A'_{p_r}B_{p_r} - (|C_r|^2)' = \frac{3}{4}B_{p_r} + \left(\frac{9}{8} + \frac{c^2_{p_r p_r}}{2}\right) \frac{B_{p_r}|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \geq 0\,.$$

This guarantees

$$A'_{p_r} = \frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2}\right) \leq \frac{3}{4}\left(1 + \frac{K}{u_r(0)}\right)\,,$$

where $K > 0$ such that $|C_r|^2 \leq K$, for all $t \geq 0$. This implies the long-time existence. As regards the last part of the statement, it is sufficient to prove that

$$\lim_{t \to +\infty} \frac{|C_r|^2}{u_r} = 0\,.$$

But,

$$u_r' \geq \frac{3}{4}B_{p_r}\,.$$

So,

$$u_r \geq \frac{3}{4}B_{p_r}t + u_r(0) \to +\infty\,, \quad t \to +\infty\,.$$

Then,

$$\lim_{t \to +\infty} u_r(t) = +\infty\,,$$

and, since $|C_r|^2$ is bounded, the assertion follows. □

We are now ready to state and prove the main theorem of this section.

Theorem 3.1.8. *Let ω be a left-invariant SKT Hermitian metric on an Oeljeklaus-Toma manifold M. Then, the pluriclosed flow starting from ω has a long-time solution ω_t such that $(M, \frac{\omega_t}{1+t})$ converges to (T^s, d) in the Gromov-Hausdorff sense. Moreover, ω lifts to an expanding algebraic soliton on the universal covering of M if and only if it is of the form (3.25). Finally, $(\mathbf{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges to $(\mathbf{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$ in the Cheeger-Gromov sense, where $\tilde{\omega}_\infty$ is an algebraic soliton.*

Proof. Let ω be a left-invariant SKT metric on an Oeljeklaus-Toma manifold. Corollary 3.1.7 implies that pluriclosed flow starting from ω has a long-time solution ω_t such that

$$\frac{\omega_t}{1+t} \to 3\omega_\infty \quad \text{as} \quad t \to \infty\,.$$

We show that $\frac{\omega_t}{1+t}$ satisfies Items 1 to 3 in Proposition 3.1.1. Here we denote by $|\cdot|_t$ the norm induced by ω_t. First of all, taking into account that

$$\omega_t|_{\mathfrak{I}\oplus\mathfrak{I}} = \omega_0|_{\mathfrak{I}\oplus\mathfrak{I}}\,,$$

Item 2 follows. Now, thanks to the fact that Item 2 holds,

$$\omega_t|_{\mathfrak{h}\oplus\mathfrak{h}} = \sum_{i=1}^{s} A_i(t)\omega^i \wedge \bar{\omega}^i$$

with $\frac{A_i(t)}{1+t} \to \frac{3}{4}$ as $t \to \infty$ and there exist $C, T > 0$ such that, for every vector $v \in \mathfrak{h}$,

$$\frac{1}{\sqrt{1+t}}|v|_t \leq C|v|_0 \,,$$

for every $t \geq T$,giving that Item 1 is satisfied.

In order to prove Item 3, let $\varepsilon, \ell > 0$ and let γ be a curve in M tangent to $\mathcal{H}$ which is parametrized by arclength with respect to $3\omega_\infty$ and such that $L_\infty(\gamma) < \ell$. Let $v = \dot\gamma$ and $T > 0$ such that

$$\left|\frac{A_i(t)}{1+t} - \frac{3}{4}\right| \leq \frac{3\varepsilon^2}{4\ell^2} \,,$$

for $t \geq T$. Then,

$$\left|\frac{1}{1+t}|v|_t^2 - |v|_\infty^2\right| \leq \sum_{i=1}^{s}\left|\frac{A_i(t)}{1+t} - \frac{3}{4}\right||v_i|^2 \leq \frac{\varepsilon^2}{\ell^2}$$

and

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left|\frac{1}{\sqrt{1+t}}|\dot\gamma|_t - |\dot\gamma|_\infty\right| da \leq \frac{\varepsilon}{\ell} b \leq \varepsilon \,,$$

since $b \leq \ell$.

Now we show the last part of the statement, using the same argument as in Proposition 3.1.2, and we prove that $(\mathbf{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbf{H}^s \times \mathbb{C}^s, \tilde\omega_\infty)$ where $\tilde\omega_\infty$ is an algebraic soliton. Again, here we are identifying ω_t with its pull-back onto $\mathbf{H}^s \times \mathbb{C}^s$ and we are fixing as base point the identity element of $\mathbf{H}^s \times \mathbb{C}^s$. It is enough to construct a 1-parameter family of biholomorphisms $\{\varphi_t\}$ of $\mathbf{H}^s \times \mathbb{C}^s$ such that

$$\varphi_t^* \frac{\omega_t}{1+t} \to \tilde\omega_\infty \,.$$

As we already observed, since $\mathfrak{J}$ is abelian the endomorphism represented by the matrix

$$D = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathfrak{J}} \end{pmatrix}$$

is a derivation of $\mathfrak{g}$ that commutes with the complex structure J. Then, we can consider

$$d\varphi_t = \exp(s(t)D) = \begin{pmatrix} I_{\mathfrak{h}} & 0 \\ 0 & e^{s(t)} I_{\mathfrak{J}} \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}, J)$$

where $s(t) = \log(\sqrt{1+t})$. Using $d\varphi_t$, we can define

$$\varphi_t \in \mathrm{Aut}(\mathbf{H}^s \times \mathbb{C}^s, J) \,.$$

For $i = 1, \ldots, s$ we have

$$\begin{aligned}
\frac{1}{1+t}(\varphi_t^*\omega_t)(Z_i, \bar Z_i) &= \frac{1}{1+t}\omega_t(Z_i, \bar Z_i) \to \frac{3}{4}\sqrt{-1}\,, \quad \text{as } t \to \infty\,, \\
\frac{1}{1+t}(\varphi_t^*\omega_t)(Z_i, \bar W_i) &= \frac{1}{\sqrt{1+t}}\omega_t(Z_i, \bar W_i) \to 0\,, \quad \text{as } t \to \infty\,, \\
\frac{1}{1+t}(\varphi_t^*\omega_t)(W_i, \bar W_i) &= \omega_t(W_i, \bar W_i) = \sqrt{-1}B_i(0)\,.
\end{aligned}$$

Then,

$$\frac{1}{1+t}\varphi_t^*\omega_t \to \tilde\omega_\infty\,, \quad \text{as } t \to \infty\,,$$

where

$$\tilde\omega_\infty = 3\,\omega_\infty + \omega|_{\mathfrak{J}\oplus\mathfrak{J}}\,.$$

In view of Proposition 3.1.4, $\tilde\omega_\infty$ is an algebraic soliton. □

3.1.4 A generalization to semidirect product of Lie algebras

From the viewpoint of Lie groups, the algebraic structure of Oeljeklaus-Toma manifolds is quite rigid and some of the results in the previous sections can be generalized to semidirect product of Lie algebras.

In this subsection, we consider a Lie algebra $\mathfrak{g}$ which is a semidirect product of Lie algebras

$$\mathfrak{g} = \mathfrak{h} \ltimes_\lambda \mathfrak{I}\,,$$

where $\lambda\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{I})$ is a representation. We further assume that $\mathfrak{g}$ has a complex structure of the form

$$J = J_{\mathfrak{h}} \oplus J_{\mathfrak{I}}$$

where $J_{\mathfrak{h}}$ and $J_{\mathfrak{I}}$ are complex structures on $\mathfrak{h}$ and $\mathfrak{I}$, respectively.

The following assumptions are all satisfied in the case of an Oeljeklaus-Toma manifold:

i. $\mathfrak{h}$ has $(1,0)$-frame such that $\{Z_1,\dots,Z_r\}$ such that $[Z_k,\bar Z_k] = -\frac{\sqrt{-1}}{2}(Z_k+\bar Z_k)$, for all $k=1,\dots,r$ and the other brackets vanish;

ii. $\mathfrak{I}$ is a $2s$-dimensional abelian Lie algebra and $J_{\mathfrak{I}}$ is a complex structure on $\mathfrak{I}$;

iii. $\lambda(\mathfrak{h}^{1,0}) \subseteq \mathrm{End}(\mathfrak{I})^{1,0}$;

iv. $\mathfrak{I}$ has a $(1,0)$-frame $\{W_1,\dots W_s\}$ such that $\lambda(Z)\cdot \bar W_r = \lambda_r(Z)\bar W_r$, for every $r=1,\dots,s$, where $\lambda_r \in \Lambda^{1,0}(\mathfrak{h})$;

v. $\sum_{a=1}^s \mathrm{Im}(\lambda_a(Z_i))$ is constant on i.

vi. $\mathfrak{I}$ has a $(1,0)$-frame $\{W_1,\dots W_s\}$ such that $\lambda(Z)\cdot W_r = \lambda'_r(Z)W_r$, for every $r=1,\dots,s$, where $\lambda'_r \in \Lambda^{1,0}(\mathfrak{h})$ and $\sum_{a=1}^s \mathrm{Im}(\lambda'_a(Z_i))$ is constant on i.

Note that condition i. is equivalent to require that $\mathfrak{h} = \underbrace{\mathfrak{f}\oplus\cdots\oplus\mathfrak{f}}_{r\text{-times}}$ equipped with the complex structure $J_{\mathfrak{h}} = \underbrace{J_{\mathfrak{f}}\oplus\cdots\oplus J_{\mathfrak{f}}}_{r\text{-times}}$, while in condition iv. the existence of $\{W_r\}$ and λ_r is equivalent to require that

$$\lambda(Z)\circ\lambda(Z') = \lambda(Z')\circ\lambda(Z)\,,$$

for every $Z, Z' \in \mathfrak{h}^{1,0}$.

The computations in Section 3.1.3 can be used to study solutions to the flow

$$\frac{\partial}{\partial t}\omega = -(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1} \tag{3.28}$$

in semidirect products of Lie algebras (this flow coincides to the pluriclosed flow only when the initial metric is pluriclosed). We have the following

Proposition 3.1.9. *Let $\mathfrak{g} = \mathfrak{h} \ltimes_\lambda \mathfrak{I}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J = J_{\mathfrak{h}} \oplus J_{\mathfrak{I}}$ and let ω be a Hermitian metric on $\mathfrak{g}$ making $\mathfrak{h}$ and $\mathfrak{I}$ orthogonal. Then the Bismut Ricci-form of ω satisfies $(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}|_{\mathfrak{h}\oplus\mathfrak{I}} = (\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}|_{\mathfrak{I}\oplus\mathfrak{I}} = 0$.*

If $\mathrm{i}-\mathrm{iv}$ hold and $\omega|_{\mathfrak{h}\oplus\mathfrak{h}}$ is diagonal with respect to the frame $\{Z_i\}$ then the $(1,1)$-component of the Bismut-Ricci form of ω does not depend on ω and the solution to the flow (3.28) starting from ω takes the following expression:

$$\omega_t = \omega - t(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}\,.$$

If $\mathrm{i}-\mathrm{iv}$ and vi hold and $\omega|_{\mathfrak{h}\oplus\mathfrak{h}}$ is a multiple of the canonical metric with respect to the frame $\{Z_i\}$, then ω is a soliton for flow (3.28) with cosmological constant $c = \frac{1}{2} + \sum_{a=1}^s \mathrm{Im}(\lambda'_a(Z_i))$.

The previous proposition does not cover the case when properties i-iv are satisfied and the restriction to $\mathfrak{h}\oplus\mathfrak{h}$ of the initial Hermitian inner product

$$\omega=\sqrt{-1}\sum_{a,b=1}^{r} g_{a\bar b}\,\omega^a\wedge\bar\omega^b+\sqrt{-1}\sum_{a,b=1}^{s} g_{r+a\overline{r+b}}\,\gamma^a\wedge\bar\gamma^b$$

is not diagonal with respect to $\{Z_i\}$. In this case flow (3.28) evolves only the components $g_{i\bar i}$ of ω along $\omega^i\wedge\bar\omega^i$ via the ODE

$$\frac{d}{dt}g_{i\bar i}=\frac{1}{4}\sum_{a=1}^{r} g^{\bar a a}\mathrm{Re}\, g_{i\bar a}-\frac{1}{2}\sum_{c,d=1}^{s} g^{\overline{r+d}r+c}\left\{\omega([Z_i,W_c],\bar W_d)+\omega([\bar Z_i,\bar W_c],W_d)\right\}\,,$$

where $g_{i\bar i}$ depends on t. Note that the quantities $-\frac{1}{2}\sum_{c,d=1}^{s} g^{\overline{r+d}r+c}\left\{\omega([Z_i,W_c],\bar W_d)+\omega([\bar Z_i,\bar W_c],W_d)\right\}$ appearing in the evolution of $g_{i\bar i}$ are independent on t.

The same computations as in Section 3.1.2 imply the following

Proposition 3.1.10. *Let $\mathfrak{g}=\mathfrak{h}\ltimes_\lambda\mathfrak{I}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J=J_{\mathfrak{h}}\oplus J_{\mathfrak{I}}$. Assume that properties* i, ii, iii *are satisfied and let ω be a left-invariant Hermitian metric on $\mathfrak{g}$. Then*

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)|_{\mathfrak{I}\oplus\mathfrak{I}}=\mathrm{Ric}^{\mathrm{Ch}}(\omega)|_{\mathfrak{h}\oplus\mathfrak{I}}=0\,,$$

while $\mathrm{Ric}^{\mathrm{Ch}}(\omega)|_{\mathfrak{h}\oplus\mathfrak{h}}$ is diagonal with respect to $\{Z_1,\dots,Z_r\}$.

If further also iv. *holds, then*

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega)(Z_i,\bar Z_i)=-\sqrt{-1}\left(\frac{1}{2}-\sum_{a=1}^{s}\mathrm{Im}(\lambda_a(Z_i))\right),\quad i=1,\dots,r\,.$$

If, in addition, v. *holds, then ω is a soliton for the Chern-Ricci flow with cosmological constant $c=\frac{1}{2}-\sum_{a=1}^{s}\mathrm{Im}(\lambda_a(Z_i))$ if and only if $\omega|_{\mathfrak{h}\oplus\mathfrak{h}}$ is a multiple of the canonical metric on $\mathfrak{h}$ with respect to the frame $\{Z_i\}$ and $\omega|_{\mathfrak{h}\oplus\mathfrak{I}}=0$.*

3.2 Homogeneous generalized Ricci flow

The present section is devoted to the study of the long time behaviour of the generalized Ricci flow in the homogeneous setting, especially in the Lie group case. The section will be divided as follows.

Subsection 3.2.1 is dedicated to the discussion of a scaling process on ECAs, which lead to a new definition of solitons for the generalized Ricci flow allowing for non classical and non steady solitons. Moreover, we will discuss a class of examples of expanding generalized Ricci solitons.

In Subsections 3.2.2 and 3.2.3, we focus our attention to the homogeneous case highlighting how one can present the space of left-invariant generalized metrics as a homogeneous space.

Subsection 3.2.4 is devoted to show the equivalence of the classical moving generalized metrics approach and the moving Dorfman brackets one. Motivated by this, in Subsection 3.2.5 we prove that a suitably modified version of the generalized Ricci curvature tensor is related to the moment map of the action of a Lie group on the space of left-invariant Dorfman brackets.

In Subsection 3.2.6, we define a flow in the space of left-invariant Dorfman brackets which is gauge-equivalent to the generalized Ricci flow. We then use this equivalence in Subsection 3.2.7 to prove a blow-up result which lead us to show the long time existence of the homogeneous generalized Ricci flow on any solvable Lie group.

Subsection 3.2.8 is focused on the definition and the study of the generalized nilsolitons. These objects appear naturally as models for the asymptotic behaviour of the homogeneous generalized Ricci flow on nilpotent Lie groups, as we will show in Subsection 3.2.9.

Finally, in Subsection 3.2.10, we deduce results on the long time behaviour of the pluriclosed flow on solvable and nilpotent Lie groups, while in Subsection 3.2.11, we obtain a full classification of generalized nilsolitons up to dimension 4.

3.2.1 Scaled ECAs and solitons of the generalized Ricci flow

To start this section, we observe that there is a natural way of scaling exact Courant algebroids. This will play a key role in the next sections.

Lemma 3.2.1. *Let $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be an exact Courant algebroid and let $c \in \mathbb{R}\backslash\{0\}$. Then, the data $(c\langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ defines a new exact Courant algebroid structure on E.*

Proof. Trivially, $c\langle\cdot,\cdot\rangle$ is a non-degenerate symmetric bilinear form of signature (n, n), regardless of the sign of c. Moreover, all the axioms except for Item 5 in Definition 1.2.1 in the definition of ECA clearly remain true after scaling $\langle\cdot,\cdot\rangle$. As regards Item 5 in Definition 1.2.1, we can notice that the π^* in the right-hand-side involves a composition with the isomorphism $\varphi_{\langle\cdot,\cdot\rangle} : E^* \simeq E$ induced by the neutral inner product. For $c\langle\cdot,\cdot\rangle$, this isomorphism is simply

$$\varphi_{c\langle\cdot,\cdot\rangle} = c^{-1}\varphi_{\langle\cdot,\cdot\rangle}, \tag{3.29}$$

which gives the claim. □

Definition 3.2.2. Let $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be an exact Courant algebroid and $c \in \mathbb{R}\backslash\{0\}$. We define the *scaled exact Courant algebroid* $c \cdot E$ to be the one associated with the data $(E, c\langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$.

We now study the effect of scaling on the 3-form H after choosing an isotropic splitting:

Lemma 3.2.3. *Let E be an ECA with isotropic splitting σ. Then, the isomorphism $E \simeq_\sigma (T \oplus T^*)_H$, $H \in \Lambda^3 M$ closed, induces an isomorphism*

$$c \cdot E \simeq_\sigma (T \oplus T^*)_{cH}.$$

Proof. This follows immediately from Proposition 1.2.5 and (1.27). □

Let us now examine the effect of scaling on generalized metrics.

Lemma 3.2.4. *Let $(E, \mathcal{G}) \simeq_\sigma ((T \oplus T^*)_H, \mathcal{G}(g, b))$ be a metric ECA and $c > 0$. Then, $\pm\mathcal{G}$ is a generalized metric on $\pm c \cdot E$ and the isomorphism $\pm c \cdot E \simeq_\sigma (T \oplus T^*)_{\pm cH}$ gives an isometry*

$$(\pm c \cdot E, \pm\mathcal{G}) \simeq_\sigma ((T \oplus T^*)_{\pm cH}, \mathcal{G}(cg, \pm cb)).$$

Proof. The first claim is clear, since the bilinear form $\pm c\langle\pm\mathcal{G}\cdot,\cdot\rangle = c\langle\mathcal{G}\cdot,\cdot\rangle$ is symmetric and positive definite. As regards the second one, we note that the scaling on the Riemannian metric follows directly from (1.28). On the other hand, considering σ_1 to be the preferred isotropic splitting induced by $\mathcal{G}$, then $(\sigma_1 - \sigma)(X) \in \ker\pi = \mathrm{Im}(\pi^*)$, for any $X \in \Gamma(TM)$. Hence, we can write

$$b = (\pi^*)^{-1}(\sigma_1 - \sigma) \colon TM \to T^*M\,, \tag{3.30}$$

where the fact that b is a 2-form is straightforward to check, using that both σ_1 and σ are isotropic. Then, the scaling on b follows from (3.29). □

Remark 3.2.5. The isometry in the second claim in Lemma 3.2.4 can be also explicitly constructed. Assuming $c > 0$, we can use the isotropic splitting σ to obtain the isometry $(\pm c \cdot E, \mathcal{G}) \simeq_\sigma (\pm c \cdot (T \oplus T^*)_H, \mathcal{G}(g, b))$. Then, one can easily see that

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \pm c\mathrm{Id} \end{pmatrix} : (\pm c \cdot (T \oplus T^*)_H, \mathcal{G}(g, b)) \to ((T \oplus T^*)_{\pm cH}, \mathcal{G}(cg, \pm cb))$$

is an isometry. Hence, composing the two isometries above, we obtain the claim.

Motivated by Definition 3.2.2 and by the effect of scaling on the 3-form H described in Lemma 3.2.4, we give the following new definition of solitons for the generalized Ricci flow.

Definition 3.2.6. A solution $(E, \mathcal{G}(t))_{t\in[0,T)}$ to the generalized Ricci flow (1.31) is called a *generalized Ricci soliton* if there exists a one-parameter family of scalings $c(t) > 0$, $c(0) = 1$, and a one-parameter family of generalized isometries

$$F_t : (c(t) \cdot E, \mathcal{G}(0)) \longrightarrow (E, \mathcal{G}(t)).$$

Differentiating the family F_t at $t = 0$ gives rise to a pair $(X, \boldsymbol{D}) \in \Gamma(TM) \times \mathrm{End}(E)$ satisfying the following properties:

1. $\boldsymbol{D} \in \mathrm{Der}([\cdot,\cdot])$, i.e. $\boldsymbol{D}[a, b] = [\boldsymbol{D}a, b] + [a, \boldsymbol{D}b]$, for any $a, b \in \Gamma(E)$;
2. $X\langle\cdot,\cdot\rangle = \langle(\boldsymbol{D} + \lambda\mathrm{Id})\cdot,\cdot\rangle + \langle\cdot,(\boldsymbol{D} + \lambda\mathrm{Id})\cdot\rangle$, where $c'(0) = -2\lambda$;
3. $[\mathcal{R}c(\mathcal{G}(0)) + \boldsymbol{D}, \mathcal{G}(0)] = 0$.

As one can easily see, the dependence of (2) on the scalings is only through $c'(0)$. This motivates the following definition.

Definition 3.2.7. Let E be a ECA and $\lambda \in \mathbb{R}$. We say that $(X, \boldsymbol{D}) \in \Gamma(TM)\times\mathrm{End}(E)$ is a λ-derivation if Item 1 and Item 2 are satisfied. We denote by $\mathrm{Der}_\lambda([\cdot,\cdot])$ the set of all λ-derivations.

It can easily be observed that 0-derivations are precisely the derivations of the Dorfman bracket introduced in [177]. Then, λ-derivations has to be interpreted as the generalization of derivations when a scaling process is taken into account.

We can give a characterization of both the set of isomorphisms between $c\cdot(T \oplus T^*)_H$ and $(T \oplus T^*)_H$ and the set of λ-derivations which will be useful in the next sections.

Lemma 3.2.8. *Let $H \in \Lambda^3M$ be closed and $c \in \mathbb{R}\backslash\{0\}$. Then,*

$$\mathrm{Aut}(c \cdot (T \oplus T^*)_H, (T \oplus T^*)_H) = \{\bar{f}_c e^b \,|\, b \in \Lambda^2M, \quad f \in \mathrm{Diff}(M), \quad f^*H = c(H - db)\},$$

where

$$\bar{f}_c = \begin{pmatrix} df & 0 \\ 0 & c(f^{-1})^* \end{pmatrix}.$$

Moreover, if $\lambda \in \mathbb{R}$,

$$\mathrm{Der}_\lambda([\cdot,\cdot]) = \{X + b \in \Gamma(TM) \oplus \Lambda^2M \,|\, \mathcal{L}_XH = -db - 2\lambda H\}.$$

Proof. Let $(F, f) \in \mathrm{Aut}(c\cdot(T \oplus T^*)_H, (T \oplus T^*)_H)$. Since F preserves the Dorfman bracket, we have that $\pi \circ F = df \circ \pi$. This naturally gives us that if

$$F = e^B \begin{pmatrix} df & 0 \\ 0 & h^* \end{pmatrix} e^b$$

then $B = 0$. On the other hand, the condition $\langle F\cdot, F\cdot\rangle = c\langle\cdot,\cdot\rangle$ forces $b \in \Lambda^2M$ and $dh = cdf^{-1}$. Moreover, we have that, for any $X + \xi, Y + \eta \in \Gamma(T \oplus T^*)$,

$$\bar{f}_c\left[(\bar{f}_c)^{-1}(X + \xi), (\bar{f}_c)^{-1}(Y + \eta)\right] = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi + c\iota_Y\iota_X(f^{-1})^*H.$$

Composing this with the usual action of e^b, we have that if $F \in \mathrm{Aut}(c \cdot (T \oplus T^*)_H, (T \oplus T^*)_H)$ then

$$F = \bar{f}_c e^b, \qquad f^*H = c(H - db).$$

The viceversa is trivial, concluding the proof of the first claim.

Let us consider a one parameter family of scalings $c(t) > 0$ such that $c(0) = 1$ and $c'(0) = -2\lambda$ and a one parameter family $(F_t, f_t) \in \mathrm{Aut}(c(t) \cdot (T \oplus T^*)_H, (T \oplus T^*)_H)$. Differentiating (F_t, f_t) at $t = 0$ gives $X + b \in \Gamma(TM) \oplus \Lambda^2M$ such that

$$\mathcal{L}_XH = -db - 2\lambda H.$$

Viceversa, given $X + b \in \Gamma(TM) \oplus \Lambda^2 M$ such that $\mathcal{L}_X H = -db - 2\lambda H$, we can consider f_t the one parameter family of diffeomorphisms generated by X and a family of scaling $c(t) > 0$ such that $c(0) = 1$ and $c'(0) = -2\lambda$. Hence, we can define

$$\bar{b}_t = \int_0^t c(s) f_s^* b_s ds\,,$$

where $b_s \in \Lambda^2 M$ such that $\mathcal{L}_{X_s} H = -db_s + \frac{c'(s)}{c(s)} H$. Then, we have that

$$d\bar{b}_t = -\int_0^t \frac{1}{c(s)} f_s^* \left(-\frac{c'(s)}{c(s)} H + \mathcal{L}_{X_s} H \right) ds = -\int_0^t \frac{d}{ds} \left(\frac{1}{c(s)} f_s^* H \right) ds = -\frac{1}{c(t)} f_t^* H + H\,.$$

Then, $F = \bar{f}_{t,c(t)} e^{\bar{b}_t} \in \mathrm{Aut}(c(t) \cdot (T \oplus T^*)_H, (T \oplus T^*)_H)$, as we wanted. □

Definition 3.2.6 is the dynamical definition of solitons for the generalized Ricci flow. As in the classical case, we can deduce from that an equivalent and static definition in terms of classical data, as the next proposition shows.

Proposition 3.2.9. *Let $(E, \mathcal{G}(t))$ be a generalized Ricci soliton, and consider the time-independent isometry $(E, \mathcal{G}(t)) \simeq_{\sigma_0} ((T \oplus T^*)_{H_0}, \mathcal{G}(g(t), b(t)))$ induced by the isotropic splitting σ_0 associated to $\mathcal{G}(0)$. Then, $g_0 := g(0)$ and H_0 satisfy*

$$\begin{aligned} \mathrm{Ric}^{\mathrm{B}}_{g_0,H_0} &= \lambda\, g_0 + \tfrac{1}{2}\, \mathcal{L}_X g_0, \\ \Delta_{g_0} H_0 &= -2\,\lambda\, H_0 - \mathcal{L}_X H_0, \end{aligned} \tag{3.31}$$

where $\lambda = -\frac{1}{2} c'(0)$ and $c(t) > 0$ are the scalings from Definition 3.2.6. Conversely, if $((T \oplus T^*)_{H_0}, \mathcal{G}(g_0, 0))$ satisfies (3.31)*, then there exists a soliton solution $\mathcal{G}(g(t), b(t))$ to* (1.31) *on $(T \oplus T^*)_{H_0}$ with $\mathcal{G}(g(0), b(0)) = \mathcal{G}(g_0, 0)$.*

Proof. By Lemma 3.2.4, the isomorphism associated to σ_0 also induces time-independent isometries

$$(c(t) \cdot E, \mathcal{G}(0)) \simeq_{\sigma_0} \left((T \oplus T^*)_{c(t) H_0}, \mathcal{G}(c(t) g_0, 0) \right).$$

It follows that $c(t) g_0$ and $c(t) H_0$ must solve the gauged-fixed equations (1.33) and (1.34).

Thus,

$$\begin{aligned} c' g_0 &= -2\, \mathrm{Ric}^{\mathrm{B}}_{g_0,H_0} + c\, \mathcal{L}_X g_0, \\ c' H_0 &= \Delta_{g_0} H_0 + c\, \mathcal{L}_X H_0. \end{aligned}$$

Evaluating at $t = 0$ and using $c(0) = 1$, $c'(0) = -2\lambda$, (3.31) follows. □

Remark 3.2.10. When $H_0 = 0$, Proposition 3.2.9 implies that Definition 3.2.6 is equivalent to the classical definition of Ricci soliton for the Ricci flow.

On the other hand, if $c(t) \equiv 1$ is constant, we recover the definition of soliton solution given in [149, §4.4], which only allows for steady solitons (i.e. $\lambda = 0$). Recently, in [269, Definition 2.1], a definition of generalized Ricci soliton, equivalent to Definition 3.2.6, was given.

Moreover, in the non-steady case, the second equation in (3.31), together with $dH_0 = 0$ and the Cartan's formula, forces $[H_0] = 0$. In particular, if M is compact, this implies that a non-steady generalized Ricci soliton with harmonic torsion is a classical Ricci soliton.

To conclude this section, we give a family of explicit examples of expanding generalized Ricci solitons.

Example 3.2.11. Let us consider $M_0 = H_3 := \mathrm{Heis}(3,\mathbb{R})$ the 3-dimensional Heisenberg group. It is well known that on M_0 we can choose coordinates $\{x, y, z\}$ so that

$$e_1 = \frac{\partial}{\partial x}\,, \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\,, \quad e_3 = \frac{\partial}{\partial z}$$

are left-invariant vector fields, forming a frame of the Lie algebra of M_0, subjected to the following structure equation:

$$[e_1, e_2] = e_3\,.$$

We will denote with $\{e^1, e^2, e^3\}$ the dual coframe of $\{e_1, e_2, e_3\}$ which is, in particular, given by:

$$e^1 = dx\,, \quad e^2 = dy\,, \quad e^3 = dz - xdy\,.$$

We next consider the following one parameter family of diffeomorphisms of M_0:

$$\varphi_t(x, y, z) = ((1+4t)^{\frac{1}{4}}x, (1+4t)^{\frac{1}{4}}y, (1+4t)^{\frac{1}{2}}z)\,, \quad (x, y, z) \in M_0\,, \quad t \geq 0\,.$$

It is easy to see that φ_t is actually an automorphism of M_0 viewed as a Lie group. Another easy computation allows us to state that

$$d\varphi_t = (1+4t)^{\frac{1}{4}}e^1 + (1+4t)^{\frac{1}{4}}e^2 + (1+4t)^{\frac{1}{2}}e^3\,, \quad t \geq 0\,.$$

Now we consider the closed left invariant 3-form $H = dx \wedge dy \wedge dz$ and observe that H is harmonic with respect to any metric on M_0. Then, one can easily see that

$$\varphi_t^* H = (1+4t)H\,, \quad t \geq 0\,.$$

Then, denoting $c(t) = 1 + 4t > 0$ and comparing with Lemma 3.2.8, we are in the position to promote φ_t to

$$F_t = \bar{\varphi}_{t,c(t)} \in \mathrm{Aut}(c(t)\cdot(T \oplus T^*)_H, (T \oplus T^*)_H)\,, \quad t \geq 0\,. \tag{3.32}$$

Moreover, we easily compute the vector field generated by φ_t:

$$X = \frac{d}{dt}\Big|_{t=0}\varphi_t = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} = xe_1 + ye_2 + (2z - xy)e_3\,,$$

which, in particular, satisfies the following bracket relations:

$$[X, e_1] = -e_1\,, \quad [X, e_2] = -e_2\,, \quad [X, e_3] = -2e_3\,.$$

Finally, $X \in \mathrm{Der}_{-2}([\cdot,\cdot])$ satisfies

$$\mathcal{L}_X H = 4H$$

which is precisely the second equation in (3.31) with $-2\lambda = c'(0) = 4$, since H is harmonic with respect to any Riemannian metric on M_0. Now, we consider the following Riemannian metric:

$$g = e^1 \odot e^1 + e^2 \odot e^2 + e^3 \odot e^3 = dx \odot dx + (1 + x^2)dy \odot dy + dz \odot dz - 2xdy \odot dz\,,$$

which is the standard left-invariant Riemannian metric on M_0 with respect to the frame $\{e_1, e_2, e_3\}$. It is easy to see that

$$\mathrm{Ric}^{\mathrm{B}}_{g,H} = -(e^1 \odot e^1 + e^2 \odot e^2) \text{ and } \mathcal{L}_X g = 2(e^1 \odot e^1 + e^2 \odot e^2 + 2e^3 \odot e^3)\,.$$

Then,

$$\mathrm{Ric}^{\mathrm{B}}_{g,H} = -2g + \frac{1}{2}\mathcal{L}_X g$$

giving precisely the first equation in (3.31). Now, we can use Proposition 3.2.9 to infer that $((T \oplus T^*)_H, \mathcal{G}(g, 0))$ is an expanding soliton for the generalized Ricci flow. Moreover, we can also specify what actually is the one parameter family of generalized isometries in Definition 3.2.6. Indeed, considering F_t as in (3.32), we have

$$F_t \mathcal{G}(g, 0) F_t^{-1} = \mathcal{G}((1 + 4t)(\varphi_t^{-1})^* g, 0)$$

but

$$g(t) = (1 + 4t)(\varphi_t^{-1})^* g = (1 + 4t)^{\frac{1}{2}} e^1 \odot e^1 + (1 + 4t)^{\frac{1}{2}} e^2 \odot e^2 + e^3 \odot e^3$$

is precisely the solution of the generalized Ricci flow starting from g, see [254, Section 6].

In the same fashion and with a bit more effort, we can construct examples of expanding solitons for the generalized Ricci flow on $M_k := H_3 \times \mathbb{R}^k$, for any $k \geq 1$. On M_k, we consider the following left-invariant frame $\{e_1, e_2, e_3, e_4, \ldots, e_{k+3}\}$ of the Lie algebra of $H_3 \times \mathbb{R}^k$ where e_1, e_2, e_3 are the same defined above while $e_{3+j} = \frac{\partial}{\partial x_{3+j}}$ where x_{3+j} is the standard coordinate on the j-th copy of $\mathbb{R}$ in the abelian factor $\mathbb{R}^k$.

Then, it is sufficient to consider $c(t) = 1 + 4t$ as scaling,

$$\varphi_t(x, y, z, x_4, \ldots, x_{3+k}) = ((1 + 4t)^{\frac{1}{4}} x, (1 + 4t)^{\frac{1}{4}} y, (1 + 4t)^{\frac{1}{2}} z, (1 + 4t)^{\frac{1}{2}} x_4, \ldots, (1 + 4t)^{\frac{1}{2}} x_{3+k}),$$

for all $(x, y, z, x_4, \ldots, x_{3+k}) \in M_k$, as the one parameter family of diffeomorphisms of M_k , $H = e^1 \wedge e^2 \wedge e^3$ and finally the standard left-invariant metric

$$g = \sum_{i=1}^{k+3} e^i \odot e^i ,$$

with respect to the coframe $\{e^1, \ldots, e^{3+k}\}$ dual with respect to $\{e_1, \ldots, e_{k+3}\}$ and obtain an expanding soliton of the generalized Ricci flow on M_k.

3.2.2 Homogeneous generalized Geometry

In this subsection we specialize our study to the case where the underlying manifold $M = \mathsf{G}$ is a simply-connected Lie group. We will denote by e the identity of G and $\mathfrak{g} = T_e\mathsf{G}$ its Lie algebra. We also ask the ECA to be compatible with the algebraic structure in the following sense:

Definition 3.2.12. We say that a metric ECA $(E \to \mathsf{G}, \mathcal{G})$ is *left-invariant* if the action of G on itself by left-translations lifts to an action on $(E, \mathcal{G})$ by isometries (as defined in Definition 1.2.8).

The action of G on E in Definition 3.2.12 is a particular instance of a wider class of Lie group actions on ECAs called *lifted* or *extended* actions, see [36, Definition 2.4] and [65, Definition 2.6].
The action of G on a left-invariant metric ECA allows to consider G-equivariant isotropic splittings. The choice of such isotropic splittings, using Proposition 1.2.9, gives rise to left-invariant classical data as the following proposition highlights.

Proposition 3.2.13. *Let $(E \to \mathsf{G}, \mathcal{G})$ be a left-invariant metric ECA. Then, any G-equivariant isotropic splitting $\sigma \colon T\mathsf{G} \to E$ of* (1.24) *induces a G-equivariant isometry*

$$(E, \mathcal{G}) \simeq_\sigma ((T \oplus T^*)_H, \mathcal{G}(g, b)),$$

where the tensors g, H, b on M are left-invariant and H depends only on σ and not on $\mathcal{G}$.

Proof. If σ is the isotropic splitting given by $\mathcal{G}$, firstly, we note that, by Proposition 1.2.9, $(E, \mathcal{G})$ is isometric to $((T \oplus T^*)_H, \mathcal{G}(g, 0))$, for some closed 3-form H and Riemannian metric g. Using this isomorphism, we can define an action of G on $(T \oplus T^*)_H$, and by definition this action will be by isometries of $\mathcal{G}(g, 0)$. By Proposition 1.2.11, H and g must be left-invariant. It remains to prove the invariance of b. As in (3.30), the 2-form b can be expressed as

$$b = (\pi^*)^{-1}(\sigma_1 - \sigma),$$

where σ_1 is the isotropic splitting induced by $\mathcal{G}$, which is G-equivariant. Then, the claim follows directly from the fact that, for any $(F, f) \in \mathrm{Aut}(E)$, we have that $F \circ \pi^* = \pi^* \circ (f^{-1})^*$ and from the expression of b above. □

Remark 3.2.14. As said, the preferred isotropic splitting induced by $\mathcal{G}$ is G-equivariant. Thus, any homogeneous metric ECA is isometric to $((T \oplus T^*)_H, \mathcal{G}(g, 0))$ for a left-invariant Riemannian metric g, and a closed left-invariant 3-form $H \in (\Lambda^3\mathsf{G})^{\mathsf{G}}$.

Notice that as a vector bundle any left-invariant ECA is trivial, since after choosing an isotropic splitting we get an isomorphism with $(T \oplus T^*)_H$, where

$$T\mathsf{G} \oplus T^*\mathsf{G} \simeq (\mathfrak{g} \oplus \mathfrak{g}^*) \times \mathsf{G} \tag{3.33}$$

because Lie groups are parallelizable by using left-invariant vector fields. By abuse of notation, we will simply denote

$$(\mathfrak{g} \oplus \mathfrak{g}^*)_H := (T \oplus T^*)_H,$$

where $H \in (\Lambda^3\mathsf{G})^{\mathsf{G}} \simeq \Lambda^3\mathfrak{g}^*$ is a closed, left-invariant 3-form on G.

3.2.3 The space of left-invariant generalized metrics as a homogeneous space

Let $E \to \mathsf{G}$ be a left-invariant ECA, and let us fix a background left-invariant generalized metric $\overline{\mathcal{G}}$ on E. The preferred isotropic splitting $\bar{\sigma} : T\mathsf{G} \to E$ induced by $\overline{\mathcal{G}}$ yields an isometry

$$(E \to \mathsf{G}, \overline{\mathcal{G}}) \simeq_{\bar{\sigma}} ((\mathfrak{g} \oplus \mathfrak{g}^*)_{\bar{H}}, \mathcal{G}(\bar{g}, 0)),$$

for some left-invariant metric $\bar{g}$ and closed 3-form $\bar{H}$ on G. By abuse of notation we write $\overline{\mathcal{G}} = \mathcal{G}(\bar{g}, 0)$. Let us set

$$\begin{aligned} \mathcal{M}^{\mathsf{G}} &:= \{\mathcal{G} \,|\, \mathcal{G} \text{ is a left-invariant generalized metric on } (\mathfrak{g} \oplus \mathfrak{g}^*)_{\bar{H}}\} \\ &\simeq \{\mathcal{G}(g, b) \,|\, g \text{ left-invariant metric on } \mathsf{G}, b \in (\Lambda^2\mathsf{G})^{\mathsf{G}}\} \end{aligned}$$

where the last identification is due to Proposition 3.2.13. Next, we consider the following Lie subgroup of $\mathsf{O}(\mathfrak{g} \oplus \mathfrak{g}^*, \langle \cdot\,, \cdot \rangle)$:

$$\mathsf{L} := \left\{ e^b \cdot \begin{pmatrix} h & 0 \\ 0 & (h^{-1})^* \end{pmatrix} \,\middle|\, h \in \mathsf{GL}(\mathfrak{g}),\ b \in \Lambda^2\mathfrak{g}^* \right\}, \qquad e^b := \begin{pmatrix} \mathrm{Id} & 0 \\ b & \mathrm{Id} \end{pmatrix}. \tag{3.34}$$

As an abstract Lie group, L is isomorphic to the semi-direct product $\Lambda^2\mathfrak{g}^* \rtimes \mathsf{GL}(\mathfrak{g})$. Moreover, its Lie algebra is

$$\mathfrak{l} := \left\{ \begin{pmatrix} A & 0 \\ \alpha & -A^* \end{pmatrix} \,\middle|\, A \in \mathfrak{gl}(\mathfrak{g}),\ \alpha \in \Lambda^2\mathfrak{g}^* \right\}.$$

The standard representation of $\mathsf{O}(\mathfrak{g} \oplus \mathfrak{g}^*, \langle\cdot, \cdot\rangle)$ on $\mathfrak{g} \oplus \mathfrak{g}^*$ gives rise to a representation of L on $\mathfrak{g} \oplus \mathfrak{g}^*$, and this extends in a standard way to an action of L on all tensor powers of $\mathfrak{g} \oplus \mathfrak{g}^*$. In particular, L acts on generalized metrics and Dorfman brackets via the following formulas:

$$\ell \cdot \mathcal{G} := \ell \circ \mathcal{G} \circ \ell^{-1}, \qquad \ell \cdot [\cdot, \cdot] := \ell[\ell^{-1}\cdot, \ell^{-1}\cdot].$$

Recall that we also have the 'change of basis' actions of $\mathsf{GL}(\mathfrak{g})$ on the spaces of brackets and 3-forms:

$$h \cdot \mu(\cdot, \cdot) = h\mu(h^{-1}\cdot, h^{-1}\cdot), \qquad h \cdot \omega(\cdot, \cdot, \cdot) = \omega(h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot), \qquad h \in \mathsf{GL}(\mathfrak{g}),$$

for $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ and $\omega \in \Lambda^3\mathfrak{g}^*$.
The next proposition allows to present the space of left-invariant metric on E as a homogeneous space.

Proposition 3.2.15. *The action of* L *on the space of left-invariant generalized metrics* $\mathcal{M}^{\mathsf{G}}$ *is transitive.*

Proof. For any $\ell \simeq (b, h) \in \mathsf{L}$, its action on the background metric is given by

$$\ell \cdot \overline{\mathcal{G}} = e^b \begin{pmatrix} h & 0 \\ 0 & (h^{-1})^* \end{pmatrix} \begin{pmatrix} 0 & \bar{g}^{-1} \\ \bar{g} & 0 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 \\ 0 & h^* \end{pmatrix} e^{-b} = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b},$$

where

$$g = (h^{-1})^* \bar{g} h^{-1} = h \cdot \bar{g}(\cdot, \cdot).$$

Since $\mathsf{GL}(\mathfrak{g})$ acts transitively on left-invariant metrics, the claim now follows from Proposition 3.2.13. □

3.2.4 Moving generalized metrics is equivalent to moving Dorfman brackets

In this subsection, we will define and study the main properties of the space of left-invariant Dorfman brackets. A similar treatment of them can also be found in [254]. Then, we will focus on describing the effect of the L-action and its infinitesimal version on Dorfman brackets.

From the Item 5 of Definition 1.2.1, the Dorfman bracket associated with a left-invariant ECA is skew-symmetric. Thus, we consider the vector space $\Lambda^2(\mathfrak{g} \oplus \mathfrak{g}^*)^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)$ of all skew-symmetric bilinear maps

$$\boldsymbol{\mu} : (\mathfrak{g} \oplus \mathfrak{g}^*) \times (\mathfrak{g} \oplus \mathfrak{g}^*) \to (\mathfrak{g} \oplus \mathfrak{g}^*).$$

Using Proposition 1.2.5, we know that choosing an isotropic splitting σ gives rise to an isomorphism $E \simeq_\sigma (\mathfrak{g} \oplus \mathfrak{g}^*)_H$. Then, from left-invariance and (1.26), it follows that the Dorfman bracket of E gets identified with $[\cdot, \cdot]_H \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{g}^*)^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)$ given by

$$[X + \xi, Y + \eta]_H = [X, Y]_{\mathfrak{g}} - \eta \circ \mathrm{ad}_{\mathfrak{g}}(X) + \xi \circ \mathrm{ad}_{\mathfrak{g}}(Y) + \iota_Y \iota_X H, \tag{3.35}$$

where $\mathrm{ad}_{\mathfrak{g}}(X) = [X, \cdot]_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ and $H \in \Lambda^3\mathfrak{g}^*$ is closed. It is not hard to see that $\langle [\cdot, \cdot]_H, \cdot \rangle$ is totally skew-symmetric. This motivates the following.

Definition 3.2.16. The *space of Dorfman brackets* is the algebraic subset of $\Lambda^2(\mathfrak{g} \oplus \mathfrak{g}^*)^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)$ given by

$$\mathcal{D} := \{\boldsymbol{\mu} \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{g}^*)^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^*) \mid \langle \boldsymbol{\mu}(\cdot, \cdot), \cdot \rangle \in \Lambda^3(\mathfrak{g} \oplus \mathfrak{g}^*)^*, \ \boldsymbol{\mu}(\mathfrak{g}^*, \mathfrak{g}^*) = 0, \ \mathcal{J}(\boldsymbol{\mu}) = 0\}.$$

Here, $\mathcal{J}(\boldsymbol{\mu})(a, b, c) := \boldsymbol{\mu}(\boldsymbol{\mu}(a, b), c) + \boldsymbol{\mu}(\boldsymbol{\mu}(b, c), a) + \boldsymbol{\mu}(\boldsymbol{\mu}(c, a), b) = 0$ is the Jacobi identity.

We can show that all elements in $\mathcal{D}$ can be written as in (3.35) for suitable choices of the 3-form H and the Lie bracket on $\mathfrak{g}$.

Lemma 3.2.17. *We have that $\boldsymbol{\mu} \in \mathcal{D}$ if and only if there exist a Lie bracket μ on $\mathfrak{g}$ and a closed* 3*-form $H \in \Lambda^3\mathfrak{g}^*$ such that, for any $X + \xi, Y + \eta \in \mathfrak{g} \oplus \mathfrak{g}^*$,*

$$\boldsymbol{\mu}(X + \xi, Y + \eta) = \mu(X, Y) - \eta \circ \mu_X + \xi \circ \mu_Y + \iota_Y \iota_X H, \qquad \mu_X := \mu(X, \cdot). \tag{3.36}$$

Proof. Sufficiency was observed above. Regarding necessity, for each $X, Y, Z \in \mathfrak{g}$, we set

$$\mu(X, Y) := \pi \circ \boldsymbol{\mu}(X, Y), \qquad H(X, Y, Z) := 2 \langle \boldsymbol{\mu}(X, Y), Z \rangle.$$

The linear conditions on $\boldsymbol{\mu}$ imply that, for every $\xi \in \mathfrak{g}^*$, we have $\boldsymbol{\mu}(\xi, Y) \in \mathfrak{g}^*$. Thus,

$$\boldsymbol{\mu}(\xi, Y)(X) = 2 \langle \boldsymbol{\mu}(\xi, Y), X \rangle = 2 \langle \boldsymbol{\mu}(Y, X), \xi \rangle = \xi\mu(Y, X) = \xi \circ \mu_Y(X).$$

By skew-symmetry, $\boldsymbol{\mu}(X, \eta) = -\eta \circ \mu_X$, and from this (3.36) follows. A simple computation shows that the Jacobi identity for $\boldsymbol{\mu}$ implies the Jacobi identity for μ and that $d_\mu H = 0$. □

Moreover, one can notice that the following conditions hold for brackets in $\mathcal{D}$:

$$\boldsymbol{\mu}(\mathfrak{g},\mathfrak{g}) \subset \mathfrak{g}\oplus\mathfrak{g}^*, \qquad \boldsymbol{\mu}(\mathfrak{g},\mathfrak{g}^*) \subset \mathfrak{g}^*, \qquad \boldsymbol{\mu}(\mathfrak{g}^*,\mathfrak{g}^*) = 0. \tag{3.37}$$

That is, $(\mathfrak{g}\oplus\mathfrak{g}^*, \boldsymbol{\mu})$ is a Lie algebra which is a central extension of $(\mathfrak{g}, \mu)$. We thus have that the space of Dorfman brackets is contained in the linear subspace:

$$V_{\mathcal{D}} := \left\{\boldsymbol{\mu} \in \Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^* \otimes (\mathfrak{g}\oplus\mathfrak{g}^*) \,|\, \boldsymbol{\mu} \text{ satisfies (3.37) and } \langle\boldsymbol{\mu}(\cdot,\cdot),\cdot\rangle \in \Lambda^3(\mathfrak{g}\oplus\mathfrak{g}^*)^*\right\}. \tag{3.38}$$

We then have two projections $(\cdot)_{\mathfrak{g}} : \mathcal{D} \to \Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$ and $(\cdot)_{\Lambda^3} : \mathcal{D} \to \Lambda^3\mathfrak{g}^*$,

$$\boldsymbol{\mu} \mapsto \boldsymbol{\mu}_{\mathfrak{g}} = \mu \in \Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}, \qquad \boldsymbol{\mu} \mapsto \boldsymbol{\mu}_{\Lambda^3} = H \in \Lambda^3\mathfrak{g}^*, \tag{3.39}$$

where here μ is simply the Lie bracket of $\mathfrak{g}$. Furthermore, we notice that, by (3.36), these projections completely determine a Dorfman bracket $\boldsymbol{\mu} \in \mathcal{D}$.

