The quantitative isoperimetric inequality for the Hilbert–Schmidt norm of localization operators

Journal of Fourier Analysis and Applications (2025) 31:12
https://doi.org/10.1007/s00041-025-10145-y

The Quantitative Isoperimetric Inequality for the Hilbert–Schmidt Norm of Localization Operators

Fabio Nicola [1] · **Federico Riccardi** [1]

Received: 17 January 2024 / Accepted: 20 January 2025

Abstract

In this paper we study the Hilbert–Schmidt norm of time-frequency localization operators $L_\Omega\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, with Gaussian window, associated with a subset $\Omega \subset \mathbb{R}^{2d}$ of finite measure. We prove, in particular, that the Hilbert–Schmidt norm of L_Ω is maximized, among all subsets Ω of a given finite measure, when Ω is a ball and that there are no other extremizers. Actually, the main result is a quantitative version of this estimate, with sharp exponent. A similar problem is addressed for wavelet localization operators, where rearrangements are understood in the hyperbolic setting.

Keywords Short-time Fourier transform · Time-frequency localization operator · Uncertainty principle · Quantitative estimate

Mathematics Subject Classification 42B10 · 49Q20 · 49R05 · 81S30 · 94A12

1 Introduction

The short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window $\varphi \in L^2(\mathbb{R}^d)$ is defined (see e.g. Gröchenig's book [27]) as

$$\mathcal{V}_\varphi f(x,\omega) = \int_{\mathbb{R}^d} f(y)\overline{\varphi(y-x)}e^{-2\pi i\omega\cdot y}\,dy, \quad (x,\omega) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{1.1}$$

A common choice for the window is the L^2-normalized Gaussian, that is

$$\varphi(x) = 2^{d/4}e^{-\pi|x|^2}, \quad x \in \mathbb{R}^d.$$

Dedicated to Karlheinz Gröchenig, on the occasion of his 65th birthday.

Communicated by Ortega-Cerdà.

✉ Fabio Nicola
fabio.nicola@polito.it

[1] Politecnico di Torino, Dipartimento di Scienze Matematiche, Corso degli Abruzzi 24, 10129 Torino, Italy

Published online: 04 February 2025

In this note we will always consider this window and therefore we will simply set $\mathcal{V} = \mathcal{V}_\varphi$.

Since we chose φ normalized in $L^2(\mathbb{R}^d)$, we have that $\mathcal{V} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ becomes an isometry. Hence, if $\|f\|_2 = 1$, the quantity $|\mathcal{V}f(x,\omega)|^2$, known as spectrogram, can be interpreted as the *time-frequency energy density* of f around the point (x,ω) in the time-frequency space. With this in mind, it is clear why having good and meaningful estimates for the short-time Fourier transform (and in particular for the spectrogram) has always been of great importance both from a theoretical and practical point of view. One of the first and at the same time most important results in this sense was obtained by Lieb in 1978 [34] and is known today as Lieb's uncertainty inequality, namely

$$\|\mathcal{V}f\|_p^p \le \left(\frac{2}{p}\right)^d \|f\|_2^p \tag{1.2}$$

for every $f \in L^2(\mathbb{R}^d)$ and $2 \le p < \infty$ (see also [9] for the identification of the extremal functions, and [35] for generalizations). Lieb's inequality is a global estimate. In the spirit of uncertainty principles, we may be interested also in local estimates, that is, for some $\Omega \subset \mathbb{R}^{2d}$ with finite Lebesgue measure, finding bounds of the quantity

$$\frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2\, dx d\omega}{\|f\|_2^2},$$

which represents the fraction of energy of f contained in Ω. The above integral can be written in an equivalent way as follows

