Thesis outline

Asia Mainenti

The general aim of my thesis is to investigate special structures on complex manifolds, especially balanced and p-Kähler, generalizing Kähler metrics, with a particular focus on the setting of Lie groups, and to study positivity of exterior forms on a complex vector space. It contains the results of two joint works with A. Fino, and a part of a work in progress with F. Fagioli.

In the last few decades, special Hermitian metrics have been of wide interest, primarily as generalizations of Kähler ones. One of the first classes to be considered was that of balanced metrics, defined by requiring that the $(n-1)$th power of the fundamental form associated to the metric is closed, where n is the complex dimension of the manifold. Balanced manifolds have resulted to be significant in the physics realm as well. They were thoroughly studied from numerous perspectives and, similarly to the Kähler case, the balanced condition was rephrased by means of the Chern connection of the Hermitian metric, and the existence of balanced metrics was characterized in terms of currents. More recently, classification results were found, on some classes of solvable and nilpotent Lie algebra, and in low dimension.

Alessandrini and Andreatta introduced p-Kähler structures as closed (p,p)-forms, pointwise *transverse*. A real (p,p)-form on a complex vector space is called transverse if its restriction to any complex subspace of complex dimension p is a positive volume form. For the extremal values of $p = 1, n-1$, respectively, we get the well known Kähler and balanced manifolds. Nevertheless, for any other possible p, this does not hold anymore, and p-Kähler structures have no metric meaning.

The main similarities and differences between p-Kähler structures and Kähler and balanced structures were extensively examined by Alessandrini and Bassanelli. Such structures share numerous properties with Kähler and balanced ones, such as the characterization in terms of currents, or the good behavior under holomorphic submersions, pullback via holomorphic immersions, or products.

As far as examples are concerned, one can build non-Kähler, *non-compact* p-Kähler manifolds of complex dimension n, for all $p \geq 2$ and $n \geq 3$. The situation is different for the compact case, as explicit compact examples are known in each odd complex dimension $n \geq 5$. However, in the first *non-trivial* case, namely 2-Kähler structures on compact complex manifolds of complex dimension 4, there is no known example in literature. Together with A. Fino, we proved that if M is a nilmanifold, or an almost solvmanifold, or if the Lie algebra associated to M admits a J-invariant abelian ideal of real codimension 2, then, (M, J) is 2-Kähler if and only if it is Kähler.

As a matter of fact, a part of this result is a particular instance of two more general results. The first one concerns *quasi-nilpotent* invariant complex structures on nilmanifolds, defined by the property that the center of the associated Lie algebra admits a non-trivial subspace, invariant under the complex structure. Under this assumption we proved that the existence of a 2-Kähler structure implies Kählerianity, in any dimension. Furthermore, we proved that a solvmanifold endowed with an invariant complex structure J such that the associated Lie algebra admits a J-invariant abelian ideal of real codimension 2 cannot be $(n-2)$-Kähler, unless Kähler. Note that this includes the case of almost abelian solvmanifolds, by integrability of the complex structure.

The second part of our work concerns a possible application of the study of p-Kähler manifolds. The non-abelian Hodge correspondence for Kähler manifolds was recently extended to a larger class of Hermitian metrics on complex manifolds, called balanced of Hodge-Riemann type. More precisely, recently Chen and Wentworth proved that on a compact complex manifold admitting such a metric, there is a 1-1 correspondence between semisimple flat bundles and isomorphism classes of

F-polystable Higgs bundles satisfying further assumptions, where F is the fundamental form of a Hodge-Riemann balanced metric. A balanced metric F is said of *Hodge-Riemann type* if F^{n-1} can be written as $F^{n-1} = \omega \wedge \Omega$, with ω a Hermitian metric and Ω a closed $(n-2, n-2)$-form such that Ω is a *Hodge Riemann form* with respect to ω for degrees (p, q), with $p+q = 2$. The last condition in fact a positivity condition on Ω, generalizing the *Hodge-Riemann bilinear relations*, which are known to hold for $(n-2)$th powers of a Kähler metric, and for the wedge product of $(n-2)$ Kähler metrics. In fact, these choices of Ω give rise the only known examples of Hodge-Riemann balanced metrics, not necessarily Kähler, but still in the Kähler setting. This gives rise to the following problem.

Question 1. Are there any non-Kähler manifolds admitting a Hodge-Riemann balanced metric?

To tackle this, we noted that a Hodge Riemann form is in particular a $(n-2)$-Kähler structure, suggesting us a possible approach to the problem, via the study of $(n-2)$-Kähler manifolds. In particular, if $n = 3$ Hodge-Riemann balanced manifolds are Kähler, so the first non-trivial complex dimension is 4. Using the results cited above, we can exclude the existence of Hodge-Riemann balanced metrics on non-Kähler solvmanifolds endowed with an invariant complex structure J, which are nilmanifolds of complex dimension 4, or such that the associated Lie algebra admits a J-invariant abelian ideal of real codimension 2, in any dimension. If $n > 4$, no general result is known about the existence of $(n-2)$-Kähler structures on nilmanifolds, and the only known examples of such structures are on odd-dimensional complex parallelizable nilmanifolds. If the balanced Hodge-Riemann structure is invariant, the problem can be stated in terms of nilpotent Lie algebras. Making use of this reduction, we can prove that the only nilmanifolds admitting invariant balanced metrics of Hodge-Riemann type are complex torus, for every complex dimension $n \geq 3$. The problem remains open in the case of non invariant balanced structures, even when the complex structure is invariant.

More in general, if J is either bi-invariant or abelian, we proved that the existence of an invariant balanced metric of Hodge-Riemann type implies Kählerianity. Furthermore, the argument used in the case of bi-invariant complex structures allows to extend the result to non invariant metrics.

In contrast with the many obstructions arising in the compact case, we were able to give the first non-Kähler example Hodge-Riemann balanced structure, on the non-compact manifold $\mathcal{I} \times \mathbb{C}$, of complex dimension 4, where $\mathcal{I}$ is the Iwasawa manifold, namely the nilmanifold obtained as the quotient $\Gamma\backslash H$ of the complex Heisenberg group $H = \mathrm{Heis}(3, \mathbb{C})$ by the lattice $\Gamma = \mathrm{Heis}(3, \mathbb{Z}[i])$.

In all the conditions above, one of the basic ingredients was the study of different positivity notions on (p, p)-forms. Despite these conditions being quite intuitive, they are not not always easy to verify, on a given real form.

In a work by Harvey and Knapp on one can find a detailed account on positivity of exterior forms on a complex vector space, together with a thorough analysis of positivity, weak positivity and strong positivity, and their strict versions. More recently, Błocki and Pliś considered the case of weakly positive $(2, 2)$-forms in complex dimension 4. Together with F. Fagioli, we gave a generalization of one of their results, for different types of positivity, and we found further criteria for weakly positive $(2, 2)$-forms in complex dimension 4. In addition to this, we investigated the relation between positivity on any complex vector space V of dimension n, and a complex subspace $\mathfrak{h}$ of codimension 1. More precisely, we provided a way to construct positive (p, p)-forms on V, starting from positive (p, p)-forms, or $(p-1, p-1)$-forms on $\mathfrak{h}$, and the analogous version for the strict versions of positivity. Conversely, given a positive (p, p)-form on V, we showed how to build in a natural way a (p, p)-form and a $(p-1, p-1)$-form on $\mathfrak{h}$, satisfying the same positivity condition of the original form.