Let us first study how the action of L relates to the projections in (3.39):

Lemma 3.2.18. *Let $\boldsymbol{\mu} \in \mathcal{D}$ with $\mu := \boldsymbol{\mu}_{\mathfrak{g}}$, $H := \boldsymbol{\mu}_{\Lambda^3}$. Then, for any $\ell = \bar{h}\, e^{\alpha} \in \mathsf{L}$, we have*

$$(\ell\cdot\boldsymbol{\mu})_{\mathfrak{g}} = h\cdot\mu, \qquad (\ell\cdot\boldsymbol{\mu})_{\Lambda^3} = h\cdot(H - d_{\mu}\alpha). \tag{3.40}$$

Proof. First, we compute for $X+\xi, Y+\eta \in \mathfrak{g}\oplus\mathfrak{g}^*$:

$$\begin{aligned}(e^{\alpha}\cdot\boldsymbol{\mu})(X+\xi, Y+\eta) &= e^{\alpha}\boldsymbol{\mu}(X+\xi-i_X\alpha,\, Y+\eta-i_Y\alpha)\\ &= e^{\alpha}\left(\boldsymbol{\mu}(X+\xi,Y+\eta) + \alpha(Y,\mu(X,\cdot)) - \alpha(X,\mu(Y,\cdot))\right)\\ &= \boldsymbol{\mu}(X+\xi,Y+\eta) - i_Y i_X(d_{\mu}\alpha).\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\begin{pmatrix} h & 0\\ 0 & (h^{-1})^*\end{pmatrix}\cdot\boldsymbol{\mu}\right)(X+\xi,Y+\eta) &= \begin{pmatrix} h & 0\\ 0 & (h^{-1})^*\end{pmatrix}\boldsymbol{\mu}(h^{-1}X+\xi\circ h, h^{-1}Y+\eta\circ h)\\ &= (h\cdot\mu)(X,Y) - \eta\circ(h\cdot\mu)_X + \xi\circ(h\cdot\mu) + i_Y i_X(h\cdot H).\end{aligned}$$

The result follows from composing the two actions. □

Given the actions of L on Dorfman brackets on $\mathfrak{g}\oplus\mathfrak{g}^*$, and of $\mathsf{GL}(\mathfrak{g})$ on brackets and 3-forms on $\mathfrak{g}$, we denote the corresponding Lie algebra representations (or 'infinitesimal actions') by

$$\Theta : \mathfrak{l} \to \mathrm{End}(\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*)), \qquad \theta : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{End}(\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}), \qquad \rho : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{End}(\Lambda^3\mathfrak{g}^*).$$

More precisely, they are given by

$$\begin{aligned}\Theta(A)\boldsymbol{\mu}(\cdot,\cdot) &= A\boldsymbol{\mu}(\cdot,\cdot) - \boldsymbol{\mu}(A\cdot,\cdot) - \boldsymbol{\mu}(\cdot,A\cdot), \qquad A \in \mathfrak{gl}(\mathfrak{g}\oplus\mathfrak{g}^*),\\ \theta(A)\mu(\cdot,\cdot) &= A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \qquad A \in \mathfrak{gl}(\mathfrak{g}),\\ \rho(A)\omega(\cdot,\cdot,\cdot) &= -\omega(A\cdot,\cdot,\cdot) - \omega(\cdot,A\cdot,\cdot) - \omega(\cdot,\cdot,A\cdot), \qquad A \in \mathfrak{gl}(\mathfrak{g}).\end{aligned}$$

From Lemma 3.2.18 we immediately deduce:

Corollary 3.2.19. *For any $L \simeq (\alpha, A) \in \mathfrak{l} \simeq \Lambda^2\mathfrak{g}^* \rtimes \mathfrak{gl}(\mathfrak{g})$, we have*

$$(\Theta(L)\boldsymbol{\mu})_{\mathfrak{g}} = \theta(A)\mu, \qquad (\Theta(L)\boldsymbol{\mu})_{\Lambda^3} = \rho(A)H - d_{\mu}\alpha.$$

Since we will be moving the Dorfman and Lie brackets on $\mathfrak{g}\oplus\mathfrak{g}^*$ and $\mathfrak{g}$ respectively, it is convenient to introduce the following notation: given $\boldsymbol{\mu}$ a Dorfman bracket as in (3.36), we denote by

$$(\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}} := (\mathfrak{g}\oplus\mathfrak{g}^*, \langle\cdot,\cdot\rangle, \boldsymbol{\mu}, \pi)$$

the left-invariant ECA structure given by the data $(\langle\cdot,\cdot\rangle, \boldsymbol{\mu}, \pi)$, where $\langle\cdot,\cdot\rangle$ and π are defined in (1.25). Notice that in order for Item 2 of Definition 1.2.1 to be satisfied, we must necessarily have that the Lie bracket on $\mathfrak{g}$ is $\mu := \boldsymbol{\mu}_{\mathfrak{g}}$.

Proposition 3.2.20. *Let $\ell = \bar{h}\, e^{\alpha} \in \mathsf{L}$, for $h \in \mathsf{GL}(\mathfrak{g})$ and $\alpha \in \Lambda^2\mathfrak{g}^*$. Then, the bundle map*

$$(\ell, h) : \big((\mathfrak{g} \oplus \mathfrak{g}^*)_H, (\ell^{-1}) \cdot \mathcal{G}\big) \to ((\mathfrak{g} \oplus \mathfrak{g}^*)_{\ell\cdot\boldsymbol{\mu}}, \mathcal{G})$$

is an isometry. Here $\boldsymbol{\mu}_{\mathfrak{g}} = [\cdot,\cdot]_{\mathfrak{g}}$ is the original Lie bracket of $\mathfrak{g}$ and $\boldsymbol{\mu}_{\Lambda^3} = H$.

Proof. We first prove the claim for $\ell = \bar{h}$ covering $h \in \mathsf{GL}(\mathfrak{g})$. By definition of the action, the map $h : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \to (\mathfrak{g}, h \cdot [\cdot,\cdot]_{\mathfrak{g}})$ is a Lie algebra isomorphism. Set $\mu := h \cdot [\cdot,\cdot]_{\mathfrak{g}}$, let $\mathsf{G}, \mathsf{G}_\mu$ be respectively the simply-connected Lie groups with Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}, \mu)$. Then, there exists a Lie group isomorphism $\varphi : \mathsf{G} \to \mathsf{G}_\mu$ for which $d_e\varphi = h$, where $e \in \mathsf{G}$ is the identity. The corresponding bundle map $\bar{\varphi} : T\mathsf{G} \oplus T^*\mathsf{G} \to T\mathsf{G}_\mu \oplus T^*\mathsf{G}_\mu$ covering φ clearly maps left-invariant sections to left-invariant sections, and since the trivialization (3.33) is done via left-invariant sections, it follows that $\bar{\varphi}$ may be represented by an element of $\mathsf{GL}(\mathfrak{g} \oplus \mathfrak{g}^*)$. The latter is precisely $\bar{h}$, and the pair $(\bar{h}, h)$ is the infinitesimal data corresponding to the ECA isomorphism

$$\bar{\varphi} : (\mathfrak{g} \oplus \mathfrak{g}^*)_H \to (\mathfrak{g} \oplus \mathfrak{g}^*)_{\bar{h}\cdot\boldsymbol{\mu}}.$$

Indeed, $\bar{h}$ preserves the Dorfman bracket by definition of the action, and it preserves the neutral inner product because $\bar{h} \in \mathsf{O}(\mathfrak{g} \oplus \mathfrak{g}^*, \langle \cdot, \cdot \rangle)$. Finally,

$$\bar{h} \cdot (\bar{h}^{-1} \cdot \mathcal{G}) = \mathcal{G},$$

thus we have an isometry.

The proof for $\ell = e^{\alpha}$ covering $\mathrm{Id} \in \mathsf{GL}(\mathfrak{g})$ is similar (in this case, $\mathsf{G} = \mathsf{G}_\mu$). □

Let us now consider the effect of scaling when Dorfman brackets are moving. Recall that, by Lemma 3.2.4, a choice of isotropic splitting σ induces an isometry $(c \cdot E, \mathcal{G}) \simeq_\sigma ((\mathfrak{g} \oplus \mathfrak{g}^*)_{cH}, \mathcal{G}(cg, cb))$, for any $c > 0$.

Proposition 3.2.21. *Let $c > 0$ and let $\boldsymbol{\mu}$ be the Dorfman bracket determined by $\boldsymbol{\mu}_{\mathfrak{g}} = [\cdot,\cdot]_{\mathfrak{g}}$ and $\boldsymbol{\mu}_{\Lambda^3} = H$. Then, the following bundle map is an isometry:*

$$\left(\pm\overline{c^{1/2}\,\mathrm{Id}}\,,\ \pm c^{1/2}\,\mathrm{Id}\right) : ((\mathfrak{g} \oplus \mathfrak{g}^*)_{\pm cH}, \mathcal{G}(cg, \pm cb)) \longrightarrow ((\mathfrak{g} \oplus \mathfrak{g}^*)_{\pm c^{-1/2}\boldsymbol{\mu}}, \mathcal{G}(g, b)).$$

Proof. First, we see that it is an ECA isomorphism. Indeed, we have that $\pm\overline{c^{1/2}\,\mathrm{Id}} \in \mathsf{O}(\mathfrak{g} \oplus \mathfrak{g}^*, \langle \cdot, \cdot \rangle)$, and

$$\pm\overline{c^{1/2}\,\mathrm{Id}}\left[(\pm\overline{c^{1/2}\,\mathrm{Id}})^{-1}\,\cdot, (\pm\overline{c^{1/2}\,\mathrm{Id}})^{-1}\,\cdot\right]_{cH} = \pm c^{-1/2}\boldsymbol{\mu}(\cdot,\cdot)$$

can be easily verified using (3.35) and (3.36) (with $\mu = [\cdot,\cdot]_{\mathfrak{g}}$). Regarding the isometry claim, we directly compute:

$$\begin{aligned}\left(\overline{c^{1/2}\,\mathrm{Id}}\right) \mathcal{G}(cg, cb) \left(\overline{c^{1/2}\,\mathrm{Id}}\right)^{-1} &= \begin{pmatrix} c^{1/2}\,\mathrm{Id} & 0 \\ 0 & c^{-1/2}\,\mathrm{Id}\end{pmatrix} e^{cb} \begin{pmatrix} 0 & c^{-1}g^{-1} \\ cg & 0 \end{pmatrix} e^{-cb} \begin{pmatrix} c^{-1/2}\,\mathrm{Id} & 0 \\ 0 & c^{1/2}\,\mathrm{Id}\end{pmatrix} \\ &= \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \mathcal{G}(g, b).\end{aligned}$$

□

3.2.5 The moment map

In this subsection, we define a moment map for the action of L on Dorfman brackets, in the sense of real GIT (see [274, 183, 53]). In Proposition 3.2.24 we show that this moment map encodes the most complicated part of the generalized Ricci curvature of left-invariant generalized metrics.

As before, we fix a background generalized metric $\overline{\mathcal{G}}$ on a left-invariant ECA, yielding an isomorphism with $((\mathfrak{g}\oplus\mathfrak{g}^*)_{\bar H}, \mathcal{G}(\bar g, 0))$. The metric $\bar g$ gives rise to an inner product on $\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*)$, also denoted by $\bar g$. To define it, we fix a $\bar g$-orthonormal basis $\{e_i\}$ of $\mathfrak{g}$ with dual basis $\{e^i\}$ and set

$$\bar g(\boldsymbol{\mu},\boldsymbol{\nu}) := 2\sum_{i,j}\left\langle \overline{\mathcal{G}}\boldsymbol{\mu}(e_i,e_j), \boldsymbol{\nu}(e_i,e_j)\right\rangle + 4\sum_{i,j}\left\langle \overline{\mathcal{G}}\boldsymbol{\mu}(e_i,e^j), \boldsymbol{\nu}(e_i,e^j)\right\rangle + 2\sum_{i,j}\left\langle \overline{\mathcal{G}}\boldsymbol{\mu}(e^i,e^j), \boldsymbol{\nu}(e^i,e^j)\right\rangle.$$

The overall factor of 2 is due to the $\frac{1}{2}$ factor in the definition of $\langle\cdot,\cdot\rangle$. Since we are only interested in Dorfman brackets and their infinitesimal variations, we will mostly work with brackets in $V_{\mathcal{D}}$ (see (3.38)). Then, the only non-vanishing structure coefficients are:

$$\boldsymbol{\mu}(e_i,e_j) = \boldsymbol{\mu}_{ij}^k e_k + \boldsymbol{\mu}_{ijk}e^k, \qquad \boldsymbol{\mu}(e_i,e^j) = -\boldsymbol{\mu}_{ik}^j e^k = -\boldsymbol{\mu}(e^j,e_i),$$

using Einstein's summation convention (summing over all i,j,k and *not* just $i<j$). The fact that the coefficient of e^k in $\boldsymbol{\mu}(e_i,e^j)$ is related to that of e_j in $\boldsymbol{\mu}(e_i,e_k)$ follows from $\langle\boldsymbol{\mu}(\cdot,\cdot),\cdot\rangle\in\Lambda^3(\mathfrak{g}\oplus\mathfrak{g}^*)^*$. We then have

$$\bar g(\boldsymbol{\mu},\boldsymbol{\nu}) := \sum_{i,j,k}\left(3\,\boldsymbol{\mu}_{ij}^k\boldsymbol{\nu}_{ij}^k + \boldsymbol{\mu}_{ijk}\boldsymbol{\nu}_{ijk}\right) = 3\,\bar g(\mu,\nu) + \bar g(H,\tilde H), \qquad \boldsymbol{\mu},\boldsymbol{\nu}\in V_{\mathcal{D}}, \tag{3.41}$$

where $\mu=\boldsymbol{\mu}_{\mathfrak{g}}, \nu=\boldsymbol{\nu}_{\mathfrak{g}}, H=\boldsymbol{\mu}_{\Lambda^3}, \tilde H=\boldsymbol{\nu}_{\Lambda^3}$. Here we have used the extension of $\bar g$ to a positive-definite inner product on $\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$, and on $\Lambda^k\mathfrak{g}^*$, given respectively by

$$\bar g(\mu,\nu) := \sum_{i,j,k}\mu_{ij}^k\nu_{ij}^k, \qquad \mu,\nu\in\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}, \qquad \mu_{ij}^k := \bar g(\mu(e_i,e_j),e_k).$$

and

$$\bar g(\alpha,\beta) := \sum_{i_1,\dots,i_k}\alpha_{i_1\cdots i_k}\beta_{i_1\cdots i_k}, \qquad \alpha,\beta\in\Lambda^k\mathfrak{g}^*, \qquad \alpha_{i_1\cdots i_k} := \alpha(e_{i_1},\dots,e_{i_k}).$$

The metric $\bar g$ also determines a maximal compact subgroup $\mathsf{O}(\mathfrak{g},\bar g)\le\mathsf{L}$. We fix on $\mathfrak{l}$ the $\mathrm{Ad}(\mathsf{O}(\mathfrak{g},\bar g))$-invariant, positive-definite inner product $\bar g_{\mathfrak{l}}$ given by

$$\bar g_{\mathfrak{l}}\left(\begin{pmatrix}A & 0\\ \alpha & -A^*\end{pmatrix}, \begin{pmatrix}B & 0\\ \beta & -B^*\end{pmatrix}\right) := 2\,\mathrm{tr}\,AB^T + \frac{1}{6}\,\mathrm{tr}\,\beta^*\alpha = 2\sum_{i,j}A_{ij}B_{ij} + \frac{1}{6}\sum_{i,j}\alpha_{ij}\beta_{ij}, \tag{3.42}$$

where $A_{ij} := g(Ae_i,e_j)$, $\alpha_{ij}=\alpha(e_i,e_j)$, etc, for a g-orthonormal basis $\{e_i\}$. While the 1/6-factor looks arbitrary, it will play a key role in the forthcoming computations.

Definition 3.2.22. The *moment map* for the action of L on $(\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*),\bar g)$ is the map

$$\mathrm{M}:\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*)\to\mathfrak{l}, \qquad \boldsymbol{\mu}\mapsto\mathrm{M}_{\boldsymbol{\mu}},$$

implicitly defined by

$$\bar g_{\mathfrak{l}}(\mathrm{M}_{\boldsymbol{\mu}},L) = \frac{1}{6}\,\bar g(\Theta(L)\boldsymbol{\mu},\boldsymbol{\mu}), \qquad L\in\mathfrak{l}.$$

Proposition 3.2.23. *The moment map is $\mathsf{O}(\mathfrak{g},\bar g)$-equivariant. Moreover, for each $\boldsymbol{\mu}\in\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*)$ and $L\in\mathfrak{l}$, it satisfies*

$$\bar g_{\mathfrak{l}}(\mathrm{M}_{\boldsymbol{\mu}},L) = \frac{1}{12}\,\frac{d}{dt}\Big|_{t=0}|\exp(tL)\cdot\boldsymbol{\mu}|_g^2.$$

Proof. Let $K\in\mathsf{O}(\mathfrak{g},\bar g)$, for every $L\in\mathfrak{l}$, we have that

$$\bar g_{\mathfrak{l}}(\mathrm{M}_{\overline{K}\cdot\boldsymbol{\mu}},L) = \frac{1}{6}\bar g(\overline{K}^{-1}\cdot\Theta(L)(\overline{K}\cdot\boldsymbol{\mu}),\boldsymbol{\mu}) = \frac{1}{6}\bar g(\Theta(\overline{K}^{-1}L\overline{K})\boldsymbol{\mu},\boldsymbol{\mu}) = \bar g_{\mathfrak{l}}(\mathrm{M}_{\boldsymbol{\mu}},\overline{K}^{-1}L\overline{K}) = \bar g_{\mathfrak{l}}(\overline{K}\,\mathrm{M}_{\boldsymbol{\mu}}\,\overline{K}^{-1},L)$$

yielding the first claim. The second one is trivial computing the derivative on the right hand side. ☐

We now define $\mathcal{R}c_{\boldsymbol{\mu}} \in \mathrm{End}(\mathfrak{g}\oplus\mathfrak{g}^*)$ to be the generalized Ricci curvature of $((\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}}, \overline{\mathcal{G}})$ (at the identity). We also denote by $\mathrm{Ric}_\mu \in \mathrm{End}(\mathfrak{g})$ the $(1,1)$-Ricci tensor of $(\mathfrak{g},\mu,\bar g)$, and by $\mathrm{M}_\mu \in \mathrm{End}(\mathfrak{g})$ the moment map for the action of $\mathsf{GL}(\mathfrak{g})$ on $\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$:

$$\bar g(\mathrm{M}_\mu, A) = \frac{1}{4}\bar g(\theta(A)\mu,\mu), \qquad A\in\mathfrak{gl}(\mathfrak{g}), \qquad \mu\in\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}.$$

The factor $\frac14$ makes this consistent with the standard notation in the homogeneous Riemannian setting, see for instance [214]. In this notation, for $\boldsymbol{\mu}\in\mathcal{D}$ with $\boldsymbol{\mu}_{\mathfrak{g}} = \mu$, $\boldsymbol{\mu}_{\Lambda^3} = H$, we have

$$\mathcal{R}c_{\boldsymbol{\mu}} = \begin{pmatrix} \mathrm{Ric}^{\mathrm{B}}_{\mu,H} & \frac12 \bar g^{-1}(d_\mu^* H)\,\bar g^{-1} \\ -\frac12 d_\mu^* H & -(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})^* \end{pmatrix}, \qquad \mathrm{Ric}^{\mathrm{B}}_{\mu,H} := \mathrm{Ric}_\mu - \frac14\,\bar g^{-1}H^2. \tag{3.43}$$

Proposition 3.2.24. *The generalized Ricci curvature and the moment map for the action of* L *on Dorfman brackets are related by*

$$\mathcal{R}c_{\boldsymbol{\mu}} = \mathrm{M}_{\boldsymbol{\mu}} + \overline{\mathrm{Ric}_\mu} - \overline{\mathrm{M}_\mu} + A_{\boldsymbol{\mu}}\,,$$

for some $A_{\boldsymbol{\mu}} \in \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \overline{\mathcal{G}}) \cap \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \langle\cdot,\cdot\rangle)$.

Proof. Let us set

$$A_{\boldsymbol{\mu}} := \begin{pmatrix} 0 & \frac12 \bar g^{-1}(d_\mu^* H)\,\bar g^{-1} \\ \frac12 d_\mu^* H & 0 \end{pmatrix}. \tag{3.44}$$

It is not hard to see that indeed $A_{\boldsymbol{\mu}} \in \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \overline{\mathcal{G}}) \cap \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \langle\cdot,\cdot\rangle)$. We are thus left with

$$\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}} = \begin{pmatrix} \mathrm{Ric}^{\mathrm{B}}_{\mu,H} & 0 \\ -d_\mu^* H & -(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})^* \end{pmatrix} \in \mathfrak{l}.$$

Furthermore, one can easily see that (3.44) defines the unique element in $\mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \bar{\mathcal{G}}) \cap \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \langle\cdot,\cdot\rangle)$ such that $\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}} \in \mathfrak{l}$.

Moreover, for any $L \simeq (B,\beta)\in\mathfrak{l}$, we compute using the definition of $\bar g_{\mathfrak{l}}$:

$$\bar g_{\mathfrak{l}}\,(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}, L) = 2\,\mathrm{tr}(\mathrm{Ric}_\mu B) - \frac12\,\mathrm{tr}(H^2 B) - \frac16 \bar g(d_\mu^* H, \beta).$$

Furthermore, we notice that

$$\begin{aligned} \bar g(\rho(B)H, H) &= \sum_{i,j,k} (\rho(B)H)(e_i,e_j,e_k)H(e_i,e_j,e_k) = -3\sum_{i,j,k} H(Be_i,e_j,e_k)H(e_i,e_j,e_k) \\ &= -3\sum_i \bar g(\iota_{e_i}H, \iota_{Be_i}H) = -3\bar g(H^2 e_i, Be_i) = -3\,\mathrm{tr}(H^2 B), \end{aligned}$$

from which we get

$$\bar g_{\mathfrak{l}}\,(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}, L) = \bar g_{\mathfrak{l}}\left(\overline{\mathrm{Ric}_\mu}, L\right) + \frac16 \bar g(\rho(B)H - d_\mu\beta, H).$$

On the other hand, by Corollary 3.2.19 and (3.41) the moment map satisfies:

$$\begin{aligned} \bar g_{\mathfrak{l}}\,(\mathrm{M}_{\boldsymbol{\mu}}, L) = \frac16\,\bar g(\Theta(L)\boldsymbol{\mu}, \boldsymbol{\mu}) &= \frac12\,\bar g(\theta(B)\mu,\mu) + \frac16\,\bar g(\rho(B)H - d_\mu\beta, H) \\ &= 2\,\bar g(\mathrm{M}_\mu, B) + \frac16\,\bar g(\rho(B)H - d_\mu\beta, H) \\ &= \bar g_{\mathfrak{l}}\left(\overline{\mathrm{M}_\mu}, L\right) + \frac16\,\bar g(\rho(B)H - d_\mu\beta, H), \end{aligned} \tag{3.45}$$

and the proposition follows. □

Corollary 3.2.25. *If the Lie algebra* $(\mathfrak{g},\mu)$ *is nilpotent, then*

$$\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}} = \mathrm{M}_{\boldsymbol{\mu}}, \qquad A_{\boldsymbol{\mu}} \in \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \overline{\mathcal{G}}) \cap \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \langle\,\cdot\,,\cdot\,\rangle).$$

Proof. Lauret's formula for the Ricci curvature of left-invariant metrics in terms of the moment map on the variety of Lie algebras [219] yields

$$\mathrm{Ric}_\mu = \mathrm{M}_\mu - \frac{1}{2}\,B_\mu - \frac{1}{2}(\mathrm{ad}_\mu U + (\mathrm{ad}_\mu U)^t),$$

where $\bar g(B_\mu\cdot,\cdot)$ is the Killing form of $(\mathfrak{g},\mu)$, and $U\in\mathfrak{g}$ denotes the *mean curvature vector* of the metric Lie algebra $(\mathfrak{g},\mu,\bar g)$, which vanishes if and only if $(\mathfrak{g},\mu)$ is unimodular. In particular, for nilpotent Lie algebras –for which it is known that the Killing form vanishes– this gives $\mathrm{Ric}_\mu = \mathrm{M}_\mu$, and the stated formula follows from Proposition 3.2.24. □

3.2.6 A flow of Dorfman brackets

The fact that the Lie group L (see (3.34)) acts transitively on the space of generalized left-invariant metrics $\mathcal{M}^{\mathsf{G}}$ on $(\mathfrak{g}\oplus\mathfrak{g}^*)_H$ (Proposition 3.2.15) and the equivalence between acting on generalized metrics and acting on Dorfman brackets (Proposition 3.2.20) lead us to consider what would be a natural counterpart to the generalized Ricci flow of left-invariant metrics, on the space of Dorfman brackets $\mathcal{D}\subset\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*)$.

As in previous subsections, we fix a left-invariant ECA $(\mathfrak{g}\oplus\mathfrak{g}^*)_{\bar H}$ and a background left-invariant generalized metric $\overline{\mathcal{G}} = \mathcal{G}(\bar g, 0)$ on it.

Definition 3.2.26. The *generalized bracket flow* with background $\big((\mathfrak{g}\oplus\mathfrak{g}^*)_H, \overline{\mathcal{G}}\big)$ is the initial value problem for a curve of brackets $\boldsymbol{\mu}(t)$ in $\Lambda^2(\mathfrak{g}\oplus\mathfrak{g}^*)^*\otimes(\mathfrak{g}\oplus\mathfrak{g}^*)$:

$$\frac{d}{dt}\boldsymbol{\mu} = -\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu}, \qquad \boldsymbol{\mu}(0) = \boldsymbol{\mu}_0, \tag{3.46}$$

where $\mathcal{R}c_{\boldsymbol{\mu}}$ and $A_{\boldsymbol{\mu}}$ are respectively defined in (3.43) and (3.44).

Since a Dorfman bracket $\boldsymbol{\mu}\in\mathcal{D}$ (see Definition 3.2.16) is determined by its two projections $\boldsymbol{\mu}_{\mathfrak{g}}, \boldsymbol{\mu}_{\Lambda^3}$, it is natural to expect the generalized bracket flow to be equivalent to a coupled flow of Lie brackets on $\mathfrak{g}$ and of closed 3-forms. Indeed:

Proposition 3.2.27. *The generalized bracket flow* (3.46) *with initial condition a Dorfman bracket* $\boldsymbol{\mu}_0\in\mathcal{D}$ *is equivalent to the following coupled flow of brackets* $\mu(t)$ *and* 3*-forms* $H(t)$*:*

$$\begin{cases} \frac{d}{dt}\mu = -\theta(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})\mu, & \mu(0) = \mu_0 := (\boldsymbol{\mu}_0)_{\mathfrak{g}}, \\ \frac{d}{dt}H = \Delta_\mu H - \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H, & H(0) = H_0 := (\boldsymbol{\mu}_0)_{\Lambda^3}. \end{cases} \tag{3.47}$$

Proof. Let $(\mu_t, H_t)_{t\in[0,T)}$ be a solution of (3.47). Then, we define

$$\boldsymbol{\mu}_t(X+\xi, Y+\eta) = \mu_t(X,Y) - \eta\circ(\mu_t)_X + \xi\circ(\mu_t)_Y + \iota_Y\iota_X H_t\,, \quad X,Y\in\mathfrak{g}\,,\ \xi,\eta\in\mathfrak{g}^*,\ t\in[0,T).$$

Using Lemma 3.2.17, we have that $\boldsymbol{\mu}_t\in\mathcal{D}$, for all $t\in[0,T)$. Now, using (3.47) and Corollary 3.2.19, we have that

$$\frac{d}{dt}\boldsymbol{\mu}_t = -\Theta(\mathcal{R}c_{\boldsymbol{\mu}_t} - A_{\boldsymbol{\mu}_t})\boldsymbol{\mu}_t\,, \quad \boldsymbol{\mu}(0) = \boldsymbol{\mu}_0\,.$$

Then, since they solve the same ODE with the same initial datum, $(\boldsymbol{\mu}_t)_{t\in[0,T)}$ coincides with the solution of the generalized bracket flow starting from $\boldsymbol{\mu}_0$.

Viceversa, let $(\boldsymbol{\mu}_t)_{t\in[0,T')}$ be a solution of the generalized bracket flow starting from $\boldsymbol{\mu}_0$. For the sake of simplicity, we will denote $\tilde\mu_t := (\boldsymbol{\mu}_t)_{\mathfrak{g}}$ and $\tilde H_t := (\boldsymbol{\mu}_t)_{\Lambda^3}$. By definition, $\boldsymbol{\mu}_t$ belongs to the L-orbit of $\boldsymbol{\mu}_0$. Using the fact that the L-action does not change π and Corollary 3.2.19, we obtain that

$$\frac{d}{dt}\tilde\mu = \pi\frac{d}{dt}\boldsymbol{\mu} = -\pi\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu} = -\theta(\mathrm{Ric}^{\mathrm{B}}_{\tilde\mu,\tilde H})\tilde\mu\,. \tag{3.48}$$

Moreover, using again the fact that the L-action acts trivially on $\langle \cdot, \cdot \rangle$, we have, for all $X, Y, Z \in \mathfrak{g}$,

$$\begin{aligned}\frac{d}{dt}\tilde{H}(X,Y,Z) &= 2\frac{d}{dt}\langle \boldsymbol{\mu}(X,Y), Z\rangle = -2\langle \Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu}(X,Y), Z\rangle \\ &= -(\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu})_{\Lambda^3}(X,Y,Z) = (\Delta_{\tilde{\mu}}\tilde{H} - \rho(\mathrm{Ric}^{\mathrm{B}}_{\tilde{\mu},\tilde{H}})\tilde{H})(X,Y,Z)\,.\end{aligned} \tag{3.49}$$

where the last equality is due again to Corollary 3.2.19. Equation (3.49) and Equation (3.48) readily guarantees that $(\tilde{\mu}_t, \tilde{H}_t)_{t\in[0,T')}$ coincides with the solution of (3.47), concluding the proof. □

Before going deeply into the discussion on how the generalized bracket flow is connected to the generalized Ricci flow, we state the following lemma which will be useful in what follows.

Lemma 3.2.28. *Let $((\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}(t)}, \bar{\mathcal{G}})_{t\in[0,T)}$ be a solution of the generalized bracket flow starting from $\boldsymbol{\mu}$. Then,*

$$\frac{d}{dt}|d^*_\mu H|^2 = 2(\bar{g}(\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})d^*_\mu H, d^*_\mu H) - 2\bar{g}(\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H, d_\mu d^*_\mu H) - |\Delta_\mu H|^2)\,.$$

In particular, if $d^*_{\mu(0)}H(0) = 0$, then $d^*_\mu H = 0$, for any $t \in [0,T)$.

Proof. First of all, we easily see that, for all $A \in \mathfrak{gl}(\mathfrak{g})$ and brackets μ, one has

$$d^*_{\theta(A)\mu} = [d^*_\mu, \rho(A^t)]. \tag{3.50}$$

Then, we can use (3.50) to infer that

$$\frac{d}{dt}d^*_\mu H = [\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H}), d^*_\mu]H - d^*_\mu\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H + \Delta_\mu d^*_\mu H = \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})d^*_\mu H - 2d^*_\mu\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H + \Delta_\mu d^*_\mu H\,.$$

Then, we easily obtain that

$$\frac{d}{dt}|d^*_\mu H|^2 = 2(\bar{g}(\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})d^*_\mu H, d^*_\mu H) - 2\bar{g}(\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H, d_\mu d^*_\mu H) - |\Delta_\mu H|^2)\,,$$

as claimed. The second part follows trivially from the first one. □

Remark 3.2.29. A similar approach was firstly introduced by Paradiso in [254]. As a main difference from our setting, the author considered the action of $\mathsf{GL}(\mathfrak{g}) \subset \mathsf{L}$ on $\mathcal{D}$. This choice yields a coupled flow as follows:

$$\begin{cases} \frac{d}{dt}\mu = -\theta(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})\mu, & \mu(0) = \mu_0 := (\boldsymbol{\mu}_0)_{\mathfrak{g}}, \\ \frac{d}{dt}H = -\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H, & H(0) = H_0 := (\boldsymbol{\mu}_0)_{\Lambda^3}\,, \end{cases}$$

which, in view of Lemma 3.2.28, is equivalent to (3.47) only in the particular case in which H_0 is harmonic.

The following is the main result of this subsection. It shows that from a left-invariant solution of the generalized Ricci flow one can construct a generalized bracket flow solution whose generalized geometry is gauge-equivalent to the original solution, and that the converse is also true. This in particular implies that the maximal existence times for both flows coincide, and that any generalized geometry question can be studied by means of the generalized bracket flow.

Theorem 3.2.30. *Let $((\mathfrak{g}\oplus\mathfrak{g}^*)_{H_0}, \mathcal{G}_t)_{t\in[0,T)}$ be a left-invariant solution to the generalized Ricci flow equation* (1.31) *and let $(\boldsymbol{\mu}(t))_{t\in[0,T')}$ be the solution to the generalized bracket flow* (3.46) *with background $((\mathfrak{g}\oplus\mathfrak{g}^*)_{H_0}, \mathcal{G}_0)$ and initial condition $\boldsymbol{\mu}(0)$ defined by $\boldsymbol{\mu}(0)_{\mathfrak{g}} = [\cdot,\cdot]_{\mathfrak{g}}$, $\boldsymbol{\mu}(0)_{\Lambda^3} = H_0$, with both solutions defined on a maximal interval of time. Then, $T = T'$ and there exists a one-parameter family of generalized isometries*

$$F_t : ((\mathfrak{g}\oplus\mathfrak{g}^*)_{H_0}, \mathcal{G}_t) \longrightarrow \big((\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}(t)}, \mathcal{G}_0\big)\,, \qquad t \in [0,T).$$

Proof. First, given any metric ECA $((\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}},\mathcal{G})$, we define the endomorphism

$$A_{\boldsymbol{\mu},\mathcal{G}} := \mathrm{Skew}_{\mathcal{G}}(d_\mu^{*_g}(H+d_\mu b)) \in \mathfrak{o}(\mathfrak{g}\oplus\mathfrak{g}^*,\langle\mathcal{G}\cdot,\cdot\rangle),$$

where $\mu=\boldsymbol{\mu}_{\mathfrak{g}}, H:=\boldsymbol{\mu}_{\Lambda^3}$, and $g\in \mathrm{Sym}^2_+(\mathfrak{g}), b\in\Lambda^2\mathfrak{g}^*$ satisfy

$$\mathcal{G}=e^b\begin{pmatrix}0 & g^{-1}\\ g & 0\end{pmatrix}e^{-b}.$$

One readily checks that the map $(\boldsymbol{\mu},\mathcal{G})\mapsto A_{\boldsymbol{\mu},\mathcal{G}}$ is L-equivariant: $A_{F\cdot\boldsymbol{\mu},F\cdot\mathcal{G}}=F\cdot A_{\boldsymbol{\mu},\mathcal{G}}$. Then, since the generalized Ricci curvature is similarly L-equivariant, it follows from Equation (3.44) that

$$\mathcal{R}c(\boldsymbol{\mu},\mathcal{G})-A_{\boldsymbol{\mu},\mathcal{G}}\in\mathfrak{l},$$

where $\mathcal{R}c(\boldsymbol{\mu},\mathcal{G})$ is the generalized Ricci curvature of $((\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}},\mathcal{G})$.

Now, given a solution to the generalized bracket flow $(\boldsymbol{\mu}(t))_{[0,T')}$ (3.46), we define the family of left-invariant endomorphisms $(\mathcal{G}_{\boldsymbol{\mu}(t)})_{t\in[0,T')}$ as follows:

$$\mathcal{G}_{\boldsymbol{\mu}(t)}(x)=\overline{\mathrm{L}_{\boldsymbol{\mu}(t)}(x)}\cdot\mathcal{G}_0(e)\,,\quad x\in\mathsf{G}, \tag{3.51}$$

where $\overline{\mathrm{L}_{\boldsymbol{\mu}(t)}(x)}$ is the lift on $\mathfrak{g}\oplus\mathfrak{g}^*$ of the left-translation in $\mathsf{G}_{\mu(t)}$ by x while $\mathcal{G}_0(e)$ and $\mathcal{G}_{\boldsymbol{\mu}(t)}(x)$ are, respectively, the endomorphisms on the fibers in e and x. We easily see that $\mathcal{G}_{\boldsymbol{\mu}(t)}$ are left-invariant generalized metric on $\mathfrak{g}\oplus\mathfrak{g}^*$. Let us then consider

$$\frac{d}{dt}F_t=F_t(\mathcal{R}c_{\boldsymbol{\mu}(t)}-A_{\boldsymbol{\mu}(t)})\,,\quad F_0=\mathrm{Id}\,,$$

where $A_{\boldsymbol{\mu}(t)}\in\mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*,\mathcal{G}_0)\cap\mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*,\langle\cdot,\cdot\rangle)$ is defined as in (3.44). By a standard result in ODE theory, the family F_t is defined on $[0,T')$ and $(F_t)_{t\in[0,T')}\subseteq\mathsf{L}$. We then define $\boldsymbol{\lambda}(t):=F_t^{-1}\cdot\boldsymbol{\mu}(0)$ yielding that

$$\frac{d}{dt}\boldsymbol{\lambda}=-\Theta(F_t^{-1}F_t')\boldsymbol{\lambda}=-\Theta(\mathcal{R}c_{\boldsymbol{\mu}(t)}-A_{\boldsymbol{\mu}(t)})\boldsymbol{\lambda}\,. \tag{3.52}$$

Thus, $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\mu}(t)$ satisfy the same ODE with equal initial datum concluding that $\boldsymbol{\mu}(t)=F_t^{-1}\cdot\boldsymbol{\mu}(0)$. From this, we can deduce that

$$F_t\colon((\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}(t)},\mathcal{G}_{\boldsymbol{\mu}(t)})\to((\mathfrak{g}\oplus\mathfrak{g}^*)_{H_0},F_t\cdot\mathcal{G}_{\boldsymbol{\mu}(t)})\,,$$

is an isometry. So, denoting $\tilde{\mathcal{G}}_t:=F_t\cdot\mathcal{G}_{\boldsymbol{\mu}(t)}$ we obtain that, in e,

$$\tilde{\mathcal{G}}_t^{-1}\frac{d}{dt}\tilde{\mathcal{G}}_t=F_t\mathcal{G}^{-1}_{\boldsymbol{\mu}(t)}(\mathcal{R}c_{\boldsymbol{\mu}(t)}-A_{\boldsymbol{\mu}(t)})\mathcal{G}_{\boldsymbol{\mu}(t)}F_t^{-1}-F_t(\mathcal{R}c_{\boldsymbol{\mu}(t)}-A_{\boldsymbol{\mu}(t)})F_t^{-1}=-2\mathcal{R}c(\tilde{\mathcal{G}}_t)\,,$$

where the last equality follows from $\mathcal{R}c(\mathcal{G})\mathcal{G}=-\mathcal{G}\mathcal{R}c(\mathcal{G})$ and from $[A_{\boldsymbol{\mu}(t)},\mathcal{G}_{\boldsymbol{\mu}(t)}]=0$. This allows us to conclude that $\mathcal{G}_t=\tilde{\mathcal{G}}_t$, since they solve the same ODE with the same initial condition.

Conversely, we define

$$\frac{d}{dt}F_t=(\mathcal{R}c(\mathcal{G}_t)-A(\mathcal{G}_t))F_t\,,\quad F(0)=\mathrm{Id},$$

where $A(\mathcal{G}_t):=A_{\boldsymbol{\mu},\mathcal{G}_t}$. As before, we consider $\tilde{\mathcal{G}}_t:=F_t\cdot\mathcal{G}_0$ and easily note that

$$\tilde{\mathcal{G}_t}^{-1}\frac{d}{dt}\tilde{\mathcal{G}}_t=\tilde{\mathcal{G}_t}^{-1}(\mathcal{R}c(\mathcal{G}_t)-A(\mathcal{G}_t))\tilde{\mathcal{G}}_t-(\mathcal{R}c(\mathcal{G}_t)-A(\mathcal{G}_t))\,.$$

On the other hand,

$$\mathcal{G}_t^{-1}\frac{d}{dt}\mathcal{G}_t=-2\mathcal{R}c(\mathcal{G}_t)=\mathcal{G}_t^{-1}(\mathcal{R}c(\mathcal{G}_t)-A(\mathcal{G}_t))\mathcal{G}_t-(\mathcal{R}c(\mathcal{G}_t)-A(\mathcal{G}_t)),$$

since, again, $\mathcal{R}c(\mathcal{G}_t)\mathcal{G}_t = -\mathcal{G}_t\mathcal{R}c(\mathcal{G}_t)$ and $[A(\mathcal{G}_t), \mathcal{G}_t] = 0$. So, in particular, we have that

$$\frac{d}{dt}\tilde{\mathcal{G}}_t = [\mathcal{R}c(\mathcal{G}_t) - A(\mathcal{G}_t), \tilde{\mathcal{G}}_t]\,, \quad \frac{d}{dt}\mathcal{G}_t = [\mathcal{R}c(\mathcal{G}_t) - A(\mathcal{G}_t), \mathcal{G}_t]\,,$$

Then, $\tilde{\mathcal{G}}_t$ and $\mathcal{G}_t$ satisfy the same ODE with the same initial condition yielding that $\mathcal{G}_t = F_t \cdot \mathcal{G}_0$. Therefore, if we let $\boldsymbol{\lambda}(t) = F_t^{-1} \cdot \boldsymbol{\mu}(0)$,

$$F_t \colon ((\mathfrak{g} \oplus \mathfrak{g}^*)_{\boldsymbol{\lambda}(t)}, \mathcal{G}_{\boldsymbol{\lambda}(t)}) \to ((\mathfrak{g} \oplus \mathfrak{g}^*)_{H_0}, \mathcal{G}_t)$$

is an isometry. In particular, this implies that

$$F_t \mathcal{R}c_{\boldsymbol{\lambda}(t)} F_t^{-1} = \mathcal{R}c(\mathcal{G}_t)\,, \quad F_t A_{\boldsymbol{\lambda}(t)} F_t^{-1} = A(\mathcal{G}_t),$$

using the $\mathrm{Aut}(E)$-equivariance of the generalized Ricci curvature and the uniqueness of $A(\mathcal{G}) \in \mathfrak{so}(\mathfrak{g} \oplus \mathfrak{g}^*, \mathcal{G}) \cap \mathfrak{so}(\mathfrak{g} \oplus \mathfrak{g}^*, \langle \cdot\,, \cdot \rangle)$ such that $\mathcal{R}c(\mathcal{G}) - A(\mathcal{G}) \in \mathfrak{l}$. Then we have that

$$\frac{d}{dt}F_t = F_t(\mathcal{R}c_{\boldsymbol{\lambda}(t)} - A_{\boldsymbol{\lambda}(t)})\,, \quad F_0 = \mathrm{Id}$$

and repeating the computations in (3.52), we infer that $\boldsymbol{\mu}(t) = \boldsymbol{\lambda}(t)$, concluding the proof. □

Remark 3.2.31. We do not necessarily need to use $\mathcal{G}_0$ as background metric, but this simplifies the formulas.

3.2.7 Global existence on solvmanifolds

As a first application of Theorem 3.2.30, we prove in this section the long-time existence of invariant solutions on any solvmanifold. Recall from [149] that the *generalized scalar curvature* of a metric Courant algebroid $((T \oplus T^*)_H, \mathcal{G}(g, 0))$ is given by

$$\mathcal{S}_{g,H} := \mathrm{R}_g - \tfrac{1}{12}|H|^2, \tag{3.53}$$

where R_g denotes the Riemannian scalar curvature of the metric g. Let us remark that this is not the trace of the generalized Ricci curvature (which is in fact traceless), but it is obtained instead by studying Lichnerowicz formulae for certain Dirac operators [149, §3.9]. Within the moving Dorfman bracket framework, a direct consequence of [220, Lemma 4.2] allows us to infer that on a nilpotent Lie group

$$\mathcal{S}_{\boldsymbol{\mu}} = -\frac{1}{12}(3|\mu|^2 + |H|^2) = -\frac{1}{12}|\boldsymbol{\mu}|^2\,. \tag{3.54}$$

Before stating the main theorem of this section, we will need a preliminary lemma.

Lemma 3.2.32. *Let $(\boldsymbol{\mu}(t))_{t \in (\varepsilon_-, \varepsilon_+)}$ be a solution of the generalized bracket flow defined on its maximal time interval. Then, there exists a uniform constant $C > 0$ such that:*

1. *if $\varepsilon_+ < \infty$, then*
$$|\boldsymbol{\mu}(t)| \geq \frac{C}{(\varepsilon_+ - t)^{\frac{1}{2}}}\,, \quad t \in [0, \varepsilon_+)\,;$$

2. *if $\varepsilon_- > -\infty$, then*
$$|\boldsymbol{\mu}(t)| \geq \frac{C}{(t - \varepsilon_-)^{\frac{1}{2}}}\,, \quad t \in (\varepsilon_-, 0]\,.$$

Proof. For the ease of notation, we will always denote with C a uniform and positive constant which may change from line to line. First of all, we observe that, as in [213, Lemma 3.1], we have $|\mathrm{Ric}_\mu|^2 \leq C|\mu|^4$. Moreover, one can easily show that $|H^2|^2 \leq C|H|^4$. Finally, we obtain that $|d^*_\mu H|^2 \leq C|\mu|^2|H|^2 \leq C|\boldsymbol{\mu}|^4$, using that $*_\mu$ is an isometry and that $|d_\mu \alpha|^2 \leq C|\mu|^2|\alpha|^2$, for any form α and any Lie bracket μ. Then, one can deduce

$$|\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}|^2 = 2|\mathrm{Ric}^{\mathrm{B}}_{\mu,H}|^2 + \frac{1}{6}|d^*_\mu H|^2 \leq C|\boldsymbol{\mu}|^4 \,. \tag{3.55}$$

Now, we fix $t_0 \in [0, \varepsilon_+)$ and we can use (3.55) and the linearity of the infinitesimal L-action Θ to infer that

$$\frac{d}{dt}|\boldsymbol{\mu}|^2 = 2\bar{g}\left(\boldsymbol{\mu}, \frac{d}{dt}\boldsymbol{\mu}\right) \leq C|\boldsymbol{\mu}|^4 \,, \quad t \in [t_0, \varepsilon_+) \,.$$

This implies, by comparison, that

$$|\boldsymbol{\mu}|^2 \leq \frac{1}{-C(t-t_0) + |\boldsymbol{\mu}(t_0)|^{-2}} \,, \quad t \in [t_0, \varepsilon_+) \,. \tag{3.56}$$

On the other hand, (3.56) readily guarantees that $\varepsilon_+ \geq t_0 + \frac{1}{C}|\boldsymbol{\mu}(t_0)|^{-2}$, giving us the first claim. The second claim can be obtained by reversing the time variable. □

We can now state the following blow-up result:

Theorem 3.2.33. *Let $(E, \mathcal{G}_t)_{t \in (\varepsilon_-, \varepsilon_+)}$ be a left-invariant solution of the generalized Ricci flow over a simply-connected Lie group G defined in its maximal time interval. Then,*

1. *if $\varepsilon_+ < \infty$, then $\mathcal{S}_t \to \infty$, as $t \to \varepsilon_+$;*
2. *if $\varepsilon_- > -\infty$, then $\mathcal{S}_t \to -\infty$, as $t \to \varepsilon_-$.*

Proof. Fixing the preferred isotropic splitting σ_0 associate with $\mathcal{G}_0$, we have the time-varying isometry $(E, \mathcal{G}_t)_{t \in (\varepsilon_-, \varepsilon_+)} \simeq_\sigma ((\mathfrak{g} \oplus \mathfrak{g}^*)_{H_t}, \mathcal{G}(g_t, 0))_{t \in (\varepsilon_-, \varepsilon_+)}$. By [301, Proposition 1.1], if $\varphi \colon \mathsf{G} \times (\varepsilon_-, \varepsilon_+) \to \mathbb{R}$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g_t}\right)\varphi_t = \tfrac{1}{6}|H_t|^2_{g_t}; \qquad t \in (\varepsilon_-, \varepsilon_+), \tag{3.57}$$

then,

$$\left(\frac{\partial}{\partial t} - \Delta_{g_t}\right)\left(\mathcal{S}_{g_t,H_t} + 2\Delta_{g_t}\varphi_t - |\nabla_{g_t}\varphi_t|^2\right) = 2|\mathrm{Ric}^{\mathrm{B}}_{g_t,H_t} + \nabla^2_{g_t}\varphi_t - \tfrac{1}{2}(d^*_{g_t}H_t + \iota_{\nabla_{g_t}\varphi_t}H_t)|^2; \qquad t \in (\varepsilon_-, \varepsilon_+).$$

In this case, we set $\varphi_t := \int_0^t \frac{1}{6}|H_s|^2 ds$, which is constant in space and hence satisfies Equation (3.57). Thus, by left-invariance, we have

$$\frac{d}{dt}\mathcal{S}_{g_t,H_t} = 2|\mathrm{Ric}^{\mathrm{B}}_{g_t,H_t} - \tfrac{1}{2}d^*_{g_t}H_t|^2 = |\mathcal{R}c(\mathcal{G}_t)|^2. \tag{3.58}$$

The remainder of the proof follows [213] for the classical Ricci flow. By Theorem 3.2.30, $((\mathfrak{g} \oplus \mathfrak{g}^*)_{H_0}, \mathcal{G}_t)_{t \in (\varepsilon_-, \varepsilon_+)}$ is isometric to $\big((\mathfrak{g} \oplus \mathfrak{g}^*)_{\boldsymbol{\mu}(t)}, \mathcal{G}(g_0, 0)\big)_{t \in (\varepsilon_-, \varepsilon_+)}$, where $\boldsymbol{\mu}$ satisfies Equation (3.46). Let $\mathcal{S}_{\boldsymbol{\mu}(t)} = \mathcal{S}_{g_t}$ denote the generalized scalar curvature of this latter metric ECA and note that by Equation (3.58) it satisfies $\frac{d}{dt}\mathcal{S}_{\boldsymbol{\mu}(t)} = |\mathcal{R}c_{\boldsymbol{\mu}(t)}|^2$. Also observe that

$$\tfrac{d}{dt}|\boldsymbol{\mu}|^2 = -2\left\langle \Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle \leq C|\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}||\boldsymbol{\mu}|^2,$$

for some constant $C > 0$. Hence, observing that $|\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}|^2 \leq 2|\mathcal{R}c_{\boldsymbol{\mu}}|^2$, we have, for $t \in (\varepsilon_-, \varepsilon_+)$,

$$\begin{aligned}\log|\boldsymbol{\mu}(t)|^2 - \log|\boldsymbol{\mu}(0)|^2 &= \int_0^t \tfrac{d}{ds}\log|\boldsymbol{\mu}(s)|^2 ds \leq C\int_0^t |\mathcal{R}c_{\boldsymbol{\mu}(s)} - A_{\boldsymbol{\mu}(s)}| ds \leq C\int_0^t |\mathcal{R}c_{\boldsymbol{\mu}(s)}| ds \\ &\leq C\int_0^t 1 + |\mathcal{R}c_{\boldsymbol{\mu}(s)} - A_{\boldsymbol{\mu}}(s)|^2 ds = C\left(t + \mathcal{S}_{\boldsymbol{\mu}(t)} - \mathcal{S}_{\boldsymbol{\mu}(0)}\right).\end{aligned}$$

On the other hand, by Lemma 3.2.32, we have that $|\boldsymbol{\mu}(t)|^2 \to \infty$, as $t \to \varepsilon_+$, forcing $\mathcal{S}_{\boldsymbol{\mu}(t)} \to \infty$, as $t \to \varepsilon_+$. A similar argument shows the statement for $t \to \varepsilon_-$. □

Recall that left-invariant metrics on solvable Lie groups have non-positive scalar curvature, and zero scalar curvature if and only if they are flat [50]. Hence, we yield the immediate corollary:

Corollary 3.2.34. *Any left-invariant solution of the generalized Ricci flow on a solvable Lie group exists for all positive times.*

In the case of a left-invariant solution on a nilpotent group, we are able to say much more thanks to Corollary 3.2.25. In particular, we can describe the precise asymptotic behaviour of the generalized scalar curvature:

Theorem 3.2.35. *For any left-invariant solution (g_t, H_t) of the generalized Ricci flow on a nilpotent Lie group G, the generalized scalar curvature satisfies $\mathcal{S}_{g_t,H_t} \sim -\frac{1}{t}$, as $t \to \infty$.*

Proof. Let (g_t, H_t) be a solution of the generalized Ricci flow on G and $\boldsymbol{\mu}(t)$ the corresponding generalized bracket flow. Then by Equation (3.46), Corollary 3.2.25 and Definition 3.2.22, we obtain

$$\frac{d}{dt}|\boldsymbol{\mu}(t)|^2 = -2\left\langle \Theta(\mathrm{M}_{\boldsymbol{\mu}(t)})\boldsymbol{\mu}(t), \boldsymbol{\mu}(t)\right\rangle = -6\overline{g}_{\mathfrak{l}}(\mathrm{M}_{\boldsymbol{\mu}(t)}, \mathrm{M}_{\boldsymbol{\mu}(t)}) = -6|M_{\boldsymbol{\mu}(t)}|^2.$$

We now claim that $|\,\mathrm{M}_{\boldsymbol{\mu}(t)}\,|^2 \geq c|\boldsymbol{\mu}|^4$, for some small constant $c > 0$. To see this, first note that the map

$$\mathbb{S}(\Lambda^2(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes (\mathfrak{g} \oplus \mathfrak{g}^*)) \ni \boldsymbol{\mu} \mapsto |\,\mathrm{M}_{\boldsymbol{\mu}}\,|^2,$$

is never zero on the unit sphere. Indeed $\mathrm{M}_{\boldsymbol{\mu}} = 0$ implies that $0 = \mathrm{tr}(\mathrm{M}_{\boldsymbol{\mu}}) = -\frac{1}{6}|\boldsymbol{\mu}|^2 = -\frac{1}{6}$, which is a contradiction. The claim therefore follows from compactness of the unit sphere. Thus,

$$-C|\boldsymbol{\mu}(t)|^4 \leq \frac{d}{dt}|\boldsymbol{\mu}(t)|^2 \leq -c|\boldsymbol{\mu}(t)|^4,$$

so $|\boldsymbol{\mu}(t)|^2 \sim \frac{1}{t}$ by ODE comparison. The result now follows from Equation (3.54).