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2\, dx\, d\omega = \langle \chi_\Omega \mathcal{V}f, \mathcal{V}f\rangle = \langle \mathcal{V}^*\chi_\Omega\mathcal{V}f, f\rangle,$$

where the operator

$$L_\Omega := \mathcal{V}^*\chi_\Omega\mathcal{V}$$

naturally appears. This interpretation reveals a connection between time-frequency energy concentration estimates and the properties of the operator L_Ω. In particular, since Ω has finite measure, L_Ω is a compact self-adjoint nonnegative operator (see e.g. [48]) and therefore its operator norm is given by

$$\|L_\Omega\| = \max_{f \in L^2(\mathbb{R}^d)\setminus\{0\}} \frac{\langle \mathcal{V}^*\chi_\Omega\mathcal{V}f, f\rangle}{\|f\|_2^2}.$$

Hence, maximizing the norm of $\mathcal{V}^*\chi_\Omega\mathcal{V}$ corresponds to maximize the energy fraction of any function $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ on Ω. In this connection, Tilli and the first author [40] recently proved that, among all subsets Ω of a given finite measure, $\|L_\Omega\|$ is maximum when Ω is a ball and that there are no other extremizers (a more general conjecture of Abreu and Speckbacher [2] was also proved in [40]). We address to [21, 30–33, 44] for extensions of this result to other geometries –notably the hyperbolic and spherical one– and for applications in complex analysis. See also [26, 38, 39] for similar problems on locally compact Abelian groups and to [1, 2, 15, 28, 42, 46] for related work.

In general, one can also measure the time-frequency concentration by a weighted L^2-norm, hence considering, for a function $F:\mathbb{R}^{2d}\to\mathbb{C}$, the so-called *time-frequency localization operator*

$$L_F := \mathcal{V}^* F \mathcal{V}$$

(hence $L_{\chi_\Omega} = L_\Omega$, with a slight abuse of notation). Since their first appearance in [5] and [13], time-frequency localization operators were intensively studied; see, for example, [3, 12, 13, 16, 37, 48] and the references therein for general results concerning boundedness, compactness, Schatten properties and asymptotics of the eigenvalues. Also, Lieb's uncertainty inequality (1.2) can be equivalently rephrased, by duality, as

$$\|L_F\| \le (1/p')^{d/p'} \|F\|_p, \qquad 1 < p < \infty,$$

p' being the conjugate exponent. Similar estimates in case the weight F is taken in the intersection of Lebesgue spaces, with a full characterization of the extremal functions, were recently considered in [24, 41, 45].

In this paper we address similar problems for the Hilbert–Schmidt norm of time-frequency localization operators, especially of the kind L_Ω. An initial result, which follows from Riesz's rearrangement inequality, states that

$$\|L_\Omega\|_{\mathrm{HS}} \le \|L_{\Omega^*}\|_{\mathrm{HS}}, \tag{1.3}$$

where Ω^* is the (open) ball centered at 0, with the same measure as Ω, and that equality occurs if and only if Ω is (equivalent, up to a set of measure zero, to) a ball (see Proposition 3.4). In Sect. 6 we also prove an analogous result for wavelet localization operators, and also for general localization operators L_F.

However, our interest is towards a *quantitative* version of the previous estimate. In general, quantitative estimates are stability results for geometric and functional inequalities stating that if a function is "almost optimal" for some inequality then it must be "close" to the set of the corresponding optimizers. This kind of results have been proved for lots of different inequalities, such as the isoperimetric inequality, Sobolev and Gagliardo–Nirenberg inequality. For a comprehensive survey on the topic, see [17] and the references therein.

Only recently, quantitative estimates have been addressed for certain time-frequency concentration problems. Precisely the quantitative version for the above mentioned Faber–Krahn type result [40] was addressed in [25], whereas the quantitative version of Lieb's uncertainty inequality (1.2) (and for the generalized Wehrl entropy of mixed states) was proved in [20].

In this note we want to fit into this thread by focusing on the following question:

If a set $\Omega \subset \mathbb{R}^{2d}$ "almost" attains equality in (1.3) can we conclude that Ω is "almost" a ball?