□

3.2.8 Generalized nilsolitons

In this subsection, following the approach by Lauret in [222] and [217], we give the definition of algebraic generalized solitons, proving their main properties in the nilpotent case. First of all, we introduce the normalized generalized bracket flow.

Definition 3.2.36. The *ℓ-normalized generalized bracket flow* with background data $((\mathfrak{g} \oplus \mathfrak{g}^*)_H, \overline{\mathcal{G}})$ is the initial value problem for a curve of brackets $\boldsymbol{\mu}(t)$ and $\ell = \ell(t)$:

$$\frac{d}{dt}\boldsymbol{\mu} = -\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu} + \ell\boldsymbol{\mu}\,, \qquad \boldsymbol{\mu}(0) = \boldsymbol{\mu}_0\,. \tag{3.59}$$

The multiplication of $\boldsymbol{\mu}$ in (3.59) by the factor ℓ has to be interpreted using Lemma 3.2.4 and Proposition 3.2.21. On the other hand, given a function ℓ, one can obtain a solution of the ℓ-normalized generalized bracket flow from a solution of the generalized bracket flow (3.46) via a time reparametrization and a scaling.

Lemma 3.2.37. *Let $((\mathfrak{g}\oplus\mathfrak{g}^*)_H, \overline{\mathcal{G}})$ be a left-invariant metric ECA over a Lie group G. Let $\boldsymbol{\mu}^\ell(t)$ and $\boldsymbol{\mu}(t)$, respectively, be the solution of the ℓ-normalized generalized bracket flow and the solution of the generalized bracket flow starting from $\boldsymbol{\mu}_0$. Then, there exist a family of scaling $c(t) > 0$ such that $c(0) = 1$ and a time reparametrization $\tau = \tau(t)$ so that*

$$\boldsymbol{\mu}^\ell(t) = c(t)\boldsymbol{\mu}(\tau)\,.$$

Proof. Following [222], we choose

$$c' = \ell c\,,\quad c(0) = 1 \quad \text{and} \quad \tau' = c^2\,,\quad \tau(0) = 0.$$

Then, we have

$$\frac{d}{dt}(c(t)\boldsymbol{\mu}(\tau)) = -c^3\Theta(\mathcal{R}c_{\boldsymbol{\mu}(\tau)} - A_{\boldsymbol{\mu}(\tau)})\boldsymbol{\mu}(\tau) + \ell c\boldsymbol{\mu}(\tau).$$

Using that $\mathcal{R}c_{c\boldsymbol{\mu}} - A_{c\boldsymbol{\mu}} = c^2(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})$, we obtain the claim. □

The case in which we will be interested the most is when we choose $c = \frac{1}{|\boldsymbol{\mu}|}$. Applying Lemma 3.2.37, we have that $\bar{\boldsymbol{\mu}} := \frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|}$ is a solution of the ℓ-normalized generalized bracket flow with

$$\ell_{\bar{\boldsymbol{\mu}}} = \frac{\bar{g}(\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu}, \boldsymbol{\mu})}{|\boldsymbol{\mu}|^4} = 6\bar{g}(\mathcal{R}c_{\bar{\boldsymbol{\mu}}} - A_{\bar{\boldsymbol{\mu}}}, \mathrm{M}_{\bar{\boldsymbol{\mu}}})\,.$$

With this choice of the normalization, we will refer to the corresponding normalized generalized bracket flow as *scalar-normalized generalized bracket flow.* In this case, the norm of the solution of the scalar-normalized generalized bracket flow will remain constantly 1, provided the initial datum has unit norm. Indeed,

$$\frac{d}{dt}|\boldsymbol{\mu}|^2 = 2\bar{g}(\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu}, \boldsymbol{\mu})(-1 + |\boldsymbol{\mu}|^2)\,,$$

then $|\boldsymbol{\mu}|^2 = 1$, since $|\boldsymbol{\mu}_0|^2 = 1$. Recalling (3.54), this also implies that $\mathcal{S}_{\boldsymbol{\mu}} = \mathcal{S}_{\boldsymbol{\mu}_0} = -\frac{1}{12}$ along the scalar-normalized generalized bracket flow.

With this in mind, we can give the definition of generalized algebraic soliton.

Definition 3.2.38. Let $\boldsymbol{\mu} \in \mathcal{D}$. We say that $\boldsymbol{\mu}$ is an algebraic soliton for the generalized Ricci flow if it is a fixed point of the scalar-normalized generalized bracket flow, i.e. the following is satisfied

$$\Theta(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}})\boldsymbol{\mu} = \ell_{\boldsymbol{\mu}}\boldsymbol{\mu}\,. \tag{3.60}$$

We say that $\boldsymbol{\mu}$ is expanding, steady or shrinking if, respectively, $\ell_{\boldsymbol{\mu}} > 0$, $\ell_{\boldsymbol{\mu}} = 0$ or $\ell_{\boldsymbol{\mu}} < 0$.

Building from this definition, we can derive equivalent conditions for a Dorfman bracket to be an algebraic soliton for the generalized Ricci flow.

Proposition 3.2.39. *Let $\boldsymbol{\mu} \in \mathcal{D}$. Then, the following are equivalent:*

1. *$\boldsymbol{\mu}$ is an algebraic soliton;*
2. *the generalized bracket flow starting at $\boldsymbol{\mu}$ evolves only by scaling;*
3. *$\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}} = \lambda\mathrm{Id} + \boldsymbol{D}$ with $\boldsymbol{D} \in \mathrm{Der}_\lambda(\boldsymbol{\mu})$ and $\lambda \in \mathbb{R}$;*
4. *there exist $D \in \mathrm{Der}(\mu)$ and $\lambda \in \mathbb{R}$ such that*

$$\begin{cases} \mathrm{Ric}^{\mathrm{B}}_{\mu,H} = \lambda\mathrm{Id} + D\,, \\ \Delta_\mu H = \lambda H + \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H\,. \end{cases} \tag{3.61}$$

Proof. The equivalence between Item 1 and Item 2 is trivial using (3.60) and Lemma 3.2.37. On the other hand, if $\boldsymbol{\mu}$ is an algebraic soliton, the fact that $\Theta(\mathrm{Id})\boldsymbol{\mu} = -\boldsymbol{\mu}$, for all $\boldsymbol{\mu} \in \mathcal{D}$, implies $\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}} = \lambda\mathrm{Id} + \boldsymbol{D}$ with $\boldsymbol{D} \in \mathrm{Der}(\boldsymbol{\mu})$. Moreover, $\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}} \in \mathfrak{so}(\mathfrak{g} \oplus \mathfrak{g}^*, \langle\,\cdot\,,\cdot\,\rangle)$, then, $\boldsymbol{D}$ has to satisfy

$$2\lambda\langle\,\cdot\,,\cdot\,\rangle + \langle \boldsymbol{D}\cdot, \cdot\rangle + \langle\cdot, \boldsymbol{D}\cdot\rangle = 0\,,$$

which gives Item 3. Viceversa, it is easy to show that if Item 3 holds, then (3.60) is satisfied with $\ell_{\boldsymbol{\mu}} = -\lambda$. Furthermore, Corollary 3.2.19 guarantees the equivalence between (3.60) and

$$\begin{cases} \theta(\mathrm{Ric}^{\mathrm{B}}_{\mu,H} + \ell_{\boldsymbol{\mu}}\mathrm{Id})\mu = 0\,, \\ \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H = \ell_{\boldsymbol{\mu}} H \end{cases}$$

which is equivalent to (3.61) with $\lambda = -\ell_{\boldsymbol{\mu}}$, concluding the proof. □

Item 3 and Item 4 in Proposition 3.2.39 can be considered as the static definition, in terms of both generalized and classical objects, of algebraic solitons and so Proposition 3.2.39 has to be regarded as the analogue of Proposition 3.2.9 in the varying Dorfman bracket framework.

Moreover, Proposition 3.2.39 allows us to construct from algebraic solitons for the generalized Ricci flow generalized metrics which are generalized Ricci solitons as in Definition 3.2.6.

Lemma 3.2.40. *Let $\boldsymbol{\mu}$ be an algebraic soliton for the generalized Ricci flow. Then, $\mathcal{G}_{\boldsymbol{\mu}}$, defined as in* (3.51)*, is a generalized Ricci soliton.*

Proof. The first equation in (3.61) can be equivalently written, in terms of symmetric $(0,2)$-tensors, as

$$\mathrm{Ric}^{\mathrm{B}}_{g_\mu,H} = \lambda g_\mu + \frac{1}{2}(g_\mu(D\cdot,\cdot) + g_\mu(\cdot,D\cdot)) = \lambda g_\mu - \frac{1}{2}\mathcal{L}_{X_D} g_\mu\,,$$

where $X_D = \frac{d}{dt}\big|_{t=0}\varphi_t$ with $\varphi_t \in \mathrm{Aut}(\mathsf{G}_\mu)$ such that $d_e\varphi_t = e^{tD}$. Moreover, plugging the first equation of (3.61) into the second one, we have

$$\Delta_\mu H = \lambda H + \rho(\lambda \mathrm{Id} + D)H = -2\lambda H + \rho(D)H = -2\lambda H + \mathcal{L}_{X_D} H\,.$$

The claim follows using Proposition 3.2.9. □

Before going into the discussion of algebraic solitons in the nilpotent case, we will need the following preliminary lemma about λ-derivations, which is nothing but an adaptation of Lemma 3.2.8 to the invariant case.

Lemma 3.2.41. *Let $(\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}}$ be a left-invariant ECA over a Lie group G. Then, we have that, for any $c \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{R}$,*

$$\mathrm{Aut}(c\cdot(\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}}, (\mathfrak{g}\oplus\mathfrak{g}^*)_{\boldsymbol{\mu}}) = \left\{\bar{A}_c e^b \mid b \in \Lambda^2\mathfrak{g}^*\,, \quad A \in \mathrm{Aut}(\mu)\,, \quad A^{-1}\cdot H = c(H - db)\right\}$$

and

$$\mathrm{Der}_\lambda(\boldsymbol{\mu}) = \left\{ \begin{pmatrix} D & 0 \\ \alpha & -2\lambda\mathrm{Id} - D^* \end{pmatrix} \;\middle|\; D \in \mathrm{Der}(\mu), \quad \rho(D)H - d_\mu\alpha - 2\lambda H = 0 \right\}.$$

Proof. The first assertion follows directly from the discussion in the non-invariant case. Definition 3.2.7 guarantees that a λ-derivation has to be of the form

$$\boldsymbol{D} = \begin{pmatrix} D & 0 \\ \alpha & -2\lambda\mathrm{Id} - D^* \end{pmatrix},$$

where $D \in \mathrm{Der}(\mu)$ and $\rho(D)H - d\alpha - 2\lambda H = 0$. □

The moment map formulation in the nilpotent case allows us to derive many other properties concerning algebraic solitons, which, in this particular case, will be called generalized nilsolitons, mimicking the classical nomenclature, see [217].

Proposition 3.2.42. *Let $\boldsymbol{\mu} \in \mathcal{D}$ be a generalized nilsoliton. Then, it is expanding. Moreover, the derivation D satisfies the following property:*

$$\mathrm{tr}\left(\left(D + \frac{1}{4}H^2\right)D'\right) = -\lambda\mathrm{tr}(D')\,,$$

for all $D' \in \mathrm{Der}(\mu)$ defining $\boldsymbol{D'} = (D',\alpha) \in \mathrm{Der}_{\lambda'}(\boldsymbol{\mu})$, for some $\lambda' \in \mathbb{R}$. Finally, $\bar{g}(\iota_X H, d^*_\mu H) = 0$, for all $X \in \mathfrak{g}$.

Proof. We have that

$$\ell_{\boldsymbol{\mu}} = 6\bar{g}(\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}, \mathrm{M}_{\boldsymbol{\mu}}) = 6|\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}|^2 \geq 0\,,$$

where the last equality follows from Corollary 3.2.25. On the other hand, $|\mathcal{R}c_{\boldsymbol{\mu}} - A_{\boldsymbol{\mu}}|^2 = 0$ would, in particular, imply that

$$\mathrm{Ric}_\mu = \frac{1}{4}H^2 \geq 0\,,$$

which cannot happen in the nilpotent case, see [220, Lemma 4.2, (iii)], giving us the first claim.

As regards the second, fixed $\boldsymbol{D'} = (D', \alpha) \in \mathrm{Der}_{\lambda'}(\boldsymbol{\mu})$, we have that

$$\bar{g}_{\mathfrak{l}}(\lambda\mathrm{Id} + \boldsymbol{D}, \lambda'\mathrm{Id} + \boldsymbol{D'}) = \frac{1}{6}\bar{g}(\Theta(\lambda'\mathrm{Id})\boldsymbol{\mu}, \boldsymbol{\mu}) = -\frac{\lambda'}{6}|\boldsymbol{\mu}|^2 = \lambda'\left(2\mathrm{R}_\mu - \frac{1}{6}|H|^2\right)\,. \tag{3.62}$$

On the other hand, we can use Proposition 3.2.39 to deduce that $\boldsymbol{D} = (D, -d_\mu^* H)$ and then obtain

$$\bar{g}_{\mathfrak{l}}(\lambda\mathrm{Id} + \boldsymbol{D}, \lambda'\mathrm{Id} + \boldsymbol{D'}) = 2\mathrm{tr}((\lambda\mathrm{Id} + D)(\lambda'\mathrm{Id} + D')) - \frac{1}{6}\bar{g}(d_\mu^* H, \alpha)\,.$$

The fact that $\boldsymbol{D'} \in \mathrm{Der}_{\lambda'}(\boldsymbol{\mu})$ implies, thanks to Lemma 3.2.41, that $d_\mu\alpha = \rho(D')H - 2\lambda' H$. This can be used to infer that

$$\begin{aligned}
\bar{g}_{\mathfrak{l}}(\lambda\mathrm{Id} + \boldsymbol{D}, \lambda'\mathrm{Id} + \boldsymbol{D'}) &= 2\lambda'\mathrm{tr}(\mathrm{Ric}^{\mathrm{B}}_{\mu,H}) + 2\lambda\mathrm{tr}(D') + 2\mathrm{tr}(DD') - \frac{1}{6}\bar{g}(\alpha, d_\mu^* H) \\
&= 2\lambda'\left(\mathrm{R}_\mu - \frac{1}{4}|H|^2\right) + 2\lambda\mathrm{tr}(D') + 2\mathrm{tr}(DD') + \frac{1}{2}\mathrm{tr}(H^2 D') + \frac{1}{3}\lambda'|H|^2 \\
&= \lambda'\left(2\mathrm{R}_\mu - \frac{1}{6}|H|^2\right) + 2\lambda\mathrm{tr}(D') + 2\mathrm{tr}(DD') + \frac{1}{2}\mathrm{tr}(H^2 D')\,.
\end{aligned} \tag{3.63}$$

Now, it is sufficient to use (3.63) in (3.62) to obtain the claim. Finally, recall that for metric nilpotent Lie algebras, any symmetric derivation D satisfies $\mathrm{tr}(\mathrm{ad}_X D) = 0$, for all $X \in \mathfrak{g}$. Then, the soliton equation in μ implies

$$0 = \mathrm{tr}(\mathrm{ad}_X H^2) = -\frac{1}{3}\bar{g}(\rho(\mathrm{ad}_X)H, H) = -\frac{1}{3}\bar{g}(\mathcal{L}_X H, H) = -\frac{1}{3}\bar{g}(d_\mu \iota_X H, H) = -\frac{1}{3}\bar{g}(\iota_X H, d_\mu^* H),$$

as claimed. □

Motivated by the wide range of results concerning classical nilsolitons proved throughout the years, see for instance [217], many questions about generalized nilsolitons can be raised. First of all, as usual, let $(E, \overline{\mathcal{G}})$ be a left-invariant metric ECA over a nilpotent Lie group G and assume that $\overline{\mathcal{G}}$ is a generalized nilsoliton. Using the preferred isotropic splitting induced by $\overline{\mathcal{G}}$, we can identify $(E, \overline{\mathcal{G}}) \simeq ((\mathfrak{g} \oplus \mathfrak{g}^*)_H, \mathcal{G}(\bar{g}, 0))$, for some closed $H \in \Lambda^3\mathfrak{g}^*$. So, in this setting, one can wonder if $\overline{\mathcal{G}}$ is unique, up to scaling and up to the action of $\mathrm{Aut}([\cdot,\cdot])$, among left-invariant generalized metrics defining a preferred representative $H' \in [H] \in H^3(\mathfrak{g}, \mathbb{R})$.

Remark 3.2.43. Considering $[H] \in H^3(\mathsf{G}, \mathbb{R})$, the uniqueness is not true in general. Indeed, on $\mathfrak{n}_3 = \mathrm{Lie}(\mathrm{Heis}(3,\mathbb{R}))$, the following

$$\mu(e_1, e_2) = e_3\,, \quad H = e^{123}\,, \quad \lambda = -2\,, \quad D = \mathrm{diag}(1,1,2)$$

defines a generalized nilsoliton, see Proposition 3.2.63 for the proof, with $[H] = 0 \in H^3(\mathrm{Heis}(3,\mathbb{R}), \mathbb{R})$. One can easily observe that the above generalized nilsoliton defines exactly the expanding soliton for the generalized Ricci flow described in Example 3.2.11. On the other hand, $\mu(e_1, e_2) = e_3$ defines also the classical nilsoliton, see [218, Theorem 4.2], i.e. $H = 0$.

In order to address the uniqueness question, we need to understand better the geometric properties of the space of left-invariant generalized metrics.

As we saw in Proposition 3.2.15, fixed E a left-invariant ECA over a Lie group G, not necessarily nilpotent, the space of left-invariant generalized metric $\mathcal{M}^{\mathsf{G}}$ can be presented as a homogeneous space where L is acting transitively by conjugation.

Of course, Proposition 3.2.15 gives also that the action of $\mathsf{SO}(\langle\cdot,\cdot\rangle)$ is transitive on $\mathcal{M}^{\mathsf{G}}$. This allows us to present $\mathcal{M}^{\mathsf{G}}$ as a homogeneous space in a different manner. Clearly, the isotropy of $\mathsf{SO}(\langle\cdot,\cdot\rangle)$ in $\overline{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}$ is

$$\overline{\mathcal{G}}_{\mathsf{SO}(\langle\cdot,\cdot\rangle)} = \left\{F \in \mathsf{SO}(\langle\cdot,\cdot\rangle) \,|\, F\overline{\mathcal{G}}F^{-1} = \overline{\mathcal{G}}\right\} =: \mathrm{Isom}(\overline{\mathcal{G}}, \mathsf{SO})\,.$$

On the other hand, we observe that

$$\theta\colon \mathsf{SO}(\langle\cdot,\cdot\rangle) \to \mathsf{SO}(\langle\cdot,\cdot\rangle)\,, \quad \sigma(F) = \overline{\mathcal{G}}F\overline{\mathcal{G}}\,,$$

is an automorphism of $\mathsf{SO}(\langle\cdot,\cdot\rangle)$ which is clearly involutive, and the set of fixed points of θ precisely coincides with $\mathrm{Isom}(\overline{\mathcal{G}}, \mathsf{SO})$. Thus, we obtain that $(\mathsf{SO}(\langle\cdot,\cdot\rangle), \mathrm{Isom}(\overline{\mathcal{G}}, \mathsf{SO}))$ is a Riemannian symmetric pair, as defined in [184, IV 3.4]. As a consequence, we obtain a reductive decomposition of $\mathfrak{so}(\langle\cdot,\cdot\rangle) = \mathfrak{h}\oplus\mathfrak{m}$ such that $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, where, using $(E, \overline{\mathcal{G}}) \simeq ((\mathfrak{g}\oplus\mathfrak{g}^*)_H, \mathcal{G}(\bar{g}, 0))$,

$$\mathfrak{h} := \mathrm{Lie}(\mathrm{Isom}(\overline{\mathcal{G}}, \mathsf{SO})) = \mathfrak{so}(\mathfrak{g}\oplus\mathfrak{g}^*, \overline{\mathcal{G}}) = \left\{ \begin{pmatrix} A & \bar{g}^{-1}\alpha\bar{g}^{-1} \\ \alpha & -A^* \end{pmatrix} \,\middle|\, A \in \mathfrak{so}(n)\,, \quad \alpha \in \Lambda^2\mathfrak{g}^* \right\}$$

and

$$\mathfrak{m} := \ker(d\theta + \mathrm{Id}) = \left\{ \begin{pmatrix} A & -\bar{g}^{-1}\alpha\bar{g}^{-1} \\ \alpha & -A^* \end{pmatrix} \,\middle|\, A \in \mathrm{Sym}(n)\,, \quad \alpha \in \Lambda^2\mathfrak{g}^* \right\}\,.$$

As usual, we will be identifying $\mathfrak{m}$ with $T_{\overline{\mathcal{G}}}\mathcal{M}^{\mathsf{G}}$. Now, using again the identification $(E, \overline{\mathcal{G}}) \simeq ((\mathfrak{g}\oplus\mathfrak{g}^*)_H, \mathcal{G}(\bar{g}, 0))$, we can define a non degenerate bilinear form $\bar{g}$ on $\mathfrak{so}(\langle\,\cdot\,,\cdot\,\rangle)$ such that, for all $(A,\alpha), (A',\alpha') \in \mathfrak{m}$ and $(B,\beta), (B',\beta') \in \mathfrak{h}$,

$$\bar{g}((A,\alpha),(A',\alpha')) = 2\mathrm{tr}(AA') + 2\bar{g}(\alpha,\alpha')\,, \quad \bar{g}((B,\beta),(B',\beta')) = -\frac{3}{4}(\mathrm{tr}(BB') + \bar{g}(\beta,\beta'))\,.$$

It is not hard to see that $\bar{g}$ is an $\mathrm{Ad}(\mathrm{Isom}(\overline{\mathcal{G}}, \mathsf{SO}))$-invariant bilinear form which, when restricted to $\mathfrak{l}$, coincides with the inner product $\bar{g}_{\mathfrak{l}}$ defined in (3.42). Moreover, using [209, Theorem 3.5], the $\mathsf{SO}(\langle\cdot,\cdot\rangle)$-invariant Riemannian metric induced by it on $\mathcal{M}^{\mathsf{G}}$ will be naturally reductive. This readily guarantees that all the geodesics emanating from $\overline{\mathcal{G}}$ are of the form $\exp(tX)\cdot\overline{\mathcal{G}}$, for some $X \in \mathfrak{m}$, see [209, Theorem 3.3]. Finally, [209, Theorem 3.5] guarantees that $(\mathcal{M}^{\mathsf{G}}, \bar{g})$ has all non-positive sectional curvatures.

Keeping in mind this and building from [182], we can define the generalized scalar functional

$$\mathcal{S}\colon \mathcal{M}^{\mathsf{G}} \to \mathbb{R}\,,$$

which associates to a given left-invariant generalized metric its generalized scalar curvature as in (3.53). We will study some analytic properties of this functional.

Proposition 3.2.44. *Let E be a left-invariant ECA over a nilpotent Lie group G. Then, we have:*

1. *the gradient of $\mathcal{S}$ at any $\overline{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}$ is given by*

$$\nabla_{\overline{\mathcal{G}}}\mathcal{S} = \tilde{\mathrm{M}}(\overline{\mathcal{G}}) := \begin{pmatrix} \mathrm{Ric}^{\mathrm{B}}_{\bar{g},H} & \frac{1}{6}\bar{g}^{-1}d^*_{\bar{g}}H\bar{g}^{-1} \\ -\frac{1}{6}d^*_{\bar{g}}H & -(\mathrm{Ric}^{\mathrm{B}}_{\bar{g},H})^* \end{pmatrix}$$

after using $(E, \overline{\mathcal{G}}) \simeq_\sigma ((\mathfrak{g}\oplus\mathfrak{g}^*)_H, \mathcal{G}(\bar{g}, 0))$;

2. *$\mathcal{S}$ is concave along geodesics, i.e. if γ is a geodesic emanating from $\overline{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}$, then $(\mathcal{S}\circ\gamma)''(0) \leq 0$;*

3. *$(\mathcal{S}\circ\gamma)''(0) = 0$ if and only if $\gamma(t) = \exp(tX)\cdot\overline{\mathcal{G}}$ where $X \in \mathrm{Der}_0([\cdot,\cdot]) \cap \mathfrak{m}$. In this case, $\mathcal{S}\circ\gamma$ is constant.*

Proof. We fix $\overline{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}$ and, as usual, we identify $(E, \overline{\mathcal{G}}) \simeq_\sigma ((\mathfrak{g} \oplus \mathfrak{g}^*)_H, \mathcal{G}(\bar{g}, 0))$. We will make use of the moving Dorfman brackets approach. First of all, we observe that, if $\alpha \in \Lambda^2\mathfrak{g}^*$ and $\boldsymbol{\mu} \in \mathcal{D}$, we have that

$$2\left\langle \overline{\mathcal{G}}\Theta \begin{pmatrix} 0 & -\bar{g}^{-1}\alpha\bar{g}^{-1} \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}(e_i, e_j), \boldsymbol{\mu}(e_i, e_j) \right\rangle = -\boldsymbol{\mu}_{ijk}\boldsymbol{\mu}_{ij}^s \alpha_{ks}\,,$$

while

$$4\left\langle \overline{\mathcal{G}}\Theta \begin{pmatrix} 0 & -\bar{g}^{-1}\alpha\bar{g}^{-1} \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}(e_i, e^j), \boldsymbol{\mu}(e_i, e^j) \right\rangle = -2\boldsymbol{\mu}_{is}^j \boldsymbol{\mu}_{iks}\alpha_{jk}$$

which gives us that

$$\bar{g}\left(\Theta \begin{pmatrix} 0 & -\bar{g}^{-1}\alpha\bar{g}^{-1} \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}, \boldsymbol{\mu}\right) = 3\boldsymbol{\mu}_{ijk}\boldsymbol{\mu}_{ij}^s\alpha_{sk} = -\bar{g}(d_\mu\alpha, H)\,. \tag{3.64}$$

Then, combining (3.64) with (3.45), we obtain that, for $X = (A, \alpha) \in \mathfrak{m}$,

$$\bar{g}(\Theta(X)\boldsymbol{\mu}, \boldsymbol{\mu}) = \bar{g}\left(\Theta \begin{pmatrix} A & 0 \\ \alpha & -A^* \end{pmatrix} \boldsymbol{\mu}, \boldsymbol{\mu}\right) + \bar{g}\left(\Theta \begin{pmatrix} 0 & -\bar{g}^{-1}\alpha\bar{g}^{-1} \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}, \boldsymbol{\mu}\right) = 12\mathrm{tr}(\mathrm{Ric}^{\mathrm{B}}_{\mu,H}A) - 2\bar{g}(d_\mu\alpha, H)\,.$$

On the other hand, considering

$$\tilde{\mathrm{M}}_{\boldsymbol{\mu}} = \begin{pmatrix} \mathrm{Ric}^{\mathrm{B}}_{\mu,H} & \frac{1}{6}\bar{g}^{-1}d_\mu^*H\bar{g}^{-1} \\ -\frac{1}{6}d_\mu^*H & -(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})^* \end{pmatrix}$$

we then obtain that

$$\bar{g}(\tilde{\mathrm{M}}_{\boldsymbol{\mu}}, X) = 2\mathrm{tr}(\mathrm{Ric}^{\mathrm{B}}_{\mu,H}A) - \frac{1}{3}\bar{g}(\alpha, d_\mu^*H) = \frac{1}{6}\bar{g}(\Theta(X)\boldsymbol{\mu}, \boldsymbol{\mu}) = \frac{1}{12}\frac{d}{dt}\Big|_{t=0}|\exp(tX)\cdot\boldsymbol{\mu}|^2\,.$$

We now compute the gradient of $\mathcal{S}$ at $\overline{\mathcal{G}}$. If $\boldsymbol{\mu} \in \mathcal{D}$ is the given Dorfman bracket on E, we know that, for any t, the generalized metric $\exp(tX)\cdot\overline{\mathcal{G}}$ is isometric to $\mathcal{G}_{\exp(-tX)\cdot\boldsymbol{\mu}}$. So, given $X \in \mathfrak{m}$, we have that

$$d_{\overline{\mathcal{G}}}\mathcal{S}(X) = \frac{d}{dt}\Big|_{t=0}\mathcal{S}(\exp(tX)\cdot\overline{\mathcal{G}}) = -\frac{1}{12}\frac{d}{dt}\Big|_{t=0}|\exp(-tX)\cdot\boldsymbol{\mu}|^2 = \frac{1}{6}\bar{g}(\Theta(X)\boldsymbol{\mu}, \boldsymbol{\mu}) = \bar{g}(\tilde{\mathrm{M}}_{\boldsymbol{\mu}}, X) = \bar{g}(\tilde{\mathrm{M}}(\overline{\mathcal{G}}), X)\,,$$

giving us the first claim.

Let now $\gamma(t) = \exp(tX)\cdot\overline{\mathcal{G}}$, for some $X \in \mathfrak{m}$, be a geodesic emanating from $\overline{\mathcal{G}}$, we have

$$(\mathcal{S}\circ\gamma)''(0) = -\frac{1}{12}\frac{d^2}{dt^2}\Big|_{t=0}|\exp(-tX)\cdot\boldsymbol{\mu}|^2 = -\frac{1}{6}(\bar{g}(\Theta(X)\Theta(X)\boldsymbol{\mu}, \boldsymbol{\mu}) + |\Theta(X)\boldsymbol{\mu}|^2) = -\frac{1}{3}|\Theta(X)\boldsymbol{\mu}|^2 \leq 0\,, \tag{3.65}$$

giving us the second claim.

Finally, clearly, if $X \in \mathrm{Der}_0(\boldsymbol{\mu}) \cap \mathfrak{m}$, $\exp(tX) \in \mathrm{Aut}(\boldsymbol{\mu})$ which guarantees that $\mathcal{S}\circ\gamma$ is constant for all t, the viceversa holds too thanks to (3.65). □

Remark 3.2.45. The element $\tilde{\mathrm{M}}_{\boldsymbol{\mu}} \in \mathfrak{m}$ can also be viewed as the moment map of the $\mathsf{SO}(\langle\cdot,\cdot\rangle)$-action on the space of Dorfman brackets $\mathcal{D}$. Indeed, it precisely satisfies the relation:

$$\bar{g}(\tilde{\mathrm{M}}_{\boldsymbol{\mu}}, X) = \frac{1}{6}\,\bar{g}(\Theta(X)\boldsymbol{\mu}, \boldsymbol{\mu})\,, \qquad X \in \mathfrak{m}\,.$$

Unfortunately, $\tilde{\mathrm{M}}_{\boldsymbol{\mu}}$ coincides with the generalized Ricci curvature only on the subspace of Dorfman bracket with harmonic torsion.

Proposition 3.2.44 can be considered as the analogue, in the generalized setting, of [182, Lemma 3.3] specialized in the nilpotent case. So, following the steps in [217, Theorem 3.5], we can give a partial answer to the uniqueness question, proving that generalized nilsolitons with harmonic torsion are unique.

Theorem 3.2.46. *Let E be a left-invariant ECA over a nilpotent Lie group G and let $\overline{\mathcal{G}}$ be a generalized nilsoliton with harmonic torsion. Then, up to scaling and up to the action of* $\mathrm{Aut}(E)$*,* $\overline{\mathcal{G}}$ *is unique.*

Proof. Let us now consider the following closed Lie subgroup of $\mathsf{SO}(\langle\cdot,\cdot\rangle)$:

$$\mathsf{K} := \pm \prod_{c\in\mathbb{R}^+} c^{-\frac{1}{2}}\mathrm{Aut}(c\cdot E, E)\,.$$

First of all, we need some clarifications about K. The action of an element in K on $\overline{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}$ is defined as follows: an element $c^{-\frac{1}{2}}F \in \mathsf{K}$ acts on $\overline{\mathcal{G}}$ as the following composition:

$$c^{-\frac{1}{2}}F\colon (E, \langle\cdot,\cdot\rangle, c^{-\frac{1}{2}}\boldsymbol{\mu}, \overline{\mathcal{G}}) \to (c\cdot E, \overline{\mathcal{G}}) \to (E, F\cdot\overline{\mathcal{G}}) \tag{3.66}$$

where the first arrow is defined, using the isometry induced by the preferred isotropic splitting of $\overline{\mathcal{G}}$, by the transformation

$$\overline{c^{-\frac{1}{2}}\mathrm{Id}}\colon ((\mathfrak{g}\oplus\mathfrak{g}^*)_{c^{-\frac{1}{2}}\boldsymbol{\mu}}, \mathcal{G}(\bar{g},0)) \to ((\mathfrak{g}\oplus\mathfrak{g}^*)_{cH}, \mathcal{G}(c\bar{g},0))$$

which is an isometry of ECA's thanks to Proposition 3.2.21. Furthermore, we can use the explicit isometry in Remark 3.2.5 to identify

$$((\mathfrak{g}\oplus\mathfrak{g}^*)_{cH}, \mathcal{G}(c\bar{g},0)) \simeq (c\cdot E, \overline{\mathcal{G}})\,.$$

In conclusion, the first arrow in (3.66) is the composition of the above isometries and it is then induced by the transformation $c^{-\frac{1}{2}}\mathrm{Id}$. Finally, the last arrow in (3.66) is just the usual action of the element $F \in \mathrm{Aut}(c\cdot E, E)$. Moreover, by Lemma 3.2.41, any element $c^{-\frac{1}{2}}F$ can be represented with

$$c^{-\frac{1}{2}}\mathrm{Id}\circ \bar{A}_c e^b = \begin{pmatrix} c^{-\frac{1}{2}}A & 0 \\ ((c^{-\frac{1}{2}}A)^{-1})^* b & ((c^{-\frac{1}{2}}A)^{-1})^* \end{pmatrix} = \overline{c^{-\frac{1}{2}}A}e^b \in \mathsf{SO}(\langle\cdot,\cdot\rangle)\,,$$

where $b \in \Lambda^2\mathfrak{g}^*$, and $A \in \mathrm{Aut}(\mu)$ such that $A^{-1}\cdot H = c(H - db)$. Finally, $-c^{\frac{1}{2}}F$ is given by the composition of

$$\overline{-\mathrm{Id}}\colon ((\mathfrak{g}\oplus\mathfrak{g}^*)_{-c^{-\frac{1}{2}}\boldsymbol{\mu}}, \mathcal{G}(\bar{g},0)) \to ((\mathfrak{g}\oplus\mathfrak{g}^*)_{c^{-\frac{1}{2}}\boldsymbol{\mu}}, \mathcal{G}(\bar{g},0))$$

with the previously defined action. Now, we can prove that K is actually a Lie subgroup of $\mathsf{SO}(\langle\cdot,\cdot\rangle)$. Clearly, it is sufficient to prove that, if $c^{-\frac{1}{2}}F \in c^{-\frac{1}{2}}\mathrm{Aut}(c\cdot E, E)$ and $\gamma^{-\frac{1}{2}}G \in \gamma^{-\frac{1}{2}}\mathrm{Aut}(\gamma\cdot E, E)$, then $c^{-\frac{1}{2}}F(\gamma^{-\frac{1}{2}}G)^{-1} \in \mathsf{K}$. It is not hard to see that

$$c^{-\frac{1}{2}}\overline{A_c}e^b(\gamma^{-\frac{1}{2}}\overline{B_\gamma}e^{b'})^{-1} = \left(\frac{c}{\gamma}\right)^{-\frac{1}{2}}(\overline{AB^{-1}})_{\frac{c}{\gamma}}e^{\beta}\,, \quad \beta := \gamma B\cdot(b-b')\,.$$

Then, we observe that $B\cdot H = \frac{1}{\gamma}H + B\cdot db'$. This implies that

$$BA^{-1}\cdot H = c(B\cdot H - B\cdot db) = \frac{c}{\gamma}(H - d\beta)\,,$$

giving us the claim. Moreover, K is closed in $\mathsf{SO}(\langle\cdot,\cdot\rangle)$. Thanks to the structure of the subgroup K, the claim is equivalent to prove that if

$$c_i^{-\frac{1}{2}}\mathrm{Aut}(c_i\cdot E, E) \ni c_i^{-\frac{1}{2}}F_i \to F_\infty \in \mathsf{SO}(\langle\cdot,\cdot\rangle)\,, \quad i\to\infty\,,$$

then $F_\infty \in \mathsf{K}$. Following [182, Remark 3.7], let us fix a generalized metric $\overline{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}$, then

$$c_i^{-\frac{1}{2}}F_i\cdot\overline{\mathcal{G}} \to \tilde{\mathcal{G}} \in \mathcal{M}^{\mathsf{G}}.$$

But, as a consequence of (3.53), the effect of scaling on the scalar curvature and on $|H|^2$, recalling that F_i preserves the Dorfman bracket, for all i, we have that,

$$\mathcal{S}(c_i^{-\frac{1}{2}}F_i\cdot\bar{\mathcal{G}}) = c_i\mathcal{S}(F_i\cdot\bar{\mathcal{G}}) = c_i\mathcal{S}(\bar{\mathcal{G}}) \to \mathcal{S}(\tilde{\mathcal{G}})\,, \quad i\to\infty\,.$$

If $S(\tilde{\mathcal{G}}) \neq 0$, then $c_i \to c$, as $i \to \infty$. This, in particular, implies that $F_i \to \tilde{F} \in \mathsf{SO}(\langle\cdot,\cdot\rangle)$ and so $F_\infty = c^{-\frac{1}{2}}\tilde{F}$. We just need to prove that $\tilde{F} \in \mathrm{Aut}(c \cdot E, E)$. This is guaranteed by the fact that $F_i = \overline{(A_i)}_{c_i} e^{b_i} \in \mathrm{Aut}(c_i \cdot E, E)$ if and only if

$$A_i^{-1} \cdot H = c_i(H - db_i) \text{ and } A_i \in \mathrm{Aut}(\mu)$$

and passing to the limit in the last equality gives us the claim. Finally, if $\mathcal{S}(\tilde{\mathcal{G}}) = 0$, then, automatically, the Dorfman bracket on E is abelian, i.e. $H = 0$ and μ is abelian. In this case,

$$\mathrm{Aut}(c \cdot E, E) = \{\bar{A}_c e^b \,|\, A \in \mathsf{GL}(\mathfrak{g})\,, \quad b \in \Lambda^2\mathfrak{g}^*\}\,.$$

Then, $\mathsf{K} = \mathrm{Aut}(E)$, which is clearly closed.

Easily, we see that

$$T_{\overline{\mathcal{G}}}(\mathsf{K}\cdot\overline{\mathcal{G}}) = \bigoplus_{\lambda\in\mathbb{R}}(\lambda\mathrm{Id} + (\mathrm{Der}_\lambda(\boldsymbol{\mu}) \cap \mathfrak{m}))\,.$$

Moreover, we note that, using again Lemma 3.2.41

$$\lambda\mathrm{Id} + (\mathrm{Der}_\lambda(\boldsymbol{\mu}) \cap \mathfrak{m}) = \left\{ \begin{pmatrix} \lambda\mathrm{Id} + D & 0 \\ 0 & -\lambda\mathrm{Id} - D^* \end{pmatrix} \,\middle|\, D \in \mathrm{Der}(\mu) \cap \mathrm{Sym}(n), \quad \rho(D)H - 2\lambda H = 0 \right\}.$$

Now, suppose that $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}}'$ are two generalized nilsolitons with harmonic torsion which do not belong to the same K-orbit. Since $\mathcal{O} = \mathsf{K}\cdot\overline{\mathcal{G}}$ and $\mathcal{O}' = \mathsf{K}\cdot\overline{\mathcal{G}}'$ consist in generalized nilsolitons, we can assume that $d(\overline{\mathcal{G}}, \overline{\mathcal{G}}') = d(\mathcal{O}, \mathcal{O}')$ and that the unique geodesic $\gamma\colon [0,1] \to \mathcal{M}^{\mathsf{G}}$ such that $\gamma(0) = \overline{\mathcal{G}}$ and $\gamma(1) = \overline{\mathcal{G}}'$ meets $\mathcal{O}$ and $\mathcal{O}'$ orthogonally. Thanks to Remark 3.2.45, we know $\tilde{\mathrm{M}}(\overline{\mathcal{G}}) = \mathcal{R}c(\overline{\mathcal{G}}) = \lambda\mathrm{Id} + \boldsymbol{D}$, $\boldsymbol{D} \in \mathrm{Der}_\lambda(\boldsymbol{\mu}) \cap \mathfrak{m}$ and using Proposition 3.2.44 (1), we have that $0 = g(\gamma'(0), \nabla_{\overline{\mathcal{G}}}\mathcal{S}) = g(\gamma'(0), \mathcal{R}c(\overline{\mathcal{G}}))$ and $0 = g(\gamma'(1), \nabla_{\overline{\mathcal{G}}'}\mathcal{S}) = g(\gamma'(1), \mathcal{R}c(\overline{\mathcal{G}}'))$ which gives us that

$$(\mathcal{S}\circ\gamma)'(0) = (\mathcal{S}\circ\gamma)'(1) = 0\,.$$

On the other hand, using Proposition 3.2.44 (2), $\mathcal{S}$ is concave along the geodesics, so $\mathcal{S}\circ\gamma$ is constant, which, thanks to Proposition 3.2.44 (3), happens if and only if $\gamma = \exp(tX)\cdot\overline{\mathcal{G}}$ where $X \in \mathrm{Der}_0(\boldsymbol{\mu}) \cap \mathfrak{m}$, giving us a contradiction. ☐

A first consequence of Theorem 3.2.46 is the uniqueness among generalized nilsolitons within $0 \in H^3(\mathfrak{g}, \mathbb{R})$ of the classical nilsoliton, if it exists.

Unfortunately, the proof of Theorem 3.2.46 is exclusive of the harmonic torsion case. In view of this and of Theorem 3.2.46 itself, we can weaken the uniqueness question as follows.

Question 3.2.47. Let $(E, \overline{\mathcal{G}})$ a left-invariant metric ECA over a nilpotent Lie group G and assume $\overline{\mathcal{G}}$ is a generalized nilsoliton. Using $(E, \bar{\mathcal{G}}) \simeq ((\mathfrak{g}\oplus\mathfrak{g}^*)_H, \mathcal{G}(\bar{g}, 0))$, is it H $\bar{g}$-harmonic?

With last objective of addressing Question 3.2.47, we can find a equivalent condition for it to be true.

Proposition 3.2.48. *Let $\boldsymbol{\mu} \in \mathcal{D}$ be an algebraic soliton. Then, $d_\mu^* H = 0$ if and only if* $\mathrm{tr}((d_\mu^* H)^2 \mathrm{Ric}^{\mathrm{B}}_{\mu,H}) \leq 0$.

Proof. Using the fact that (3.39) completely determines $\boldsymbol{\mu} \in \mathcal{D}$ and (3.60), we have that $\boldsymbol{\mu}$ can be viewed as a pair (μ, H) solving

$$\begin{cases} \theta(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})\mu = -\lambda\mu, \\ \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H + d_\mu d_\mu^* H = -\lambda H, \end{cases}$$

for some $\lambda \in \mathbb{R}$. Applying d^* to the second equation yields, using (3.50),

$$0 = \lambda d_\mu^* H + d_\mu^*\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H + d_\mu^* d_\mu d_\mu^* H = \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})d_\mu^* H + d_\mu^* d_\mu d_\mu^* H, \tag{3.67}$$

where in the last equality we used $\theta(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})\mu = -\lambda\mu$. Now, just as in the proof of Proposition 3.2.24, we have that $\bar{g}(\rho(A)\alpha, \alpha) = -2\,\mathrm{tr}(\alpha^2 A)$, for all 2-forms $\alpha \in \Lambda^2\mathfrak{g}^*$ and $A \in \mathfrak{gl}(\mathfrak{g})$. Now taking an inner product of (3.67) with $d^*_\mu H$ gives

$$0 = \bar{g}(\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})d^*_\mu H, d^*_\mu H) + |d_\mu d^*_\mu H|^2 = -2\,\mathrm{tr}((d^*_\mu H)^2 \mathrm{Ric}^{\mathrm{B}}_{\mu,H}) + |d_\mu d^*_\mu H|^2.$$

Since $-\mathrm{tr}((d^*_\mu H)^2\mathrm{Ric}^{\mathrm{B}}_{\mu,H}) \geq 0$, then, we yield $d^*_\mu H = 0$ as required. □

Besides Question 3.2.47, one can also pose a finiteness question on generalized nilsolitons, namely:

Question 3.2.49. Let G be a nilpotent Lie group. Is the set of generalized nilsolitons on G finite?

In Section 3.2.11 we will provide, within the full classification of generalized nilsolitons up to dimension 4, a counterexample to Question 3.2.49.

Finally, it is well known, see for instance [217], that there exist nilpotent Lie algebras which do not admit any classical nilsoliton. So, one may hope to produce generalized nilsolitons even on such Lie algebras. This cannot be done in general. Indeed, on characteristically nilpotent Lie algebras, i.e. such that the Lie algebra of derivations is nilpotent, we cannot find neither classical nor generalized nilsolitons, due to the fact that any derivation is nilpotent, see [317, Theorem 5].

3.2.9 Long-time behaviour on nilmanifolds

In this section, we will recall a definition of Cheeger-Gromov convergence for exact Courant algebroids introduced in [149]. Then, we will discuss the long-time behaviour of the generalized Ricci flow on nilpotent Lie groups in the particular case in which the starting ECA has harmonic torsion, showing that, in this case, the asympotics are precisely the generalized nilsolitons defined in Subsection 3.2.8.

Given a subset $V \subset M$ of a manifold, we will denote by $i_V : V \to M$ the inclusion map.

Definition 3.2.50 ([149, Definition 5.25])**.** A sequence of pointed metric ECAs $(E_i \to M_i, \mathcal{G}_i, p_i)_{i\geq 1}$ converges to $(\overline{E} \to \overline{M}, \overline{\mathcal{G}}, \overline{p})$ in the *generalized Cheeger–Gromov* topology if there exists a sequence $\{U_i\}_{i\geq 1}$ of neighbourhoods of $\overline{p}$ exhausting $\overline{M}$ and Courant algebroid isomorphisms

$$(F_i, f_i) \colon (i^*_{U_i}\overline{E} \to U_i) \to (F_i(i^*_{U_i}\overline{E}) \to f_i(U_i)) \subset (E_i \to M_i),$$

satisfying:

1. $f_i^{-1}(p_i) \to \overline{p}$, as $i \to \infty$;
2. $F_i^{-1} \circ \mathcal{G}_i \circ F_i \to \overline{\mathcal{G}}$ in $C^\infty_{\mathrm{loc}}(\overline{E})$, as $i \to \infty$.

As usual, fixing isotropic splittings of any ECA of the sequence, we can rewrite this definition as follows.

Lemma 3.2.51. *The sequence* $((TM_i \oplus T^*M_i)_{H_i}, \mathcal{G}(g_i, b_i), p_i)_{i\geq 1}$ *converges to* $((T\overline{M} \oplus T^*\overline{M})_{\overline{H}}, \mathcal{G}(\overline{g}, \overline{b}), \overline{p})$ *in the generalized Cheeger–Gromov topology if and only if there exists an exhaustion* $\{U_i\}_{i\geq 1}$ *of* $\overline{M}$ *consisting in neighbourhoods of* $\overline{p}$*, diffeomorphisms* $f_i \colon U_i \to f_i(U_i) \subset M_i$*, and 2-forms* $a_i \in \Lambda^2(U_i)$ *such that*

1. $i^*_{U_i}\overline{H} = f_i^* H_i + da_i$*, for all* $i \geq 1$*;*
2. $f_i^{-1}(p_i) \to \overline{p}$*, as* $i \to \infty$*;*
3. $f_i^* g_i \to \overline{g}$ *in* $C^\infty_{\mathrm{loc}}(\overline{M})$*, as* $i \to \infty$*;*
4. $f_i^* b_i - a_i \to \overline{b}$ *in* $C^\infty_{\mathrm{loc}}(\overline{M})$*, as* $i \to \infty$*.*

Proof. For the first direction, we see that in these isotropic splittings, the isomorphisms in Definition 3.2.50 are given by

$$(F_i, f_i) = (\overline{f_i} \circ e^{a_i}, f_i),$$

for some 2-forms $a_i \in \Lambda^2(U_i)$ satisfying

$$i_{U_i}^* \overline{H} = f_i^* H_i + da_i.$$

In this case

$$F_i^{-1} \circ \mathcal{G}_i(g_i, b_i) \circ F_i = \mathcal{G}(f_i^* g_i, f_i^* b_i - a_i),$$

so $F_i \circ \mathcal{G}_i(g_i, b_i) \circ F_i^{-1} \to \mathcal{G}(\overline{g}, \overline{b})$ if and only if Item 3 and Item 4 hold. The converse is easily seen. □

Moreover, we can specify Lemma 3.2.51 when the preferred isotropic splitting of $\mathcal{G}_i$ are chosen.

Corollary 3.2.52. *The sequence $((TM_i \oplus T^*M_i)_{H_i}, \mathcal{G}(g_i, 0), p_i)$ converges to $((T\overline{M} \oplus T^*\overline{M})_{\overline{H}}, \mathcal{G}(\overline{g}, 0), \overline{p})$ in the generalized Cheeger–Gromov topology if and only if there is an exhaustion $\{U_i \ni \overline{p}\}$ of $\overline{M}$, diffeomorphisms $f_i \colon U_i \to f_i(U_i) \subset M_i$, and 2-forms $a_i \in \Lambda^2(U_i)$ such that*

1. *$i_{U_i}^* \overline{H} = f_i^* H_i + da_i$, for all $i \geq 1$;*
2. *$f_i^{-1}(p_i) \to \overline{p}$, as $i \to \infty$;*
3. *$f_i^* g_i \to \overline{g}$ in $C^\infty_{\mathrm{loc}}(\overline{M})$, as $i \to \infty$;*
4. *$a_i \to 0$ in $C^\infty_{\mathrm{loc}}(\overline{M})$, as $i \to \infty$.*

In particular, $f_i^* H_i \to \overline{H}$ in $C^\infty_{\mathrm{loc}}(\overline{M})$, as $i \to \infty$.