The answer is positive and is given in Proposition 4.2, where the following quantitative bound is stated:

$$c_1\beta(|\Omega|)\alpha[\Omega]^2 \le \|L_{\Omega^*}\|_{\mathrm{HS}}^2 - \|L_\Omega\|_{\mathrm{HS}}^2, \tag{1.4}$$

where β is given by

$$\beta(t) = \begin{cases} t^{2+\frac{1}{d}}, & \text{for } 0 < t \le 1 \\ t^2 e^{-c_2 t^{1/d}}, & \text{for } t > 1 \end{cases}$$

with $c_1, c_2 > 0$, and $\alpha[\Omega]$ is the *Fraenkel asymmetry index*, which is defined as

$$\alpha[\Omega] := \inf\left\{\frac{|\Omega\triangle B|}{|\Omega|} :\ B \subset \mathbb{R}^{2d} \text{ is a ball of measure } |B| = |\Omega|\right\}. \tag{1.5}$$

Here $\Omega\triangle B = (\Omega \setminus B) \cup (B \setminus \Omega)$ is the symmetric difference of Ω and B. We observe that $\alpha[\Omega]$ is a dimensionless quantity, and that, by compactness, the above infimum is achieved by some (not necessarily unique) ball.

In our proof we use a quantitative version of Riesz's rearrangement inequality proved by Christ in [10] and we follow the strategy used by Frank and Lieb in [22] to address an optimization problem for the potential energy functional in $\mathbb{R}^d$ with interaction kernel $|x|^{-\lambda}$, $0 < \lambda < d$. This connection with physically relevant problems is actually not that surprising, since our issue can be seen as a similar problem for a potential energy with Gaussian interaction (cf. (4.1) below). We observe that this type of isoperimetric problems dates back at least to Poincaré [43] and still represents a challenging and very active research field (see, for example, [7, 8, 18, 22] and the references therein).

In Sect. 5 we analyze the optimality of (1.4) and in particular we prove that the exponent 2 of $\alpha[\Omega]$ and the behavior of $\beta(t)$ for $t \to 0^+$ are sharp, while for $t \to +\infty$ we conjecture that the estimate actually could hold with $\beta(t) = t^{2-1/2d}$ (Conjecture 5.1). This leads us to another conjecture (Conjecture 5.2), that is a refinement of Christ's result and seems of independent interest.

We now know, in particular, that both the operator norm and the Hilbert–Schmidt norm of L_Ω are maximized, among all subsets Ω of a given finite measure, when Ω is a ball. It is natural to wonder whether the same holds for other Schatten-von Neumann norms. We plan to investigate this issue, together with the above mentioned conjectures, in a subsequent work.

2 Notation and Preliminaries

In the following, we are going to denote the ball with center 0 and radius r in $\mathbb{R}^d$ or $\mathbb{R}^{2d}$ (depending on the context) as B_r. The d-dimensional Lebesgue measure of a subset $\Omega \subset \mathbb{R}^d$ will be denoted as $|\Omega|$. We set $\|f\|_p$ for the L^p norm of a function f. We define $L^1(\mathbb{R}^{2d}) + L^2(\mathbb{R}^{2d}) = \{f_1 + f_2 :\ f_1 \in L^1(\mathbb{R}^{2d}),\ f_2 \in L^2(\mathbb{R}^{2d})\}$. The Fourier transform of a function f will be denoted by $\widehat{f}$, according to the following definition:

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\omega\cdot x}\,dx.$$

Given a subset $\Omega \subset \mathbb{R}^d$ of finite measure, we denote by Ω^* the *symmetric rearrangement of the set* Ω, that is the open ball with center 0 and such that $|\Omega| = |\Omega^*|$. In the spirit of the layer cake representation, given a measurable function f on $\mathbb{R}^d$ we can also define the *symmetric decreasing rearrangement of* f as

$$f^*(x) = \int_0^{+\infty} \chi_{\{|f|>t\}^*}(x)\, dt.$$

The symmetric decreasing rearrangement f^* is the only function of the type $f^*(x) = g(|x|)$, with $g : [0 + \infty) \to [0, +\infty]$ decreasing and right-continuous (see [36]; here and throghout the paper by decreasing we mean nonincreasing), that has the same distribution function of f, and appears in various optimization problems and inequalities. In the following, we are going to use one of the main results in this sense, that is the *Riesz rearrangement inequality*.