We now describe how such convergence is achieved at the level of Lie brackets. Fix a vector space $\mathfrak{g}$ and a generalized metric $\overline{\mathcal{G}} = \mathcal{G}(\overline{g}, 0)$ on $\mathfrak{g} \oplus \mathfrak{g}^*$. For a Dorfman bracket $\boldsymbol{\mu}$ on $\mathfrak{g} \oplus \mathfrak{g}^*$, denote by $((\mathfrak{g} \oplus \mathfrak{g})_{\boldsymbol{\mu}}, \mathsf{G}_{\boldsymbol{\mu}_\mathfrak{g}}, \mathcal{G}_{\boldsymbol{\mu}})$ the corresponding metric ECA.

Theorem 3.2.53. *Let $(\boldsymbol{\mu}_i)_{i\geq 1}$ be a sequence of nilpotent Dorfman brackets converging to $\overline{\boldsymbol{\mu}}$ on $\mathfrak{g} \oplus \mathfrak{g}^*$. Then, $\left((\mathfrak{g} \oplus \mathfrak{g}^*)_{\boldsymbol{\mu}_i}, \mathsf{G}_{(\boldsymbol{\mu}_i)_\mathfrak{g}}, \mathcal{G}_{\boldsymbol{\mu}_i}, e\right)_{i\geq 1}$ converges to $\left((\mathfrak{g} \oplus \mathfrak{g}^*)_{\overline{\boldsymbol{\mu}}}, \mathsf{G}_{\overline{\boldsymbol{\mu}}_\mathfrak{g}}, \mathcal{G}_{\overline{\boldsymbol{\mu}}}, e\right)$ in the generalized Cheeger–Gromov topology.*

Proof. As usual, we consider $\mathcal{G}(g, 0)$ as a background generalized metric, hence

$$\left((\mathfrak{g} \oplus \mathfrak{g}^*)_{\boldsymbol{\mu}_i}, \mathsf{G}_{(\boldsymbol{\mu}_i)_\mathfrak{g}}, \mathcal{G}_{\boldsymbol{\mu}_i}, e\right) = \left((\mathfrak{g} \oplus \mathfrak{g}^*)_{H_i}, \mathsf{G}_{\mu_i}, \mathcal{G}(g_{\mu_i}, 0), e\right),$$

where $H_i = \boldsymbol{\mu}_{\Lambda^3}$ and $\mu_i = \boldsymbol{\mu}_\mathfrak{g}$. We also set $\overline{\mu} = \overline{\boldsymbol{\mu}}_\mathfrak{g}$ and $\overline{H} = \overline{\boldsymbol{\mu}}_{\Lambda^3}$. Then by nilpotence, the exponential maps of G_{μ_i} and $\mathsf{G}_{\overline{\mu}}$ are diffeomorphisms. We consider

$$f_i := \exp_{\mu_i} \circ \exp_{\overline{\mu}}^{-1} \colon \mathsf{G}_{\overline{\mu}} \to \mathsf{G}_{\mu_i},$$

which is a diffeomorphism with $f_i(e) = e$. Recall (see for instance [221]) that for $X \in \mathfrak{g}$, the exponential map $\exp_\mu \colon \mathfrak{g} \to \mathsf{G}_\mu$ of the Lie bracket $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ satisfies

$$(d\exp_\mu)_X = (dL_{\exp_\mu(X)})_e \circ \frac{\mathrm{Id} - e^{-\mathrm{ad}_\mu X}}{\mathrm{ad}_\mu X},$$

where

$$\frac{\mathrm{Id} - e^{-\mathrm{ad}_\mu X}}{\mathrm{ad}_\mu X} = \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_\mu X)^k.$$

Hence, df_i depends continuously on μ_i, and an argument analogous to [221, Corollary 6.10] shows that $f_i^* g_{\mu_i} \to g_{\overline{\mu}}$ and $f_i^* H_i \to \overline{H}$ smoothly and uniformly on compact subsets. Then, we define

$$a_i := \iota_{\mathrm{Id}}(\overline{H} - f_i^* H_i),$$

where $\mathrm{Id} \in \Gamma(T\mathbb{R}^n)$ is the identity vector field on $\mathbb{R}^n$. By Cartan's formula,

$$da_i = \mathcal{L}_{\mathrm{Id}}(\overline{H} - f_i^* H_i) = \overline{H} - f_i^* H_i.$$

Hence, $\overline{H} = f_i^* H_i + da_i$ with $a_i \to 0$ smoothly and uniformly on compact subsets. Thus, the result follows by Corollary 3.2.52. □

With the technical details established, we now turn to studying the long-time behaviour of the generalized Ricci flow on nilpotent Lie groups when the initial metric ECA has harmonic torsion.

Proposition 3.2.54. *Let $(\boldsymbol{\mu}(t))_{t\in[0,\infty)}$ be a solution to the generalized bracket flow* (3.46) *such that $\boldsymbol{\mu}(0)_{\mathfrak{g}}$ is a nilpotent Lie bracket and $\boldsymbol{\mu}(0)_{\Lambda^3}$ is harmonic. Then, the rescaled brackets $\frac{\boldsymbol{\mu}(t)}{|\boldsymbol{\mu}(t)|}$ converge as $t \to \infty$ to an algebraic soliton bracket $\boldsymbol{\mu}_\infty \neq 0$.*

Proof. Long-time existence follows immediately from Corollary 3.2.34. Up to a time-reparametrization, the rescaled brackets $\overline{\boldsymbol{\mu}}(t) = \frac{\boldsymbol{\mu}(t)}{|\boldsymbol{\mu}(t)|}$ solve (3.59) with $\ell = \ell_{\overline{\boldsymbol{\mu}}}$, defined in Subsection 3.2.8. That is, the scalar-normalized bracket flow. Since the brackets are nilpotent, Corollary 3.2.25 implies that

$$\frac{d}{dt}\overline{\boldsymbol{\mu}} = -\Theta(M_{\overline{\boldsymbol{\mu}}} + 6|M_{\overline{\boldsymbol{\mu}}}|^2 \,\mathrm{Id})\overline{\boldsymbol{\mu}}.$$

On the other hand, using the same strategy as in [53, Lemma 7.2], Lemma 3.2.28 and similar computations as in the proof of Proposition 3.2.44, one can easily see that this is, up to scaling, the negative gradient flow of the real-analytic functional

$$\Lambda^2(\mathfrak{g} \oplus \mathfrak{g}^*)^* \otimes (\mathfrak{g} \oplus \mathfrak{g}^*) \ni \boldsymbol{\mu} \longmapsto \frac{|M_{\boldsymbol{\mu}}|^2}{|\boldsymbol{\mu}|^4} \in \mathbb{R}.$$

By compactness and Łojasiewicz's Theorem on real-analytic gradient flows [237], the ω-limit consists of a unique fixed point $\boldsymbol{\mu}_\infty$ to the scalar-normalized bracket flow, and hence it is an algebraic soliton. □

Let us remark some important facts. First of all, we should observe that the group L is not reductive. This does not allow us, in general, to use directly [53, Lemma 7.2] to deduce the fact that the scalar-normalized bracket flow is a gradient flow. Moreover, it is easy to be shown that, removing the harmonicity of the initial torsion, the scalar-normalized bracket flow is not the negative gradient flow the above functional. Finally, this gives an alternative and dynamical proof of the uniqueness of generalized nilsolitons with harmonic torsion.

Let us now prove the main convergence results on nilpotent Lie groups.

Theorem 3.2.55. *Let $(E \to \mathsf{G}, \mathcal{G}_t)$ be a left-invariant solution of the generalized Ricci flow on a left-invariant ECA over a simply-connected, nilpotent, non-abelian Lie group G. Assume that the preferred representative of the Ševera class of E determined by $\mathcal{G}_0$ is harmonic. Then, the scaled ECA's $(-\mathcal{S}(\mathcal{G}_t) \cdot E, \mathcal{G}_t)$ converge in the generalized Cheeger–Gromov sense to a left-invariant, nilpotent, non-abelian, generalized Ricci soliton.*

Proof. After choosing an isotropic splitting, the left-invariant solution is equivalent to the bracket flow (3.46) by Theorem 3.2.30. Denoting this solution by $(\boldsymbol{\mu}(t))_{t\in[0,\infty)}$, we have by (3.54) and Proposition 3.2.54 that the rescaled brackets $\frac{\boldsymbol{\mu}(t)}{(-\mathcal{S}(\mathcal{G}_t))^{1/2}}$ converge to a nilpotent, non-abelian algebraic soliton bracket. These rescaled brackets correspond to the rescaled ECA's $(-\mathcal{S}(\mathcal{G}_t) \cdot E, \mathcal{G}_t)$ by Proposition 3.2.21 and Lemma 3.2.4. Finally, by Theorem 3.2.53, we obtain convergence of $(-\mathcal{S}(\mathcal{G}_t) \cdot E, \mathcal{G}_t)$ to an algebraic soliton. □

From a more classical perspective, we obtain the following slightly weaker statement:

Corollary 3.2.56. *Let (G, H_t, g_t) be a left-invariant solution of the generalized Ricci flow on a simply-connected, non-abelian, nilpotent Lie group G and assume that H_0 is g_0-harmonic. Then, the rescaled family $(\mathsf{G}, -\mathcal{S}(H_t, g_t)H_t, -\mathcal{S}(H_t, g_t)g_t)$ converges to a nilpotent, generalized soliton $(\overline{\mathsf{G}}, \overline{H}, \overline{g})$ in the following sense: for every sequence of times, there is a subsequence $t_k \to \infty$, and diffeomorphisms $f_k \colon \overline{\mathsf{G}} \to \mathsf{G}$ fixing the identity, such that $(f_k^* H_{t_k}, f_k^* g_{t_k}) \to (\overline{H}, \overline{g})$, as $k \to \infty$, in $C^\infty_{loc}(\overline{\mathsf{G}})$.*

3.2.10 Connections to the pluriclosed flow

In this subsection, we show how all the machinery developed in the previous subsections applies to the special case of pluriclosed flow.

First of all, as a direct consequence of Proposition 1.1.60, Remark 1.2.14 and Corollary 3.2.34, we have the following.

Corollary 3.2.57. *Let (G, J, g) be a solvable Lie group endowed with a left-invariant SKT structure. Then, the solution of the pluriclosed flow starting from g is immortal.*

Corollary 3.2.57 recovers many known results in the literature such as those in [54] on compact complex surfaces, in [99] on SKT nilmanifolds, in [32] on almost-abelian SKT solvmanifolds and on Oeljeklaus-Toma manifolds, as saw in Section 3.1.

The gauge-equivalence between pluriclosed and generalized Ricci flow allows us also to give an equivalent condition for a generalized Ricci soliton to be a pluriclosed soliton.

Theorem 3.2.58. *Let (M, J) be a complex manifold endowed with a SKT Hermitian metric g. Then, a soliton (g, H, X) for the generalized Ricci flow is a pluriclosed soliton if and only if $H = d^c\omega$ and $X + \theta^\sharp$ is holomorphic.*

Proof. We recall that (ω, X) is a pluriclosed soliton if and only if

$$(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1} = \lambda\omega + \frac{1}{2}\mathcal{L}_X\omega\,, \quad X \text{ holomorphic}\,, \quad \lambda \in \mathbb{R}\,.$$

By means of the holomorphicity of X, we have that $[\mathcal{L}_X, J] = 0$ on any form on M, yielding that $d^c\mathcal{L}_X\omega = \mathcal{L}_X d^c\omega$. Then, using [305, Proposition 6.4] and defining $H = d^c\omega$, we obtain

$$\frac{1}{2}\Delta_g H - \frac{1}{2}\mathcal{L}_{\theta^\sharp}H = -d^c(\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1} = -\lambda H - \frac{1}{2}\mathcal{L}_X H\,,$$

which gives us

$$\Delta_g H = -2\lambda H - \mathcal{L}_{X-\theta^\sharp}H\,.$$

On the other hand, we have that

$$\mathrm{Ric}^{\mathrm{B}}_{g,H} + \frac{1}{2}\mathcal{L}_{\theta^\sharp}g = (\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}(\cdot, J\cdot) = \lambda g + \frac{1}{2}\mathcal{L}_X g.$$

This guarantees that

$$\mathrm{Ric}^{B}_{g,H} = \lambda g + \frac{1}{2}\mathcal{L}_{X-\theta^\sharp}g\,,$$

giving us the first claim. The converse can be proved following the same steps backwards. □

In view of Proposition 1.1.60 and Remark 1.2.14, we readily obtain the gauge-equivalence between the generalized bracket flow and the bracket flow for the pluriclosed flow introduced in [32] and [99]. First of all, we recall the definition of the latter.

Definition 3.2.59. Let (G, μ_0) be a Lie group endowed with left-invariant complex structure J and a left-invariant SKT metric ω. The bracket flow is the following evolution equation:

$$\frac{d}{dt}\mu = -\frac{1}{2}\theta(P)\mu\,, \quad \mu(0) = \mu_0 \tag{3.68}$$

where $P \in \mathfrak{gl}(\mathfrak{g}, J)$ defined by $\omega(P\cdot, \cdot) = (\mathrm{Ric}^{\mathrm{B}}(\omega))^{1,1}(\cdot, \cdot)$.

Combining Theorem 3.2.30, Proposition 1.1.60, Remark 1.2.14 and [99, Theorem 4.2], we have the following.

Corollary 3.2.60. *Up to a time reparametrization by* 2*, the bracket flow* (3.68) *is gauge equivalent to the generalized bracket flow* (3.46)*.*

Finally, Theorem 3.2.58 can be specialized in the case of algebraic solitons, as defined in Definition 3.2.38 and in [32, Definition 2.7]. For the sake of clarity, we quickly recall the definition of algebraic solitons for the pluriclosed flow.

Definition 3.2.61 ([32], Definition 2.7)**.** A Lie group G endowed with a left-invariant SKT structure (J, g) is called algebraic pluriclosed soliton if there exists $D \in \mathrm{Der}(\mathfrak{g}) \cap \mathfrak{gl}(\mathfrak{g}, J)$ and $\lambda \in \mathbb{R}$ such that

$$P = \lambda \mathrm{Id} + \frac{1}{2}(D + D^t).$$

Corollary 3.2.62. *Let G be a Lie group endowed with a left-invariant SKT structure (J, g). Then, an algebraic soliton (g, H, D) for the generalized Ricci flow is an algebraic pluriclosed soliton if and only if $D - \mathrm{ad}_{\theta^\sharp} \in \mathrm{Der}(\mathfrak{g}) \cap \mathfrak{gl}(\mathfrak{g}, J)$ and $H = d^c\omega$.*

Proof. By Lemma 3.2.40, (g, H, D) is an algebraic soliton for the generalized Ricci flow if and only if $(g, H, -X_D)$ is a soliton for the generalized Ricci flow where X_D is the vector field defined by

$$X_D = \frac{d}{dt}\Big|_{t=0} \varphi_t \text{ with } \varphi_t \in \mathrm{Aut}(\mathsf{G}) \text{ such that } d_e\varphi_t = e^{tD}\,.$$

On the other hand, by Theorem 3.2.58, $(g, H, -X_D)$ is a pluriclosed soliton if and only if $H = d^c\omega$ and $-X_D + \theta^\sharp$ is holomorphic. But, $-X_D + \theta^\sharp$ is holomorphic if and only if $D - \mathrm{ad}_{\theta^\sharp} \in \mathfrak{gl}(\mathfrak{g}, J)$, giving the claim. □

3.2.11 Classification of solitons up to dimension 4

In this subsection, we will provide a classification of generalized nilsolitons on nilpotent Lie algebras of dimension up to 4. As it is clear from the discussion in Subsection 3.2.8, we can only hope to classify generalized nilsolitons up to scaling and up to generalized automorphisms of the ECA we are considering. Classical nilsolitons were classified in [218]. So, we will focus on classifying the non-classical ones.

As it is well known, besides the abelian ones, nilpotent Lie algebras of dimension up to 4 are 3: $\mathfrak{n}_3$, the Lie algebra associated to the Heisenberg group $\mathrm{Heis}(3, \mathbb{R})$, which is the only non-abelian nilpotent Lie algebra of dimension 3, $\mathfrak{n}_3 \oplus \mathbb{R}$ and $\mathfrak{n}_4$, the only indecomposable non-abelian nilpotent Lie algebra of dimension 4.

We will go through the classification case by case using (3.61).

In dimension 3, there is only one generalized soliton which is non-classic. This is mainly due to the fact that $H^3(\mathfrak{n}_3, \mathbb{R}) = \mathbb{R}$.

Proposition 3.2.63. *On $\mathfrak{n}_3$, the unique non-classic generalized nilsoliton is given by:*

$$\mu(e_1, e_2) = e_3\,, \quad H = e^{123}\,, \quad \lambda = -2\,, \quad D = \mathrm{diag}(1, 1, 2)\,.$$

Proof. Given a left-invariant metric on $\mathrm{Heis}(3, \mathbb{R})$, thanks to [242, pag 305, (4.2)], we can always find an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{n}_3$ such that

$$\mu(e_1, e_2) = ae_3\,, \quad a \in \mathbb{R}\backslash\{0\}\,.$$

Moreover, thanks to [242, Theorem 4.3], we know that

$$\mathrm{Ric}_\mu = \frac{1}{2}a^2\mathrm{diag}(-1, -1, 1)\,.$$

Now, every $0 \neq H \in \Lambda^3\mathfrak{n}_3^*$, it will be of the form $H = be^{123}$ with $b \neq 0$ and it will be trivially closed and harmonic. On the other hand, we have that

$$H^2 = 2b^2\mathrm{Id}$$

which implies that

$$\mathrm{Ric}^{\mathrm{B}}_{\mu,H} = \frac{1}{2}\mathrm{diag}(-(a^2+b^2), -(a^2+b^2), a^2-b^2)\,.$$

Moreover, every $D \in \mathrm{Der}(\mathfrak{n}_3,\mu)$ has to satisfy

$$D^3_3 = D^1_1 + D^2_2\,.$$

This guarantees that

$$\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda\mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_3,\mu)$$

if and only if

$$\lambda = -\frac{1}{2}(3a^2+b^2)\,.$$

On the other hand, by straightforward computations we have that

$$0 = \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = -(a^2-b^2)H\,,$$

which is satisfied if and only if $a^2 = b^2$, giving us the claim, using Proposition 3.2.39. □

Now, we will focus on the 4-dimensional case. A first difference between the 3-dimensional one is a larger third cohomology group which allows a wider possibility for a generalized metric to be a generalized nilsoliton.

We will, first of all, analyse the case of $\mathfrak{n}_3 \oplus \mathbb{R}$. An important remark to be done is that this Lie algebra is the only one among the 4-dimensional ones which admits left-invariant complex structures, see for instance [238] or [279]. The complex structure on $\mathfrak{n}_3 \oplus \mathbb{R}$ is unique up to biholomorphisms and quotients by co-compact lattices give rise to the so-called *primary Kodaira surfaces*, the only compact complex surfaces with zero Kodaira dimension and first Betti number equal to 3. Clearly, every left-invariant Hermitian metric on $\mathfrak{n}_3 \oplus \mathbb{R}$ is SKT. So, using Corollary 3.2.62, we will be able to detected which generalized nilsoliton on $\mathfrak{n}_3 \oplus \mathbb{R}$ is an algebraic pluriclosed soliton.

Proposition 3.2.64. *On $\mathfrak{n}_3 \oplus \mathbb{R}$, the only non-classic generalized nilsolitons are the following:*

1. $\mu(e_1,e_2) = e_3\,,\quad H = e^{123}\,,\quad \lambda = -2\,,\quad D = \mathrm{diag}(1,1,2,2)\,.$
2. $\mu(e_1,e_2) = e_3\,,\quad H = \lambda_1 e^{234} + \lambda_2 e^{134}\,,\quad \lambda_1^2+\lambda_2^2 = 1\,,\quad \lambda = -\frac{3}{2}\,,\quad D = \mathrm{diag}\left(1-\frac{\lambda_2^2}{2}, 1-\frac{\lambda_1^2}{2}, \frac{3}{2}, 1\right).$

In particular, the first one is the unique algebraic pluriclosed soliton on $\mathfrak{n}_3 \oplus \mathbb{R}$.

Proof. Thanks to [328, Theorem 3.1], every inner product on $\mathfrak{n}_3 \oplus \mathbb{R}$ is isometric to an inner product such that an orthonormal basis $\{e_1,\dots,e_4\}$ satisfies the following

$$\mu(e_1,e_2) = ae_3\,,\quad a \neq 0\,.$$

We know that $D \in \mathrm{Der}(\mathfrak{n}_3 \oplus \mathbb{R},\mu) \cap \mathrm{Sym}(\mathfrak{n}_3 \oplus \mathbb{R})$ if and only if

$$D = \begin{pmatrix} a_1 & a_3 & 0 & a_5 \\ a_3 & a_2 & 0 & a_6 \\ 0 & 0 & a_1+a_2 & 0 \\ a_5 & a_6 & 0 & a_4 \end{pmatrix}\,,\quad a_i \in \mathbb{R}\,,\quad i = 1,\dots,6\,,$$

with respect to the frame $\{e_1,\dots,e_4\}$. Moreover, using [220, Lemma 4.2], it is easy to see that

$$\mathrm{Ric}_\mu = \frac{1}{2}a^2\mathrm{diag}(-1,-1,1,0)\,.$$

It is straightforward to check that a generic closed $H \in \Lambda^3(\mathfrak{n}_3 \oplus \mathbb{R})^*$ is of the form:

$$H = \lambda_4 e^{123} + \lambda_3 e^{124} + \lambda_2 e^{134} + \lambda_1 e^{234}\,, \quad \lambda_i \in \mathbb{R}\,, \quad i = 1, \ldots, 4\,.$$

Consequently, we have that $\Delta_\mu H = -a^2\lambda_3 e^{124}$ and

$$H^2 = 2\begin{pmatrix} \lambda_4^2+\lambda_2^2+\lambda_3^2 & \lambda_2\lambda_1 & -\lambda_3\lambda_1 & \lambda_4\lambda_1 \\ \lambda_2\lambda_1 & \lambda_4^2+\lambda_1^2+\lambda_3^2 & \lambda_2\lambda_3 & -\lambda_4\lambda_2 \\ -\lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_1^2+\lambda_2^2+\lambda_4^2 & \lambda_4\lambda_3 \\ \lambda_4\lambda_1 & -\lambda_4\lambda_2 & \lambda_4\lambda_3 & \lambda_1^2+\lambda_2^2+\lambda_3^2 \end{pmatrix}. \tag{3.69}$$

This readily implies that we need to impose that $\lambda_3 = 0$ or $\lambda_4 = \lambda_2 = \lambda_1 = 0$ in order to hope for $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda\mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_3 \oplus \mathbb{R}, \mu)$.

First of all, we suppose $\lambda_3 = 0$. In this case, $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda\mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_3 \oplus \mathbb{R}, \mu)$ if and only if

$$\lambda = -\frac{1}{2}(\lambda_4^2 + 3a^2)\,.$$

The condition

$$\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = 0$$

is equivalent to

$$\begin{cases} \lambda_4(-2a^2 + 3\lambda_2^2 + 2\lambda_4^2 + 3\lambda_1^2) = 0\,, \\ \lambda_2(-3a^2 + 3\lambda_2^2 + 2\lambda_4^2 + 3\lambda_1^2) = 0\,, \\ \lambda_1(-3a^2 + 3\lambda_2^2 + 2\lambda_4^2 + 3\lambda_1^2) = 0\,. \end{cases} \tag{3.70}$$

However, the first equation implies that $a^2 = \frac{1}{2}\left(3\lambda_2^2 + 2\lambda_4^2 + 3\lambda_1^2\right)$ or $\lambda_4 = 0$. On the other hand, if $a^2 = \frac{1}{2}(3\lambda_2^2 + 2\lambda_4^2 + 3\lambda_1^2)$, we have that

$$-3a^2 + 3\lambda_2^2 + 2\lambda_4^2 + 3\lambda_1^2 = -a^2 \neq 0\,.$$

From this, we obtain that $\lambda_1 = \lambda_2 = 0$, giving us Item 1.

Now let $\lambda_4 = \lambda_3 = 0$. In this case, the second equation in (3.70) implies that $\lambda_2 = 0$ or $\lambda^2 = \lambda_1^2 + \lambda_2^2$. If the last condition holds, then, the third equation is satisfied too, giving us Item 2. It remains to analyse when $\lambda_4 = \lambda_3 = \lambda_2 = 0$. In this case, the third equation in (3.70) gives us that $a^2 = \lambda_1^2$, since $\lambda_1 \neq 0$ if we want non-classical generalized nilsoliton. Thus, we recover a particular generalized nilsoliton belonging to Item 2.

It remains to analyse when $\lambda_3 \neq 0$ and $\lambda_4 = \lambda_2 = \lambda_1 = 0$. In this case, $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda\mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_3 \oplus \mathbb{R}, \mu)$ if and only if $\lambda = -\frac{1}{2}(3a^2 + 2\lambda_3^2)$. On the other hand, the condition

$$\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = 0$$

is equivalent to $\lambda_3(a^2 + \lambda_3^2) = 0$ which cannot happen unless $\lambda_3 = 0$, concluding the classification.
To address the last part of the statement, we define $Je_1 = e_2$, $Je_4 = e_3$ and consider

$$\omega^1 = \frac{1}{\sqrt{2}}(e^1 + \sqrt{-1}e^2)\,, \quad \omega^2 = \frac{1}{\sqrt{2}}(e^4 + \sqrt{-1}e^3)$$

obtaining that

$$d\omega^1 = 0\,, \quad d\omega^2 = \frac{a}{\sqrt{2}}\omega^{1\bar{1}}\,,$$

which, in particular, gives us the integrability of J. The fundamental form of the metric is then $\omega = \sqrt{-1}(\omega^{1\bar{1}} + \omega^{2\bar{2}})$, which is SKT, while the torsion of the Bismut connection is given by

$$H := d^c\omega = \sqrt{-1}(\bar{\partial}\omega - \partial\omega) = -\frac{1}{\sqrt{2}}a(\omega^{1\bar{1}\bar{2}} - \omega^{1\bar{1}2}) = ae^{123}\,.$$

On the other hand, we have that

$$d\omega = \frac{\sqrt{-1}}{\sqrt{2}} a(\omega^{1\bar{1}2} + \omega^{1\bar{1}\bar{2}}) = ae^{124}\,.$$

Then, it is easy to see that Lee form associated to ω has the following form:

$$\theta = \frac{a}{\sqrt{2}}(\omega^2 + \omega^{\bar{2}}) = ae^4\,.$$

This gives easily that $\mathrm{ad}_\mu \theta^\sharp = 0$. Moreover, given a diagonal derivation D, it will commute with J if and only if it is of the form

$$D = b\mathrm{diag}(1,1,2,2)\,, \quad b \in \mathbb{R}\,.$$

But, the derivation in Item 1 is of the above form. Then, applying Theorem 3.2.58, we obtain the claim, concluding the proof. ☐

Item 2 in Proposition 3.2.64 in particular provides the first example of nilpotent Lie group admitting a infinite family of generalized nilsolitons, answering negatively to Question 3.2.49. Now, we focus on the study of generalized nilsolitons in the case of $\mathfrak{n}_4$.

Proposition 3.2.65. *On $\mathfrak{n}_4$, the unique non-classic generalized nilsolitons are the following:*

1. $\mu(e_1,e_2) = e_3\,,\quad \mu(e_1,e_3) = \frac{\sqrt{3}}{2}e_4\,,\quad H = \frac{\sqrt{3}}{2}e^{134}\,,\quad \lambda = -\frac{3}{2}\,,\quad D = \mathrm{diag}\left(\frac{2}{3},1,\frac{5}{4},\frac{3}{2}\right)\,,$
2. $\mu(e_1,e_2) = e_3\,,\quad \mu(e_1,e_3) = e_4\,,\quad H = e^{234}\,,\quad \lambda = -\frac{3}{2}\,,\quad D = \mathrm{diag}\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2}\right)\,.$

Proof. Thanks to [328, Theorem 3.2], up to isometries, given any inner product on $\mathfrak{n}_4$, we can always find an orthonormal frame $\{e_1,\dots,e_4\}$ of $\mathfrak{n}_4$ such that

$$\mu(e_1,e_2) = ae_3 + be_4\,,\quad \mu(e_1,e_3) = ce_4\,,\quad a,c>0\,.$$

In this framework, a symmetric derivation D is of the form

$$D = \mathrm{diag}(\alpha,\beta,\alpha+\beta,2\alpha+\beta)\,,\quad \beta,\alpha\in\mathbb{R}\,,$$

see for instance [315, Lemma 1]. Moreover, the Ricci endomorphism, using again [220], takes the following form:

$$\mathrm{Ric}_\mu = \frac{1}{2}\begin{pmatrix} -(a^2+b^2+c^2) & 0 & 0 & 0\\ 0 & -(a^2+b^2) & -bc & 0\\ 0 & -bc & a^2-c^2 & ab\\ 0 & 0 & ab & b^2+c^2\end{pmatrix}\,. \tag{3.71}$$

Again, a generic closed $H\in\Lambda^3\mathfrak{n}_4^*$ will be of the following form:

$$H = \lambda_4 e^{123} + \lambda_3 e^{124} + \lambda_2 e^{134} + \lambda_1 e^{234}\,,\quad \lambda_i\in\mathbb{R}\,,\quad i=1,\dots,4\,.$$

Consequently, it is easy to see that

$$\Delta_\mu H = (-\lambda_4(c^2+b^2) + \lambda_3 ab)e^{123} + (-\lambda_3 a^2 + \lambda_4 ab)e^{124}$$

and the endomorphism H^2 will have the same form as in (3.69). Combining (3.71) and (3.69), we can see that, in order for $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda\mathrm{Id}$ to be a derivation, we need to impose that

$$\lambda_2\lambda_1 = \lambda_3\lambda_1 = \lambda_4\lambda_1 = \lambda_4\lambda_2 = 0\,,\quad \lambda_4\lambda_3 = ab\,,\quad \lambda_2\lambda_3 = -bc\,,$$

which gives us that $b = 0$ and just one among λ_i's can be non-zero.

First of all, we consider $\lambda_1 \neq 0$. In this particular case, we have that $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda \mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_3 \oplus \mathbb{R}, \mu)$ if and only if

$$\lambda = -\frac{3}{2}a^2 \quad \text{and} \quad c^2 = a^2\,.$$

On the other hand,

$$0 = \rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = \frac{3}{2}\lambda_1\left(\lambda_1^2 - a^2\right)e^{234}$$

which is satisfied if and only if $\lambda_1^2 = a^2$, giving us Item 2.

Now, we assume $\lambda_2 \neq 0$. In this case, we have that $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda \mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_4, \mu)$ if and only if

$$\lambda = -\frac{3}{2}a^2 \quad \text{and} \quad 3c^2 + \lambda_2^2 = 3a^2\,. \tag{3.72}$$

On the other hand,

$$\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = 0$$

is equivalent to

$$\frac{\lambda_2}{2}(c^2 + 3\lambda_2^2 - 3a^2) = 0$$

which gives that $3a^2 = c^2 + 3\lambda_2^2$. Combining this last equality with the second relation in (3.72), we obtain that $c^2 = \lambda_1^2 = \frac{3}{4}a^2$, giving us Item 1. Now, we consider the case in which $\lambda_3 \neq 0$. Then, we have that $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda \mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_4, \mu)$ if and only if

$$\lambda = -\frac{1}{2}(2\lambda_3^2 + 3a^2) \quad \text{and} \quad c^2 = a^2 + \frac{2}{3}\lambda_3^2\,.$$

Moreover, the condition

$$\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = 0$$

is equivalent to

$$\frac{1}{2}\lambda_3\left(a^2 + \lambda_3^2\right) = 0$$

which is impossible since we are assuming that $\lambda_3 \neq 0$.

The left case to analyse is that in which $\lambda_4 \neq 0$. In this hypothesis, we have that $\mathrm{Ric}^{\mathrm{B}}_{\mu,H} - \lambda \mathrm{Id} \in \mathrm{Der}(\mathfrak{n}_3 \oplus \mathbb{R}, \mu)$ if and only if

$$\lambda = -\frac{1}{2}(3a^2 + \lambda_4^2) \quad \text{and} \quad a^2 = \frac{1}{3}(\lambda_4^2 + 3c^2)\,. \tag{3.73}$$

Moreover,

$$\rho(\mathrm{Ric}^{\mathrm{B}}_{\mu,H})H - \Delta_\mu H + \lambda H = 0$$

is equivalent to

$$\lambda_4(-a^2 + 2c^2 + \lambda_4^2) = 0$$

which is satisfied if and only if $2c^2 + \lambda_4^2 = a^2$. On the other hand, combining this with the second relation in (3.73) gives that $c^2 = -\frac{2}{3}\lambda_4^2$ which is not possible, concluding the proof.

□

Chapter 4

Special metrics in hypercomplex Geometry

As mentioned in the previous chapters, the study of special Hermitian metrics has gained great importance throughout the last decades. This is the case also in the hypercomplex setting. As we saw in Subsection 1.3.2, a great amount of conditions on a hyperHermitian metric can be imposed in order for it to be considered special. This chapter has the purpose of giving a detailed treatment of many of the notions arising in the hypercomplex scenario. The present chapter is divided as follows.

Section 4.1 is devoted to the introduction and study of two canonical forms arising in hyperHermitian Geometry which have a close relation with the Lee form.

In Section 4.2, we define an analogue of the first Bott-Chern class in the hypercomplex setting, highlighting the main consequences of the vanishing of this invariant.

Sections 4.3, 4.4 and 4.5 are dedicated, respectively, to the study of the main properties of quaternionic Gauduchon, quaternionic balanced and strong HKT metrics. Among them, we prove sufficient conditions for a quaternionic Gauduchon metric to exist in a fixed conformal class, see Theorem 4.3.9. Moreover, we obtain an incompatibility result between strong HKT metrics and balanced hyperhermitian metrics, see Theorem 4.5.3, ultimately proving Theorem E.

In Section 4.6, a relevant Einstein condition in the hypercomplex setting is defined and discussed. This condition coupled with the HKT one is the object of a conjecture mimicking the analogous conjecture concerning Fano Kähler-Einstein metrics.

Finally, in Section 4.7, we collect several examples of compact hypercomplex manifolds admitting a type of special metrics but not stronger ones. We study, moreover, a construction by Arroyo and Nicolini and another by Barberis and Fino, and we prove that Joyce's examples admit Chern-Einstein hyperHermitian metrics.

The following chapter is a collection of results proved in a joint work with Giovanni Gentili, see [139].

4.1 Canonical forms in hyperHermitian geometry

In this section, we recover some results found in the literature and prove new ones about hyperHermitian metrics making use of two new 1-forms: α and β. The section will be divided in four subsections. Subsection 4.1.1 will be dedicated to the definition of the forms α and β and to their basic properties. Among all, we will rewrite the Lee form and the Bismut and Chern-Ricci forms in terms of α and β. Subsection 4.1.2 will be concerned in the study of the Obata connection using the form α. Then, in Subsection 4.1.3, we will study Chern and Bismut scalar curvature, showing that they do not depend on the preferred choice of the pair of anti-commuting complex structure we consider. Finally, in the last subsection, we will write in terms of hyperHermitian data the classical definition of special metric coming from complex Geometry.

4.1.1 The forms α and β

Let us immediately start this subsection with the definition of the forms α and β.

Definition 4.1.1. Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. We define the forms $\alpha_\Omega, \beta_\Omega \in \Lambda_I^{1,0}M$ via the relations:

$$\partial\bar{\Omega}^n = \alpha_\Omega \wedge \bar{\Omega}^n\,, \qquad \partial\Omega^{n-1} = \beta_\Omega \wedge \Omega^{n-1}\,.$$

To lighten the notation, whenever there is no confusion about the hyperHermitian metric in use, we shall write α and β in place of α_Ω and β_Ω. Let us explain why such forms are well-defined. Regarding α, it is well-known that $\alpha + \bar{\alpha}$ is the connection 1-form η_I of the Obata connection on the canonical bundle $K_{(M,I)}$, see, for instance, [331]. Since for any $L \in \mathsf{H}$ the volume induced by Ω_L is the same, we see that

$$\eta_I \otimes \frac{\Omega_I^n \wedge \bar{\Omega}_I^n}{(n!)^2} = \nabla^{\mathrm{Ob}}\left(\frac{\Omega_L^n \wedge \bar{\Omega}_L^n}{(n!)^2}\right) = \eta_L \otimes \frac{\Omega_L^n \wedge \bar{\Omega}_L^n}{(n!)^2} = \eta_L \otimes \frac{\Omega_I^n \wedge \bar{\Omega}_I^n}{(n!)^2}\,.$$

Therefore the connection 1-form $\eta_I = \alpha + \bar{\alpha}$ actually does not depend on I. This fact will be very important in what follows and for this reason we shall drop the reference to the complex structure and simply denote it η.
On the other hand, using Proposition 1.3.20, we see that β is well-defined.

Lemma 4.1.2. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then, α is ∂-closed and $\partial_J\alpha$ is q-real.*

Proof. The proof can be found in [329, Section 10.1]. It boils down to expanding the identities $\partial^2\bar{\Omega}^n = 0$ and $(\partial\partial_J + \partial_J\partial)\bar{\Omega}^n = 0$. □

Remark 4.1.3. Observe that β does not satisfy the same properties as α, in general. The same argument above with Ω^{n-1} in place of $\bar{\Omega}^n$ only shows that $\mathrm{tr}_\Omega(\partial\beta) = 0$ and $\mathrm{tr}_\Omega(\partial_J\beta) = \mathrm{tr}_\Omega(\partial J\bar{\beta})$.

The next lemma gives an alternative definition of α and β providing explicit expressions for them in terms of Λ, defined in Definition 1.3.19.

Lemma 4.1.4. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then*

$$\alpha = \bar{\Lambda}(\partial\bar{\Omega})\,, \quad \beta = \Lambda(\partial\Omega)\,.$$

Proof. First of all, we fix $Z \in \Gamma(T_I^{1,0}M)$. Then, we have

$$\begin{aligned} \iota_Z\bar{\Lambda}(\partial\bar{\Omega}) &= g(\iota_Z\partial\bar{\Omega}, \bar{\Omega}) = *\left(\iota_Z\partial\bar{\Omega} \wedge \frac{\Omega^n \wedge \bar{\Omega}^{n-1}}{n!(n-1)!}\right) \\ &= -*\left(\partial\bar{\Omega} \wedge \frac{\iota_Z\Omega^n \wedge \bar{\Omega}^{n-1}}{n!(n-1)!}\right) = -*\left(\alpha \wedge \frac{\iota_Z\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}\right) = \iota_Z\alpha\,. \end{aligned}$$

Then, $\alpha = \bar{\Lambda}(\partial\bar{\Omega})$. In a similar fashion, we have

$$\begin{aligned} \iota_Z\Lambda(\partial\Omega) &= g(\iota_Z\partial\Omega, \Omega) = *\left(\iota_Z\partial\Omega \wedge \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{n!(n-1)!}\right) = -*\left(\partial\Omega \wedge \iota_Z\Omega \wedge \frac{\Omega^{n-2} \wedge \bar{\Omega}^n}{n!(n-2)!}\right) \\ &= *\left(\iota_Z\Omega \wedge \beta \wedge \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{n!(n-1)!}\right) = -*\left(\beta \wedge \frac{\iota_Z\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}\right) = \iota_Z\beta\,, \end{aligned}$$

concluding the proof. □

The interest of the forms α and β, as it turns out, is that they are strictly related to other well-known quantities.

Proposition 4.1.5. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then, the following hold:*

(a) the Lee forms of ω_L, for $L \in \mathsf{H}$, all coincide and they are equal to

$$\theta_\Omega := \alpha + \bar\alpha + \beta + \bar\beta\,;$$

(b) the first Chern-Ricci form of ω_I is

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) = dI(\alpha+\bar\alpha) = dI\eta\,;$$

(c) the Bismut-Ricci form of ω_I is

$$\mathrm{Ric}^{\mathrm{B}}(\omega_I) = -dI(\beta+\bar\beta)\,;$$

(d) the Ricci tensor of the Obata connection is

$$\mathrm{Ric}^{\mathrm{Ob}} = d(\alpha+\bar\alpha) = d\eta\,,$$

in particular the Obata scalar curvature always vanishes.

Proof. To prove (a) let $\theta_L = -Ld^*\omega_L$ be the Lee form of ω_L. We first observe that

$$d^*\Omega = -*\partial\left(\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}\right) = -*\left((\alpha+\beta)\wedge\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}\right) = J(\bar\alpha+\bar\beta)\,.$$

Hence

$$J\theta_J = d^*\omega_J = d^*(\Omega+\bar\Omega) = J(\bar\alpha+\bar\beta+\alpha+\beta)$$

and similarly

$$K\theta_K = d^*\omega_K = -\sqrt{-1}d^*(\Omega-\bar\Omega) = K(\bar\alpha+\bar\beta+\alpha+\beta)\,,$$

therefore $\theta_J = \theta_K$. The same argument replacing J and K with K and I respectively, shows that $\theta_K = \theta_I$. Thus we have $\theta_I = \theta_J = \theta_K =: \theta_\Omega$ and actually, for any $L = aI + bJ + cK \in \mathsf{H}$

$$\theta_L = -Ld^*\omega_L = -Ld^*(a\omega_I + b\omega_J + c\omega_K) = L(aI^{-1} + bJ^{-1} + cK^{-1})\theta_\Omega = \theta_\Omega\,,$$

using the fact that $aI^{-1} + bJ^{-1} + cK^{-1} = L^{-1}$.

To show (b), we fix local I-holomorphic coordinates and, for simplicity, set $P := \mathrm{pf}(\Omega_{ij})$ and $G := \det(g_{r\bar s})$. From (1.41), it follows that $\bar\partial P = P\bar\alpha$. Then, from (1.42), we get

$$\bar\partial G = \overline{P}\bar\partial P + P\overline{\partial P} = G\bar\alpha + P\overline{\partial P}\,.$$

Hence

$$\begin{aligned}
\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) &= -\sqrt{-1}\partial\left(G^{-1}\bar\partial G\right) = -\sqrt{-1}\left(\partial\bar\alpha + \partial(G^{-1}P\overline{\partial P})\right)\\
&= \sqrt{-1}\left(-\partial\bar\alpha + G^{-2}\partial G\wedge(P\overline{\partial P}) - G^{-1}\partial(P\overline{\partial P})\right)\\
&= \sqrt{-1}\left(-\partial\bar\alpha + G^{-1}P\alpha\wedge\overline{\partial P} + G^{-1}\partial P\wedge\overline{\partial P} - G^{-1}\partial P\wedge\overline{\partial P} + G^{-1}P\bar\partial\overline{\partial P}\right)\\
&= \sqrt{-1}\left(-\partial\bar\alpha + G^{-1}P\alpha\wedge\overline{\partial P} + G^{-1}P\bar\partial(\overline{P}\alpha)\right) = \sqrt{-1}\left(\bar\partial\alpha - \partial\bar\alpha\right).
\end{aligned}$$

Now (c) follows from (a) and (b) together with the well-known identity

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) - \mathrm{Ric}^{\mathrm{B}}(\omega_I) = dI\theta_\Omega\,.$$

It remains to prove (d). Let ∇^{Ob} and $\tilde\nabla^{\mathrm{Ob}}$ be the Obata connections on TM and $K_{(M,I)}$ respectively. Here the Obata-Ricci tensor is $\mathrm{Ric}^{\mathrm{Ob}}(X,Y) = \mathrm{tr}(Z\mapsto R^{\nabla^{\mathrm{Ob}}}(Z,X)Y)$, where

$$R^{\nabla^{\mathrm{Ob}}}(Z,X)Y = \left(\nabla_Z\nabla_X - \nabla_X\nabla_Z - \nabla_{[Z,X]}\right)Y\,,\quad X,Y,Z\in\Gamma(TM)\,.$$

We know that $\eta = \alpha + \bar{\alpha}$ is the connection 1-form of $\tilde{\nabla}^{\mathrm{Ob}}$, therefore the curvature endomorphism can be expressed as

$$R^{\tilde{\nabla}^{\mathrm{Ob}}} = d\eta + \eta \wedge \eta = d\eta \,.$$

On the other hand, for every $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} R^{\tilde{\nabla}^{\mathrm{Ob}}}(X,Y) &= -\mathrm{tr}^{\mathbb{C}}(R^{\nabla^{\mathrm{Ob}}}(X,Y)) = -\frac{1}{2}\mathrm{tr}^{\mathbb{R}}(R^{\nabla^{\mathrm{Ob}}}(X,Y)) \\ &= \frac{1}{2}\left(\mathrm{Ric}^{\mathrm{Ob}}(X,Y) - \mathrm{Ric}^{\mathrm{Ob}}(Y,X)\right) \,, \end{aligned}$$

where the last identity follows from the Bianchi identity, which holds because ∇^{Ob} is torsion-free. To conclude, we only need to observe that the Ricci tensor of the Obata connection is skew-symmetric, see [6, Theorem 5.6]. □

Remark 4.1.6. Some of the descriptions obtained in Proposition 4.1.5 were previously known only in the special case when the hyperHermitian metric is HKT. In detail, the fact that the Lee forms coincide was due to Ivanov and Papadopoulos [196], and their relation with the Obata connection 1-form was shown in [47, Lemma 2.2]; finally, the formula for the Obata-Ricci tensor was obtained by Alekseevsky and Marchiafava [5, Proposition 15] in the general case and rediscovered by Ivanov and Petkov [197, Proposition 4.1] in the HKT case.

Remark 4.1.7. Again, the role of I is not preferential: the formulae for the Ricci forms in Proposition 4.1.5 hold more generally for any complex structure $L \in \mathsf{H}$, in other words

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega_L) = dL\eta \,, \quad \mathrm{Ric}^{\mathrm{B}}(\omega_L) = -dL(\beta_{\Omega_L} + \bar{\beta}_{\Omega_L}) \,,$$

where complex conjugation is now intended with respect to L.

For convenience, we write here down some formulae for conformal changes. For the ease of notation, we write $\alpha_f = \alpha_{\mathrm{e}^f\Omega}$, $\beta_f = \beta_{\mathrm{e}^f\Omega}$, $\theta_f = \theta_{\mathrm{e}^f\Omega}$ and $\omega_f = \mathrm{e}^f\omega_I$, where $f \in C^\infty(M, \mathbb{R})$. It is easy to check that

$$\alpha_f = \alpha + n\partial f \,, \qquad \beta_f = \beta + (n-1)\partial f \,. \tag{4.1}$$

From these, it follows that

$$\begin{aligned} \theta_f &= \theta_\Omega + (2n-1)df \,, \\ \mathrm{Ric}^{\mathrm{Ch}}(\omega_f) &= \mathrm{Ric}^{\mathrm{Ch}}(\omega_I) - 2n\sqrt{-1}\partial\bar{\partial} f \,, &(4.2) \\ \mathrm{Ric}^{\mathrm{B}}(\omega_f) &= \mathrm{Ric}^{\mathrm{B}}(\omega_I) + 2(n-1)\sqrt{-1}\partial\bar{\partial} f \,. &(4.3) \end{aligned}$$

4.1.2 Holonomy of the Obata connection

We now derive some interesting consequences stemming from the formulae we have found. To begin, we recall the following general lemma:

Lemma 4.1.8. *A connection on a trivial line bundle is flat if and only if the connection 1-form is closed. Furthermore, the bundle admits a global parallel section if and only if the connection 1-form is exact.*

Recall that the canonical bundle of a hypercomplex manifold (M, H) is always topologically trivial as a trivialization is given by the top wedge power of the $(2,0)$-form Ω corresponding to any hyperHermitian metric.

Proposition 4.1.9. *Let (M^n, H) be a hypercomplex manifold. Then, the following are equivalent:*

1. *the restricted holonomy group of the Obata connection is contained in $\mathsf{SL}(n, \mathbb{H})$;*
2. *the Obata-Ricci tensor vanishes, for any hyperHermitian metric Ω on (M, H);*

3. *the Obata-connection 1-form on $K_{(M,L)}$ is closed, for all $L \in \mathsf{H}$, i.e. $d\eta = 0$.*

Proof. The restricted holonomy group of the Obata connection is contained in $\mathsf{SL}(n, \mathbb{H})$ if and only if the Obata connection on the canonical bundle $K_{(M,I)} = \Lambda^{2n,0}_I M$ is flat (see e.g. [331]) and by Lemma 4.1.8 this happens if and only if the Obata-Ricci tensor computed with respect to any hyperHermitian metric vanishes: $\mathrm{Ric}^{\mathrm{Ob}} = d\eta = 0$. □

The equivalence of (1) and (2) in Proposition 4.1.9 is well-known and goes back to Alekseevsky and Marchiafava [6, Theorem 5.6]. We now prove the global counterpart of Proposition 4.1.9, which can be seen as a generalization of [197, Theorem 2.2].

Proposition 4.1.10. *Let (M^n, H) be a hypercomplex manifold. Then, the following are equivalent:*

1. *the holonomy group of the Obata connection is contained in $\mathsf{SL}(n, \mathbb{H})$;*
2. *the $(1,0)$-form α_Ω is ∂-exact, for any hyperHermitian metric Ω on (M, H);*
3. *in any hyperHermitian conformal class there exists a unique (up to scaling) metric Ω on (M, H) such that $\alpha_\Omega = 0$.*

Proof. The holonomy group of the Obata connection ∇^{Ob} is contained in $\mathsf{SL}(n, \mathbb{H})$ if and only if there exists a global ∇^{Ob}-parallel section and by Lemma 4.1.8 this is equivalent to (2). Obviously (3) implies (2); to see the converse suppose $\alpha_\Omega = \partial f$ for some $f \in C^\infty(M, \mathbb{R})$, then the conformally rescaled metric $\Omega_f = \mathrm{e}^{-\frac{f}{n}}\Omega$ satisfies $\alpha_{\Omega_f} = 0$, thanks to (4.1). The uniqueness of such a metric is clear. □

4.1.3 Scalar curvatures

In this subsection, we will focus on scalar curvatures of hyperHermitian metrics. From (1.44) and Lemma 1.3.7 we infer that

$$\Lambda_\Omega(\partial_J \alpha) = \Lambda_{\omega_I}(\sqrt{-1}\bar\partial\alpha) = \frac{1}{2}\Lambda_{\omega_I}(\sqrt{-1}\bar\partial\alpha - \sqrt{-1}\partial\bar\alpha) = \frac{1}{2}\Lambda_{\omega_I}\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) = \frac{1}{2}s^{\mathrm{Ch}}(\omega_I)\,,$$

where we also used Proposition 4.1.5 (b). In the same way one shows that the Bismut scalar curvature satisfies

$$s^{\mathrm{B}}(\omega_I) = -2\Lambda_\Omega(\partial_J\beta)\,.$$

Proposition 4.1.11. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. The Chern scalar curvatures of any $L \in \mathsf{H}$ coincide. The same property holds for the Bismut scalar curvatures.*

Proof. We prove the proposition for Chern scalar curvatures, for Bismut scalar curvatures the argument is analogous. Observe that it is enough to identify $s^{\mathrm{Ch}}(\omega_P)$ with $s^{\mathrm{Ch}}(\omega_L)$ for all $L \in \mathsf{H}$ that anti-commute with P and by symmetry we may assume $P = I$. To see this, recall that each complex structure in H corresponds to a point on the sphere S^2 and, supposing P corresponds to the north pole of S^2, then the L's correspond to points on the equator. Assume we are able to prove that the relative Chern scalar curvatures are equal, then the same argument repeated replacing P with all L's allows to cover the entire sphere, thus identifying all Chern scalar curvatures.