Theorem 2.1 *[Riesz's rearrangement inequality]([36, Theorems 3.7, 3.9]) Let f, g and h be three nonnegative measurable functions on $\mathbb{R}^d$. Then we have*

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} & f(x)g(x-y)h(y)\, dxdy \\ & \le \int_{\mathbb{R}^d\times\mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)\, dxdy,\end{aligned} \tag{2.1}$$

with the understanding that if the left-hand side is $+\infty$ then also the right-hand side is. If, in addition, g is strictly symmetric decreasing and f and h are not zero and the above integrals are finite, equality occurs if and only $f(x) = f^*(x-y)$ and $h(x) = h^*(x-y)$ for almost every $x \in \mathbb{R}^d$ and some $y \in \mathbb{R}^d$.

By saying that g is strictly symmetric decreasing we mean that $g(x) = g^*(x) = \tilde{g}(|x|)$ for a.e. $x \in \mathbb{R}^d$ and that $\tilde{g} \colon [0, \infty) \to [0, \infty]$ is strictly decreasing.

Rearrangements are possible not only in the Euclidean setting, but also for other geometries. In the following, we will consider the Poincaré upper half-plane $\mathbb{R}\times\mathbb{R}_+ = \{(x, s) \in \mathbb{R}^2 \colon s > 0\} \simeq \mathbb{C}_+$, endowed with the hyperbolic distance

$$d_H(z, w) = 2\,\mathrm{arctanh}\left|\frac{z-w}{z-\overline{w}}\right|, \quad z, w \in \mathbb{C}_+$$

and the hyperbolic measure given by $d\nu = dxds/s^2$, that is the left Haar measure of $\mathbb{R}\times\mathbb{R}_+$ regarded as the affine ("$b + ax$") group. Recalling that the unit of the group is $(0, 1)$ we can easily define the symmetric rearrangement of a subset $E \subset \mathbb{R}\times\mathbb{R}_+$, like in the Euclidean case, as the ball $E^* = \{z \in \mathbb{C}_+ \colon d_H(z, (0, 1)) < r\}$, where r is chosen so that $\nu(E) = \nu(E^*)$. We point out that hyperbolic balls with center $(0, 1)$ and radius R are, as subsets of $\mathbb{R}^2$, Euclidean balls with center $(0, \cosh(R))$ and radius $\sinh(R)$. Then, given a nonnegative measurable function f on $\mathbb{C}_+$ we can define its symmetric decreasing rearrangement exactly as we have done in the Euclidean case.

In Sect. 6 we are going to need the hyperbolic version of Theorem 2.1, which holds with the proper adjustments also in the hyperbolic setting (see [4, Sect. 7.6]).

Theorem 2.2 *Let f, h be two nonnegative measurable functions on $\mathbb{C}_+$ and let $g\colon [0,+\infty) \to [0,+\infty)$ be decreasing. Then we have*

$$\int_{\mathbb{C}_+\times\mathbb{C}_+} f(z)g(d_H(z,w))h(w)\,d\nu(z)d\nu(w)$$
$$\leq \int_{\mathbb{C}_+\times\mathbb{C}_+} f^*(z)g(d_H(z,w))h^*(w)\,d\nu(z)d\nu(w),$$

with the understanding that if the left-hand side is $+\infty$ then also the right-hand side is. If, in addition, g is strictly symmetric decreasing and f and h are not zero and the above integrals are finite, equality occurs if and only $f(z) = f^*(az+b)$ and $h(z) = h^*(az+b)$ for almost every $z \in \mathbb{C}_+$ and some $a > 0$ and $b \in \mathbb{R}$.

Observe that $az + b$ is just the product (in the affine group) of (b, a) and z, where z is regarded as the element $(\mathrm{Re}z, \mathrm{Im}z)$.