Let then $L = aJ + bK \in \mathsf{H}$ be the generic complex structure that anti-commutes with I. Note that

$$\omega_L = a\omega_J + b\omega_K = a(\Omega + \bar\Omega) - \sqrt{-1}b(\Omega - \bar\Omega) = (a - \sqrt{-1}b)\Omega + (a + \sqrt{-1}b)\bar\Omega \tag{4.4}$$

and set $w = a - \sqrt{-1}b$ for simplicity. Keeping in mind that $|w| = 1$, we compute

$$\omega_L^{2n-1} = \binom{2n-1}{n}\bar w\Omega^{n-1} \wedge \bar\Omega^n + \binom{2n-1}{n} w\Omega^n \wedge \bar\Omega^{n-1}\,,$$

$$\omega_L^{2n} = \binom{2n}{n}\Omega^n \wedge \bar\Omega^n\,.$$

Thus for any 2-form ξ we have

$$\Lambda_{\omega_L}\xi = n\bar{w}\frac{\xi^{2,0}\wedge\Omega^{n-1}\wedge\bar{\Omega}^n}{\Omega^n\wedge\bar{\Omega}^n} + nw\frac{\xi^{0,2}\wedge\Omega^n\wedge\bar{\Omega}^{n-1}}{\Omega^n\wedge\bar{\Omega}^n} = \bar{w}\,\Lambda_\Omega(\xi^{2,0}) + w\,\Lambda_{\bar{\Omega}}(\xi^{0,2})\,.$$

Now, to conclude the proof we only need to compute the $(2,0)$ and $(0,2)$ parts of $\mathrm{Ric}^{\mathrm{Ch}}(\omega_L)$. We observe, by expanding the Chern-Ricci forms of ω_J and ω_K, that:

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega_J) = J\mathrm{Ric}^{\mathrm{Ch}}(\omega_J) = JdJ\eta = \partial_J\alpha + \partial_J\bar{\alpha} + \bar{\partial}_J\alpha + \overline{\partial_J\alpha}\,, \tag{4.5}$$
$$\mathrm{Ric}^{\mathrm{Ch}}(\omega_K) = IJ^{-1}dJI(\alpha+\bar{\alpha}) = \sqrt{-1}\left(-\partial_J\alpha - \partial_J\bar{\alpha} + \bar{\partial}_J\alpha + \overline{\partial_J\alpha}\right)\,. \tag{4.6}$$

Then, since $\mathrm{Ric}^{\mathrm{Ch}}(\omega_L) = dL\eta = a\mathrm{Ric}^{\mathrm{Ch}}(\omega_J) + b\mathrm{Ric}^{\mathrm{Ch}}(\omega_K)$, we get

$$(\mathrm{Ric}^{\mathrm{Ch}}(\omega_L))^{2,0} = w\partial_J\alpha\,, \quad (\mathrm{Ric}^{\mathrm{Ch}}(\omega_L))^{0,2} = \overline{w\partial_J\alpha}\,. \tag{4.7}$$

Finally, we conclude

$$s^{\mathrm{Ch}}(\omega_L) = \Lambda_{\omega_L}\mathrm{Ric}^{\mathrm{Ch}}(\omega_L) = \bar{w}\,\Lambda_\Omega\left((\mathrm{Ric}^{\mathrm{Ch}}(\omega_L))^{2,0}\right) + w\,\Lambda_{\bar{\Omega}}\left((\mathrm{Ric}^{\mathrm{Ch}}(\omega_L))^{0,2}\right) = s^{\mathrm{Ch}}(\omega_I)\,.$$

□

In view of Proposition 4.1.11 we will omit the reference to the $(1,1)$-form with respect to which we are considering scalar curvatures and simply denote them by s^{Ch} and s^{B}. To avoid ambiguity, if we need to specify the hyperHermitian metric Ω with respect to which scalar curvatures are considered, we shall write them as $s^{\mathrm{Ch}}(\Omega)$ and $s^{\mathrm{B}}(\Omega)$.

4.1.4 Special metrics

We shall now study equivalent conditions, in terms of the hyperHermitian data, for a hyperHermitian metric to satisfy the Gauduchon condition, see Definition 1.1.34, and the balanced one, see Definition 1.1.47.

The following lemma identifies conditions on the canonical $(2,0)$-form Ω in order for ω_L to be Gauduchon.

Lemma 4.1.12. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then, the following are equivalent:*

1. *ω_I is Gauduchon;*
2. *ω_L is Gauduchon for any $L \in \mathsf{H}$;*
3. *$s^{\mathrm{Ch}} - s^{\mathrm{B}} - 2|\alpha+\beta|^2 = 0$;*
4. *$\partial^*\partial_J^*\Omega = 0$;*
5. *$\partial\partial_J(\Omega^{n-1}\wedge\bar{\Omega}^n) = 0$.*

Proof. By definition, if ω_I is Gauduchon then $d^*\theta_{\omega_I} = 0$. Hence, the equivalence of (1) and (2) follows from Proposition 4.1.5 (a). Next, we compute

$$\begin{aligned} d^*(\alpha+\beta) &= *d\left(J(\bar{\alpha}+\bar{\beta})\wedge\frac{\Omega^{n-1}\wedge\bar{\Omega}^n}{n!(n-1)!}\right) = \mathrm{tr}_\Omega(\partial_J(\alpha+\beta)) - |\alpha+\beta|^2 \\ &= \frac{s^{\mathrm{Ch}}}{2} - \frac{s^{\mathrm{B}}}{2} - |\alpha+\beta|^2\,, \end{aligned}$$

where we used q-realness of $\partial_J\alpha$ and Remark 4.1.3. Therefore

$$d^*\theta_\Omega = 2\mathrm{Re}(d^*(\alpha+\beta)) = s^{\mathrm{Ch}} - s^{\mathrm{B}} - 2|\alpha+\beta|^2\,,$$

giving the equivalence between (1) and (3). On the other hand, we also have

$$\partial^*\partial_J^*\Omega = -*\partial\partial_J\left(\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}\right) = *\partial\left(J(\bar\alpha+\bar\beta)\wedge\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}\right) = d^*(\alpha+\beta)\,,$$

concluding the proof. □

The following is an adaptation of Lemma 4.1.12 to the balanced case. The proof is analogous.

Lemma 4.1.13. *Let (M^n,H,Ω) be a hyperHermitian manifold. The following are equivalent*

1. *ω_I is balanced;*
2. *ω_L is balanced for any $L\in\mathsf{H}$;*
3. *$\alpha+\beta=0$;*
4. *$\partial^*\Omega=0$;*
5. *$\partial(\Omega^{n-1}\wedge\bar\Omega^n)=0$.*

The following lemma will be useful later.

Lemma 4.1.14. *Let (M^n,H,Ω) be a compact hyperHermitian manifold with non-negative Bismut scalar curvature. Then Ω cannot have negative Chern scalar curvature, furthermore Ω is balanced if and only if it is Chern scalar flat.*

Proof. Let $\Omega_{\mathrm{G}}=\mathrm{e}^f\Omega$ be the Gauduchon metric in the conformal class of Ω. Then

$$\begin{aligned}0 &= s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - s^{\mathrm{B}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}}+\beta_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}}\\ &= \mathrm{e}^{-f}\left(s^{\mathrm{Ch}}(\Omega)-s^{\mathrm{B}}(\Omega)-2|\alpha_\Omega+\beta_\Omega+(2n-1)\partial f|^2_\Omega-2(2n-1)\Delta_\Omega f\right)\end{aligned}$$

Since Ω has non-negative Bismut scalar curvature we deduce

$$2(2n-1)\Delta_\Omega f\le s^{\mathrm{Ch}}(\Omega)-2|\alpha_\Omega+\beta_\Omega+(2n-1)\partial f|^2_\Omega\,. \tag{4.8}$$

If we had $s^{\mathrm{Ch}}(\Omega)<0$ the maximum principle would imply that f is constant yielding the inequality $s^{\mathrm{Ch}}(\Omega)\ge 2|\alpha_\Omega+\beta_\Omega|^2_\Omega$ which contradicts the negativity of $s^{\mathrm{Ch}}(\Omega)$. Furthermore, it is clear from (4.8) that if $s^{\mathrm{Ch}}(\Omega)=0$ then $\theta_\Omega=\alpha_\Omega+\beta_\Omega=0$. Conversely, if Ω is balanced, $s^{\mathrm{Ch}}(\Omega)=s^{\mathrm{B}}(\Omega)\ge0$. On the other hand, we have that

$$\int_M s^{\mathrm{Ch}}(\Omega)\frac{\Omega^n\wedge\bar\Omega^n}{(n!)^2}=2\int_M\partial_J\alpha_\Omega\wedge\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}=0\,,$$

integrating by parts and using Lemma 4.1.13, but thus is possible only if $s^{\mathrm{Ch}}(\Omega)=0$, giving the claim. □

By using Definition 1.1.49 and Proposition 1.1.50, in [110, Proof of Theorem 3.1], the authors show that if M is equipped with an invariant hypercomplex structure H and a compatible HKT metric, then it is also equipped with a compatible invariant HKT metric. The same holds for all the other kind of special metrics we are interested in.

Theorem 4.1.15. *Let $(M^n:=\mathsf{G}/\Gamma,\mathsf{H})$ be a compact quotient of a Lie group by a lattice equipped with a left-invariant hypercomplex structure. If (M,H) admits a metric which is weak HKT (resp. strong HKT, quaternionic balanced, quaternionic strongly Gauduchon, quaternionic Gauduchon) then it admits an invariant one.*

Proof. Let Ω be a hyperHermitian metric on (M,H). Since μ commutes with I,J and d it also commutes with ∂ and ∂_J. Moreover, if $\mu(\Omega^{n-1})$ is a q-positive $(2n-2,0)$-form, then by Lemma 1.3.4 there exists an invariant hyperHermitian metric $\tilde\Omega$ such that $\tilde\Omega^{n-1}=\mu(\Omega^{n-1})$. From this point the conclusion is straightforward. □

4.2 The first quaternionic Bott-Chern class

In this section, we will introduce and study an invariant of the hypercomplex structure which we call *first quaternionic Bott-Chern class*. It will play an important role in what follows.

Let E be a I-holomorphic line bundle over a hypercomplex manifold (M^n, H). The curvature R of any I-Hermitian metric h on E is a closed real $(1,1)$-form on M and we shall consider the q-real, $(2,0)$-form corresponding to the J-anti-invariant part of R:

$$\Phi\left(\frac{R-JR}{2}\right) = \frac{1}{4}\left(\sqrt{-1}R(JX,Y) + \sqrt{-1}R(X,JY) - R(KX,Y) - R(X,KY)\right) .$$

It is well-known that R is locally $\partial\bar{\partial}$-exact. This guarantees that $\Phi(\frac{R-JR}{2})$ is locally $\partial\partial_J$-exact. We then are led to give the following definition.

Definition 4.2.1. Let (M^n, H) be a hypercomplex manifold and (E, h) be a I-Hermitian line bundle over (M^n, H) with curvature R. The *first quaternionic Bott-Chern class of E with respect to J* is the Bott-Chern cohomology class of $\Phi(\frac{R-JR}{2})$ in the quaternionic sense:

$$c_1^{\mathrm{qBC}}(E, J) := \left[\Phi\left(\frac{R-JR}{2}\right)\right]_{\mathrm{qBC}} \in H^{2,0}_{\mathrm{qBC}}(M) .$$

This class does not depend on the choice of h. Indeed, any other I-Hermitian metric on E has curvature $R' = R + \sqrt{-1}\partial\bar{\partial}f$, for some $f \in C^\infty(M, \mathbb{R})$, hence the corresponding $(2,0)$-forms are related by $\Phi(\frac{R'-JR'}{2}) = \Phi(\frac{R-JR}{2}) + \frac{1}{2}\partial\partial_J f$, thanks to Lemma 1.3.7.

If $E = -K_{(M,I)}$ is the anticanonical bundle of (M, I) we call $c_1^{\mathrm{qBC}}(-K_{(M,I)}, J)$ the first quaternionic Bott-Chern class of M with respect to (I, J) and denote it $c_1^{\mathrm{qBC}}(M, I, J)$. Note that for any hyperHermitian metric Ω on (M, H) with corresponding $(1,1)$-form ω_I, we can choose a Hermitian metric on $-K_{(M,I)}$ so that $R = \mathrm{Ric}^{\mathrm{Ch}}(\omega_I)$. As a matter of fact, we can show that $\partial_J\alpha$ is a representative of $c_1^{\mathrm{qBC}}(M, I, J)$.

Lemma 4.2.2. *Let (M, H, Ω) be a hyperHermitian manifold. Then, under the bijection of Lemma 1.3.6, the J-anti-invariant part of the Chern-Ricci form $\mathrm{Ric}^{\mathrm{Ch}}(\omega_I)$ corresponds to $\partial_J\alpha$.*

Proof. From Item b of Proposition 4.1.5, we obtain the identity

$$\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) - J\mathrm{Ric}^{\mathrm{Ch}}(\omega_I)}{2} = \frac{\sqrt{-1}}{2}\left(\bar{\partial}\alpha - \partial\bar{\alpha} - J\bar{\partial}\alpha + J\partial\bar{\alpha}\right) = \sqrt{-1}\left(\bar{\partial}\alpha - J\bar{\partial}\alpha\right) ,$$

where we used that $\bar{\partial}\alpha - J\bar{\partial}\alpha = -\partial\bar{\alpha} + J\partial\bar{\alpha}$ which follows from Lemma 1.3.7 and the fact that $\partial_J\alpha$ is q-real. Therefore using again Lemma 1.3.7 and applying the bijection of Lemma 1.3.6 we obtain precisely $\partial_J\alpha$. ☐

We may actually give an equivalent definition of the first quaternionic Bott-Chern class as follows. Let $\Theta \in \Lambda_I^{2n,0}M$ be any q-positive section of the canonical bundle, then there exists $\alpha_\Theta \in \Lambda_I^{1,0}M$ such that

$$\partial\bar{\Theta} = \alpha_\Theta \wedge \bar{\Theta} .$$

Similarly as before, the quaternionic Bott-Chern cohomology class of $\partial_J\alpha_\Theta$ does not depend on the choice of Θ. Indeed, if Θ' is another q-positive $(2n,0)$-form, there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $\Theta' = f\Theta$. Therefore, $\partial_J\alpha_{\Theta'} = \partial_J\alpha_\Theta - \partial\partial_J f$, which shows that $[\partial_J\alpha_{\Theta'}]_{\mathrm{BC}} = [\partial_J\alpha_\Theta]_{\mathrm{BC}}$.

In particular, we observe that $c_1^{\mathrm{qBC}}(M, I, J) = 0$ if and only if there exists a metric in each hyperHermitian conformal class such that $\partial_J\alpha = 0$, equivalently by Lemma 4.2.2 the Chern-Ricci form of ω_I is J-invariant. Clearly, the definition of the first quaternionic Bott-Chern class depends on the choice of a basis (I, J) for the hypercomplex structure. However, it turns out that the vanishing of such class is independent from it. For this reason, when this occurs, we will unambiguously write $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$.

Proposition 4.2.3. *Let (M, H) be a hypercomplex manifold. If $c_1^{\mathrm{qBC}}(M, I, J) = 0$ then also $c_1^{\mathrm{qBC}}(M, L, P) = 0$, for any other pair of anti-commuting complex structures $L, P \in H$.*

Proof. First of all, note that by the same symmetry argument at the beginning of the proof of Proposition 4.1.11, it is enough to show the claim for $P = I$ and $L = aJ + bK \in \mathsf{H}$. Then, for any hyperHermitian metric Ω, keeping in mind the identities (4.7), we have

$$\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_L) - I\mathrm{Ric}^{\mathrm{Ch}}(\omega_L)}{2} = (\mathrm{Ric}^{\mathrm{Ch}}(\omega_L))^{2,0} + (\mathrm{Ric}^{\mathrm{Ch}}(\omega_L))^{0,2} = (a - \sqrt{-1}b)\partial_J\alpha + (a + \sqrt{-1}b)\overline{\partial_J\alpha}\,. \quad (4.9)$$

Observe that, since $c_1^{\mathrm{qBC}}(M, I, J) = 0$, we can choose Ω so that $\partial_J\alpha = 0$, which allows us to conclude that $c_1^{\mathrm{qBC}}(M, L, I) = 0$. □

We now observe that c_1^{qBC} is additive with respect to the tensor product of line bundles. Indeed, for any pair of I-holomorphic line bundles E and F over a hypercomplex manifold (M, H) equipped with connections ∇^E and ∇^F respectively, the induced connection ∇ on $E \otimes F$ has curvature $R^\nabla = R^{\nabla^E} + R^{\nabla^F}$. Then clearly

$$c_1^{\mathrm{qBC}}(E \otimes F, J) = c_1^{\mathrm{qBC}}(E, J) + c_1^{\mathrm{qBC}}(F, J)\,.$$

The first consequence of the vanishing of the first quaternionic Bott-Chern class that we prove is the following.

Proposition 4.2.4. *Let (M, H) be a compact hypercomplex manifold. If $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$, then $\kappa(M, L) \leq 0$ for all $L \in \mathsf{H}$, where $\kappa(M, L)$ denotes the Kodaira dimension of (M, L), recall Definition 1.1.10. Moreover, $\kappa(M, L) = 0$ if and only if $K_{(M,L)}$ is holomorphically torsion.*

Proof. Without loss of generality we prove the statement for $L = I$. Let Ω be any hyperHermitian metric on (M, H). Since $c_1^{\mathrm{qBC}}(K_{(M,I)}, J) = -c_1^{\mathrm{qBC}}(M, I, J) = 0$, there exists an I-Hermitian metric h on $K_{(M,I)}$ such that $\Phi(\frac{R_h - JR_h}{2}) = 0$. Now, for any $k \geq 1$ and any section $\psi \in H^0(M, K_{(M,I)}^{\otimes k})$, a straightforward computation gives

$$\Delta_{\omega_I}|\psi|^2 = |\nabla\psi|^2 - k|\psi|^2 \mathrm{tr}_{\omega_I}(R_h) = |\nabla\psi|^2 - 2k|\psi|^2 \Lambda_\Omega\left(\Phi\left(\frac{R_h - JR_h}{2}\right)\right) = |\nabla\psi|^2 \geq 0\,,$$

where $|\cdot|^2$ and ∇ are the pointwise squared norm and the Chern connection with respect to the metric h^k induced on the power $K_{(M,I)}^{\otimes k}$, respectively. The strong maximum principle now implies that $|\psi|^2$ is constant, whence $\nabla\psi \equiv 0$. Consequently, for any $k \geq 0$ we have $\dim H^0(M, K_{(M,I)}^{\otimes k}) \leq 1$ from which it follows $\kappa(M, I) \leq 0$. Indeed, any non-trivial $\psi_1, \psi_2 \in H^0(M, K_{(M,I)}^{\otimes k})$ are parallel and nowhere vanishing. Thus, fixed any point $x \in M$, there exists $c \in \mathbb{C} \setminus \{0\}$ such that $\psi_1(x) = c\,\psi_2(x)$. Now, the section $\psi_1 - c\,\psi_2$ is parallel and vanishes at x implying $\psi_1 \equiv c\,\psi_2$.

Finally, we have $\kappa(M, I) = 0$ if and only if there is at least a power $k \geq 1$ such that

$$\dim H^0(M, K_{(M,I)}^{\otimes k}) = 1\,,$$

i.e. there exists a global holomorphic section of $K_{(M,I)}^{\otimes k}$, which is then parallel and nowhere vanishing. □

Be aware that, in general, for two different complex structures in H the corresponding Kodaira dimensions need not be equal (see Subsection 4.7.9).

Lemma 4.2.5. *Let (M, H, Ω) be a compact hyperHermitian manifold with $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$. Then the following are equivalent:*

1. *$s^{\mathrm{Ch}} = 0$;*
2. *s^{Ch} is constant;*

3. $\partial_J \alpha = 0$.

Proof. The assumption $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$ implies $\partial_J \alpha = \partial\partial_J f$, for some $f \in C^\infty(M, \mathbb{R})$. But then

$$s^{\mathrm{Ch}} = 2\Lambda_\Omega(\partial_J \alpha) = 2\Lambda_\Omega(\partial\partial_J f) = 2\Delta_\Omega f\,.$$

By the maximum principle, if s^{Ch} is constant then so is f, implying $\partial_J \alpha = 0$ and $s^{\mathrm{Ch}} = 0$. □

Remark 4.2.6. Let (M^n, H) be a hypercomplex manifold. In [331, Claim 1.2], Verbitsky observed that being $\mathsf{SL}(n, \mathbb{H})$ implies that $K_{(M,I)}$ is holomorphically trivial, for any $L \in \mathsf{H}$. This can also be deduced from Proposition 4.1.10. Of course, the holomorphic triviality if $K_{(M,I)}$ implies that it is holomorphically torsion. Finally, in [320, Proposition 1.1], it is shown that the assumption that $K_{(M,I)}$ is holomorphically torsion implies $c_1^{\mathrm{BC}}(M, I) = 0$. We now show that the vanishing of the first Bott-Chern class forces the quaternionic Bott-Chern class to vanish as well. For simplicity we show it for $L = I$. If $c_1^{\mathrm{BC}}(M, I) = 0$, for any hyperHermitian metric Ω on (M, H), we have $\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) = \sqrt{-1}\partial\bar\partial f$, for some $f \in C^\infty(M, \mathbb{R})$. The hyperHermitian metric $\Omega_f = \mathrm{e}^{\frac{f}{2n}}\Omega$ is Chern-Ricci flat, thanks to (4.2), then clearly $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$. In general, all the implications above cannot be reversed, counterexamples can be found in [17, Example 6.3] for the first and second implications, in [320, Examples 3.1, 3.2] for the third, and we refer to Example 4.7.9 for the last one. However, as we shall see in the next section, there is a quite broad class of hypercomplex manifolds for which these conditions are actually equivalent.

4.3 Quaternionic Gauduchon metrics

In this section we will study in detail quaternionic Gauduchon metrics, recall Definition 1.3.31, on a compact hypercomplex manifold, providing equivalent and sufficient conditions for the existence of such metrics.
Firstly, we characterize the quaternionic Gauduchon condition in terms of the $(1, 0)$-form β.

Lemma 4.3.1. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. Then, Ω is quaternionic Gauduchon if and only if $s^{\mathrm{B}} + 2|\beta|^2 = 0$.*

Proof. Since

$$\partial\partial_J \Omega^{n-1} = (-\partial_J \beta + \beta \wedge J^{-1}\bar\beta) \wedge \Omega^{n-1} = \frac{1}{n}\left(\frac{1}{2}s^{\mathrm{B}} + |\beta|^2\right)\Omega^n\,,$$

we clearly have the lemma. □

Note that, by Proposition 4.1.5 (a) and the fact that the Obata connection 1-form $\eta = \alpha + \bar\alpha$ is independent from the choice of the complex structure in H, the same is true for $\beta + \bar\beta$. In particular, thanks to Proposition 4.1.11, we see that the quaternionic Gauduchon condition holds on Ω_I if and only if it holds replacing I, J with any other pair of anti-commuting complex structures in H. We shall observe in Example 4.7.11 that this is no longer true for the quaternionic strongly Gauduchon condition.

As a first difference with the compact complex case, where Gauduchon metrics exist in any conformal class, see Theorem 1.1.35, we emphasize that there are examples of compact hypercomplex manifolds which do not admit any quaternionic Gauduchon metric, see Example 4.7.9.
We can use Lemma 4.3.1 to give an alternative proof of the following result by Grantcharov, Lejmi and Verbitsky, which determines a sufficient condition to the existence of quaternionic Gauduchon metrics.

Lemma 4.3.2 ([172, Proposition 16])**.** *Let (M^n, H) be a compact $\mathsf{SL}(n, \mathbb{H})$-manifold. Then, there exist a unique, up to scaling, quaternionic Gauduchon metric in any hyperHermitian conformal class.*

Proof. Let Ω_{G} be a Gauduchon hyperHermitian metric which then, by Lemma 4.1.12, satisfies $s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - s^{\mathrm{B}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}} + \beta_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}} = 0$. Since (M, H) is $\mathsf{SL}(n, \mathbb{H})$, applying Proposition 4.1.10, we have $\alpha_{\Omega_{\mathrm{G}}} =$

$(n-1)\partial f$, for some $f \in C^\infty(M,\mathbb{R})$ and thus $s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) = -2(n-1)\Delta_{\Omega_{\mathrm{G}}} f$. Therefore, the conformally rescaled metric $\Omega = \mathrm{e}^f \Omega_{\mathrm{G}}$ satisfies

$$\begin{aligned} s^{\mathrm{B}}(\Omega) + 2|\beta_\Omega|^2_\Omega &= \mathrm{e}^{-f}\left(s^{\mathrm{B}}(\Omega_{\mathrm{G}}) + 2(n-1)\Delta_{\Omega_{\mathrm{G}}} f + 2|\beta_{\Omega_{\mathrm{G}}} + (n-1)\partial f|^2_{\Omega_{\mathrm{G}}}\right) \\ &= \mathrm{e}^{-f}\left(s^{\mathrm{B}}(\Omega_{\mathrm{G}}) - s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) + 2|\beta_{\Omega_{\mathrm{G}}} + \alpha_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}}\right) = 0\,, \end{aligned}$$

i.e. it is quaternionic Gauduchon. □

In the invariant setting we can make the above stronger.

Lemma 4.3.3. *Let $(M := \mathsf{G}/\Gamma, \mathsf{H})$ be a $\mathsf{SL}(n,\mathbb{H})$ compact quotient of a Lie group by a lattice equipped with a left-invariant hypercomplex structure. Then, every invariant hyperHermitian metric is Gauduchon and quaternionic Gauduchon.*

Proof. Let Ω be an invariant hyperHermitian metric on M. We know that Ω is conformal to a Gauduchon metric Ω_{G} and, applying Definition 1.1.49 and Proposition 1.1.50, we see that $\mu(\Omega_{\mathrm{G}})$ is an invariant Gauduchon metric which is a constant multiple of Ω. Therefore, Ω itself is Gauduchon. Now, the $\mathsf{SL}(n,\mathbb{H})$ condition implies that $\alpha_\Omega = \partial f$, for some $f \in C^\infty(M,\mathbb{R})$, but by invariance we must have $\alpha_\Omega = 0$. In conclusion, the quaternionic Gauduchon condition $s^{\mathrm{B}}(\Omega) + 2|\beta_\Omega|^2_\Omega = 0$ is equivalent to the Gauduchon one. □

Observe that quaternionic Gauduchon metrics exist on hypercomplex manifolds that are not $\mathsf{SL}(n,\mathbb{H})$, indeed there even exist HKT non-$\mathsf{SL}(n,\mathbb{H})$ manifolds such as the ones constructed by Joyce (see Subsection 4.7.3) or Swann [310].

We now study some interesting consequences of the existence of a quaternionic Gauduchon metric.

Proposition 4.3.4. *Let $(M^n, \mathsf{H}, \Omega)$ be a compact hyperHermitian manifold admitting a compatible quaternionic Gauduchon metric. Then, the following are equivalent:*

1. $\alpha = 0$;
2. $\mathrm{Ric}^{\mathrm{Ch}}(\omega_L) = 0$, *for all* $L \in \mathsf{H}$;
3. $\partial_J \alpha = 0$.

Proof. From Proposition 4.1.5 (b), we know that $\alpha = 0$ always implies Chern-Ricci flatness and this in turn implies $\partial_J\alpha = 0$ by Lemma 4.2.2. Therefore, we only need to show that $\partial_J \alpha = 0$ implies $\alpha = 0$. Let $\tilde\Omega$ be a quaternionic Gauduchon metric on (M, H). We compute

$$\partial\partial_J \bar\Omega^n = \left(-\partial_J\alpha + \alpha \wedge J^{-1}\bar\alpha\right) \wedge \bar\Omega^n\,. \tag{4.10}$$

Assuming $\partial_J\alpha = 0$ and wedging with $\frac{\tilde\Omega^{n-1}}{n!(n-1)!}$, we get

$$\frac{\tilde\Omega^{n-1} \wedge \partial\partial_J\bar\Omega^n}{n!(n-1)!} = |\alpha|^2_{\tilde\Omega}\,\frac{\tilde\Omega^n \wedge \bar\Omega^n}{(n!)^2}$$

and integrating by parts yields

$$0 = \int_M \frac{\partial\partial_J\tilde\Omega^{n-1} \wedge \bar\Omega^n}{n!(n-1)!} = \int_M \frac{\tilde\Omega^{n-1} \wedge \partial\partial_J\bar\Omega^n}{n!(n-1)!} = \int_M |\alpha|^2_{\tilde\Omega}\,\frac{\tilde\Omega^n \wedge \bar\Omega^n}{(n!)^2}\,,$$

showing that we must have $\alpha = 0$. □

From Proposition 4.3.4 we obtain the following corollary.

Corollary 4.3.5. *A compact hypercomplex manifold (M^n, H) is $\mathsf{SL}(n,\mathbb{H})$ if and only if it admits a compatible quaternionic Gauduchon metric and $c_1^{\mathrm{qBC}}(M,\mathsf{H}) = 0$.*

Proof. We already know that the $\mathsf{SL}(n,\mathbb{H})$ condition implies $c_1^{\mathrm{qBC}}(M,\mathsf{H}) = 0$ and the existence of a quaternionic Gauduchon metric. Let us show the converse. The assumption $c_1^{\mathrm{qBC}}(M,\mathsf{H}) = 0$ implies that any hyperHermitian conformal class contains a hyperHermitian metric such that $\partial_J\alpha = 0$, therefore $\alpha = 0$ thanks to Proposition 4.3.4 and then (M,H) is $\mathsf{SL}(n,\mathbb{H})$, by Lemma 4.1.10. □

Thus, on a compact quaternionic Gauduchon manifold (M,H,Ω) all the implications in Remark 4.2.6 can be reversed.

It is clear that the $\partial\partial_J$-Lemma implies $c_1^{\mathrm{qBC}}(M,\mathsf{H}) = 0$ and if (M,H) admits a HKT metric then the $\partial\partial_J$-Lemma is implied by the $\mathsf{SL}(n,\mathbb{H})$ condition (see [172, Theorem 6]). Therefore the following corollary follows from Corollary 4.3.5.

Corollary 4.3.6. *Let (M^n,H) be a compact hypercomplex manifold admitting a compatible HKT metric. Then, the $\mathsf{SL}(n,\mathbb{H})$ condition and the $\partial\partial_J$-Lemma are equivalent.*

Note that the HKT assumption cannot be weakened, as there are examples of quaternionic balanced $\mathsf{SL}(n,\mathbb{H})$-manifolds without HKT metrics on which the $\partial\partial_J$-Lemma does not hold (see Example 4.7.1). We observe here an important difference with the complex setting. In general, the $\partial\bar\partial$-Lemma does not force neither the Chern nor the Bott-Chern class to vanish. Of course, examples of this are Fano or anti-Fano manifolds.

Lemma 4.3.7. *Let (M^n,H) be a compact hypercomplex manifold satisfying the $\partial\partial_J$-Lemma. Then, the following are equivalent:*

1. *(M,H) is a $\mathsf{SL}(n,\mathbb{H})$-manifold;*

2. *There exists a quaternionic Gauduchon metric on (M,H);*

3. *There exists a quaternionic strongly Gauduchon metric on (M,H).*

Proof. The fact that (1) implies (2) follows from Lemma 4.3.2 and the converse is a consequence of Corollary 4.3.5. We only need to prove that (2) implies (3). Let Ω be a quaternionic Gauduchon metric, in other words $\partial\partial_J\Omega^{n-1} = 0$. This means that $\partial_J\Omega^{n-1}$ is ∂-closed and ∂_J-exact. Thanks to the $\partial\partial_J$-Lemma, it is therefore $\partial\partial_J$-exact so that Ω is quaternionic strongly Gauduchon. □

By conjunction of Lemma 4.3.7, Corollary 4.3.6 and [232, Theorem 10.1] we infer

Corollary 4.3.8. *Let (M^2,H) be a compact hypercomplex manifold. Then, any two of the following conditions implies the third:*

1. *(M,H) is $\mathsf{SL}(2,\mathbb{H})$;*

2. *There exists a HKT metric on (M,H);*

3. *The $\partial\partial_J$-Lemma holds on (M,H).*

We should mention that the combination of [172, Theorem 25] and [172, Theorem 1] already gives that, assuming (1), (2) and (3) are equivalent.

In view of these results, it becomes particularly relevant to provide characterizations to the existence of quaternionic Gauduchon metrics on compact hypercomplex manifolds. Our next result fulfils this purpose for what regards conformal classes. Before we state the theorem, we want to highlight some easy necessary conditions for a quaternionic Gauduchon metric to exist. So, first of all, let Ω be a quaternionic Gauduchon metric and let Ω_{G} be the Gauduchon metric with unit volume in the conformal class of Ω. Now, we have that

$$0 = \int_M \frac{\partial\partial_J\Omega^{n-1}\wedge\bar\Omega_{\mathrm{G}}^n}{n!(n-1)!} = \int_M \frac{\Omega^{n-1}\wedge\partial\partial_J\bar\Omega_{\mathrm{G}}^n}{n!(n-1)!}\,.$$

But, keeping in mind (4.10), we obtain

$$0=\int_M(\partial_J\alpha_{\Omega_{\rm G}}-\alpha_{\Omega_{\rm G}}\wedge J^{-1}\bar\alpha_{\Omega_{\rm G}})\wedge\frac{\Omega^{n-1}\wedge\bar\Omega^n_{\rm G}}{n!(n-1)!}=\int_M\left(\Lambda_\Omega(\partial_J\alpha_{\Omega_{\rm G}})-|\alpha_{\Omega_{\rm G}}|^2_\Omega\right)\frac{\Omega^n\wedge\bar\Omega^n_{\rm G}}{(n!)^2}\,.$$

On the other hand, since $\Omega_{\rm G}$ and Ω are conformal, we can find $f\in C^\infty(M,\mathbb{R})$ such that $\Omega=\mathrm{e}^{\frac{f}{n-1}}\Omega_{\rm G}$. Then

$$\int_M\mathrm{e}^f\left(s^{\rm Ch}(\Omega_{\rm G})-2|\alpha_{\Omega_{\rm G}}|^2_{\Omega_{\rm G}}\right)\frac{\Omega^n_{\rm G}\wedge\bar\Omega^n_{\rm G}}{(n!)^2}=0\,. \tag{4.11}$$

Identity (4.11) tells us that if a quaternionic Gauduchon metric exists in the conformal class of Ω then

$$A=\left\{f\in C^\infty(M,\mathbb{R})\;\middle|\;\int_M\mathrm{e}^f\left(s^{\rm Ch}(\Omega_{\rm G})-2|\alpha_{\Omega_{\rm G}}|^2_{\Omega_{\rm G}}\right)\frac{\Omega^n_{\rm G}\wedge\bar\Omega^n_{\rm G}}{(n!)^2}=0\right\}\neq\emptyset\,.$$

One can easily show that the above condition is equivalent to the following

$$s^{\rm Ch}(\Omega_{\rm G})-2|\alpha_{\Omega_{\rm G}}|^2_{\Omega_{\rm G}}\text{ either vanishes identically or it changes sign}\,. \tag{4.12}$$

Notice that on a compact quotient of a Lie group, either locally homogeneous or homogeneous, we can only hope for a quaternionic Gauduchon metric in the conformal class of an invariant Gauduchon metric when

$$s^{\rm Ch}(\Omega_{\rm G})=2|\alpha_{\Omega_{\rm G}}|^2_{\Omega_{\rm G}}\,.$$

This, in particular, forces the Gauduchon degree of the conformal class of $\Omega_{\rm G}$ to be non-negative. On the other hand, when the latter is zero, automatically, we obtain $\alpha_{\Omega_{\rm G}}=0$ which in particular implies that the manifold is $\mathsf{SL}(n,\mathbb{H})$.

Since we will be searching for a quaternionic Gauduchon metric in the conformal class of a given hyperHermitian metric, we can always consider the latter as the unique Gauduchon metric $\Omega_{\rm G}$ in there. Using Lemma 4.3.1, if $\Omega_f=\mathrm{e}^{\frac{f}{n-1}}\Omega_{\rm G}$ is quaternionic Gauduchon, for some $f\in C^\infty(M,\mathbb{R})$, we have

$$\Lambda_{\Omega_f}(\partial_J\beta_{\Omega_f})-|\beta_{\Omega_f}|^2_{\Omega_f}=0\,,$$

which is readily seen to be equivalent to

$$\Delta_{\Omega_{\rm G}}f+|\beta_{\Omega_{\rm G}}+\partial f|^2+\frac12 s^{\rm B}(\Omega_{\rm G})=0\,. \tag{4.13}$$

Integrating (4.13) with respect to the volume induced by $\Omega_{\rm G}$ gives

$$\Gamma^{\rm B}(\{\Omega_{\rm G}\})=\int_M s^{\rm B}(\Omega_{\rm G})\frac{\Omega^n_{\rm G}\wedge\bar\Omega^n_{\rm G}}{(n!)^2}\leq 0\,, \tag{4.14}$$

namely the Gauduchon-Bismut degree of the conformal class of $\Omega_{\rm G}$, introduced in [39], has to be non-positive. Conditions (4.12) and (4.14) are thus necessary in order for a quaternionic Gauduchon metric to exist in the conformal class of $\Omega_{\rm G}$. We are now in the position to prove that they are also sufficient.

Theorem 4.3.9. *Let $(M^n,\mathsf{H},\Omega_{\rm G})$ be a compact hyperHermitian manifold and $\Omega_{\rm G}$ be a Gauduchon metric. Suppose (4.12) and (4.14) are satisfied, then, there exists a unique quaternionic Gauduchon metric of unit volume in the conformal class of $\Omega_{\rm G}$.*

Proof. First of all, we show the uniqueness part. Assume Ω and $\Omega_f:=\mathrm{e}^{\frac{f}{n-1}}\Omega$ are both quaternionic Gauduchon metrics with unit volume, then

$$\begin{aligned}0=s^{\rm B}(\Omega_f)+2|\beta_{\Omega_f}|^2_{\Omega_f}&=\mathrm{e}^{-\frac{f}{n-1}}\left(s^{\rm B}(\Omega)+2\Delta_\Omega f+2|\beta_\Omega+\partial f|^2_\Omega\right)\\&=2\mathrm{e}^{-\frac{f}{n-1}}\left(\Delta_\Omega f+2\mathrm{Re}(g(\beta_\Omega,\partial f))+|\partial f|^2_\Omega\right)\,.\end{aligned}$$

Since we have $\Delta_\Omega f + 2\mathrm{Re}(g(\beta_\Omega, \partial f)) = -|\partial f|^2_\Omega \leq 0$, we can regard f as a supersolution of the linear equation

$$\Delta_\Omega \varphi + 2\mathrm{Re}(g(\beta_\Omega, \partial\varphi)) = 0\,.$$

Applying the minimum principle, we obtain that f must be constant. On the other hand, the fact that both Ω_f and Ω have unit volume guarantees that $f = 0$.

We now prove the existence part of the theorem. Let Ω_{G} be a Gauduchon metric of unit volume and fix $h \in A$. Up to addition of a constant, we may and do assume that h has zero mean with respect to Ω_{G}. We consider $\Omega_h = \mathrm{e}^{\frac{h}{n-1}}\Omega_{\mathrm{G}}$ and, as done above, we have that

$$\mathrm{e}^{\frac{h}{n-1}}\left(\frac{1}{2}s^{\mathrm{B}}(\Omega_h) + |\beta_{\Omega_h}|^2_{\Omega_h}\right) = \frac{1}{2}s^{\mathrm{B}}(\Omega_{\mathrm{G}}) + \Delta_{\Omega_{\mathrm{G}}}h + |\beta_{\Omega_{\mathrm{G}}} + \partial h|^2_{\Omega_{\mathrm{G}}}\,. \tag{4.15}$$

Now, the equation to be solved is (4.13). We perform the classical method of continuity. We consider the following family of equations for $t \in [0,1]$

$$\Delta_{\Omega_{\mathrm{G}}} f + |\beta_{\Omega_{\mathrm{G}}} + \partial f|^2_{\Omega_{\mathrm{G}}} + \frac{1}{2}s^{\mathrm{B}}(\Omega_{\mathrm{G}}) = (1-t)\mathrm{e}^{\frac{h}{n-1}}\left(\frac{1}{2}s^{\mathrm{B}}(\Omega_h) + |\beta_{\Omega_h}|^2_{\Omega_h}\right)\,. \tag{4.16}$$

We will search for solutions in A with zero mean with respect to Ω_{G}. For $t = 0$, we easily observe that h is a solution. Now, consider $t \in [0,1]$ such that the corresponding equation admits a solution $f \in C^{2,\alpha}(M,\mathbb{R})$ and define the operator $F_t\colon A_0^{2,\alpha} \to \mathbb{R}$ such that

$$F_t(\varphi) := \Delta_{\Omega_{\mathrm{G}}}\varphi + |\beta_{\Omega_{\mathrm{G}}} + \partial\varphi|^2_{\Omega_{\mathrm{G}}} + \frac{1}{2}s^{\mathrm{B}}(\Omega_{\mathrm{G}}) - (1-t)\mathrm{e}^{\frac{h}{n-1}}\left(\frac{1}{2}s^{\mathrm{B}}(\Omega_h) + |\beta_{\Omega_h}|^2_{\Omega_h}\right)\,,$$

for any $t \in [0,1]$, where

$$A_0^{2,\alpha} = \left\{\varphi \in C_0^{2,\alpha}(M,\mathbb{R}) \;\middle|\; \int_M \mathrm{e}^{\varphi}\left(s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}}\right)\frac{\Omega_{\mathrm{G}}^n \wedge \bar{\Omega}_{\mathrm{G}}^n}{(n!)^2} = 0\right\}\,.$$

Here $C_0^{2,\alpha}(M,\mathbb{R})$ denotes the functions in $C^{2,\alpha}(M,\mathbb{R})$ with zero mean with respect to Ω_{G}. It is easy to show that $A_0^{2,\alpha}$ is a Banach manifold. Moreover, we can infer that, for any $f \in A_0^{2,\alpha}$,

$$T_f A_0^{2,\alpha} = \left\{v \in C_0^{2,\alpha}(M,\mathbb{R}) \;\middle|\; \int_M v\,\mathrm{e}^{f}\left(s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}}\right)\frac{\Omega_{\mathrm{G}}^n \wedge \bar{\Omega}_{\mathrm{G}}^n}{(n!)^2} = 0\right\}.$$

Then, the linearization of F_t at f is

$$d_f F_t(v) = \Delta_{\Omega_{\mathrm{G}}} v + 2\mathrm{Re}(g(\beta_{\Omega_{\mathrm{G}}} + \partial f, \partial v))\,, \quad v \in T_f A_0^{2,\alpha}\,,$$

which is a second order linear elliptic operator whose kernel, thanks to the maximum principle, is zero when imposing the zero mean condition on v. On the other hand, its index is equal to that of the Laplacian, implying that it is invertible. Applying the Implicit Function Theorem, we can conclude openness of the set of $t \in [0,1]$ for which (4.16) is solvable.

Now, in order to show closedness and conclude the proof, we need some a priori estimates. First of all, we prove an L^2-gradient estimate. Integrating (4.16) and using (4.15) we get:

$$\|\beta_{\Omega_{\mathrm{G}}} + \partial f\|^2_{L^2(\Omega_{\mathrm{G}})} = -\frac{t}{2}\Gamma^{\mathrm{B}}(\{\Omega_{\mathrm{G}}\}) + (1-t)\|\beta_{\Omega_{\mathrm{G}}} + \partial h\|^2_{L^2(\Omega_{\mathrm{G}})} \leq C\,,$$

thanks to the fact that h is a datum. Here and in what follows C will be a positive constant that does not depend on f and t, which may change value from line to line. We may and do assume that $\|\partial f\|^2_{L^2(\Omega_{\mathrm{G}})} \geq ||\beta_{\Omega_{\mathrm{G}}}||^2_{L^2(\Omega_{\mathrm{G}})}$, otherwise we are done. Consequently

$$\|\partial f\|^2_{L^2(\Omega_{\mathrm{G}})} - \|\beta_{\Omega_{\mathrm{G}}}\|^2_{L^2(\Omega_{\mathrm{G}})} = \left|\|\partial f\|^2_{L^2(\Omega_{\mathrm{G}})} - \|\beta_{\Omega_{\mathrm{G}}}\|^2_{L^2(\Omega_{\mathrm{G}})}\right| \leq \|\partial f + \beta_{\Omega_{\mathrm{G}}}\|^2_{L^2(\Omega_{\mathrm{G}})} \leq C\,,$$

which gives the desired estimate

$$\|\partial f\|^2_{L^2(\Omega_G)} \leq C\,. \tag{4.17}$$

Using the Poincaré inequality we then get an estimate on the L^2-norm of the solution f. Now, setting

$$\psi := -\left(\frac{1}{2}s^{\mathrm{B}}(\Omega_{\mathrm{G}}) + |\beta_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}} + |\partial f|^2_{\Omega_{\mathrm{G}}}\right) + (1-t)\mathrm{e}^{\frac{h}{n-1}}\left(\frac{1}{2}s^{\mathrm{B}}(\Omega_h) + |\beta_{\Omega_h}|^2_{\Omega_h}\right)$$

and rearranging equation (4.16), we can consider f as a solution of the linear equation

$$\Delta_{\Omega_{\mathrm{G}}}\varphi + 2\mathrm{Re}(g(\beta_{\Omega_{\mathrm{G}}}, \partial\varphi)) = \psi\,.$$

This allows us to use the Calderón-Zygmund inequality [163, Theorem 9.11] to deduce

$$\|f\|_{W^{2,2}(\Omega_{\mathrm{G}})} \leq C\left(\|f\|_{L^2(\Omega_{\mathrm{G}})} + \|\psi\|_{L^2(\Omega_{\mathrm{G}})}\right) \leq C\,,$$

thanks to (4.17). We can obtain higher-order estimates by bootstrapping. Using the Sobolev embeddings gives us the estimates we were looking for, concluding the existence part of the theorem. □

Remark 4.3.10. If (M^n, H) is a compact $\mathsf{SL}(n,\mathbb{H})$-manifold we can recover Lemma 4.3.2 as a corollary of Theorem 4.3.9. Indeed, the $\mathsf{SL}(n,\mathbb{H})$ condition implies that $\alpha_{\Omega_{\mathrm{G}}} = \partial f$, for some $f \in C^\infty(M,\mathbb{R})$, where Ω_{G} is any Gauduchon hyperHermitian metric on (M,H). The metric Ω_{G} then satisfies

$$s^{\mathrm{B}}(\Omega_{\mathrm{G}}) = s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}} + \beta_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}} = -2\Delta_{\Omega_{\mathrm{G}}} f - 2|\partial f + \beta_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}}$$

which, integrated gives (4.14). Observe that for this argument is actually enough to impose $c_1^{\mathrm{qBC}}(M,\mathsf{H}) = 0$. Also, suppose $s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}} = -2\Delta_{\Omega_{\mathrm{G}}} f - 2|\partial f|^2_{\Omega_{\mathrm{G}}}$ has a sign, then the same is true for

$$\mathrm{e}^f\left(|\partial f|^2_{\Omega_{\mathrm{G}}} + \Delta_{\Omega_{\mathrm{G}}} f\right) = \Delta_{\Omega_{\mathrm{G}}}(\mathrm{e}^f)\,.$$

By the maximum principle $s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) - 2|\alpha_{\Omega_{\mathrm{G}}}|^2_{\Omega_{\mathrm{G}}}$ must vanish identically, so that (4.12) is satisfied.

Remark 4.3.11. In the previous remark we observed that the vanishing of the first quaternionic Bott-Chern class implies (4.14). Therefore, invoking Theorem 4.3.9, we can rephrase Corollary 4.3.5 by stating that the $\mathsf{SL}(n,\mathbb{H})$ condition is equivalent to $c_1^{\mathrm{qBC}}(M,\mathsf{H}) = 0$ together with the fact that (4.12) holds in some conformal class.

One can observe that the search for a quaternionic Gauduchon metric within a conformal class can be considered as a particular instance of the problem of prescribing the Bismut scalar curvature on a compact complex manifold. As far as the author is aware, this problem was taken into account only in the constant case, i.e. the Bismut-Yamabe problem, in [39]. On the other hand, many results about the related problem of prescribing the Chern scalar curvature can be found in the literature, see for instance [23, 138, 343] and the references therein.

We now give another characterization of the existence of a quaternionic Gauduchon metric in terms of currents.

Proposition 4.3.12. *Let (M^n, H) be a compact hypercomplex manifold with $n \geq 2$. Then, M admits no quaternionic Gauduchon metric if and only if there exists a non-zero, $\partial\partial_J$-exact, q-real and q-positive $(2, 2n)$-current.*

Proof. We consider the following spaces:

$$W_1 = \{\varphi \in \Lambda_I^{2n-2,0}M \mid J\bar\varphi = \varphi\,,\, \varphi \text{ } q\text{-positive}\}\,,\quad W_2 = \{\psi \in \Lambda_I^{2n-2,0}M \mid \partial\partial_J\psi = 0\}\,.$$

On the other hand, every $\varphi \in W_1$ can be written as $\varphi = \Omega^{n-1}$ for some hyperHermitian metric Ω. Then, it easy to see that the non existence of quaternionic Gauduchon metrics on M is equivalent to

$W_1 \cap W_2 = \emptyset$. Thus, thanks to the Hahn-Banach Theorem, see for instance [172, Theorem 31], we can find a current $T \in \mathcal{D}_I^{2,2n}(M)$ such that

$$T|_{W_1} > 0\,, \quad T|_{W_2} = 0\,.$$

The fact that T is positive on W_1 guarantees that T is both q-real and q-positive. On the other hand, we have that, for any $\gamma \in \Lambda_I^{2n-3,0}M$,

$$\partial T(\gamma) = T(\partial\gamma) = 0\,,$$

since $\partial\gamma \in W_2$ and $T|_{W_2} = 0$. Then, $\partial T = 0$ and analogously $\partial_J T = 0$. Hence, T defines a cohomology class

$$[T]_{\mathrm{qBC}} \in H'^{2,2n}_{\mathrm{qBC}}(M,\mathbb{R}) := \frac{\{T \in \mathcal{D}_I^{2,2n}(M) \mid \partial T = \partial_J T = 0\}}{\partial\partial_J \mathcal{D}_I^{0,2n}(M)}\,.$$

Now, the claim is equivalent to prove that $[T]_{\mathrm{BC}} = 0$. In order to do this, we identify $T \in \Lambda_I^{2,2n}M \otimes \mathcal{D}^0(M)$, where $\mathcal{D}^0(M)$ are the distributions on M. This identification is compatible with ∂ and ∂_J and $H'^{2,2n}_{\mathrm{BC}}(M,\mathbb{R}) \simeq H^{2,2n}_{\mathrm{BC}}(M,\mathbb{R})$, the quaternionic Bott-Chern cohomology. Moreover, we have that

$$\langle\cdot,\cdot\rangle \colon H^{2,2n}_{\mathrm{qBC}}(M) \times H^{2n-2,0}_{\mathrm{qA}}(M) \to \mathbb{R} \qquad \langle[\varphi]_{\mathrm{qBC}}, [\psi]_{\mathrm{qA}}\rangle = \int_M \varphi \wedge \psi$$

is well-defined and non-degenerate, To see the non-degeneracy, fix a hyperHermitian metric g on M, so that

$$H^{*,*}_{\mathrm{qA}}(M,\mathbb{R}) \simeq \mathcal{H}_{\mathrm{qA}} := \ker\Delta_{\mathrm{qA}}\,, \quad H^{*,*}_{\mathrm{qBC}}(M,\mathbb{R}) \simeq \mathcal{H}_{\mathrm{qBC}} := \ker\Delta_{\mathrm{qBC}}\,,$$

where Δ_{qA} and Δ_{qBC} are, respectively, the so-called quaternionic Aeppli and Bott-Chern Laplacian. Δ_{qA} and Δ_{qBC} are fourth-order elliptic operators and their explicit expression can be found in [172]. Moreover, we have that, on compact hypercomplex manifolds,

$$\mathcal{H}_{\mathrm{qA}} = \ker\partial^* \cap \ker\partial_J^* \cap \ker\partial\partial_J\,, \quad \mathcal{H}_{\mathrm{qBC}} = \ker\partial \cap \ker\partial_J \cap \ker\partial^*\partial_J^*\,.$$

In view of this, it is easy to check that if $\varphi \in \mathcal{H}_{\mathrm{qBC}}$, then $*\varphi \in \mathcal{H}_{\mathrm{qA}}$. Consequently, for every $[\varphi]_{\mathrm{qBC}} \in H^{2,2n}_{\mathrm{qBC}}(M)$, we have

$$\langle[\varphi]_{\mathrm{qBC}}, [*\varphi]_{\mathrm{qA}}\rangle = \int_M \varphi \wedge *\varphi = \|\varphi\|^2_{L^2}\,,$$

giving the non-degeneracy of the pairing $\langle\cdot,\cdot\rangle$. Now, since $\langle[T]_{\mathrm{qBC}}, [\psi]_{\mathrm{qA}}\rangle = 0$, for any $\partial\partial_J$-closed $(2n-2,0)$-form ψ, then $[T]_{\mathrm{qBC}} = 0$, proving the claim. □

As highlighted by Theorem 4.3.9 and Example 4.7.9, it is not always possible to find quaternionic Gauduchon metrics on a given compact hypercomplex manifold. This can also be understood from Proposition 4.3.12. Indeed, comparing with the complex case, the characterization of the hypothetical non-existence of Gauduchon metrics on a compact complex manifold would be equivalent to the existence of a non-zero, positive $\partial\bar\partial$-exact $(1,1)$-current. Of course, such a current does not exist due to the fact that, on compact complex manifolds, any plurisubharmonic function is constant. On the other hand, this phenomenon might not happen in the hypercomplex setting. Indeed, chosen $\Theta \in \Lambda_I^{2n,0}M$ to be a q-positive volume form on M, any $\partial\partial_J$-exact, q-real and q-positive $(2,2n)$-current can be written as

$$T = \partial\partial_J(f\bar\Theta) = \partial\partial_J f \wedge \bar\Theta + \partial f \wedge \partial_J\bar\Theta - \partial_J f \wedge \partial\bar\Theta + f\partial\partial_J\bar\Theta\,, \quad f \in \mathcal{D}^0(M,\mathbb{C})\,, \tag{4.18}$$

which, in general, does not allow us to use the theory of quaternionic plurisubharmonic functions. On the other hand, when M is $\mathsf{SL}(n,\mathbb{H})$, we can always choose Θ to be holomorphic. This, together with the fact that quaternionic plurisubharmonic functions on a compact hypercomplex manifold are constant, gives another proof of Lemma 4.3.2. Although we are choosing Θ to be q-positive, the distribution f might be complex valued. However, if we further impose $\partial\partial_J\bar\Theta = 0$, then f can be chosen, up to additive constants, to admit only real values. Indeed, imposing q-realness of T in these hypothesis, we obtain that, defined $h = -\frac{\sqrt{-1}}{2}(f - \bar{f})$,

$$\partial\partial_J h \wedge \bar{\Theta} + \partial h \wedge \partial_J \bar{\Theta} - \partial_J h \wedge \partial\bar{\Theta} = 0\,.$$

Then, $h \in \mathcal{D}^0(M, \mathbb{R})$ satisfies a second order elliptic equation without zero order terms. So, applying standard elliptic regularity, $h \in C^\infty(M, \mathbb{R})$, then, by the maximum principle, h is constant. On the other hand, we can observe that, if c is a constant, then, of course, $\partial\partial_J(c\bar{\Theta}) = 0$. Thanks to this, we can add a suitable constant to f in order for it to be real-valued, as claimed.
Stemming from this discussion, we can give a weaker sufficient condition for the existence of quaternionic Gauduchon metrics.

Proposition 4.3.13. *Let (M^n, H) be a compact hypercomplex manifold. If there exists a q-positive volume form $\Theta \in \Lambda^{2n,0}_I M$ such that $\partial\partial_J\bar{\Theta} = 0$, then, there exists a quaternionic Gauduchon metric.*

Proof. Thanks to Proposition 4.3.12, the existence of a quaternionic Gauduchon metric is equivalent to the fact that any $\partial\partial_J$-exact, q-real and q-positive $(2, 2n)$-current is zero. Now, as above, any such current T can be written as in (4.18). Suppose $\partial\partial_J\bar{\Theta} = 0$ and consider the following closed convex cone

$$C = \{f \in \mathcal{D}^0(M, \mathbb{R}) \quad | \quad \partial\partial_J(f\bar{\Theta}) \geq 0\}\,. \tag{4.19}$$

It is not so hard to prove that $C \cap C^\infty(M, \mathbb{R})$ is dense in C in the weak sense of distributions. So, fixed $f \in C$, we can find $\{f_n\}_n \subseteq C \cap C^\infty(M, \mathbb{R})$ such that $f_n \to f$ in the weak sense of distributions. On the other hand, f_n satisfies

$$\partial\partial_J f_n \wedge \bar{\Theta} + \partial f_n \wedge \partial_J\bar{\Theta} - \partial_J f_n \wedge \partial\bar{\Theta} \geq 0$$

and so we can apply the maximum principle obtaining that f_n is constant. This implies that f is constant, giving the claim. □

We conclude this section remarking that the hypothesis of q-positivity of Θ cannot be removed. Indeed, we can consider a non-$\mathsf{SL}(n, \mathbb{H})$-manifold (M, H) with $K_{(M,I)}$ holomorphically trivial. Then, we can find an holomorphic volume form Θ which is, in particular, $\bar{\partial}\partial_J$-closed but not q-positive, since the manifold is not $\mathsf{SL}(n, \mathbb{H})$. On the other hand, Proposition 4.3.13 will guarantee the existence of a quaternionic Gauduchon metric. Then, applying Corollary 4.3.5, we will obtain that (M, H) is $\mathsf{SL}(n, \mathbb{H})$, reaching a contradiction. An example of a non-$\mathsf{SL}(n, \mathbb{H})$-manifold with holomorphically trivial canonical bundle can be found in Example 4.7.9.

4.4 Quaternionic balanced metrics

Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold. The quaternionic balanced condition $\partial\Omega^{n-1} = 0$ does not imply that the metric is HKT or even that M admits any HKT metric, see Examples 4.7.1,4.7.2 and 4.7.3. On the contrary, for any $1 \leq k < n-1$, requiring $\partial\Omega^k = 0$ would imply that the metric is HKT thanks to Proposition 1.3.20. Moreover, the form β_Ω is ∂-exact if and only if there exists a quaternionic balanced metric in the conformal class of Ω which is unique, up to scaling. Note that, as it happens for quaternionic Gauduchon metrics, the quaternionic balanced conditions do not depend on the choice of the complex structure in H.

On $\mathsf{SL}(n, \mathbb{H})$-manifolds, the existence of quaternionic balanced metrics turns out to be equivalent to the existence of balanced metrics.

Lemma 4.4.1. *Let (M^n, H) be a $\mathsf{SL}(n, \mathbb{H})$-manifold. Then, there exists a quaternionic balanced metric if and only if there exists a balanced hyperHermitian metric and the two metrics are conformal. Moreover, if M is a compact quotient of a Lie group by a lattice and the hypercomplex structure H is left-invariant, then an invariant hyperHermitian metric is quaternionic balanced if and only if it is balanced.*

Proof. In order to prove the first assertion, we simply note that, by Lemma 4.1.13, a hyperHermitian metric is balanced if and only if $\alpha+\beta=0$ and, by the $\mathsf{SL}(n,\mathbb{H})$ condition, this implies that β is ∂-exact. Conversely, if β is ∂-exact we can find a hyperHermitian balanced metric performing a conformal change.

Now, suppose M is a compact quotient of a Lie group by a lattice and let Ω be an invariant hyper-Hermitian metric. As shown in Lemma 4.3.3, the invariance of Ω and the $\mathsf{SL}(n,\mathbb{H})$ condition allow to use the symmetrization and deduce $\alpha=0$. Consequently, the Lee form is $\theta_\Omega=\beta+\bar\beta$, thus Ω is quaternionic balanced if and only if it is balanced. □

Lemma 4.4.1 can be considered as a generalization of [43, Proposition 4.11] and [156, Theorem 5.1].

Furthermore, Theorem 4.3.9 can be specialized in the quaternionic balanced case to give the following characterization.

Corollary 4.4.2. *Let $(M,\mathsf{H},\Omega_{\rm G})$ be a compact hyperHermitian manifold, where $\Omega_{\rm G}$ is a Gauduchon metric. Then, there exists a unique quaternionic balanced metric of unit volume in the conformal class of $\Omega_{\rm G}$ if and only if* (4.12) *holds and $\Gamma^{\rm B}(\{\Omega_{\rm G}\})=0$.*

Proof. Firstly, suppose that there exists a quaternionic balanced metric of unit volume in the conformal class of $\Omega_{\rm G}$. Then, it is, in particular, quaternionic Gauduchon and thus (4.12) holds. On the other hand, quaternionic balanced metrics are Bismut-Ricci flat. Hence, using (4.3), we see that the Bismut-Ricci form of $\Omega_{\rm G}$ with respect to I is $\partial\bar\partial$-exact giving $\Gamma^{\rm B}(\{\Omega_{\rm G}\})=0$. Conversely, using Theorem 4.3.9, we can find a solution $f\in C^\infty(M,\mathbb{R})$ to (4.13). On the other hand, integrating again (4.13) and using that $\Gamma^{\rm B}(\{\Omega_{\rm G}\})=0$ we obtain that $\beta_{\Omega_{\rm G}}=-\partial f$. This gives us the claim. □

Next, we give a characterization of the existence of quaternionic balanced metrics in terms of currents which can be considered as the analogue of [240, Proposition 4.5] for balanced metrics.

Proposition 4.4.3. *Let (M^n,H) be a compact hypercomplex manifold with $n\geq 2$. Then, M admits no quaternionic balanced metrics if and only if there exists a non-zero, ∂-exact, q-real and q-positive $(2,2n)$-current.*

Proof. The strategy of the proof follows that of Proposition 4.3.12. Indeed, it is sufficient to choose

$$W_1=\{\varphi\in\Lambda_I^{2n-2,0}M\mid J\bar\varphi=\varphi\,,\ \varphi\ q\text{-positive}\}\,,\quad W_2=\{\psi\in\Lambda_I^{2n-2,0}M\mid \partial\psi=0\}$$

and use the non-degenerate pairing

$$\langle\cdot,\cdot\rangle\colon H^{2,2n}_\partial(M)\times H^{2n-2,0}_\partial(M)\to\mathbb{R}\,,\qquad \langle[\varphi],[\psi]\rangle=\int_M\varphi\wedge\psi$$

to conclude. □

We shall use Proposition 4.4.3 to provide examples of compact quaternionic strongly Gauduchon manifolds not admitting quaternionic balanced metrics, see Examples 4.7.4, 4.7.5 and 4.7.6.

Combining Lemma 4.1.14 with Proposition 4.1.5 (b) and Proposition 4.3.4 we get the following equivalences, generalizing [47, Lemma 2.2].

Corollary 4.4.4. *Let (M^n,H,Ω) be a compact quaternionic balanced manifold. Then Ω cannot have negative Chern scalar curvature. Furthermore, the following are equivalent:*

1. $s^{\rm Ch}=0$*;*
2. Ω *is balanced;*
3. $\alpha=0$*;*
4. $\mathrm{Ric}^{\rm Ch}(\omega_L)=0$ *, for all* $L\in\mathsf{H}$*;*

5. $\partial_J \alpha = 0$.

The quaternionic balanced condition also restricts the possibilities for the Kodaira dimension.

Proposition 4.4.5. *Let (M, H) be a compact hypercomplex manifold admitting a compatible quaternionic balanced metric Ω. Then, $\kappa(M, L) \leq 0$, for all $L \in \mathsf{H}$. Moreover, if the Gauduchon metric in the conformal class of Ω is not balanced then $\kappa(M, L) = -\infty$ for all $L \in \mathsf{H}$.*

Proof. It follows from the fact that the quaternionic balanced condition implies Bismut-Ricci flatness and [14, Proposition 3.1]. □

Corollary 4.4.6. *Let (M^n, H) be a compact hypercomplex manifold admitting a compatible quaternionic balanced metric. Then, $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$ if and only if $\kappa(M, L) = 0$ for all $L \in \mathsf{H}$, and $c_1^{\mathrm{qBC}}(M, \mathsf{H}) \neq 0$ if and only if $\kappa(M, L) = -\infty$, for all $L \in \mathsf{H}$.*

Proof. We know from Corollary 4.3.5 that $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$ is equivalent to $K_{(M,L)}$ being holomorphically torsion for all $L \in \mathsf{H}$ and, by Proposition 4.2.4, this implies $\kappa(M, L) = 0$, for all $L \in \mathsf{H}$. On the other hand, if $c_1^{\mathrm{qBC}}(M, \mathsf{H}) \neq 0$ the Gauduchon metric Ω_{G} in the conformal class of any quaternionic balanced metric cannot be balanced, otherwise the manifold would be $\mathsf{SL}(n, \mathbb{H})$. Indeed, the form $\beta_{\Omega_{\mathrm{G}}}$ must be ∂-exact, but then, if Ω_{G} was balanced, α_{G} would also be ∂-exact, implying the $\mathsf{SL}(n, \mathbb{H})$ condition. □

By very well-know results, recall Theorem 1.1.52 and Theorem 1.1.53, the class of compact balanced manifolds is closed under products, proper holomorphic submersions and proper holomorphic modifications. In what follows, we will study some of the same closedness properties for the class of compact quaternionic balanced manifolds. First of all, we will recall the definition of hyperHermitian submersion, see for instance [6, 192, 193].

Definition 4.4.7. Let (M, H, g) and (M', H', g') be two hypercomplex manifolds. A map $f : M \to M'$ is called *hypercomplex* if, for any $L \in \mathsf{H}$, there exists $L' \in \mathsf{H}'$ such that $f : (M, L) \to (M', L')$ is holomorphic. If f is also a Riemannian submersion, then f will be called a *hyperHermitian submersion.*

Examples of hyperHermitian submersions between compact manifolds can be produced standardly looking, for instance, at finite coverings of compact hyperHermitian manifolds. Moreover, the projection onto the quotient of a Lie group by a lattice, endowed with a left-invariant hypercomplex structure, gives an example of hypercomplex map between a compact and a hypercomplex manifold, not necessarily compact. This can also be seen as a hyperHermitian submersion for a suitable choice of the hyperHermitian metrics, for instance a left-invariant hyperHermitian metric on the quotient and its pullback on the group. Another example can be found in [192, Section 4].

Proposition 4.4.8. *Let (M, H) and (M', H') be two quaternionic balanced manifolds and (M'', H'') be a hypercomplex manifold. Then, $(M \times M', \mathsf{H} \oplus \mathsf{H}')$ is quaternionic balanced. Moreover, if $f : (M, \mathsf{H}) \to (M'', \mathsf{H}'')$ is a proper hyperHermitian submersion, then (M'', H'') is quaternionic balanced.*

Proof. The first claim is trivial. Indeed, if Ω_M and $\Omega_{M'}$ are quaternionic balanced metrics on (M, H) and (M', H') respectively, then $\Omega = \Omega_M + \Omega_{M'}$ is quaternionic balanced on $(M \times M', \mathsf{H} \oplus \mathsf{H}')$.

Let us now prove the second assertion. Let n and m be the quaternionic dimensions of M and M'' respectively. We fix Ω_M to be a quaternionic balanced metric on M. For the sake of simplicity, we will also fix basis (I, J) and (I'', J'') for the hypercomplex structures H and H'', respectively, such that f is both (I, I'') and (J, J'')-holomorphic. Then, since f is proper we can consider

$$\gamma = f_* \Omega_M^{n-1} \,,$$

which is the $(2m-2)$-form given by integration of Ω_M^{n-1} along the fibers of f. Now, since f is (I, I'')-holomorphic and $\partial \Omega_M^{n-1} = 0$, then $\gamma \in \Lambda_{I''}^{2m-2,0} M''$ and $\partial \gamma = 0$. Thus, it is sufficient to prove that γ is q-real and q-positive. Indeed, if this is true, we know that there will exist a hyperHermitian metric $\Omega_{M''}$ such that $\gamma = \Omega_{M''}^{m-1}$, giving the claim. First of all, we note that q-realness is trivial because f is

(J, J'')-holomorphic and then $J'' f_* = f_* J$. Fixed a point $x \in M$ and any $(1,0)$-frame $\{Z_1, \ldots, Z_{2n}\}$ of $T_x M$ such that $J Z_{2i-1} = \bar{Z}_{2i}$, we can write at x

$$\Omega_M^{n-1} = a_{ij} \zeta^1 \wedge \zeta^2 \wedge \ldots \wedge \hat{\zeta}^i \wedge \ldots \wedge \hat{\zeta}^j \wedge \ldots \wedge \zeta^{2n} ,$$

where $\{\zeta^i\}$ is the dual coframe with respect to $\{Z_i\}$ and (a_{ij}) is a skew-symmetric matrix which is positive in the sense that $a_{ij} \zeta^i \wedge \zeta^j$ is q-positive. Now, fix $p \in M''$ and choose $\{\tilde{Z}_1, \ldots, \tilde{Z}_{2m}\}$ a $(1,0)$-frame with respect to I'' of $T_p M''$ such that $J'' \tilde{Z}_{2i-1} = \bar{\tilde{Z}}_{2i}$, for all $i = 1, \ldots, m$. We then consider $x \in F := f^{-1}(p)$ and, using that f is hypercomplex, we can lift $\{\tilde{Z}_1, \ldots, \tilde{Z}_{2m}\}$ to $\{Z_1, \ldots, Z_{2m}\} \subset T^{1,0}_{x,I} M$ such that $J Z_{2i-1} = \bar{Z}_{2i}$, for all $i = 1, \ldots, m$ and complete it to a $(1,0)$-frame $\{Z_1, \ldots, Z_{2n}\}$ of $T_x M$ such that $J Z_{2j-1} = \bar{Z}_{2j}$, for all $j = m+1, \ldots, n$. We can moreover assume that $\mathrm{vol}_F = \zeta^{m+1} \wedge \zeta^{m+2} \wedge \ldots \wedge \zeta^{2n}$ is a volume form when restricted to F. Now, we easily see that

$$\gamma = \tilde{a}_{ij} \tilde{\zeta}^1 \wedge \tilde{\zeta}^2 \wedge \ldots \hat{\tilde{\zeta}}^i \wedge \ldots \wedge \hat{\tilde{\zeta}}^j \wedge \ldots \wedge \tilde{\zeta}^{2m} , \quad \tilde{a}_{ij} = \int_F a_{ij} \mathrm{vol}_F$$

which is again skew-symmetric and positive, concluding the proof. □

A direct consequence of the second statement is that if $f \colon M \to M''$ is a finite covering, then M is quaternionic balanced if and only if M'' is quaternionic balanced.

As regards the closedness under modifications, we leave the question open. Before going into the problem, it is necessary to understand whether a smooth proper modification, compatible with the hypercomplex structures, of a HKT manifold stays HKT or not. The analogue of this problem in the Kähler setting was settled by Hironaka in [185] where a smooth proper modification of a Kähler manifold which is non-Kähler is provided. This example is nothing but a blow-up of $\mathbb{CP}^3$ along a singular curve. However, it is not even clear how one can blow-up a hypercomplex manifold obtaining another one or which operation can substitute the blow-up.

4.5 Strong HKT metrics

In this section, we will study properties of compact strong HKT manifolds, recall Definition 1.3.29, establishing Theorem E, namely, that on a compact hypercomplex manifold a strong HKT metric and a balanced hyperHermitian one cannot coexist without forcing the manifold to be hyperKähler.

First of all, we prove two preliminary formulae which hold in the hyperHermitian setting. One of these can be considered as the quaternionic analogue of [14, Formula (2.13)].

Proposition 4.5.1. *Let $(M^n, \mathsf{H}, \Omega)$ be a hyperHermitian manifold with $n \geq 2$. Then, for any $Z \in \Gamma(T_I^{1,0} M)$, we have*

$$\partial_J \alpha(Z, J\bar{Z}) = |\iota_Z \partial \bar{\Omega}|^2 + |\iota_{J\bar{Z}} \partial \bar{\Omega}|^2 - n \frac{\iota_{J\bar{Z}} \iota_Z \left(\partial \partial_J \bar{\Omega} \right) \wedge \bar{\Omega}^{n-1}}{\bar{\Omega}^n} . \tag{4.20}$$

Moreover,

$$\frac{1}{2} s^{\mathrm{Ch}}(\Omega) + g(\partial \partial_J \bar{\Omega}, \Omega \wedge \bar{\Omega}) - |\partial \bar{\Omega}|^2 = 0 . \tag{4.21}$$

Proof. We have that

$$\partial \partial_J \bar{\Omega}^n = n \partial \partial_J \bar{\Omega} \wedge \bar{\Omega}^{n-1} + n(n-1) \partial \bar{\Omega} \wedge \partial_J \bar{\Omega} \wedge \bar{\Omega}^{n-2} .$$

Now, we fix $Z \in \Gamma(T_I^{1,0} M)$ and compute

$$\iota_{J\bar{Z}} \iota_Z \left(\partial \bar{\Omega} \wedge \partial_J \bar{\Omega} \wedge \bar{\Omega}^{n-2} \right) = (\iota_Z \partial \bar{\Omega}) \wedge (\iota_{J\bar{Z}} \partial_J \bar{\Omega}) \wedge \bar{\Omega}^{n-2} - (\iota_{J\bar{Z}} \partial \bar{\Omega}) \wedge (\iota_Z \partial_J \bar{\Omega}) \wedge \bar{\Omega}^{n-2} .$$

Using (1.46) and Lemma 4.1.4, we infer that

$$\begin{aligned}\iota_{J\bar{Z}}\iota_Z\left(\partial\bar\Omega\wedge\partial_J\bar\Omega\wedge\bar\Omega^{n-2}\right) &= \frac{1}{n(n-1)}\left(|\alpha(Z)|^2+|\alpha(J\bar Z)|^2-|\iota_Z\partial\bar\Omega|^2-|\iota_{J\bar Z}\partial\bar\Omega|^2\right)\bar\Omega^n\\ &= \frac{1}{n(n-1)}\left((\alpha\wedge J^{-1}\bar\alpha)(Z,J\bar Z)-|\iota_Z\partial\bar\Omega|^2-|\iota_{J\bar Z}\partial\bar\Omega|^2\right)\bar\Omega^n\,.\end{aligned}$$

This, together with (4.10), gives (4.20).

Finally, choosing I-holomorphic coordinates $(z^1,\dots,z^{2n})$ at a point where Ω takes the expression $\Omega=\sum_{i=1}^n dz^{2i-1}\wedge dz^{2i}$, we may choose $Z=\frac{\partial}{\partial z^{2i-1}}$ in (4.20) and sum over i to deduce (4.21). □

The next theorem we want to present goes in the direction of the so-called Fino-Vezzoni conjecture, see Conjecture 1.1.57.

It is clear from Definition 1.3.29 that a HKT metric Ω is strong if and only if ω_L are all simultaneously SKT, for $L\in\mathsf{H}$, recall Definition 1.1.56. Hence, the question of whether strong HKT and balanced hyperHermitian metrics can coexist on a non-hyperKähler manifold or not can be viewed as a particular instance of the Fino-Vezzoni conjecture.

Before we prove the announced Theorem E we need the following preliminary incompatibility result.

Proposition 4.5.2. *Let (M^n,H) be a hypercomplex manifold with $n\geq 2$. If Ω is a strong HKT metric compatible with H which is not hyperKähler, then $\partial_J\alpha_\Omega\geq 0$ and $\partial_J\alpha_\Omega\neq 0$. In particular, $c_1^{\mathrm{qBC}}(M,I,J)$ admits a q-semipositive representative and, if M is compact, it is non zero.*

Proof. From (4.20) using that $\partial\partial_J\bar\Omega=0$, we see that

$$\partial_J\alpha_\Omega(Z,J\bar Z)=|\iota_Z\partial\bar\Omega|^2+|\iota_{J\bar Z}\partial\bar\Omega|^2\geq 0\,,\quad Z\in\Gamma(T_I^{1,0}M)$$

and since Ω is not hyperKähler there exists at least one $Z\in T_I^{1,0}M$ such that $\partial_J\alpha_\Omega(Z,J\bar Z)>0$, proving the first statement.

Furthermore, suppose M is compact and $c_1^{\mathrm{qBC}}(M,\mathsf{H})=0$, then there would be a function $f\in C^\infty(M,\mathbb{R})$ such that

$$0\leq\partial_J\alpha_\Omega=\partial\partial_J f.$$

The above equation, in particular, implies that f is a quaternionic plurisubharmonic function. By this, since we are working on a compact manifold, f must be constant, forcing $\partial_J\alpha_\Omega=0$ which is impossible, thanks to the first part of the proof. □

The existence of a balanced HKT metric forces a compact hypercomplex manifold (M^n,H) to be $\mathsf{SL}(n,\mathbb{H})$ and thus $c_1^{\mathrm{qBC}}(M,\mathsf{H})=0$. Hence, *a fortiori*, a non-hyperKähler strong HKT manifold admits no balanced HKT metrics. We are now ready to prove Theorem E.

Theorem 4.5.3. *Let $(M^n,\mathsf{H},\tilde\Omega)$ be a compact non-hyperKähler strong HKT manifold. Then, there is no balanced hyperHermitian metric on (M,H).*

Proof. If $n=1$ balanced metrics coincide with Kähler metrics, thus the result is obvious.

Suppose $n\geq 2$ and assume by contradiction that there is a balanced hyperHermitian metric Ω on M compatible with H, then

$$\int_M \mathrm{s}^{\mathrm{Ch}}(\Omega)\frac{\Omega^n\wedge\bar\Omega^n}{(n!)^2}=2\int_M\partial_J\alpha_\Omega\wedge\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}=2\int_M\alpha_\Omega\wedge\partial_J\left(\frac{\Omega^{n-1}\wedge\bar\Omega^n}{n!(n-1)!}\right)=0$$

which implies that there exists $f\in C^\infty(M,\mathbb{R})$ such that $\mathrm{s}^{\mathrm{Ch}}(\Omega)=\Delta_\Omega f$. On the other hand, we know that $\partial_J\alpha_{\tilde\Omega}=\partial_J\alpha_\Omega+\partial\partial_J\varphi$, for some $\varphi\in C^\infty(M,\mathbb{R})$. Tracing this last relation with respect to Ω yields

$$\mathrm{tr}_\Omega(\partial_J\alpha_{\tilde\Omega})=\frac12 s^{\mathrm{Ch}}(\Omega)+\Delta_\Omega\varphi=\frac12\Delta_\Omega(f+2\varphi)\,.$$

But then Proposition 4.5.2 gives $\Delta_\Omega(f+2\varphi) \geq 0$ and from the maximum principle we get $\mathrm{tr}_\Omega(\partial_J \alpha_{\tilde{\Omega}}) = 0$. We claim that this entails $\partial_J \alpha_{\tilde{\Omega}} = 0$ contradicting Proposition 4.5.2. To see this, take I-holomorphic local coordinates $(z^1, \ldots, z^{2n})$ at a point where

$$\Omega = \sum_{i=1}^{n} dz^{2i-1} \wedge dz^{2i}\,, \quad \partial_J \alpha_{\tilde{\Omega}} = \sum_{i=1}^{n} \lambda_i dz^{2i-1} \wedge dz^{2i}\,,$$

where $\lambda_i \geq 0$ by q-semipositivity of $\partial_J \alpha_{\tilde{\Omega}}$. Then $0 = \mathrm{tr}_\Omega(\partial_J \alpha_{\tilde{\Omega}}) = \sum_{i=1}^n \lambda_i$ as claimed, concluding the proof. □

From Corollary 4.4.6 we also deduce the following fact about the Kodaira dimension.

Corollary 4.5.4. *Let (M, H) be a hypercomplex manifold admitting a compatible strong HKT metric. Then, $\kappa(M, L) = -\infty$, for all $L \in \mathsf{H}$.*

4.6 Chern-Einstein hyperHermitian metrics

Let (M, J, ω) be a Hermitian manifold. The metric is called *first Chern-Einstein* if the $(1,1)$-form ω satisfies

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega) = \lambda\omega\,, \quad \lambda \in C^\infty(M, \mathbb{R})\,.$$

The function λ is called the *Einstein factor*. On compact complex manifolds the problem of finding a Hermitian metric such that the Chern-Ricci form is a multiple of the metric itself is essentially understood [24]. First Chern-Einstein metrics with non-identically-zero Einstein factor exist if and only if they are conformal to a Kähler metric in $\pm c_1(M)$. Furthermore, such a metric is unique, up to scaling, in its conformal class. On the other hand, assuming the necessary condition $c_1^{\mathrm{BC}}(M, J) = 0$, a Chern-Ricci flat metric always exists in any conformal class, and it is unique up to scaling.

Let us show how, in the hypercomplex setting, a natural Einstein condition arises by looking at a suitable geometric flow.

Definition 4.6.1. Let $(M, \mathsf{H}, \Omega_0)$ be a hyperHermitian manifold. The *hyperHermitian Chern-Ricci flow* is the following evolution equation:

$$\begin{cases} \frac{\partial}{\partial t}\Omega = -\partial_J \alpha_\Omega\,, \\ \Omega(0) = \Omega_0\,. \end{cases} \tag{4.22}$$

Such a flow is strictly related to the quaternionic Monge-Ampère equation and it was already considered in [47]. Indeed, [47, Equation (5)] is equivalent to (4.22) by means of Lemma 4.2.2. This new formulation of the flow makes evident that the HKT condition is preserved along it.

Definition 4.6.2. Let (M, H) be a hypercomplex manifold. A hyperHermitian metric Ω is *Chern-Einstein* if it is a static point of (4.22), namely if it is satisfied the following:

$$\partial_J \alpha_\Omega = \lambda\Omega \tag{4.23}$$

for some $\lambda \in C^\infty(M, \mathbb{R})$. If moreover Ω is HKT we say that it is *HKT-Einstein*.

We observe that, tracing (4.23), λ is proportional to the Chern scalar curvature:

$$s^{\mathrm{Ch}} = 2n\lambda\,.$$

With the next lemma we rephrase the definition in terms of the forms ω_L, for $L \in \mathsf{H}$.

Lemma 4.6.3. *Let (M, H, Ω) be a hyperHermitian manifold. Then, the following are equivalent:*

1. *Ω is Chern-Einstein with Einstein factor λ;*

2. ω_I *satisfies*

$$\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) - J\mathrm{Ric}^{\mathrm{Ch}}(\omega_I)}{2} = \lambda\omega_I\,;$$

3. For any pair of anti-commuting complex structures $L, P \in \mathsf{H}$,

$$\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_L) - P\mathrm{Ric}^{\mathrm{Ch}}(\omega_L)}{2} = \lambda\omega_L\,.$$

Proof. We already observed that the equivalence of (1) and (2) follows from Lemma 4.2.2. The fact that (1) is equivalent to (3) can be obtained using the symmetry argument as in the proof of Proposition 4.1.11, thanks to which it is sufficient to assume $P = I$ and $L = aJ + bK \in \mathsf{H}$. On the other hand, this case follows from (4.4) and (4.9). □

Remark 4.6.4. It has been pointed out to us by M. Lejmi that a hyperHermitian metric Ω is Chern-Einstein if and only if

$$\left(\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_J) + \sqrt{-1}\mathrm{Ric}^{\mathrm{Ch}}(\omega_K)}{2}\right)^{(2,0)} = \lambda\Omega\,.$$

This follows directly from (4.5) and (4.6).

4.6.1 The 1-dimensional case

We start with compact hyperHermitian manifolds of quaternionic dimension 1. These are always HKT for dimensional reasons and they have been classified by Boyer in [56]. Up to conformal equivalence, the complete list is the following:

- Tori with the flat metric;
- K3 surfaces with a hyperKähler metric;
- Quaternionic Hopf surfaces, recall Example 1.3.2, with the standard locally conformally flat metric, see (1.38).

The first two classes are hyperKähler while Hopf surfaces are HKT non-Kähler, therefore they are the right candidate to check the HKT-Einstein condition. Kato [207] has described all complex Hopf surfaces admitting a hypercomplex structure.

We may endow the Hopf surface with the hyperHermitian metric Ω corresponding to ω_I as in (1.38). It is easy to check that ω_I is SKT. We claim that it is also HKT-Einstein. We have

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) = 2\omega_I - \frac{2\sqrt{-1}}{|z|^4}(\bar{z}^1 dz^1 + \bar{z}^2 dz^2) \wedge (z^1 d\bar{z}^1 + z^2 d\bar{z}^2)$$

concluding

$$\frac{\mathrm{Ric}^{\mathrm{Ch}}(\omega_I) - J\mathrm{Ric}^{\mathrm{Ch}}(\omega_I)}{2} = \omega_I\,.$$

Then, ω_I is HKT-Einstein with Einstein factor identically equal to 1.

From this perspective the classification result of Boyer can be regarded as a hypercomplex version of the classical uniformization Theorem. The fact that there is no representative with negative Einstein constant is not an accident. As a matter of fact we will see that this peculiar fact happens in any dimension for compact HKT-Einstein manifolds.

4.6.2 The case $\lambda \not\equiv 0$

Let (M^n, H) be a hypercomplex manifold. If $c_1^{\mathrm{qBC}}(M, I, J)$ has a sign, it admits a representative $\pm\Omega$ where Ω is a hyperHermitian metric which is necessarily HKT. Notice that this implies that $c_1^{\mathrm{qBC}}(M, P, L)$ has the same sign as $c_1^{\mathrm{qBC}}(M, I, J)$ for any pair of anti-commuting $P, L \in \mathsf{H}$. For this reason we will write $c_1^{\mathrm{qBC}}(M, \mathsf{H}) \gtrless 0$. Now, by definition $[\partial_J \alpha_\Omega]_{\mathrm{qBC}} = [\pm\Omega]_{\mathrm{qBC}}$, thus there exists $f \in C^\infty(M, \mathbb{R})$ such that

$$\partial_J \alpha_\Omega = \pm\Omega + \partial\partial_J f \, .$$

It is then easy to show that Ω is conformal to a hyperHermitian Einstein metric. Indeed, $\Omega_f = \mathrm{e}^{\frac{f}{n}}\Omega$ satisfies

$$\partial_J \alpha_{\Omega_f} = \partial_J \alpha_\Omega - \partial\partial_J f = \pm\Omega = \pm \mathrm{e}^{-\frac{f}{n}}\Omega_f \, .$$

We shall however observe that the condition $c_1^{\mathrm{qBC}}(M, \mathsf{H}) < 0$ can never occur in the compact setting. Indeed, this would give the contradiction

$$0 > -\int_M \frac{\Omega^n \wedge \bar\Omega_f^n}{(n!)^2} = \int_M \partial_J \alpha_{\Omega_f} \wedge \frac{\Omega^{n-1} \wedge \bar\Omega_f^n}{(n!)^2} = \int_M \alpha_{\Omega_f} \wedge \frac{\Omega^{n-1} \wedge \partial_J \bar\Omega_f^n}{(n!)^2}$$
$$= \frac{1}{n}\int_M |\alpha_{\Omega_f}|^2_\Omega \frac{\Omega^n \wedge \bar\Omega_f^n}{(n!)^2} \, .$$

It is however important to observe that this behaviour is exclusive of the compact case. Indeed, in Subsection 4.7.4, we will provide examples of HKT-Einstein metrics with negative Einstein factor on suitable solvable Lie groups.
Let us now consider the case $c_1^{\mathrm{qBC}}(M, \mathsf{H}) > 0$ and the conformal rescaling $\Omega_{\mathrm{G}} = \mathrm{e}^{\frac{h}{n}}\Omega$ where Ω_{G} is the Gauduchon metric with unit volume. Then,

$$\partial_J \alpha_{\Omega_{\mathrm{G}}} = \partial_J \alpha_\Omega - \partial\partial_J h = \Omega + \partial\partial_J (f-h) = \mathrm{e}^{-\frac{h}{n}}\Omega_{\mathrm{G}} + \partial\partial_J(f-h) \, .$$

Tracing and integrating against the Gauduchon volume, it implies that

$$\Gamma(\{\Omega\}) := \int_M s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) \frac{\Omega_{\mathrm{G}}^n \wedge \bar\Omega_{\mathrm{G}}^n}{(n!)^2} = 2\int_M \mathrm{e}^{-\frac{h}{n}} \frac{\Omega_{\mathrm{G}}^n \wedge \bar\Omega_{\mathrm{G}}^n}{n!(n-1)!} > 0 \, .$$

Now, we shall show that quaternionic Gauduchon metrics conformal to hyperHermitian Einstein ones with non-identically zero Einstein factor are actually HKT.

Proposition 4.6.5. *Let $(M^n, \mathsf{H}, \Omega)$ be a compact hyperHermitian manifold with $n \geq 2$ and admitting a quaternionic Gauduchon metric $\tilde\Omega$ conformal to Ω. If*

$$\partial_J \alpha = \lambda\Omega$$

for some $\lambda \in C^\infty(M, \mathbb{R})$, $\lambda \not\equiv 0$, then $\tilde\Omega$ is HKT. Moreover, in this case, $c_1^{\mathrm{qBC}}(M, \mathsf{H}) > 0$.

Proof. Let $f \in C^\infty(M, \mathbb{R})$ be such that $\tilde\Omega = \mathrm{e}^f\Omega$. Applying ∂_J to the Einstein equation, we obtain that

$$\partial_J(\lambda \mathrm{e}^{-f}) \wedge \tilde\Omega + \lambda \mathrm{e}^{-f}\partial_J\tilde\Omega = 0 \, . \tag{4.24}$$

Let us now set $\psi = (\lambda\mathrm{e}^{-f})^{n-1}$. Then, (4.24) implies $\partial_J\psi \wedge \tilde\Omega^{n-1} + \psi\partial_J\tilde\Omega^{n-1} = 0$. Applying ∂ to this identity, we get

$$\partial\partial_J\psi \wedge \tilde\Omega^{n-1} - \partial_J\psi \wedge \partial\tilde\Omega^{n-1} + \partial\psi \wedge \partial_J\tilde\Omega^{n-1} + \psi\partial\partial_J\tilde\Omega^{n-1} = 0 \, .$$

Using the quaternionic Gauduchon condition and the definition of $\beta_{\tilde\Omega}$, we obtain

$$\Delta_{\tilde\Omega}\psi + \tilde g(d\psi, \beta_{\tilde\Omega} + \bar\beta_{\tilde\Omega}) = 0 \, .$$

Consequently, by the maximum principle, ψ must be constant. Now also $\lambda\mathrm{e}^{-f}$ is a non-zero constant and (4.24) reveals that $\tilde\Omega$ is HKT. □

4.6.3 The HKT case

Now we move on to study the HKT case. Let $(M^n, \mathsf{H}, \Omega)$ be compact HKT-Einstein of quaternionic dimension $n \geq 2$. Let λ be the Einstein factor. Since α and Ω are ∂-closed we have

$$0 = \partial\partial_J\alpha = \partial\lambda \wedge \Omega$$

therefore, for $n \geq 2$

$$0 = \partial\lambda \wedge \partial_J\lambda \wedge \Omega^{n-1} = \frac{1}{n}|\partial\lambda|^2\Omega^n$$

showing that λ is constant. Moreover, since $c_1^{\mathrm{qBC}}(M, \mathsf{H})$ cannot be negative, then $\lambda \geq 0$. Furthermore, we can deduce an explicit formula for it. To see this, we simply integrate by parts

$$\lambda = \frac{1}{\int_M \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}} \int_M \lambda \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2} = \frac{1}{\int_M \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}} \int_M \partial_J\alpha \wedge \frac{\Omega^{n-1} \wedge \bar{\Omega}^n}{(n!)^2} = \frac{1}{n} \frac{1}{\int_M \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}} \int_M |\alpha|^2 \frac{\Omega^n \wedge \bar{\Omega}^n}{(n!)^2}$$

This is a striking difference with the usual behaviour of the Einstein factor in other settings. We summarize this as follows.

Proposition 4.6.6. *Let (M, H) be a compact hypercomplex manifold. Then, a Chern-Einstein hyperHermitian metric with non-identically zero Einstein factor has constant Chern scalar curvature if and only if it is HKT. When this is the case, the Chern scalar curvature is necessarily non-negative.*

It is fairly easy to come up with examples of compact hyperHermitian Einstein manifolds with $\lambda = 0$ but not admitting compatible HKT metrics. For instance, any hypercomplex nilmanifold M with a non-abelian hypercomplex structure H, by [43, Theorem 4.6], does not have any compatible HKT metric. However, since hypercomplex nilmanifolds are $\mathsf{SL}(n, \mathbb{H})$, as in the proof of Lemma 4.3.3, any left-invariant hyperHermitian metric Ω satisfies $\alpha = 0$. In Subsection 4.7.3, we will provide examples of HKT-Einstein metrics with strictly positive Einstein factor.

Within our framework, the quaternionic Calabi conjecture formulated by Alesker and Verbitsky in [9, Conjecture 1.5] can be phrased as follows: on a compact HKT manifold (M, H, Ω), for any representative $\Psi \in c_1^{\mathrm{qBC}}(M, \mathsf{H})$ it should always be possible to find another HKT metric $\Omega_\varphi := \Omega + \partial\partial_J\varphi$, for some $\varphi \in C^\infty(M, \mathbb{R})$, such that $\partial_J\alpha_{\Omega_\varphi} = \Psi$. If this turns out to be true, when $c_1^{\mathrm{qBC}}(M, \mathsf{H}) = 0$ we would be able to find HKT-Einstein metrics with vanishing Einstein factor (cf. [332]), i.e. that are balanced and Chern-Ricci flat.
In a similar fashion, one is led to speculate on the existence of HKT-Einstein metrics on compact HKT manifolds with positive first quaternionic Bott-Chern class. More precisely, we wonder:

Question 4.6.7. Let $(M^n, \mathsf{H}, \Omega)$ be a compact HKT manifold such that $c_1^{\mathrm{qBC}}(M, \mathsf{H}) > 0$. Does it always exist $\varphi \in C^\infty(M, \mathbb{R})$ such that Ω_φ is a HKT-Einstein metric?

Proceeding similarly to the Kähler case, the question turns out to be equivalent to the solvability of the following quaternionic Monge-Ampère equation:

$$(\Omega + \partial\partial_J\varphi)^n = \mathrm{e}^{f-\varphi}\Omega^n\,, \quad \Omega + \partial\partial_J\varphi > 0\,,$$

where $\varphi \in C^\infty(M, \mathbb{R})$ is the unknown and $f \in C^\infty(M, \mathbb{R})$ is the datum. Unfortunately, this case is the quaternionic analogue of the Fano case in the Kähler setting, hence, when approaching the problem with the classical method of continuity, the same difficulties arise. Furthermore, it is natural to expect that certain obstructions may emerge in our context, similar to those found by Futaki [142] and Matsushima [239]. It is extremely likely that more sophisticated tools are necessary to approach this problem.
As a final remark, since in the compact HKT case the Einstein factor is non-negative, the quaternionic Monge-Ampère equation corresponding to negative Chern scalar curvature can never be solved.

4.7 Examples and constructions

In this section, we collect several examples and two interesting constructions, one inspired by Arroyo and Nicolini [33, Section 5], the other due to Barberis and Fino [44]. The purpose of the first part of this section is to exhibit examples admitting one type of metrics listed in Definition 1.3.31 but none of the one immediately stronger. We shall provide several explicit examples with different dimensions, because these will serve the purpose of "building blocks" in Subsection 4.7.1 to provide examples in any quaternionic dimension for which it is possible. In Subsection 4.7.3 we study the examples constructed by Joyce [201] and Spindel, Sevrin, Troost, Van Proeyen [291] showing that they are HKT-Einstein with positive Einstein constant. Finally, Subsection 4.7.4 is devoted to present some non-compact HKT-Einstein manifolds, two of which have negative Einstein factor.

In what follows, we will work on Lie algebras endowed with a hypercomplex structure. So, we will always fix a coframe $\{e^1,\dots,e^{4n}\}$ for the Lie algebra and, unless otherwise stated, use the following hypercomplex structure: for all $k=1,\dots,n$,

$$Ie^{4k-3}=-e^{4k-2}\,,\quad Ie^{4k-1}=-e^{4k}\,,\quad Je^{4k-3}=-e^{4k-1}\,,\quad Je^{4k-2}=e^{4k}\,,$$

In view of this, the $(1,0)$-coframe $\{\zeta^1,\dots,\zeta^{2n}\}$ with respect to I will be $\zeta^j=e^{2j-1}+\sqrt{-1}e^{2j}$, for all $j=1,\dots,2n$, and J will act on it as $J\zeta^{2i-1}=-\bar\zeta^{2i}$, for all $i=1,\dots,n$.

We start our inspection from manifolds which admit quaternionic balanced metrics but not HKT ones. An example of such manifold can be found, for instance, in [112]. Here we present three other such examples. Note that in quaternionic dimension 2 the quaternionic balanced condition coincides with the HKT condition, hence we need to work in dimension at least 3. The examples we provide are, $3,4$ and 5-dimensional respectively.

Example 4.7.1. We consider the nilpotent Lie algebra in [232, Example 3] with structure constants:

$$de^i=0\,,\quad i=1,\dots,8\,,$$

$$de^9=e^1\wedge e^5\,,\quad de^{10}=e^1\wedge e^6\,,\quad de^{11}=e^1\wedge e^7\,,\quad de^{12}=e^1\wedge e^8\,.$$

The $(1,0)$-basis $(\zeta^1,\dots,\zeta^6)$ with respect to I satisfies:

$$d\zeta^i=0\,,\quad i=1,\dots,4\,,\quad d\zeta^5=\frac{1}{2}(\zeta^1\wedge\zeta^3+\bar\zeta^1\wedge\zeta^3)\,,\quad d\zeta^6=\frac{1}{2}(\zeta^1\wedge\zeta^4+\bar\zeta^1\wedge\zeta^4)\,.$$

Since the hypercomplex structure is not abelian, by [43, Theorem 4.6] the nilmanifold N does not admit any HKT metric. However, we shall show that N admits an invariant quaternionic balanced metric. Indeed, the invariant hyperHermitian metric on N that makes the coframe unitary, which corresponds to the q-real, q-positive $(2,0)$ form

$$\Omega=\zeta^1\wedge\zeta^2+\zeta^3\wedge\zeta^4+\zeta^5\wedge\zeta^6\,,$$

satisfies $\partial\Omega^2=0$ meaning that it is quaternionic balanced.
Observe that this example does not satisfy the $\partial\partial_J$-Lemma because of [230, Theorem 5] and the observation of Lejmi and Weber [232, Example 3] that it is not C^∞-pure nor C^∞-full.

Example 4.7.2. Consider the nilpotent Lie algebra with structure equations:

$$de^i=0\,,\quad i=1,\dots,12\,,$$

$$de^{13}=e^1\wedge e^5+e^1\wedge e^9\,,\qquad de^{15}=e^1\wedge e^7+e^1\wedge e^{11}\,,$$

$$de^{14}=e^1\wedge e^6+e^1\wedge e^{10}\,,\qquad de^{16}=e^1\wedge e^8+e^1\wedge e^{12}\,.$$

It is easy to show that the hypercomplex structure is non-abelian. The $(1,0)$-basis $(\zeta^1,\dots,\zeta^8)$ with respect to I satisfies:

$$d\zeta^i = 0\,,\quad i=1,\dots,6\,,$$

$$d\zeta^7 = \frac{1}{2}\left(\zeta^1\wedge\zeta^3+\bar\zeta^1\wedge\zeta^3+\zeta^1\wedge\zeta^5+\bar\zeta^1\wedge\zeta^5\right)\,,$$
$$d\zeta^8 = \frac{1}{2}\left(\zeta^1\wedge\zeta^4+\bar\zeta^1\wedge\zeta^4+\zeta^1\wedge\zeta^6+\bar\zeta^1\wedge\zeta^6\right)\,.$$

Again, the metric that makes the coframe unitary is quaternionic balanced.

Example 4.7.3. Consider the nilpotent Lie algebra with structure equations:

$$de^i = 0\,,\quad i=1,\dots,16\,,$$

$$de^{17} = e^1\wedge e^5 + e^9\wedge e^{13}\,,\qquad de^{19} = e^1\wedge e^7+e^9\wedge e^{15}\,,$$
$$de^{18} = e^1\wedge e^6 + e^9\wedge e^{14}\,,\qquad de^{20} = e^1\wedge e^8+e^9\wedge e^{16}\,.$$

Again, the hypercomplex structure is non-abelian. The $(1,0)$-basis $(\zeta^1,\dots,\zeta^{10})$ with respect to I satisfies:

$$d\zeta^i = 0\,,\quad i=1,\dots,8\,,$$
$$d\zeta^9 = \frac{1}{2}\left(\zeta^1\wedge\zeta^3+\bar\zeta^1\wedge\zeta^3+\zeta^5\wedge\zeta^7+\bar\zeta^5\wedge\zeta^7\right)\,,$$
$$d\zeta^{10} = \frac{1}{2}\left(\zeta^1\wedge\zeta^4+\bar\zeta^1\wedge\zeta^4+\zeta^5\wedge\zeta^8+\bar\zeta^5\wedge\zeta^8\right)\,.$$

Also in this case, the hyperHermitian metric with respect to which the given frame is unitary satisfies the quaternionic balanced condition.

We now exhibit examples of compact quaternionic strongly Gauduchon nilmanifolds on which no quaternionic balanced metric exists. Notice that the minimum quaternionic dimension for which it is possible to provide such an example as a nilmanifold is 3, indeed all hypercomplex nilmanifolds are $\mathsf{SL}(n,\mathbb{H})$, see [43, Corollary 3.3]. Therefore, in real dimension 8, they admit a quaternionic strongly Gauduchon metric if and only if they admit a HKT metric by [232, Theorem 10.1].

Example 4.7.4. Consider the nilpotent Lie algebra with structure equations:

$$de^i = 0\,,\quad i=1,\dots,8\,,$$
$$de^9 = e^1\wedge e^3\,,\quad de^{10} = e^1\wedge e^4+e^7\wedge e^8\,,\quad de^{11} = e^5\wedge e^7\,,\quad de^{12} = -e^3\wedge e^4+e^5\wedge e^8\,.$$

The structure equations can be rewritten in terms of the $(1,0)$-coframe $(\zeta^1,\dots,\zeta^6)$ as follows:

$$d\zeta^i = 0\,,\quad i=1,\dots,4\,,$$

$$d\zeta^5 = \frac{1}{2}(\zeta^1\wedge\zeta^2+\bar\zeta^1\wedge\zeta^2-\zeta^4\wedge\bar\zeta^4)\,,\quad d\zeta^6 = \frac{1}{2}(\zeta^3\wedge\zeta^4+\bar\zeta^3\wedge\zeta^4+\zeta^2\wedge\bar\zeta^2)\,.$$

In particular, we have that:

$$\partial\zeta^1=\partial\zeta^2=\partial\zeta^3=\partial\zeta^4=0\,,\qquad \partial\zeta^5=\frac{1}{2}\zeta^1\wedge\zeta^2\,,\qquad \partial\zeta^6=\frac{1}{2}\zeta^3\wedge\zeta^4\,,$$
$$\partial_J\zeta^1=\partial_J\zeta^2=\partial_J\zeta^3=\partial_J\zeta^4=0\,,\qquad \partial_J\zeta^5=-\frac{1}{2}\zeta^3\wedge\zeta^4\,,\qquad \partial_J\zeta^6=\frac{1}{2}\zeta^1\wedge\zeta^2\,.$$

The invariant hyperHermitian metric Ω that renders the coframe unitary is quaternionic strongly Gauduchon as

$$\partial\Omega^2 = 2\partial_J(\zeta^3\wedge\zeta^4\wedge\zeta^5\wedge\zeta^6-\zeta^1\wedge\zeta^2\wedge\zeta^5\wedge\zeta^6)\,.$$

Since the structure constants are rational, the Lie algebra admits a lattice and the metric above descends to the compact quotient N. On the other hand, we claim that there are no quaternionic balanced metrics on the nilmanifold N. Indeed, the form $\zeta^1 \wedge \zeta^2 = 2\partial\zeta^5$ induces a q-positive, ∂-exact, $(2, 2n)$-current T given by integration

$$T(\gamma) := \int_N \zeta^1 \wedge \zeta^2 \wedge \gamma \wedge \bar{\Theta}\,, \quad \gamma \in \Lambda_I^{2n-2,0} N\,,$$

where Θ is any holomorphic volume form in $\Lambda_I^{2n,0} N$. We conclude applying Proposition 4.4.3.

Example 4.7.5. Consider the nilpotent Lie algebra with structure equations:

$$de^i = 0\,, \quad i = 1, \ldots, 12\,,$$

$$de^{13} = e^1 \wedge e^3\,, \qquad de^{15} = e^5 \wedge e^7 + e^9 \wedge e^{11}\,,$$
$$de^{14} = e^1 \wedge e^4 + e^7 \wedge e^8 + e^{11} \wedge e^{12}\,, \qquad de^{16} = -e^3 \wedge e^4 + e^5 \wedge e^8 + e^9 \wedge e^{12}\,.$$

The complex structure equations are:

$$d\zeta^i = 0\,, \quad i = 1, \ldots, 6\,,$$
$$d\zeta^7 = \frac{1}{2}(\zeta^1 \wedge \zeta^2 + \bar{\zeta}^1 \wedge \zeta^2 - \zeta^4 \wedge \bar{\zeta}^4 - \zeta^6 \wedge \bar{\zeta}^6)\,,$$
$$d\zeta^8 = \frac{1}{2}(\zeta^3 \wedge \zeta^4 + \bar{\zeta}^3 \wedge \zeta^4 + \zeta^5 \wedge \zeta^6 + \bar{\zeta}^5 \wedge \zeta^6 + \zeta^2 \wedge \bar{\zeta}^2)\,.$$

The invariant hyperHermitian metric that makes such coframe unitary is quaternionic strongly Gauduchon. On the other hand, as in the previous example, it is easy to see that there are no quaternionic balanced metrics.

Example 4.7.6. Consider the nilpotent Lie algebra with structure equations:

$$de^i = 0\,, \quad i = 1, \ldots, 16\,,$$

$$de^{17} = e^1 \wedge e^3 + e^5 \wedge e^7\,, \qquad de^{18} = e^1 \wedge e^4 + e^5 \wedge e^8 + e^{11} \wedge e^{12} + e^{15} \wedge e^{16}\,,$$
$$de^{19} = e^9 \wedge e^{11} + e^{13} \wedge e^{15}\,, \qquad de^{20} = -e^3 \wedge e^4 - e^7 \wedge e^8 + e^9 \wedge e^{12} + e^{13} \wedge e^{16}\,.$$

The structure equations can be rewritten in terms of a $(1,0)$-coframe with respect to I as follows:

$$d\zeta^i = 0\,, \quad i = 1, \ldots, 8\,,$$
$$d\zeta^9 = \frac{1}{2}(\zeta^1 \wedge \zeta^2 + \bar{\zeta}^1 \wedge \zeta^2 + \zeta^3 \wedge \zeta^4 + \bar{\zeta}^3 \wedge \zeta^4 - \zeta^6 \wedge \bar{\zeta}^6 - \zeta^8 \wedge \bar{\zeta}^8)\,,$$
$$d\zeta^{10} = \frac{1}{2}(\zeta^5 \wedge \zeta^6 + \bar{\zeta}^5 \wedge \zeta^6 + \zeta^7 \wedge \zeta^8 + \bar{\zeta}^7 \wedge \zeta^8 + \zeta^2 \wedge \bar{\zeta}^2 + \zeta^4 \wedge \bar{\zeta}^4)\,.$$

The invariant hyperHermitian metric that makes the $(1,0)$-coframe unitary is quaternionic strongly Gauduchon and the same argument of Example 4.7.4 shows that there exists no quaternionic balanced metric.

Remark 4.7.7. Both the quaternionic balanced condition and the strongly Gauduchon condition are not preserved under small deformations of the hypercomplex structure. Indeed, in [110], the authors provide an example of an 8-dimensional nilmanifold on which the existence of HKT metrics is not preserved along a deformation of the hypercomplex structure. However, HKT metrics in quaternionic dimension 2 coincide with quaternionic balanced ones while, as remarked above, on hypercomplex nilmanifolds of real dimension 8 the existence of HKT metrics is equivalent to the existence of quaternionic strongly Gauduchon metrics.
It would be interesting to know if the same phenomenon occurs for the quaternionic Gauduchon condition.

We now focus on manifolds admitting quaternionic Gauduhcon metric but no strongly quaternionic Gauduchon ones.

We already mentioned that any 8-dimensional hypercomplex nilmanifold with non-abelian hypercomplex structure admits no quaternionic strongly Gauduchon metric. On the other hand it admits a quaternionic Gauduchon metric by Lemma 4.3.2. Notice that, as a consequence of Lemma 4.3.7, no example in this class can satisfy the $\partial\partial_J$-Lemma.
Here we present an example in each dimension.

Example 4.7.8. For $n \geq 2$, consider the $4n$-dimensional nilpotent Lie algebra with structure equations:

$$de^i = 0\,, \quad i = 1, \dots, 4n-3\,,$$

$$de^{4n-2} = \sum_{k=1}^{n-1} e^{4k-3} \wedge e^{4k-2}\,, \quad de^{4n-1} = \sum_{k=1}^{n-1} e^{4k-3} \wedge e^{4k-1}\,, \quad de^{4n} = \sum_{k=1}^{n-1} e^{4k-3} \wedge e^{4k}\,.$$

The $(1,0)$-basis $(\zeta^1, \dots, \zeta^{2n})$ with respect to I satisfies:

$$d\zeta^i = 0\,, \quad i = 1, \dots, 2n-2\,,$$

$$d\zeta^{2n-1} = -\frac{1}{2}\sum_{k=1}^{n-1} \zeta^{2k-1} \wedge \bar\zeta^{2k-1}\,, \quad d\zeta^{2n} = \frac{1}{2}\sum_{k=1}^{n-1} (\zeta^{2k-1} \wedge \zeta^{2k} + \bar\zeta^{2k-1} \wedge \zeta^{2k})\,.$$

We shall check that the generic invariant hyperHermitian metric

$$\Omega = A_{ij}\zeta^i \wedge \zeta^j\,, \quad A_{ij} \in \mathbb{C}\,,$$

is not quaternionic strongly Gauduchon. We compute

$$\begin{aligned}
\partial\Omega^{n-1} &= \partial\left(A_{ij}\zeta^i \wedge \zeta^j\right)^{n-1} \\
&= (n-1)!\,\partial\left(\mathrm{pf}(A(i,j))\zeta^1 \wedge \dots \wedge \hat\zeta^i \wedge \dots \wedge \hat\zeta^j \wedge \dots \wedge \zeta^{2n}\right) \\
&= -(n-1)! \sum_{1 \leq i < j \leq 2n-1} \mathrm{pf}(A(i,j))\zeta^1 \wedge \dots \wedge \hat\zeta^i \wedge \dots \wedge \hat\zeta^j \wedge \dots \wedge \zeta^{2n-1} \wedge \partial\zeta^{2n} \\
&= -\frac{(n-1)!}{2}\left(\sum_{k=1}^{n-1} \mathrm{pf}(A(2k-1,2k))\right)\zeta^1 \wedge \dots \wedge \zeta^{2n-1}\,,
\end{aligned}$$

where the hat symbols means that the term is missing and $A(i,j)$ denotes the matrix obtained from the skew-symmetric matrix $A = (A_{rs})$ removing the i^{th} and j^{th} rows and columns. Then $\partial\Omega^{n-1}$ is non-zero, because by q-positivity of Ω we must have $\mathrm{pf}(A(2k-1,2k)) > 0$, for all $k = 1, \dots, n$. Moreover, it is not ∂_J-exact since

$$\partial_J\zeta^i = 0\,, \quad i \neq 2n-1\,, \quad \partial_J\zeta^{2n-1} = -\frac{1}{2}\sum_{k=1}^{n-1} \zeta^{2k-1} \wedge \zeta^{2k}\,.$$

On the other hand, these are all nilpotent Lie algebras with rational structure constants, therefore the corresponding nilmanifolds are $\mathsf{SL}(n,\mathbb{H})$ and the existence of a quaternionic Gauduchon metric is guaranteed by Lemma 4.3.2.

Finally, we discuss an example of hypercomplex manifold which do not admit any quaternionic Gauduchon metrics.

Example 4.7.9. The example we describe appeared in [17, Example 6.3] to provide a negative answer to a question posed by Verbitsky. It is constructed as a compact quotient $S = \mathsf{G}/\Gamma$ of a solvable Lie group by a lattice Γ equipped with an invariant hypercomplex structure H. Andrada and Tolcachier show that

the canonical bundle of (S, I) is holomorphically trivial but that of (S, J) is not, indeed, they even show that $c_1^{\mathrm{BC}}(S, J) \neq 0$. On the other hand, the holomorphic triviality of $K_{(S,I)}$ implies $c_1^{\mathrm{qBC}}(S, \mathsf{H}) = 0$ (cf. Remark 4.2.6). We then conclude that a compatible quaternionic Gauduchon metric cannot exists from Corollary 4.3.5.
For the sake of completeness we also show this in another way. Thanks to the symmetrization process, it is enough to show that (S, H) does not admit invariant quaternionic Gauduchon metrics. We may easily conclude by means of Theorem 4.3.9. Indeed, (S, H) cannot satisfy condition (4.12) in any conformal class (although it does satisfy condition (4.14)). For any invariant Gauduchon hyperHermitian metric Ω_{G}, we must have $s^{\mathrm{Ch}}(\Omega_{\mathrm{G}}) = 0$ by the triviality of $K_{(S,I)}$ and so if (4.12) were to hold, it would imply $\alpha_{\Omega_{\mathrm{G}}} = 0$ which contradicts the fact that (S, H) is not $\mathsf{SL}(n, \mathbb{H})$.

Remark 4.7.10. This example, together with Proposition 4.2.4 also shows that $\kappa(S, I) = 0$ while $\kappa(S, J) = -\infty$, so, in general, it may happen that different complex structures in the same hypercomplex structure yield different Kodaira dimensions.
From this, it also follows that the complex deformation

$$I_t = \cos(\pi t) I + \sin(\pi t) J\,, \quad t \in [0, 1/2]\,,$$

does not preserve the Kodaira dimension, a fact that is in contrast with the projective case as shown by Siu [286]. We are grateful to G. Grantcharov for this observation.

We conclude this section showing that the quaternionic strongly Gauduchon condition depends on the choice of the pair of anti-commuting complex structures in the same hypercomplex structure

Example 4.7.11. We consider Example 4.7.4. Let us show that with respect to the pair (J, I) there are no quaternionic strongly Gauduchon metrics. Consider the $(1, 0)$-coframe with respect to J given by $w^{2k-1} = e^{4k-3} + \sqrt{-1} e^{4k-1}$ and $w^{2k} = e^{4k-2} - \sqrt{-1} e^{4k}$ for $k = 1, 2, 3$, and let $d = \partial + \bar{\partial}$ be the splitting of the exterior differential with respect to J. Then

$$\begin{aligned} \partial w^i &= 0\,, \quad i = 1, \dots, 5\,, & \partial w^6 &= \frac{1}{2}(\sqrt{-1} w^1 \wedge w^2 + w^3 \wedge w^4)\,, \\ I^{-1}\bar{\partial} I w^i &= 0\,, \quad i \neq 5\,, & I^{-1}\bar{\partial} I w^5 &= \frac{1}{2}(\sqrt{-1} w^1 \wedge w^2 - w^3 \wedge w^4)\,. \end{aligned}$$

It is now easy to check that the generic invariant hyperHermitian metric Ω cannot be quaternionic strongly Gauduchon, since $\partial \Omega^2$ is proportional to $w^1 \wedge w^2 \wedge w^3 \wedge w^4 \wedge w^5$ which is never $I^{-1}\bar{\partial} I$-exact.

4.7.1 Arroyo-Nicolini's construction

In [33, Section 5], Arroyo and Nicolini give a procedure to construct nilpotent SKT Lie algebras starting from previous ones. Here we adapt their argument to build new hyperHermitian nilpotent Lie algebras.

Let $(\mathfrak{g}_1, [\cdot,\cdot]_1, \mathsf{H}_1)$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2, \mathsf{H}_2)$ be two nilpotent Lie algebras equipped with a hypercomplex structure generated by (I_1, J_1) and (I_2, J_2) respectively. Provided we have $\dim[\mathfrak{g}_i, \mathfrak{g}_i] < \dim Z(\mathfrak{g}_i)$, we can choose $e_i \in Z(\mathfrak{g}_i) \setminus [\mathfrak{g}_i, \mathfrak{g}_i]$, where $Z(\mathfrak{g}_i)$ is the center of $\mathfrak{g}_i$. Then, we can define the new Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \langle X, Y, Z, W \rangle$ with Lie bracket

$$[\cdot,\cdot]|_{\mathfrak{g}_i \times \mathfrak{g}_i} := [\cdot,\cdot]_i\,, \quad [X, Y] = -[Z, W] := e_1 + e_2$$

and hypercomplex structure H determined by:

$$\begin{aligned} I|_{\mathfrak{g}_i} &= I_i\,, & IX &= Y\,, & IZ &= W\,, \\ J|_{\mathfrak{g}_i} &= J_i\,, & JX &= Z\,, & JY &= -W\,. \end{aligned}$$

Observe that the nilpotency step of $\mathfrak{g}$ is the maximum of the nilpotency steps of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Theorem 4.7.12. *The hypercomplex Lie algebra* $(\mathfrak{g}, \mathsf{H})$ *admits HKT (resp. quaternionic balanced, quaternionic strongly Gauduchon) metrics if and only if* $(\mathfrak{g}_1, \mathsf{H}_1)$ *and* $(\mathfrak{g}_2, \mathsf{H}_2)$ *do.*

Proof. Clearly, H is abelian if and only if both H_1 and H_2 are, therefore, thanks to [43, Theorem 4.6], the theorem is true for HKT metrics. We can be more precise: Ω_i is a HKT metric on $\mathfrak{g}_i$ for $i = 1, 2$ if and only if

$$\Omega := \Omega_1 + \Omega_2 + \zeta^1 \wedge \zeta^2 \tag{4.25}$$

is HKT, where (ζ^1, ζ^2) is the $(1,0)$ coframe with respect to I dual to $(X - \sqrt{-1}Y, Z - \sqrt{-1}W)$. This is because $\partial(\zeta^1 \wedge \zeta^2) = 0$ and the only modification that occurs in the structure constants of $\mathfrak{g}_i$ is that now $de^i = 2\sqrt{-1}(-\zeta^1 \wedge \bar{\zeta}^1 + \zeta^2 \wedge \bar{\zeta}^2)$, which before was zero. In particular, $\partial e^i = 0$ is unaltered.

The same phenomenon occurs for the other kinds of special metrics. Let Ω_i be a hyperHermitian metric on $\mathfrak{g}_i$ and define Ω on $\mathfrak{g}$ as in (4.25). Let n_i be the quaternionic dimension of $\mathfrak{g}_i$ and let $n = n_1 + n_2 + 1$ be that of $\mathfrak{g}$. Since

$$\Omega^{n-1} = \binom{n}{n_1}\Omega_1^{n_1} \wedge \Omega_2^{n_2} + n_2\binom{n}{n_1}\Omega_1^{n_1} \wedge \Omega_2^{n_2-1} \wedge \zeta^1 \wedge \zeta^2 + n_1\binom{n}{n_2}\Omega_1^{n_1-1} \wedge \Omega_2^{n_2} \wedge \zeta^1 \wedge \zeta^2\,,$$

we see that

$$\partial\Omega^{n-1} = n_2\binom{n}{n_1}\Omega_1^{n_1} \wedge \partial\Omega_2^{n_2-1} \wedge \zeta^1 \wedge \zeta^2 + n_1\binom{n}{n_2}\partial\Omega_1^{n_1-1} \wedge \Omega_2^{n_2} \wedge \zeta^1 \wedge \zeta^2\,.$$

Therefore, Ω is quaternionic balanced or quaternionic strongly Gauduchon if and only if Ω_1 and Ω_2 are. □

Assume that $(\mathfrak{g}_i, \mathsf{H}_i)$ are equipped with metrics g_i that makes them *indecomposable*, that amounts to say that they cannot be written as a g_i-orthogonal sum of H_i-invariant ideals for $i = 1, 2$. If we choose $e_i \in Z(\mathfrak{g}_i) \cap [\mathfrak{g}_i, \mathfrak{g}_i]^{\perp_{g_i}}$, the resulting hypercomplex Lie algebra $(\mathfrak{g}, \mathsf{H})$ is then also indecomposable, a fact that can be seen with the same argument of Arroyo and Nicolini [33, Subsection 5.1]. Furthermore, it is not hard to check that $\dim[\mathfrak{g}, \mathfrak{g}] < \dim Z(\mathfrak{g})$ so this Lie algebra can be used again in the process to obtain higher-dimensional indecomposable examples.

Note that, if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have rational structure constants the same is true for $\mathfrak{g}$, therefore we can find lattices. This discussion, together with the observation that the examples produced in Examples 4.7.1, 4.7.2, 4.7.3, 4.7.4, 4.7.5, 4.7.6, 4.7.8 are indecomposable nilpotent Lie algebras with rational structure constants, shows that we can use them as building blocks to construct compact nilmanifolds carrying a certain type of special metric but no metrics of the class immediately stronger among those we have studied. This can be achieved in any quaternionic dimension except for those low dimensions for which we explained that this cannot happen.

4.7.2 Barberis-Fino's construction

In this subsection, we study the behaviour of the metric conditions under a construction due to Barberis and Fino [44]. The idea is to take a hypercomplex Lie algebra and build a new one via a quaternionic representation.

Let $\mathfrak{g}$ be a $4n$-dimensional Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(k, \mathbb{H})$ a Lie algebra homomorphism. Define on $T_\rho\,\mathfrak{g} := \mathfrak{g} \ltimes_\rho \mathbb{H}^k$ the Lie bracket

$$[(X, U), (Y, V)] := ([X, Y], \rho_X(V) - \rho_Y(U))$$

for every $X, Y \in \mathfrak{g}$ and $U, V \in \mathbb{H}^k$. If $\mathfrak{g}$ is endowed with a hypercomplex structure H

$$\tilde{L}(X, U) := (LX, lU)\,,$$

defines a hypercomplex structure $\tilde{\mathsf{H}}$ on $T_\rho\,\mathfrak{g}$, where if $L = aI + bJ + cK \in \mathsf{H}$ then $l = ai + bj + ck$, being i, j, k the quaternion units in $\mathbb{H}$. Finally, if g is a hyperHermitian metric on $(\mathfrak{g}, \mathsf{H})$, the metric $\tilde{g}$

induced by g and the natural metric on $\mathbb{H}^k$ in such a way that $\mathfrak{g}$ is orthogonal to $\mathbb{H}^k$ is hyperHermitian on $(T_\rho\,\mathfrak{g}, \tilde{\mathsf{H}})$. We will denote objects on $(T_\rho\,\mathfrak{g}, \tilde{\mathsf{H}})$ with a tilde, so, for instance, Ω and $\tilde{\Omega}$ will be the respective $(2,0)$-forms with respect to I and $\tilde{I}$.

Theorem 4.7.13 ([44]). *Let $(\mathfrak{g}, \mathsf{H}, g)$ and $(T_\rho\,\mathfrak{g}, \tilde{\mathsf{H}}, \tilde{g})$ be as above and denote $p\colon T_\rho\,\mathfrak{g} \to \mathfrak{g}$ the orthogonal projection. Then,*

1. *$\tilde{\nabla}^{\mathrm{Ob}}\tilde{g} = 0$ if and only if $\nabla^{\mathrm{Ob}} g = 0$ and $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$.*
2. *$\tilde{\Omega}$ is HKT if and only if Ω is HKT and $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$;*
3. *if $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$ then $T^{\tilde{L}} = T^L \circ p$ and $dT^{\tilde{L}} = dT^L \circ p$, where $T^{\tilde{L}}$ and T^L are the torsions of the Bismut connections of $(\tilde{g}, \tilde{L})$ and (g, L) respectively, for $L \in \mathsf{H}$;*
4. *if $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$ then $\theta_{\tilde{\Omega}} = \theta_\Omega \circ p$.*

In particular, when $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$, $(T_\rho\,\mathfrak{g}, \tilde{\mathsf{H}}, \tilde{\Omega})$ is strong HKT (resp. weak HKT, hyperKähler, balanced) if and only if $(\mathfrak{g}, \mathsf{H}, \Omega)$ is.

We can refine this result by proving the following result.

Theorem 4.7.14. *Let $(\mathfrak{g}, \mathsf{H}, \Omega)$ and $(T_\rho\,\mathfrak{g}, \tilde{\mathsf{H}}, \tilde{\Omega})$ be as above, denote $p\colon T_\rho\,\mathfrak{g} \to \mathfrak{g}$ the orthogonal projection and assume $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$. Then, $\alpha_{\tilde{\Omega}} = \alpha_\Omega \circ p$, $\beta_{\tilde{\Omega}} = \beta_\Omega \circ p$, $\mathrm{Ric}^{\mathrm{Ch}}(\omega_{\tilde{L}}) = \mathrm{Ric}^{\mathrm{Ch}}(\omega_L) \circ p$ and $\mathrm{Ric}^{\mathrm{B}}(\omega_{\tilde{L}}) = \mathrm{Ric}^{\mathrm{B}}(\omega_L) \circ p$. In particular, $(T_\rho\,\mathfrak{g}, \tilde{\mathsf{H}}, \tilde{\Omega})$ is quaternionic balanced (resp. quaternionic Gauduchon) if and only if $(\mathfrak{g}, \mathsf{H}, \Omega)$ is.*

Proof. Let Ω' be the standard metric on $\mathbb{H}^k$. Then

$$\tilde{\Omega}((X,U),(Y,V)) = \Omega(X,Y) + \Omega'(U,V)\,.$$

It follows that

$$\begin{aligned} d\tilde{\Omega}((X,U),(Y,V),(Z,W)) = d\Omega(X,Y,Z) &- \Omega'((\rho_X + \rho_X^*)V, W) - \Omega'((\rho_Y + \rho_Y^*)W, U) \\ &- \Omega'((\rho_Z + \rho_Z^*)U, V)\,. \end{aligned}$$

Therefore, since we assumed $\rho\colon \mathfrak{g} \to \mathfrak{sp}(k)$ we get $d\tilde{\Omega} = d\Omega \circ p$. Therefore, taking $(3,0)$-parts and applying Λ yields

$$\beta_{\tilde{\Omega}}(X,U) = (\Lambda_{\tilde{\Omega}}\partial\tilde{\Omega})(X,U) = (\Lambda_\Omega \partial\Omega)(X) = \beta_\Omega(X)\,. \tag{4.26}$$

Now, using Proposition 4.1.5 (a), Theorem 4.7.13 (4) and (4.26), we deduce the claim for $\alpha_{\tilde{\Omega}}$. This, in turn, implies $\mathrm{Ric}^{\mathrm{Ch}}(\omega_{\tilde{L}}) = \mathrm{Ric}^{\mathrm{Ch}}(\omega_L) \circ p$. Similarly, we also have $\mathrm{Ric}^{\mathrm{B}}(\omega_{\tilde{L}}) = \mathrm{Ric}^{\mathrm{B}}(\omega_L) \circ p$. □

As a consequence of the above, we notice that

$$\partial_{\tilde{J}}\alpha_{\tilde{\Omega}} = \partial_J \alpha_\Omega \circ p\,.$$

Hence, if $\tilde{\Omega}$ is Einstein

$$\partial_J \alpha_\Omega(X,Y) = \partial_{\tilde{J}}\alpha_{\tilde{\Omega}}((X,U),(Y,V)) = \lambda\tilde{\Omega}((X,U),(Y,V)) = \lambda\Omega(X,Y) + \lambda\Omega'(U,V)\,,$$

for all $X, Y \in \mathfrak{g}$ and $U, V \in \mathbb{H}^k$, which is possible if and only if $\lambda = 0$.

4.7.3 Joyce's examples

In this subsection, we shall prove the following theorem.

Theorem 4.7.15. *Let G be a compact semisimple Lie group and $k \geq 0$ an integer such that the product $T^k \times \mathsf{G}$ admits an invariant hypercomplex structure H as defined in [201]. Then, $(T^k \times \mathsf{G}, \mathsf{H})$ admits an invariant strong HKT-Einstein metric with positive Einstein constant.*

To do so, it will be useful to briefly overview the construction of Joyce [201]. Let G be a compact semisimple Lie group of rank r and fix a maximal torus H in G. Within this framework, structure theory can be performed, which allows to obtain a decomposition of $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ of the following form:

$$\mathfrak{g} = \mathfrak{b} \oplus \bigoplus_{j=1}^{m} \mathfrak{d}_j \oplus \bigoplus_{j=1}^{m} \mathfrak{f}_j \,,$$

where $\mathfrak{b}$ is abelian of dimension $r - m$, $\mathfrak{d}_j \subseteq \mathfrak{g}$ are subalgebras isomorphic to $\mathfrak{su}(2)$, and $\mathfrak{f}_j \subseteq \mathfrak{g}$ are subspaces satisfying the following properties:

(J1) $[\mathfrak{d}_j, \mathfrak{b}] = 0$, for any $j = 1, \dots, m$, and $\mathfrak{h} := \mathrm{Lie}(H) \subseteq \mathfrak{b} \oplus \bigoplus_{j=1}^{m} \mathfrak{d}_j$;

(J2) $[\mathfrak{d}_j, \mathfrak{d}_i] = 0$, for $j \neq i$;

(J3) $[\mathfrak{d}_j, \mathfrak{f}_i] = 0$, for $j < i$;

(J4) $[\mathfrak{d}_j, \mathfrak{f}_j] \subseteq \mathfrak{f}_j$, for any $j = 1, \dots, m$, and this Lie bracket action is isomorphic to the direct sum of a finite amount of copies of the $\mathfrak{su}(2)$-action on $\mathbb{C}^2$ by left matrix multiplication.

Such a decomposition will be called a *Joyce decomposition* of the Lie algebra $\mathfrak{g}$.
Now, denote $T^{2m-r} \cong \mathsf{U}(1)^{2m-r}$ the $(2m - r)$-dimensional torus, so that the Lie algebra of $T^{2m-r} \times \mathsf{G}$ decomposes as

$$(2m - r)\mathfrak{u}(1) \oplus \mathfrak{g} \cong \mathbb{R}^m \oplus \bigoplus_{j=1}^{m} \mathfrak{d}_j \oplus \bigoplus_{j=1}^{m} \mathfrak{f}_j \,.$$

We define a hypercomplex structure $I, J, K \in \mathrm{End}((2m - r)\mathfrak{u}(1) \oplus \mathfrak{g})$ in the following manner. For every $j = 1, \dots, m$, denote $(e_1^j, e_2^j, e_3^j, e_4^j)$ a basis of $\mathbb{R} \oplus \mathfrak{d}_j$ such that $(e_1^1, e_1^2, \dots, e_1^m)$ is the standard basis of $\mathbb{R}^m$ and e_2^j, e_3^j, e_4^j satisfy the commutation relations:

$$[e_2^j, e_3^j] = 2e_4^j \,, \quad [e_4^j, e_2^j] = 2e_3^j \,, \quad [e_3^j, e_4^j] = 2e_2^j \,. \tag{4.27}$$

We can regard $\mathbb{R} \oplus \mathfrak{d}_j$ as a copy of the space of quaternions.

(a) For any $j = 1, \dots, m$, let I, J, K act on $\mathbb{R} \oplus \mathfrak{d}_j$ as:

$$Ie_1^j = e_2^j \,, \quad Ie_3^j = e_4^j \,, \quad Je_1^j = e_3^j \,, \quad Je_2^j = -e_4^j \,, \quad Ke_1^j = e_4^j \,, \quad Ke_2^j = e_3^j \,.$$

We obviously further require $I^2 = J^2 = -\mathrm{Id}$.

(b) For any $j = 1, \dots, m$, let I, J, K act on $\mathfrak{f}_j$ as:

$$If = [e_2^j, f] \,, \quad Jf = [e_3^j, f] \,, \quad Kf = [e_4^j, f] \,,$$

for each $f \in \mathfrak{f}_j$.

It is clear that I, J, K induce a hypercomplex structure H on $\mathbb{R}^m \oplus \bigoplus_{j=1}^{m} \mathfrak{d}_j$, the fact that it is also a hypercomplex structure on $\bigoplus_{j=1}^{m} \mathfrak{f}_j$ follows from (J3). At this point Joyce uses an argument due to Samelson [280] to prove that I and J must be integrable.

Since the group G is semisimple, the opposite of the Cartan-Killing form $-B$ is a positive-definite inner product on the Lie algebra $\mathfrak{g}$. We now follow the argumentation of Grantcharov and Poon [174] in order to show that $-B$ can be extended to a HKT metric on G. This observation is originally due to Opfermann and Papadopoulos [251], who also generalized the construction to certain homogeneous spaces.
It can be seen that the Joyce decomposition is orthogonal with respect to B, see [174, Lemma 2]. Let then H_j, X_j, Y_j be an orthogonal basis for $\mathfrak{d}_j$ such that $H_j \in \mathfrak{h}$ and

$$B(H_j,H_j) = B(X_j,X_j) = B(Y_j,Y_j) = -\lambda_j^2,$$

for some $\lambda_j \in \mathbb{R}$. Since $(2m-r)\mathfrak{u}(1) \oplus \mathfrak{b} \cong \mathbb{R}^m$, we can extend B to $\mathbb{R}^m$ simply by setting $B(e_i,e_j) = -\delta_{ij}\lambda_j^2$, where $(e_1,\ldots,e_m)$ is the canonical basis of $\mathbb{R}^m$. An easy inspection shows that this extension is a hyperHermitian metric on $(2m-r)\mathfrak{u}(1) \oplus \mathfrak{g}$. Let g' be the metric induced on $T^{2m-r} \times \mathsf{G}$ by such hyperHermitian inner product. Consider the left-invariant connection ∇ on $T^{2m-r} \times \mathsf{G}$ such that all left-invariant vector fields are parallel. Since the metric g' and the hypercomplex structure H on $T^{2m-r} \times \mathsf{G}$ are left-invariant, they are preserved by ∇. The torsion tensor is $T^\nabla(X,Y) = -[X,Y]$, therefore

$$T^\nabla(X,Y,Z) := g'(T^\nabla(X,Y),Z) = B([X,Y],Z) \in \Lambda^3(T^{2m-r} \times \mathsf{G}). \tag{4.28}$$

It follows that the Bismut connections of I and J coincide with ∇. Therefore, the metric g' is HKT and it is easy to see that it is also strong.

We shall now show that such a metric can be appropriately rescaled at each level of the Joyce decomposition in order to obtain an HKT-Einstein metric.

Theorem 4.7.16. *Let G be a compact semisimple Lie group of rank r and H an invariant hypercomplex structure defined on $T^{2m-r} \times \mathsf{G}$ as above. Then, $(T^{2m-r} \times \mathsf{G}, \mathsf{H})$ admits an invariant strong HKT-Einstein metric.*

Proof. Here we can work at the algebra level. Let

$$\mathfrak{g} = \mathfrak{b} \oplus \bigoplus_{j=1}^m \mathfrak{d}_j \oplus \bigoplus_{j=1}^m \mathfrak{f}_j$$

be a Joyce decomposition of the Lie algebra $\mathfrak{g}$ and let B be the Cartan-Killing form on $\mathfrak{g}$. Denote $d_j = \dim_{\mathbb{H}}(\mathfrak{f}_j)$ and set $\mu_j = \frac{1}{\sqrt{2(1+d_j)}}$, for any $j = 1,\ldots,m$. Then, we claim that

$$g = -\sum_{j=1}^m \mu_j^2 B|_{\mathbb{R}\oplus\mathfrak{d}_j\oplus\mathfrak{f}_j}$$

is an HKT-Einstein metric on $(2m-r)\mathfrak{u}(1) \oplus \mathfrak{g}$ satisfying

$$\partial_J \alpha = \Omega\,, \tag{4.29}$$

where Ω denotes the corresponding $(2,0)$ form with respect to I.

First of all, it is clear that g is a hyperHermitian metric with respect to H. Second, the same discussion as above shows that the metric induced by g on $T^{2m-r} \times \mathsf{G}$ is strong HKT. Since on both sides of (4.29) we have q-real $(2,0)$-forms, it is enough to show the identity along the diagonal, i.e.

$$\partial_J\alpha(X,JX) = \lambda\Omega(X,JX)\,, \quad X \in (2m-r)\mathfrak{u}(1) \oplus \mathfrak{g}\,. \tag{4.30}$$

Let $\theta = \theta_\Omega$ be the Lee form of Ω. Then, we have

$$\partial_J\alpha = \partial_J\theta^{1,0} = \frac{1}{4}J^{-1}(d - \sqrt{-1}d_I^c)(J\theta - \sqrt{-1}K\theta)\,,$$

whence

$$4\partial_J\alpha(X,JX) = J\theta([JX,X]) - J\theta([KX,IX]) - \sqrt{-1}K\theta([JX,X]) + \sqrt{-1}K\theta([KX,IX])\,.$$

On a HKT manifold the Lee form satisfies

$$\theta(X) = -\frac{1}{2}\sum_{i=1}^{4n} T^{\nabla}(IX,e_i,Ie_i) = -\frac{1}{2}\sum_{i=1}^{4n} T^{\nabla}(JX,e_i,Je_i) = -\frac{1}{2}\sum_{i=1}^{4n} T^{\nabla}(KX,e_i,Ke_i)\,,$$

where $(e_1,\ldots,e_{4n})$ is an orthonormal basis of the tangent bundle (see [196]). In this setting, the torsion 3-form is defined as in (4.28) with respect to g, therefore, we obtain

$$\begin{aligned}8\partial_J\alpha(X,JX) = \sum_{i=1}^{4n}\big(&T^{\nabla}([JX,X],e_i,Je_i) - T^{\nabla}([KX,IX],e_i,Je_i)\\ &- \sqrt{-1}T^{\nabla}([JX,X],e_i,Ke_i) + \sqrt{-1}T^{\nabla}([KX,IX],e_i,Ke_i)\big)\,.\end{aligned}$$

But, since

$$T^{\nabla}([X,Y],Z,W) = T^{\nabla}([Z,W],X,Y)\,,$$

we get

$$\begin{aligned}8\partial_J\alpha(X,JX) = \sum_{i=1}^{4n}\big(&-g([[e_i,Je_i],JX],X) + g([[e_i,Je_i],KX],IX)\\ &+ \sqrt{-1}g([[e_i,Ke_i],JX],X) - \sqrt{-1}g([[e_i,Ke_i],KX],IX)\big)\,.\end{aligned} \tag{4.31}$$

At this point, we choose the orthonormal basis $(e_1,\ldots,e_{4n})$ to be the following: for any $j=1,\ldots,m$, let $(e_1^j,\ldots,e_4^j)$ be an orthonormal basis of $\mathbb{R}\oplus\mathfrak{d}_j$ with only non-zero commutators:

$$[e_2^j,e_3^j] = 2\mu_j e_4^j\,,\quad [e_4^j,e_2^j] = 2\mu_j e_3^j\,,\quad [e_3^j,e_4^j] = 2\mu_j e_2^j\,.$$

On $\mathfrak{f}_j$, we fix any orthonormal basis $\{f_1^j,\ldots,f_{4d_j}^j\}$ such that $f_{4k-2}^j = If_{4k-3}^j$, $f_{4k-1}^j = Jf_{4k-3}^j$, $f_{4k}^j = Kf_{4k-3}^j$ for $k=1,\ldots,d_j$. Notice that $(e_1^j,\ldots,e_4^j)$ is, in general, different from the basis chosen in (4.27) and, with this new orthonormal basis, the definition of H on $\mathfrak{f}_j$ is:

$$If = \frac{1}{\mu_j}[e_2^j,f]\,,\quad Jf = \frac{1}{\mu_j}[e_3^j,f]\,,\quad Kf = \frac{1}{\mu_j}[e_4^j,f]\,.$$

Thus, we have

$$[e_1^j,Je_1^j] + [e_2^j,Je_2^j] + [e_3^j,Je_3^j] + [e_4^j,Je_4^j] = 4\mu_j e_3^j\,,$$

$$[e_1^j,Ke_1^j] + [e_2^j,Ke_2^j] + [e_3^j,Ke_3^j] + [e_4^j,Ke_4^j] = 4\mu_j e_4^j\,.$$

On the other hand, we claim that, for any $j=1,\ldots,m$,

$$\sum_{i=1}^{4d_j}[f_i^j,Jf_i^j] = 4d_j\mu_j e_3^j\,. \tag{4.32}$$

Indeed, we observe that

$$\sum_{r=0}^{3}[f_{4k-r}^j,Jf_{4k-r}^j] = 2\left([f_{4k-3}^j,f_{4k-1}^j] - [f_{4k-2}^j,f_{4k}^j]\right)\,.$$

By definition of (I, J, K) on $\mathfrak{f}_j$ and using Jacobi's identity, we have

$$\begin{aligned}[f^j_{4k-2}, f^j_{4k}] &= [f^j_{4k-2}, Kf^j_{4k-3}] = \frac{1}{\mu_j}[[f^j_{4k-2}, e^j_4], f^j_{4k-3}] + \frac{1}{\mu_j}[e^j_4, [f^j_{4k-2}, f^j_{4k-3}]] \\ &= -[Kf^j_{4k-2}, f^j_{4k-3}] + \frac{1}{\mu_j}[e^j_4, [f^j_{4k-2}, f^j_{4k-3}]] \\ &= [f^j_{4k-3}, f^j_{4k-1}] + \frac{1}{\mu_j}[e^j_4, [f^j_{4k-2}, f^j_{4k-3}]] \,,\end{aligned}$$

hence

$$\sum_{i=1}^{4d_j}[f^j_i, Jf^j_i] = -\frac{2}{\mu_j}\sum_{k=1}^{d_j}[e^j_4, [f^j_{4k-2}, f^j_{4k-3}]] \,.$$

To prove the claim, we fix $k \in \{1, \dots, d_j\}$ and consider the components of $[f^j_{4k-2}, f^j_{4k-3}]$ with respect to the Joyce decomposition:

$$[f^j_{4k-2}, f^j_{4k-3}] = \sum_{p=1}^{m}\left(D^p_1 e^p_1 + D^p_2 e^p_2 + D^p_3 e^p_3 + D^p_4 e^p_4 + \sum_{q=1}^{4d_p} F^p_q f^p_q\right) .$$

Then, we have that $[e^j_4, D^p_1 e^p_1] = 0$, for $p = 1, \dots, m$ and $[e^j_4, D^p_q e^p_q] = 0$, for $p \neq j$ and $q \in \{2, 3, 4\}$, by properties (J1) and (J2). Moreover, since the chosen basis is orthonormal, we have

$$D^j_2 = g([f^j_{4k-2}, f^j_{4k-3}], e^j_2) = g(f^j_{4k-2}, [f^j_{4k-3}, e^j_2]) = -\mu_j g(f^j_{4k-2}, If^j_{4k-3}) = -\mu_j \,,$$

while, proceeding similarly,

$$D^j_3 = -\mu_j g(f^j_{4k-2}, f^j_{4k-1}) = 0 \;\text{ and }\; D^j_4 = -\mu_j g(f^j_{4k-2}, f^j_{4k}) = 0 \,.$$

Furthermore, for $p < j$ and $q \in \{1, \dots, 4d_p\}$, we have

$$\begin{aligned}F^p_q &= -g([f^j_{4k-2}, f^j_{4k-3}], I^2 f^p_q) = -\frac{1}{\mu_j}g([f^j_{4k-2}, f^j_{4k-3}], [e^p_2, If^p_q]) \\ &= -\frac{1}{\mu_j}g([[f^j_{4k-2}, f^j_{4k-3}], e^p_2], If^p_q) \\ &= -\frac{1}{\mu_j}g([[f^j_{4k-2}, e^p_2], f^j_{4k-3}] + [f^j_{4k-2}, [f^j_{4k-3}, e^p_2]], If^p_q) = 0 \,,\end{aligned}$$

where in the last equality we used (J3). For $p = j$, as above, we have that

$$\begin{aligned}F^j_q &= -\frac{1}{\mu_j}g([[f^j_{4k-2}, e^j_2], f^j_{4k-3}] + [f^j_{4k-2}, [f^j_{4k-3}, e^j_2]], If^j_q) \\ &= g([If^j_{4k-2}, f^j_{4k-3}] + [f^j_{4k-2}, If^j_{4k-3}], If^j_q) \\ &= g(-[f^j_{4k-3}, f^j_{4k-3}] + [f^j_{4k-2}, f^j_{4k-2}], If^j_q) = 0 \,.\end{aligned}$$

Finally, $[e^j_4, F^p_q f^p_q] = 0$, for $p > j$ and $q \in \{1, \dots, 4d_p\}$, by (J3). Then, putting everything together we infer

$$[e^j_4, [f^j_{4k-2}, f^j_{4k-3}]] = -\mu_j[e^j_4, e^j_2] = -2\mu_j^2 e^j_3$$

and summing up over k we obtain (4.32), as claimed. In the same way, one can verify that

$$\sum_{i=1}^{4d_j}[f^j_i, Kf^j_i] = 4d_j\mu_j e^j_4 \;.$$

Therefore

$$\sum_{i=1}^{4n}[e_i, Je_i] = 4\sum_{j=1}^{m}(1+d_j)\mu_j e_3^j\,, \quad \sum_{i=1}^{4n}[e_i, Ke_i] = 4\sum_{j=1}^{m}(1+d_j)\mu_j e_4^j$$

and (4.31) becomes

$$\begin{aligned}\partial_J\alpha(X, JX) = \frac{1}{2}\sum_{j=1}^{m}(1+d_j)\mu_j\big(&-g([e_3^j, JX], X) + g([e_3^j, KX], IX)\\ &+\sqrt{-1}g([e_4^j, JX], X) - \sqrt{-1}g([e_4^j, KX], IX)\big)\,.\end{aligned}$$

We have that, for any X in the chosen basis, $\Omega(X, JX) = \frac{1}{2}$. So, we aim to show that $\partial_J\alpha(X, JX) = \frac{1}{2}$ which will conclude the proof. We will prove it case by case. Suppose, firstly, that $X = e_1^k$, for some $k = 1, \ldots, m$. Then

$$\partial_J\alpha(X, JX) = \frac{1}{2}(1+d_k)\mu_k g([e_3^k, e_4^k], e_2^k) = (1+d_k)\mu_k^2 = \frac{1}{2}\,.$$

Moreover, if $X = e_2^k$, for some $k = 1, \ldots, m$, then

$$\partial_J\alpha(X, JX) = \frac{1}{2}(1+d_k)\mu_k g([e_3^k, e_4^k], e_2^k) = \frac{1}{2}\,.$$

Furthermore, if $X = e_3^k$, for some $k = 1, \ldots, m$, then

$$\partial_J\alpha(X, JX) = -\frac{1}{2}(1+d_k)\mu_k g([e_3^j, e_2^k], e_4^k) = \frac{1}{2}\,.$$

If $X = e_4^k$, for some $k = 1, \ldots, m$, then

$$\partial_J\alpha(X, JX) = -\frac{1}{2}(1+d_k)\mu_k g([e_3^j, e_2^k], e_4^k) = \frac{1}{2}\,.$$

Finally, if $X \in \mathfrak{f}_k$, for some $k = 1, \ldots, m$, the terms in the sum for $j \neq k$ vanish because

$$\begin{aligned}g([e_3^j, JX], X) &= -\frac{1}{\mu_j}g([e_3^j, [e_2^k, KX]], X)\\ &= -\frac{1}{\mu_j}g([[e_3^j, e_2^k], KX], X) - \frac{1}{\mu_j}g([e_2^k, [e_3^j, KX]], X)\\ &= \frac{1}{\mu_j}g([e_3^j, KX], [e_2^k, X]) = g([e_3^j, KX], IX)\,,\end{aligned}$$

where in the second to last equality we used (J2). In the same way one can obtain that $g([e_4^j, JX], X) = g([e_4^j, KX], IX)$. Therefore we only need to consider the terms for $k = j$, which yields

$$\begin{aligned}\partial_J\alpha(X, JX) = \frac{1}{2}(1+d_k)\mu_k^2\big(&-g(J^2X, X) + g(JKX, IX)\\ &+\sqrt{-1}g(KJX, X) - \sqrt{-1}g(K^2X, IX)\big) = \frac{1}{2}\end{aligned}$$

and the proof is concluded. □

4.7.4 Some non-compact HKT-Einstein manifolds

We conclude this section providing other examples of invariant HKT-Einstein manifolds. This time the examples are non-compact and they are simply-connected solvable Lie groups. We shall consider all 4-dimensional solvable Lie algebras, which are classified in [40]. By dimensional reasons they all are HKT and we show here that they actually are HKT-Einstein. Except for the abelian Lie algebra we have three cases to consider. Again, we adopt the convention explained at the beginning of the section regarding the coframe and the hypercomplex structure.

Example 4.7.17. Here we take into account the Lie algebra $\mathfrak{aff}(\mathbb{C})$ of the affine motion group of $\mathbb{C}$. The structure equations are

$$de^1 = -e^1 \wedge e^4 + e^2 \wedge e^3 \,, \quad de^2 = 0 \,, \quad de^3 = e^1 \wedge e^2 - e^3 \wedge e^4 \,, \quad de^4 = 0 \,,$$

which translate to the complex coframe to

$$d\zeta^1 = \frac{\sqrt{-1}}{2}(\bar{\zeta}^1 \wedge \zeta^2 - \zeta^1 \wedge \bar{\zeta}^2) \,, \quad d\zeta^2 = \frac{\sqrt{-1}}{2}(\zeta^1 \wedge \bar{\zeta}^1 - \zeta^2 \wedge \bar{\zeta}^2) \,.$$

The diagonal metric $\Omega = \zeta^1 \wedge \zeta^2$ satisfies $\alpha = -\sqrt{-1}\zeta^2$ and thus $\partial_J \alpha = 0$, so that the metric Ω is HKT-Einstein with vanishing Einstein constant.

Example 4.7.18. Consider the 4-dimensional solvable Lie algebra with structure equations

$$de^1 = 0 \,, \quad de^2 = -e^1 \wedge e^2 \,, \quad de^3 = -e^1 \wedge e^3 \,, \quad de^4 = -e^1 \wedge e^4 \,.$$

In terms of the complex $(1,0)$-coframe we have

$$d\zeta^1 = \frac{1}{2}\zeta^1 \wedge \bar{\zeta}^1 \,, \quad d\zeta^2 = -\frac{1}{2}(\zeta^1 \wedge \zeta^2 + \bar{\zeta}^1 \wedge \zeta^2) \,.$$

The diagonal metric $\Omega = \zeta^1 \wedge \zeta^2$ satisfies $\alpha = -\zeta^1$ and so $\partial_J \alpha = -\frac{1}{2}\Omega$, showing that Ω induces on the corresponding simply-connected solvable Lie group an invariant HKT-Einstein metric with negative Einstein constant.

Example 4.7.19. We conclude with the solvable Lie algebra with structure equations

$$de^1 = 0 \,, \quad de^2 = -e^1 \wedge e^2 + 2e^3 \wedge e^4 \,, \quad de^3 = -\frac{1}{2}e^1 \wedge e^3 \,, \quad de^4 = -\frac{1}{2}e^1 \wedge e^4 \,,$$

equivalently,

$$d\zeta^1 = \frac{1}{2}\zeta^1 \wedge \bar{\zeta}^1 - \zeta^2 \wedge \bar{\zeta}^2 \,, \quad d\zeta^2 = -\frac{1}{4}(\zeta^1 \wedge \zeta^2 + \bar{\zeta}^1 \wedge \zeta^2) \,.$$

The diagonal metric $\Omega = \zeta^1 \wedge \zeta^2$ satisfies $\alpha = -\frac{3}{4}\zeta^1$ and so $\partial_J \alpha = -\frac{3}{16}\Omega$, which shows that Ω is HKT-Einstein with negative Einstein constant.

Bibliography

[1] E. Abbena and A. Grassi, *Hermitian left invariant metrics on complex Lie groups and cosymplectic Hermitian manifolds*, Boll. Unione Mat. Ital., VI. Ser., A **5** (1986), 371–379.

[2] S. Agmon, *Lectures on elliptic boundary value problems*, reprint of the 1965 original ed., Providence, RI: AMS Chelsea Publishing, 2010.

[3] D. V. Alekseevsky and B. N. Kimel'fel'd, *Structure of homogeneous Riemann spaces with zero Ricci curvature*, Funct. Anal. Appl. **9** (1975), 97–102.

[4] D. V. Alekseevsky and S. Marchiafava, *Quaternionic-like structures on a manifold. I: 1-integrability and integrability conditions*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. **4** (1993), no. 1, 43–52.

[5] ———, *Hypercomplex structures on quaternionic manifolds*, pp. 1–19, Springer Netherlands, Dordrecht, 1996.

[6] ———, *Quaternionic structures on a manifold and subordinated structures*, Ann. Mat. Pura Appl. (4) **171** (1996), 205–273.

[7] S. Alesker, *Solvability of the quaternionic Monge-Ampère equation on compact manifolds with a flat hyperKähler metric*, Adv. Math. **241** (2013), 192–219.

[8] S. Alesker and M. Verbitsky, *Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry*, J. Geom. Anal. **16** (2006), no. 3, 375–399.

[9] ———, *Quaternionic Monge-Ampère equation and Calabi problem for HKT-manifolds*, Isr. J. Math. **176** (2010), 109–138.

[10] L. Alessandrini and G. Bassanelli, *Smooth proper modifications of compact Kähler manifolds*, Complex analysis, Proc. Int. Workshop Ded. H. Grauert, Aspects Math. E, **17** (1991), 1–7.

[11] ———, *Small deformations of a class of compact non-Kähler manifolds*, Proc. Am. Math. Soc. **109** (1990), no. 4, 1059–1062.

[12] ———, *Metric properties of manifolds bimeromorphic to compact Kähler spaces*, J. Differ. Geom. **37** (1993), no. 1, 95–121.

[13] ———, *Modifications of compact balanced manifolds*, C. R. Acad. Sci., Paris, Sér. I **320** (1995), no. 12, 1517–1522.

[14] B. Alexandrov and S. Ivanov, *Vanishing theorems on Hermitian manifolds*, Differ. Geom. Appl. **14** (2001), no. 3, 251–265.

[15] L. Álvarez-Cónsul, A. D. A. de La Hera, and M. Garcia-Fernandez, *Vertex algebras from the Hull-Strominger system*, arXiv preprint arXiv:2305.06836 (2023), 1–72.

[16] A. Andrada and M. L. Barberis, *Hypercomplex almost abelian solvmanifolds*, J. Geom. Anal. **33** (2023), no. 7, no. 213, 31.

[17] A. Andrada and A. Tolcachier, *On the canonical bundle of complex solvmanifolds and applications to hypercomplex geometry*, arXiv preprint arXiv:2307.16673 (2023), 1–32.

[18] ______, *Harmonic complex structures and special Hermitian metrics on products of Sasakian manifolds*, J. Geom. Anal. **34** (2024), no. 6, 42.

[19] A. Andrada and R. Villacampa, *Abelian balanced Hermitian structures on unimodular Lie algebras*, Transform. Groups **21** (2016), no. 4, 903–927.

[20] B. Andreas and M. Garcia-Fernandez, *Heterotic non-Kähler geometries via polystable bundles on Calabi-Yau threefolds*, J. Geom. Phys. **62** (2012), no. 2, 183–188.

[21] ______, *Solutions of the Strominger system via stable bundles on Calabi-Yau threefolds*, Comm. Math. Phys. **315** (2012), no. 1, 153–168.

[22] B. Andrews and C. Hopper, *The Ricci flow in Riemannian geometry. A complete proof of the differentiable 1/4-pinching sphere theorem*, Lect. Notes Math., vol. 2011, Berlin: Springer, 2011.

[23] D. Angella, S. Calamai, and C. Spotti, *On the Chern-Yamabe problem*, Math. Res. Lett. **24** (2017), no. 3, 645–677.

[24] ______, *Remarks on Chern-Einstein Hermitian metrics*, Math. Z. **295** (2020), no. 3-4, 1707–1722.

[25] D. Angella, A. Dubickas, A. Otiman, and J. Stelzig, *On metric and cohomological properties of Oeljeklaus-Toma manifolds*, Publ. Mat., Barc. **68** (2024), no. 1, 219–239.

[26] D. Angella, V. Guedj, and C. H. Lu, *Plurisigned Hermitian metrics*, Trans. Amer. Math. Soc. **376** (2023), no. 7, 4631–4659.

[27] D. Angella and V. Tosatti, *Leafwise flat forms on Inoue-Bombieri surfaces*, J. Funct. Anal. **285** (2023), no. 5, 34.

[28] V. Apostolov and M. Gualtieri, *Generalized Kähler manifolds, commuting complex structures, and split tangent bundles*, Comm. Math. Phys. **271** (2007), no. 2, 561–575.

[29] V. Apostolov, J. Streets, and Y. Ustinovskiy, *Generalized Kähler-Ricci flow on toric Fano varieties*, Trans. Amer. Math. Soc. **375** (2022), no. 6, 4369–4409.

[30] ______, *Variational structure and uniqueness of generalized Kähler-Ricci solitons*, Peking Math. J. **6** (2023), no. 2, 307–351.

[31] C. Arezzo and F. Pacard, *Blowing up and desingularizing constant scalar curvature Kähler manifolds*, Acta Math. **196** (2006), no. 2, 179–228.

[32] R. M. Arroyo and R. A. Lafuente, *The long-time behavior of the homogeneous pluriclosed flow*, Proc. Lond. Math. Soc. (3) **119** (2019), no. 1, 266–289.

[33] R. M. Arroyo and M. Nicolini, *SKT structures on nilmanifolds*, Math. Z. **302** (2022), no. 2, 1307–1320.

[34] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95.

[35] B. Banos and A. Swann, *Potentials for hyper-Kähler metrics with torsion*, Classical Quantum Gravity **21** (2004), no. 13, 3127–3135.

[36] D. Baraglia and P. Hekmati, *Transitive Courant algebroids, string structures and T-duality*, Adv. Theor. Math. Phys. **19** (2015), no. 3, 613–672.

[37] G. Barbaro, *Global stability of the pluriclosed flow on compact simply connected simple lie groups of rank two*, Transform. Groups (2022), 1–19.

[38] ______, *Bismut Hermitian Einstein metrics and the stability of the pluriclosed flow*, arXiv preprint arXiv:2307.10207 (2023), 1–18.

[39] ______, *On the curvature of the Bismut connection: Bismut-Yamabe problem and Calabi-Yau with torsion metrics*, J. Geom. Anal. **33** (2023), no. 5, 23.

[40] M. L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*, Proc. Am. Math. Soc. **125** (1997), no. 4, 1043–1054.

[41] ______, *Abelian hypercomplex structures on central extensions of H-type Lie algebras*, J. Pure Appl. Algebra **158** (2001), no. 1, 15–23.

[42] M. L. Barberis and I. Dotti, *Abelian complex structures on solvable Lie algebras*, J. Lie Theory **14** (2004), no. 1, 25–34.

[43] M. L. Barberis, I. Dotti, and M. Verbitsky, *Canonical bundles of complex nilmanifolds, with applications to hypercomplex geometry*, Math. Res. Lett. **16** (2009), no. 2-3, 331–347.

[44] M. L. Barberis and A. Fino, *New HKT manifolds arising from quaternionic representations*, Math. Z. **267** (2011), no. 3-4, 717–735.

[45] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differ. Geom. **18** (1983), no. 4, 755–782.

[46] L. Bedulli, G. Gentili, and L. Vezzoni, *A parabolic approach to the Calabi-Yau problem in HKT geometry*, Math. Z. **302** (2022), no. 2, 917–933.

[47] ______, *The parabolic quaternionic Calabi-Yau equation on hyperKähler manifolds*, arXiv preprint arXiv:2303.02689 (2023), 1–16.

[48] L. Bedulli and L. Vezzoni, *A parabolic flow of balanced metrics*, J. Reine Angew. Math. **723** (2017), 79–99.

[49] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40.

[50] L. Bérard-Bergery, *Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 543–576.

[51] O. Biquard and V. Minerbe, *A Kummer construction for gravitational instantons*, Comm. Math. Phys. **308** (2011), no. 3, 773–794.

[52] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. **284** (1989), no. 4, 681–699.

[53] C. Böhm and R. A. Lafuente, *Real geometric invariant theory*, Differential geometry in the large, London Math. Soc. Lecture Note Ser., vol. 463, Cambridge Univ. Press, Cambridge, 2021, pp. 11–49.

[54] J. Boling, *Homogeneous solutions of pluriclosed flow on closed complex surfaces*, J. Geom. Anal. **26** (2016), no. 3, 2130–2154.

[55] M. Bordoni, *Spectral estimates for submersions with fibers of basic mean curvature*, An. Univ. Vest Timiş. Ser. Mat.-Inform. **44** (2006), no. 1, 23–36.

[56] C. Boyer, *A note on hyperhermitian four-manifolds*, Proc. Am. Math. Soc. **102** (1988), no. 1, 157–164.

[57] C. Boyer, K. Galicki, and B. M. Mann, *Some new examples of compact inhomogeneous hypercomplex manifolds*, Math. Res. Lett. **1** (1994), no. 5, 531–538.

[58] ———, *Hypercomplex structures on Stiefel manifolds*, Ann. Global Anal. Geom. **14** (1996), no. 1, 81–105.

[59] ———, *Hypercomplex structures from 3-Sasakian structures*, J. Reine Angew. Math. **501** (1998), 115–141.

[60] B. Brienza and A. Fino, *Generalized Kähler manifolds via mapping tori*, arXiv preprint arXiv:2305.11075 (2023), 1–27.

[61] B. Brienza, A. Fino, and G. Grantcharov, *CYT and SKT manifolds with parallel Bismut torsion*, arXiv preprint arXiv:2401.07800 (2024), 1–20.

[62] R. L. Bryant, *Ricci flow solitons in dimension three with $SO(3)$-symmetries*, Unpubblished (2005), 1–24.

[63] N. Buchdahl, *On compact Kähler surfaces*, Ann. Inst. Fourier **49** (1999), no. 1, 287–302.

[64] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Grad. Stud. Math., vol. 33, Providence, RI: American Mathematical Society (AMS), 2001.

[65] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Adv. Math. **211** (2007), no. 2, 726–765.

[66] T.H. Buscher, *A symmetry of the string background field equations*, Physics Letters B **194** (1987), no. 1, 59–62.

[67] E. Calabi, *Extremal Kähler metrics*, Eugenio Calabi—collected works, Springer, Berlin, 2020, pp. 549–580.

[68] ———, *Métriques kählériennes et fibrés holomorphes*, Eugenio Calabi—collected works, Springer, Berlin, 2020, pp. 485–510.

[69] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, *Strings in background fields*, Nuclear Phys. B **262** (1985), no. 4, 593–609.

[70] K. Cao and F. Zheng, *Fino–Vezzoni conjecture on Lie algebras with abelian ideals of codimension two*, Math. Z. **307** (2024), no. 2, 31.

[71] É. Cartan and J. A. Schouten, *On Riemannian geometries admitting an absolute parallelism*, Proc. Akad. Wet. Amsterdam **29** (1926), 933–946.

[72] ———, *On the geometry of the group-manifold of simple and semi-simple groups*, Proc. Akad. Wet. Amsterdam **29** (1926), 803—815.

[73] A. Cattaneo and A. Tomassini, *$\partial\bar{\partial}$-lemma and p-Kähler structures on families of solvmanifolds*, arXiv preprint arXiv:2403.10126 (2024), 1–14.

[74] X. X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.

[75] ———, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π*, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.

[76] ———, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches* 2π *and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.

[77] X. X. Chen and W. Y. He, *On the Calabi flow*, Am. J. Math. **130** (2008), no. 2, 539–570.

[78] S.-S. Chern, *Characteristic classes of Hermitian manifolds*, Ann. Math. (2) **47** (1946), 85–121.

[79] I. Chiose, *Obstructions to the existence of Kähler structures on compact complex manifolds*, Proc. Am. Math. Soc. **142** (2014), no. 10, 3561–3568.

[80] I. Chiose, R. Răsdeaconu, and I. Şuvaina, *Balanced manifolds and SKT metrics*, Ann. Mat. Pura Appl. (4) **201** (2022), no. 5, 2505–2517.

[81] C. Ciulică, A. Otiman, and M. Stanciu, *Special non-Kähler metrics on Endo-Pajitnov manifolds*, arXiv preprint arXiv:2403.15618 (2024), 1–16.

[82] A. Coimbra, C. Strickland-Constable, and D. Waldram, *Supergravity as generalised geometry I: type II theories*, J. High Energy Phys. (2011), no. 11, 091, 35.

[83] T. Collins, S. Picard, and S.-T. Yau, *The Strominger system in the square of a Kähler class*, arXiv preprint arXiv:2211.03784 (2022), 1–15.

[84] T. J. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 631–661.

[85] P. del Pezzo, *Sulle superficie del* n^{mo} *ordine immerse nello spazio di* n *dimensioni*, Rend. Circ. Mat. Palermo **1** (1887), 241–271 (Italian).

[86] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.

[87] J.-P. Demailly, *Complex analytic and differential geometry*, available at https://www-fourier. ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf (2012), 1–455.

[88] R. Dervan and L. M. Sektnan, *Extremal metrics on fibrations*, Proc. Lond. Math. Soc. (3) **120** (2020), no. 4, 587–616.

[89] S. Dinew and M. Sroka, *On the Alesker-Verbitsky conjecture on hyperKähler manifolds*, Geom. Funct. Anal. **33** (2023), no. 4, 875–911.

[90] S. Donaldson, *Remarks on gauge theory, complex geometry and* 4*-manifold topology*, Fields Medallists' lectures, World Sci. Ser. 20th Century Math., **5** (1997), 384–403.

[91] ———, *Scalar curvature and stability of toric varieties*, J. Differ. Geom. **62** (2002), no. 2, 289–349.

[92] I. Dotti and A. Fino, *Abelian hypercomplex 8-dimensional nilmanifolds*, Ann. Global Anal. Geom. **18** (2000), no. 1, 47–59.

[93] ———, *HyperKähler torsion structures invariant by nilpotent Lie groups*, Classical Quantum Gravity **19** (2002), no. 3, 551–562.

[94] ———, *Hypercomplex eight-dimensional nilpotent Lie groups*, J. Pure Appl. Algebra **184** (2003), no. 1, 41–57.

[95] P. Eberlein, *Geometry of* 2*-step nilpotent groups with a left invariant metric, II*, Trans. Amer. Math. Soc. **343** (1994), no. 2, 805–828.

[96] G. Edwards, *The Chern-Ricci flow on primary Hopf surfaces*, Math. Z. **299** (2021), no. 3-4, 1689–1702.

[97] T. Eguchi and A. J. Hanson, *Self-dual solutions to Euclidean gravity*, Ann. Physics **120** (1979), no. 1, 82–106.

[98] N. Enrietti, A. Fino, and L. Vezzoni, *Tamed symplectic forms and strong Kähler with torsion metrics*, J. Symplectic Geom. **10** (2012), no. 2, 203–223.

[99] ———, *The pluriclosed flow on nilmanifolds and tamed symplectic forms*, J. Geom. Anal. **25** (2015), no. 2, 883–909.

[100] S. Fang, V. Tosatti, B. Weinkove, and T. Zheng, *Inoue surfaces and the Chern-Ricci flow*, J. Funct. Anal. **271** (2016), no. 11, 3162–3185.

[101] S. Fedoruk, E. Ivanov, and A. Smilga, $\mathcal{N} = 4$ *mechanics with diverse (4, 4, 0) multiplets: explicit examples of hyper-Kähler with torsion, Clifford-Kähler with torsion, and octonionic Kähler with torsion geometries*, J. Math. Phys. **55** (2014), no. 5, no. 052302, 29.

[102] T. Fei, *A construction of non-Kähler Calabi-Yau manifolds and new solutions to the Strominger system*, Adv. Math. **302** (2016), 529–550.

[103] ———, *Some torsional local models for heterotic strings*, Comm. Anal. Geom. **25** (2017), no. 5, 941–968.

[104] T. Fei, Z. Huang, and S. Picard, *A construction of infinitely many solutions to the Strominger system*, J. Differ. Geom. **117** (2021), no. 1, 23–39.

[105] T. Fei and D. H. Phong, *Unification of the Kähler-Ricci and anomaly flows*, Differential geometry, Calabi-Yau theory, and general relativity. Lectures given at conferences celebrating the 70th birthday of Shing-Tung Yau at Harvard University, Cambridge, MA, USA, May 2019, Somerville, MA: International Press, 2020, pp. 89–103.

[106] T. Fei and S.-T. Yau, *Invariant solutions to the Strominger system on complex Lie groups and their quotients*, Comm. Math. Phys. **338** (2015), no. 3, 1183–1195.

[107] A. Fernández, S. Ivanov, L. Ugarte, and D. Vassilev, *Non-Kähler heterotic string solutions with non-zero fluxes and non-constant Dilaton*, J. High Energy Phys. **2014** (2014), no. 6, 22.

[108] J. Fine, *Constant scalar curvature Kähler metrics on fibred complex surfaces*, J. Differ. Geom. **68** (2004), no. 3, 397–432.

[109] ———, *Fibrations with constant scalar curvature Kähler metrics and the CM-line bundle*, Math. Res. Lett. **14** (2007), no. 2, 239–247.

[110] A. Fino and G. Grantcharov, *Properties of manifolds with skew-symmetric torsion and special holonomy*, Adv. Math. **189** (2004), no. 2, 439–450.

[111] A. Fino, G. Grantcharov, and E. Perez, *The pluriclosed flow for T^2-invariant Vaisman metrics on the Kodaira-Thurston surface*, J. Geom. Phys. **201** (2024), 11.

[112] A. Fino, G. Grantcharov, and M. Verbitsky, *Special Hermitian structures on suspensions*, arXiv preprint arXiv:2208.12168 (2022), 1–24.

[113] A. Fino, G. Grantcharov, and L. Vezzoni, *Astheno-Kähler and balanced structures on fibrations*, Int. Math. Res. Not. **2019** (2019), no. 22, 7093–7117.

[114] ———, *Solutions to the Hull-Strominger system with torus symmetry*, Comm. Math. Phys. **388** (2021), no. 2, 947–967.

[115] A. Fino, H. Kasuya, and L. Vezzoni, *SKT and tamed symplectic structures on solvmanifolds*, Tôhoku Math. J. (2) **67** (2015), no. 1, 19–37.

[116] A. Fino, A. Otal, and L. Ugarte, *Six-dimensional solvmanifolds with holomorphically trivial canonical bundle*, Int. Math. Res. Not. **2015** (2015), no. 24, 13757–13799.

[117] A. Fino and F. Paradiso, *Generalized Kähler almost abelian Lie groups*, Ann. Mat. Pura Appl. (4) **200** (2021), no. 4, 1781–1812.

[118] ———, *Hermitian structures on a class of almost nilpotent solvmanifolds*, J. Algebra **609** (2022), 861–925.

[119] ———, *Balanced Hermitian structures on almost abelian Lie algebras*, J. Pure Appl. Algebra **227** (2023), no. 2, 25.

[120] ———, *Hermitian structures on six-dimensional almost nilpotent solvmanifolds*, arXiv preprint arXiv:2306.03485 (2023), 1–41.

[121] A. Fino, M. Parton, and S. Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv. **79** (2004), no. 2, 317–340.

[122] A. Fino, N. Tardini, and L. Vezzoni, *Pluriclosed and Strominger Kähler-like metrics compatible with abelian complex structures*, Bull. Lond. Math. Soc. **54** (2022), no. 5, 1862–1872.

[123] A. Fino and A. Tomassini, *Blow-ups and resolutions of strong Kähler with torsion metrics*, Adv. Math. **221** (2009), no. 3, 914–935.

[124] ———, *Non-Kähler solvmanifolds with generalized Kähler structure*, J. Symplectic Geom. **7** (2009), no. 2, 1–14.

[125] A. Fino and L. Vezzoni, *Special Hermitian metrics on compact solvmanifolds*, J. Geom. Phys. **91** (2015), 40–53.

[126] ———, *On the existence of balanced and SKT metrics on nilmanifolds*, Proc. Am. Math. Soc. **144** (2016), no. 6, 2455–2459.

[127] A. Freibert and A. Swann, *Two-step solvable SKT shears*, Math. Z. **299** (2021), no. 3-4, 1703–1739.

[128] M. Freibert and A. Swann, *Compatibility of balanced and SKT metrics on two-step solvable Lie groups*, Transform. Groups (2023), 1–31.

[129] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Universitext, New York, NY: Springer, 1998.

[130] J. Fu, Z. Wang, and D. Wu, *Form-type Calabi-Yau equations*, Math. Res. Lett. **17** (2010), no. 5, 887–903.

[131] J. Fu, X. Xu, and D. Zhang, *The parabolic quaternionic Monge-Ampère type equation on hyperKähler manifolds*, arXiv preprint arXiv:2310.09225 (2023), 1–18.

[132] ———, *The Monge–Ampère equation for $(n-1)$-quaternionic PSH functions on a hyperKähler manifold*, Math. Z. **307** (2024), no. 2, 1–25.

[133] J.-X. Fu, J. Li, and S.-T. Yau, *Balanced metrics on non-Kähler Calabi-Yau threefolds*, J. Differ. Geom. **90** (2012), no. 1, 81–129.

[134] J.-X. Fu, L.-S. Tseng, and S.-T. Yau, *Local heterotic torsional models*, Comm. Math. Phys. **289** (2009), no. 3, 1151–1169.

[135] J.-X. Fu and S.-T. Yau, *A Monge-Ampère-type equation motivated by string theory*, Comm. Anal. Geom. **15** (2007), no. 1, 29–76.

[136] ______, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation*, J. Differ. Geom. **78** (2008), no. 3, 369–428.

[137] A. Fujiki, *Moduli space of polarized algebraic manifolds and Kähler metrics*, Sugaku Expositions **5** (1992), no. 2, 173–191.

[138] E. Fusi, *The prescribed Chern scalar curvature problem*, J. Geom. Anal. **32** (2022), no. 6, 21.

[139] E. Fusi and G. Gentili, *Special metrics in hypercomplex geometry*, arXiv preprint arXiv:2401.13056 (2024), 1–50.

[140] E. Fusi, R. A. Lafuente, and J. Stanfield, *The homogeneous generalized Ricci flow*, arXiv preprint arXiv:2404.15749 (2024), 1–40.

[141] E. Fusi and L. Vezzoni, *On the pluriclosed flow on Oeljeklaus-Toma manifolds*, Canad. J. Math. **76** (2024), no. 1, 39–65.

[142] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), 437–443.

[143] M. Garcia-Fernandez, *Lectures on the Strominger system*, Travaux mathématiques. Vol. XXIV, Trav. Math., vol. 24, Fac. Sci. Technol. Commun. Univ. Luxemb., Luxembourg, 2016, pp. 7–61.

[144] ______, *Ricci flow, Killing spinors, and T-duality in generalized geometry*, Adv. Math. **350** (2019), 1059–1108.

[145] ______, *T-dual solutions of the Hull-Strominger system on non-Kähler threefolds*, J. Reine Angew. Math. **766** (2020), 137–150.

[146] M. Garcia-Fernandez, J. Jordan, and J. Streets, *Non-Kähler Calabi-Yau geometry and pluriclosed flow*, J. Math. Pures Appl. (9) **177** (2023), 329–367.

[147] M. Garcia-Fernandez and R. G. Molina, *Futaki Invariants and Yau's Conjecture on the Hull-Strominger system*, arXiv preprint arXiv:2303.05274 (2023), 1–38.

[148] ______, *Harmonic metrics for the Hull-Strominger system and stability*, arXiv preprint arXiv:2301.08236 (2023), 1–26.

[149] M. Garcia-Fernandez and J. Streets, *Generalized Ricci flow*, University Lecture Series, vol. 76, American Mathematical Society, Providence, RI, 2021.

[150] S.J. Gates, C.M. Hull, and M. Roček, *Twisted multiplets and new supersymmetric non-linear σ-models*, Nuclear Phys. B **248** (1984), no. 1, 157–186.

[151] P. Gauduchon, *Fibres hermitiens à endomorphisme de Ricci non négatif*, Bull. Soc. Math. Fr. **105** (1977), 113–140.

[152] ______, *Le théorème de l'excentricite nulle*, C. R. Acad. Sci., Paris, Sér. A **285** (1977), 387–390.

[153] ______, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), 495–518.

[154] ______, *Hermitian connections and Dirac operators*, Boll. Unione Mat. Ital., VII. Ser., B **11** (1997), no. 2, 257–288.

[155] ______, *Calabi's extremal Kähler metrics: An elementary introduction*, Preprint (2010), 1–310.

[156] G. Gentili and N. Tardini, *HKT Manifolds: Hodge Theory, Formality and Balanced Metrics*, Q. J. Math. **75** (2024), no. 2, 413–435.

[157] G. Gentili and L. Vezzoni, *The quaternionic Calabi conjecture on abelian hypercomplex nilmanifolds viewed as tori fibrations*, Int. Math. Res. Not. **2022** (2022), no. 12, 9499–9528.

[158] ______, *A remark on the quaternionic Monge-Ampère equation on foliated manifolds*, Proc. Am. Math. Soc. **151** (2023), no. 3, 1263–1275.

[159] G. Gentili and J. Zhang, *Fully non-linear elliptic equations on compact manifolds with a flat hyperKähler metric*, J. Geom. Anal. **32** (2022), no. 9, 38.

[160] ______, *Fully non-linear parabolic equations on compact manifolds with a flat hyperKähler metric*, arXiv preprint arXiv:2204.12232 (2022), 1–29.

[161] G. W. Gibbons and S. W. Hawking, *Classification of gravitational instanton symmetries*, Comm. Math. Phys. **66** (1979), no. 3, 291–310.

[162] G. W. Gibbons, G. Papadopoulos, and K. S. Stelle, *HKT and OKT geometries on soliton black hole moduli spaces*, Nuclear Phys. B **508** (1997), no. 3, 623–658.

[163] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.

[164] M. Gill, *Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds*, Comm. Anal. Geom. **19** (2011), no. 2, 277–303.

[165] S. Gindi and J. Streets, *Structure of collapsing solutions of generalized Ricci flow*, J. Geom. Anal. **31** (2021), no. 4, 4253–4286.

[166] ______, *Four-dimensional generalized Ricci flows with nilpotent symmetry*, Comm. Contemp. Math. **25** (2023), no. 7, no. 2250025, 28.

[167] F. Giusti and F. Podestà, *Real semisimple Lie groups and balanced metrics*, Rev. Mat. Iberoam. **39** (2023), no. 2, 711–729.

[168] F. Giusti and C. Spotti, *A Kümmer construction for Chern-Ricci flat balanced manifolds*, arXiv preprint arXiv:2309.12909 (2023), 1–30.

[169] E. Goldstein and S. Prokushkin, *Geometric model for complex non-Kähler manifolds with* SU(3) *structure*, Comm. Math. Phys. **251** (2004), no. 1, 65–78.

[170] D. Grantcharov, G. Grantcharov, and Y. S. Poon, *Calabi-Yau connections with torsion on toric bundles*, J. Differ. Geom. **78** (2008), no. 1, 13–32.

[171] F. Grantcharov, H. Pedersen, and Y. S. Poon, *Deformations of hypercomplex structures associated to Heisenberg groups*, Q. J. Math. **59** (2008), no. 3, 335–362.

[172] G. Grantcharov, M. Lejmi, and M. Verbitsky, *Existence of HKT metrics on hypercomplex manifolds of real dimension 8*, Adv. Math. **320** (2017), 1135–1157.

[173] G. Grantcharov, G. Papadopoulos, and Y. S. Poon, *Reduction of HKT-structures*, J. Math. Phys. **43** (2002), no. 7, 3766–3782.

[174] G. Grantcharov and Y. S. Poon, *Geometry of hyper-Kähler connections with torsion*, Comm. Math. Phys. **213** (2000), no. 1, 19–37.

[175] P. Griffiths and J. Harris, *Principles of algebraic geometry*, 2nd ed. ed., New York, NY: John Wiley & Sons Ltd., 1994.

[176] J. Grover, J. Gutowski, C. A. R. Herdeiro, and W. Sabra, *HKT geometry and de Sitter supergravity*, Nuclear Phys. B **809** (2009), no. 3, 406–425.

[177] M. Gualtieri, *Generalized complex geometry*, Ann. Math. (2) **174** (2011), no. 1, 75–123.

[178] Y. Guo and F. Zheng, *Hermitian geometry of Lie algebras with abelian ideals of codimension 2*, Math. Z. **304** (2023), no. 3, no. 51, 24.

[179] V. Gutiérrez, *Generalized Ricci flow on aligned homogeneous spaces*, arXiv preprint arXiv:2401.03332 (2024), 1–14.

[180] J. Gutowski and G. Papadopoulos, *The dynamics of very special black holes*, Physics Letters B **472** (2000), no. 1, 45–53.

[181] J. Gutowski and W. Sabra, *HKT geometry and fake five-dimensional supergravity*, Classical Quantum Gravity **28** (2011), no. 17, 175023, 11.

[182] J. Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), no. 2, 279–352.

[183] P. Heinzner and G. W. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), no. 1, 197–232.

[184] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001.

[185] H. Hironaka, *On the theory of birational blowing up*, PhD thesis, Harvard (1960), 1–179.

[186] ———, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I*, Ann. Math. (2) **79** (1964), 109–203.

[187] ———, *Resolution of singularities of an algebraic variety over a field of characteristic zero. II*, Ann. Math. (2) **79** (1964), 205–326.

[188] ———, *Flattening theorem in complex-analytic geometry*, Am. J. Math. **97** (1975), 503–547.

[189] P.S. Howe and G. Papadopoulos, *Twistor spaces for hyperKähler manifolds with torsion*, Physics Letters B **379** (1996), no. 1, 80–86.

[190] C. M. Hull, *Superstring compactifications with torsion and spacetime supersymmetry*, Superunification and extra dimensions (Torino, 1985), World Sci. Publishing, Singapore, 1986, pp. 347–375.

[191] D. Huybrechts, *Complex geometry: an introduction*, Universitext, Springer-Verlag, Berlin, 2005.

[192] S. Ianuş, A. M. Ionescu, R. Mocanu, and G. E. Vîlcu, *Riemannian submersions from almost contact metric manifolds*, Abh. Math. Semin. Univ. Hambg. **81** (2011), no. 1, 101–114.

[193] S. Ianuş, R. Mazzocco, and G. E. Vîlcu, *Harmonic maps between quaternionic Kähler manifolds*, J. Nonlinear Math. Phys. **15** (2008), no. 1, 1–8.

[194] M. Inoue, *On surfaces of Class* VII_0, Invent. Math. **24** (1974), 269–310.

[195] S. Ivanov and I. Minchev, *Quaternionic Kähler and hyperKähler manifolds with torsion and twistor spaces*, J. Reine Angew. Math. **567** (2004), 215–233.

[196] S. Ivanov and G. Papadopoulos, *Vanishing theorems and string backgrounds*, Classical Quantum Gravity **18** (2001), no. 6, 1089–1110.

[197] S. Ivanov and A. Petkov, *HKT manifolds with holonomy* $\mathrm{SL}(n, H)$, Int. Math. Res. Not. (2012), no. 16, 3779–3799.

[198] S. Jordan and J. Streets, *On a Calabi-type estimate for pluriclosed flow*, Adv. Math. **366** (2020), 107097, 18.

[199] J. Jost and S.-T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math. **170** (1993), no. 2, 221–254.

[200] D. Joyce, *The hypercomplex quotient and the quaternionic quotient*, Math. Ann. **290** (1991), no. 2, 323–340.

[201] ———, *Compact hypercomplex and quaternionic manifolds*, J. Differ. Geom. **35** (1992), no. 3, 743–761.

[202] ———, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[203] ———, *Asymptotically locally Euclidean metrics with holonomy SU(m)*, Ann. Global Anal. Geom. **19** (2001), no. 1, 55–73.

[204] ———, *Riemannian holonomy groups and calibrated geometry*, Oxf. Grad. Texts Math., vol. 12, Oxford: Oxford University Press, 2007.

[205] D. Kaledin and M. Verbitsky, *Non-Hermitian Yang-Mills connections*, Sel. Math., New Ser. **4** (1998), no. 2, 279–320.

[206] H. Kasuya, *Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds*, Bull. Lond. Math. Soc. **45** (2013), no. 1, 15–26.

[207] M. Kato, *Compact differentiable* 4*-folds with quaternionic structures*, Math. Ann. **248** (1980), no. 1, 79–96.

[208] F. Keller and S. Waldmann, *Formal deformations of Dirac structures*, J. Geom. Phys. **57** (2007), no. 3, 1015–1036.

[209] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996.

[210] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. III: Stability theorems for complex structures*, Ann. Math. (2) **71** (1960), 43–76.

[211] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differ. Geom. **29** (1989), no. 3, 665–683.

[212] ———, *A Torelli-type theorem for gravitational instantons*, J. Differ. Geom. **29** (1989), no. 3, 685–697.

[213] R. A. Lafuente, *Scalar curvature behavior of homogeneous Ricci flows*, J. Geom. Anal. **25** (2015), no. 4, 2313–2322.

[214] R. A. Lafuente and J. Lauret, *Structure of homogeneous Ricci solitons and the Alekseevskii conjecture*, J. Differ. Geom. **98** (2014), no. 2, 315–347.

[215] A. Lamari, *Courants kählériens et surfaces compactes. (Kähler currents and compact surfaces)*, Ann. Inst. Fourier **49** (1999), no. 1, 263–285.

[216] A. Latorre and L. Ugarte, *On non-Kähler compact complex manifolds with balanced and astheno-Kähler metrics*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 1, 90–93.

[217] J. Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), no. 4, 715–733.

[218] ———, *Finding Einstein solvmanifolds by a variational method*, Math. Z. **241** (2002), no. 1, 83–99.

[219] ———, *A canonical compatible metric for geometric structures on nilmanifolds*, Ann. Global Anal. Geom. **30** (2006), no. 2, 107–138.

[220] ———, *The Ricci flow for simply connected nilmanifolds*, Comm. Anal. Geom. **19** (2011), no. 5, 831–854.

[221] ———, *Convergence of homogeneous manifolds*, J. Lond. Math. Soc. (2) **86** (2012), no. 3, 701–727.

[222] ———, *Ricci flow of homogeneous manifolds*, Math. Z. **274** (2013), no. 1-2, 373–403.

[223] ———, *Curvature flows for almost-Hermitian Lie groups*, Trans. Amer. Math. Soc. **367** (2015), no. 10, 7453–7480.

[224] ———, *The search for solitons on homogeneous spaces*, Geometry, Lie theory and applications. Proceedings of the Abel symposium 2019, Ålesund, Norway June 24–28, 2019, Cham: Springer, 2022, pp. 147–170.

[225] J. Lauret and E. A. Rodríguez Valencia, *On the Chern-Ricci flow and its solitons for Lie groups*, Math. Nachr. **288** (2015), no. 13, 1512–1526.

[226] J. Lauret and C. Will, *Bismut Ricci flat generalized metrics on compact homogeneous spaces*, Trans. Amer. Math. Soc. **376** (2023), no. 10, 7495–7519.

[227] C. LeBrun, *Counter-examples to the generalized positive action conjecture*, Comm. Math. Phys. **118** (1988), no. 4, 591–596.

[228] K.-H. Lee, *The stability of generalized Ricci solitons*, J. Geom. Anal. **33** (2023), no. 9, no. 273, 52.

[229] ———, *The stability of non-Kähler Calabi-Yau metrics*, arXiv preprint arXiv:2401.06867 (2024), 1–25.

[230] M. Lejmi and N. Tardini, *On the invariant and anti-invariant cohomologies of hypercomplex manifolds*, Transform. Groups (2023), 1–25.

[231] M. Lejmi and P. Weber, *Cohomologies on hypercomplex manifolds*, Complex and symplectic geometry, Springer INdAM Ser., vol. 21, Springer, Cham, 2017, pp. 107–121.

[232] ———, *Quaternionic Bott-Chern cohomology and existence of HKT metrics*, Q. J. Math. **68** (2017), no. 3, 705–728.

[233] X. Li and Y. Ye, *On the shrinking solitons of generalized Ricci flow*, arXiv preprint arXiv:2404.06141 (2024), 1–20.

[234] Y. Li and F. Zheng, *Fino-Vezzoni conjecture in Hermitian geometry (in chinese)*, Sci. Sin. Math. **54** (2024).

[235] K. Liu and X. Yang, *Ricci curvatures on Hermitian manifolds*, Trans. Amer. Math. Soc. **369** (2017), no. 7, 5157–5196.

[236] Z.-J. Liu, A. Weinstein, and P. Xu, *Manin triples for Lie bialgebroids*, J. Differ. Geom. **45** (1997), no. 3, 547–574.

[237] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Les Équations aux Dérivées Partielles (Paris, 1962), Colloq. Internat. CNRS, vol. No. 117, CNRS, Paris, 1963, pp. 87–89.

[238] T. B. Madsen and A. Swann, *Invariant strong KT geometry on four-dimensional solvable Lie groups*, J. Lie Theory **21** (2011), no. 1, 55–70.

[239] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne*, Nagoya Math. J. **11** (1957), 145–150.

[240] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), no. 3-4, 261–295.

[241] J. Michelson and A. Strominger, *Superconformal multi-black hole quantum mechanics*, J. High Energy Phys. (1999), no. 9, Paper 5, 16.

[242] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), no. 3, 293–329.

[243] R. Moraru and M. Verbitsky, *Stable bundles on hypercomplex surfaces*, Cent. Eur. J. Math. **8** (2010), no. 2, 327–337.

[244] D. R. Morrison and G. Stevens, *Terminal quotient singularities in dimensions three and four*, Proc. Am. Math. Soc. **90** (1984), 15–20.

[245] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. Math. (2) **65** (1957), 391–404.

[246] M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jpn. J. Math. **26** (1956), 43–77.

[247] K. Oeljeklaus and M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier **55** (2005), no. 1, 161–171.

[248] K. G. O'Grady, *Desingularized moduli spaces of sheaves on a* $K3$, J. Reine Angew. Math. **512** (1999), 49–117.

[249] ———, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), no. 3, 435–505.

[250] T. Oliynyk, V. Suneeta, and E. Woolgar, *A gradient flow for worldsheet nonlinear sigma models*, Nuclear Phys. B **739** (2006), no. 3, 441–458.

[251] A. Opfermann and G. Papadopoulos, *Homogeneous HKT and QKT manifolds*, arXiv preprint math-ph/9807026 (1998), 1–33.

[252] A. Otal, L. Ugarte, and R. Villacampa, *Invariant solutions to the Strominger system and the heterotic equations of motion*, Nuclear Phys. B **920** (2017), 442–474.

[253] A. Otiman, *Special Hermitian metrics on Oeljeklaus-Toma manifolds*, Bull. Lond. Math. Soc. **54** (2022), no. 2, 655–667.

[254] F. Paradiso, *Generalized Ricci flow on nilpotent Lie groups*, Forum Math. **33** (2021), no. 4, 997–1014.

[255] H. Pedersen and Y. S. Poon, *Deformations of hypercomplex structures*, J. Reine Angew. Math. **499** (1998), 81–99.

[256] ———, *Inhomogeneous hypercomplex structures on homogeneous manifolds*, J. Reine Angew. Math. **516** (1999), 159–181.

[257] H. Pedersen, Y. S. Poon, and A. Swann, *Hypercomplex structures associated to quaternionic manifolds*, Differ. Geom. Appl. **9** (1998), no. 3, 273–292.

[258] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv preprint math/0211159 (2002), 1–39.

[259] ______, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv preprint math/0307245 (2003), 1–7.

[260] ______, *Ricci flow with surgery on three manifolds*, arXiv preprint math.DG/0303109 (2003), 1–22.

[261] D. H. Phong, S. Picard, and X. Zhang, *The Fu-Yau equation with negative slope parameter*, Invent. Math. **209** (2017), no. 2, 541–576.

[262] ______, *The anomaly flow and the Fu-Yau equation*, Ann. PDE **4** (2018), no. 2, 60.

[263] ______, *Anomaly flows*, Comm. Anal. Geom. **26** (2018), no. 4, 955–1008.

[264] ______, *Geometric flows and Strominger systems*, Math. Z. **288** (2018), no. 1-2, 101–113. MR 3774405

[265] ______, *The anomaly flow on unimodular Lie groups*, Advances in complex geometry. Contributions from the JHU-UMD complex geometry seminar, John Hopkins University, Baltimore, MD, USA and University of Maryland, College Park, MD, USA, 2015–2018, Providence, RI: American Mathematical Society (AMS), 2019, pp. 217–237.

[266] F. Podestà, *Homogeneous Hermitian manifolds and special metrics*, Transform. Groups **23** (2018), no. 4, 1129–1147.

[267] F. Podestà and A. Raffero, *Bismut Ricci flat manifolds with symmetries*, Proc. Roy. Soc. Edinburgh Sect. A **153** (2023), no. 4, 1371–1390.

[268] ______, *Infinite families of homogeneous Bismut Ricci flat manifolds*, Comm. Contemp. Math. **26** (2024), no. 2, no. 2250075, 17.

[269] ______, *Three-dimensional positively curved generalized Ricci solitons with $SO(3)$-symmetries*, arXiv preprint arXiv:2401.05028 (2024), 1–21.

[270] Y. S. Poon and A. Swann, *Potential functions of HKT spaces*, Classical Quantum Gravity **18** (2001), no. 21, 4711–4714.

[271] D. Popovici, *Deformation limits of projective manifolds: Hodge numbers and strongly gauduchon metrics*, Invent. Math. **194** (2013), no. 3, 515–534.

[272] M. Pujia and L. Vezzoni, *A remark on the Bismut-Ricci form on 2-step nilmanifolds*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 2, 222–226.

[273] A. Raffero and L. Vezzoni, *On the dynamical behaviour of the generalized Ricci flow*, J. Geom. Anal. **31** (2021), no. 10, 10498–10509.

[274] R. W. Richardson and P. J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. (2) **42** (1990), no. 3, 409–429.

[275] S.-s. Roan, *Minimal resolutions of Gorenstein orbifolds in dimension three*, Topology **35** (1996), no. 2, 489–508.

[276] S. Rollenske, *Dolbeault cohomology of nilmanifolds with left-invariant complex structure*, Complex and differential geometry, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011, pp. 369–392.

[277] X. Rong, *Convergence and collapsing theorems in Riemannian geometry*, Handbook of geometric analysis. No. 2, Somerville, MA: International Press; Beijing: Higher Education Press, 2010, pp. 193–299.

[278] R. Rubio and C. Tipler, *The Lie group of automorphisms of a Courant algebroid and the moduli space of generalized metrics*, Rev. Mat. Iberoam. **36** (2020), no. 2, 485–536.

[279] S. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 311–333.

[280] H. Samelson, *A class of complex-analytic manifolds*, Portugal. Math. **12** (1953), 129–132.

[281] P. Ševera, *Letters to Alan Weinstein about Courant algebroids*, arXiv preprint arXiv:1707.00265 (2017), 1–29.

[282] P. Ševera and F. Valach, *Ricci flow, Courant algebroids, and renormalization of Poisson-Lie T-duality*, Lett. Math. Phys. **107** (2017), no. 10, 1823–1835.

[283] ———, *Courant algebroids, Poisson-Lie T-duality, and type II supergravities*, Comm. Math. Phys. **375** (2020), no. 1, 307–344.

[284] T. Sferruzza and A. Tomassini, *Dolbeault and Bott-Chern formalities: deformations and $\partial\bar{\partial}$-lemma*, J. Geom. Phys. **175** (2022), no. 104470, 19.

[285] S. R. Simanca, *Kähler metrics of constant scalar curvature on bundles over* $\mathbf{C}\mathrm{P}_{n-1}$, Math. Ann. **291** (1991), no. 2, 239–246.

[286] Y.-T. Siu, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type*, Complex geometry. Collection of papers dedicated to Hans Grauert on the occasion of his 70th birthday, Berlin: Springer, 2002, pp. 223–277.

[287] P. J. Slodowy, *Simple singularities and simple algebraic groups*, Lect. Notes Math., vol. 815, Springer, Cham, 1980.

[288] A. Soldatenkov, *Holonomy of the Obata connection on $SU(3)$*, Int. Math. Res. Not. **2012** (2012), no. 15, 3483–3497.

[289] A. Soldatenkov and M. Verbitsky, *Holomorphic Lagrangian fibrations on hypercomplex manifolds*, Int. Math. Res. Not. **2015** (2015), no. 4, 981–994.

[290] A. J. Sommese, *Quaternionic manifolds*, Math. Ann. **212** (1975), 191–214.

[291] P. Spindel, A. Sevrin, W. Troost, and A. Van Proeyen, *Extended supersymmetric σ-models on group manifolds. I. The complex structures*, Nuclear Phys. B **308** (1988), no. 2-3, 662–698.

[292] M. Sroka, *Sharp uniform bound for the quaternionic Monge-Ampère equation on hyperhermitian manifolds*, Calc. Var. Partial Differential Equations **63** (2024), no. 4, no. 102.

[293] J. Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. **221** (2009), no. 4, 1397–1408.

[294] J. Streets, *Pluriclosed flow, Born-Infeld geometry, and rigidity results for generalized Kähler manifolds*, Comm. Partial Differential Equations **41** (2016), no. 2, 318–374.

[295] ———, *Pluriclosed flow on manifolds with globally generated bundles*, Complex Manifolds **3** (2016), no. 1, 222–230.

[296] ———, *Generalized geometry, T-duality, and renormalization group flow*, J. Geom. Phys. **114** (2017), 506–522.

[297] ———, *Pluriclosed flow on generalized Kähler manifolds with split tangent bundle*, J. Reine Angew. Math. **739** (2018), 241–276.

[298] ______, *Classification of solitons for pluriclosed flow on complex surfaces*, Math. Ann. **375** (2019), no. 3-4, 1555–1595.

[299] ______, *Pluriclosed flow and the geometrization of complex surfaces*, Geometric analysis—in honor of Gang Tian's 60th birthday, Progr. Math., vol. 333, Birkhäuser/Springer, Cham, 2020, pp. 471–510.

[300] ______, *Ricci-Yang-Mills flow on surfaces and pluriclosed flow on elliptic fibrations*, Adv. Math. **394** (2022), 31.

[301] ______, *Scalar curvature, entropy, and generalized Ricci flow*, Int. Math. Res. Not. (2023), no. 11, 9481–9510.

[302] J. Streets and G. Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. (2010), no. 16, 3101–3133.

[303] ______, *Hermitian curvature flow*, J. Eur. Math. Soc. **13** (2011), no. 3, 601–634.

[304] ______, *Generalized Kähler geometry and the pluriclosed flow*, Nuclear Phys. B **858** (2012), no. 2, 366–376.

[305] ______, *Regularity results for pluriclosed flow*, Geom. Topol. **17** (2013), no. 4, 2389–2429.

[306] J. Streets and Y. Ustinovskiy, *Classification of generalized Kähler-Ricci solitons on complex surfaces*, Comm. Pure Appl. Math. **74** (2021), no. 9, 1896–1914.

[307] J. Streets and M. Warren, *Evans-Krylov estimates for a nonconvex Monge-Ampère equation*, Math. Ann. **365** (2016), no. 1-2, 805–834.

[308] A. Strominger, *Superstrings with torsion*, Nuclear Phys. B **274** (1986), no. 2, 253–284.

[309] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math., Inst. Hautes Étud. Sci. **47** (1977), 269–331.

[310] A. Swann, *Twisting Hermitian and hypercomplex geometries*, Duke Math. J. **155** (2010), no. 2, 403–431.

[311] G. Székelyhidi, *On blowing up extremal Kähler manifolds*, Duke Math. J. **161** (2012), no. 8, 1411–1453.

[312] ______, *An introduction to extremal Kähler metrics*, Graduate Studies in Mathematics, vol. 152, American Mathematical Society, Providence, RI, 2014.

[313] ______, *Blowing up extremal Kähler manifolds II*, Invent. Math. **200** (2015), no. 3, 925–977.

[314] G. Székelyhidi, V. Tosatti, and B. Weinkove, *Gauduchon metrics with prescribed volume form*, Acta Math. **219** (2017), no. 1, 181–211.

[315] A. Thompson, *Bach flow of simply connected nilmanifolds*, Adv. Geom. **24** (2024), no. 1, 127–139.

[316] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37.

[317] S. Tôgô, *On the derivation algebras of Lie algebras*, Canad. J. Math. **13** (1961), 201–216.

[318] A. Tomberg, *Twistor spaces of hypercomplex manifolds are balanced*, Adv. Math. **280** (2015), 282–300.

[319] P. Topping, *Lectures on the Ricci flow*, Lond. Math. Soc. Lect. Note Ser., vol. 325, Cambridge: Cambridge University Press, 2006.

[320] V. Tosatti, *Non-Kähler Calabi-Yau manifolds*, Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong, Contemp. Math., vol. 644, Amer. Math. Soc., Providence, RI, 2015, pp. 261–277.

[321] ______, *KAWA lecture notes on the Kähler-Ricci flow*, Ann. Fac. Sci. Toulouse Math. (6) **27** (2018), no. 2, 285–376.

[322] V. Tosatti and B. Weinkove, *The Chern-Ricci flow on complex surfaces*, Compos. Math. **149** (2013), no. 12, 2101–2138.

[323] ______, *On the evolution of a Hermitian metric by its Chern-Ricci form*, J. Differ. Geom. **99** (2015), no. 1, 125–163.

[324] ______, *The Monge-Ampère equation for $(n-1)$-plurisubharmonic functions on a compact Kähler manifold*, J. Amer. Math. Soc. **30** (2017), no. 2, 311–346.

[325] L. Ugarte, *Hermitian structures on six-dimensional nilmanifolds*, Transform. Groups **12** (2007), no. 1, 175–202.

[326] L. Ugarte and R. Villacampa, *Balanced Hermitian geometry on 6-dimensional nilmanifolds*, Forum Math. **27** (2015), no. 2, 1025–1070.

[327] Y. Ustinovskiy, *The Hermitian curvature flow on manifolds with non-negative Griffiths curvature*, Am. J. Math. **141** (2019), no. 6, 1751–1775.

[328] S. Van Thuong, *Metrics on 4-dimensional unimodular Lie groups*, Ann. Global Anal. Geom. **51** (2017), no. 2, 109–128.

[329] M. Verbitsky, *HyperKähler manifolds with torsion, supersymmetry and Hodge theory*, Asian J. Math. **6** (2002), no. 4, 679–712.

[330] ______, *HyperKähler manifolds with torsion obtained from hyperholomorphic bundles*, Math. Res. Lett. **10** (2003), no. 4, 501–513.

[331] ______, *Hypercomplex manifolds with trivial canonical bundle and their holonomy*, Moscow Seminar on Mathematical Physics. II, Amer. Math. Soc. Transl. Ser. 2, vol. 221, Amer. Math. Soc., Providence, RI, 2007, pp. 203–211.

[332] ______, *Balanced HKT metrics and strong HKT metrics on hypercomplex manifolds*, Math. Res. Lett. **16** (2009), no. 4, 735–752.

[333] ______, *Positive forms on hyperKähler manifolds*, Osaka J. Math. **47** (2010), no. 2, 353–384.

[334] S. Verbitsky, *Surfaces on Oeljeklaus-Toma manifolds*, arXiv preprint arXiv:1306.2456 (2013), 1–8.

[335] L. Vezzoni, *A note on canonical Ricci forms on 2-step nilmanifolds*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 325–333.

[336] H.-C. Wang, *Complex parallelisable manifolds*, Proc. Am. Math. Soc. **5** (1954), 771–776.

[337] Q. Wang, B. Yang, and F. Zheng, *On Bismut flat manifolds*, Trans. Amer. Math. Soc. **373** (2020), no. 8, 5747–5772.

[338] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[339] ______, *Open problems in geometry*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Part 1, Amer. Math. Soc., Providence, RI, 1993, pp. 1–28.

[340] ———, *Metrics on complex manifolds*, Sci. China Math. **53** (2010), no. 3, 565–572.

[341] Y. Ye, *Derivative estimates of pluriclosed flow*, Adv. Math. **447** (2024), 33.

[342] ———, *Pluriclosed flow and Hermitian-symplectic structures*, J. Geom. Anal. **34** (2024), no. 4, 18.

[343] W. Yu, *Prescribed Chern scalar curvatures on compact Hermitian manifolds with negative Gauduchon degree*, J. Funct. Anal. **285** (2023), no. 2, no. 109948, 27.

[344] T. Zheng, *The Chern-Ricci flow on Oeljeklaus-Toma manifolds*, Canad. J. Math. **69** (2017), no. 1, 220–240.