3 Hilbert–Schmidt Norm of Localization Operators

3.1 A General Setting for Localization Operators

In this section we define localization operators in a general setting.

Given a σ-finite measure space (X, μ) and a separable Hilbert space $\mathcal{H}$, with norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$, we suppose to have a map $X \ni x \mapsto \varphi_x \in \mathcal{H}$ such that:

(i) The map $X \ni x \mapsto \langle f, \varphi_x\rangle$ is measurable for every $f \in \mathcal{H}$;
(ii) $\|\varphi_x\| \leq c_1$ for a.e. $x \in X$ and some constant $c_1 > 0$.

Then, for $F \in L^1(X)$, we consider the corresponding localization operator $L_F\colon \mathcal{H} \to \mathcal{H}$ defined (weakly) as

$$\langle L_F f, g\rangle = \int_X F(x)\langle f, \varphi_x\rangle\langle \varphi_x, g\rangle\,d\mu(x), \quad f, g \in \mathcal{H}. \tag{3.1}$$

From (ii) and since we supposed $F \in L^1(X)$ it is immediate to see that L_F is bounded and that $\|L_F\| \leq c_1^2\|F\|_{L^1(X)}$. In fact, it turns out that L_F is a trace class operator.

Proposition 3.1 *Under the assumptions (i) and (ii), if $F \in L^1(X)$ then L_F is trace class and its trace is given by*

$$\mathrm{tr}L_F = \int_X F(x)\|\varphi_x\|^2\,d\mu(x). \tag{3.2}$$

Moreover, for its Hilbert–Schmidt norm we have the formula

$$\|L_F\|^2_{\mathrm{HS}} = \int_{X\times X} F(x)|\langle\varphi_x, \varphi_y\rangle|^2\overline{F(y)}\,d\mu(x)d\mu(y). \tag{3.3}$$

Proof We start proving that L_F is a trace class operator and that (3.2) holds true when F is nonnegative.

Given an orthonormal basis f_j, $j = 1, 2, \dots,$ of $\mathcal{H}$, we have

$$\sum_{j=1}^{\infty} \langle L_F f_j, f_j \rangle = \int_X F(x) \sum_{j=1}^{\infty} |\langle f_j, \varphi_x \rangle|^2 \, d\mu(x)$$
$$= \int_X F(x) \|\varphi_x\|^2 \, d\mu(x) \overset{\text{(ii)}}{\leq} c_1^2 \int_X F(x) d\mu(x) = c_1^2 \|F\|_{L^1(X)},$$

where the exchange between summation and integral is allowed since everything is positive. The general case easily follows since (3.2) is linear in F and this can be decomposed as $F = [(\mathrm{Re} F)_+ - (\mathrm{Re} F)_-] + i[(\mathrm{Im} F)_+ - (\mathrm{Im} F)_+]$, where $(\cdot)_+$ and $(\cdot)_-$ denote the positive and negative part, respectively.

Now we prove (3.3). We have

$$\begin{aligned}
\|L_F\|_{\mathrm{HS}}^2 &= \sum_{j=1}^{\infty} \langle L_F f_j, L_F f_j \rangle \\
&= \sum_{j=1}^{\infty} \int_X F(x) \langle f_j, \varphi_x \rangle \langle \varphi_x, L_F f_j \rangle \, d\mu(x) \\
&= \sum_{j=1}^{\infty} \int_X F(x) \langle f_j, \varphi_x \rangle \overline{\int_X F(y) \langle f_j, \varphi_y \rangle \langle \varphi_y, \varphi_x \rangle \, d\mu(y)} \, d\mu(x) \\
&= \sum_{j=1}^{\infty} \int_{X \times X} \langle f_j, \varphi_x \rangle \overline{\langle f_j, \varphi_y \rangle} \langle \varphi_x, \varphi_y \rangle F(x) \overline{F(y)} \, d\mu(x) d\mu(y),
\end{aligned}$$

and the desired results follows by exchanging the summation and the integrals and noticing that

$$\sum_{j=1}^{\infty} \langle f_j, \varphi_x \rangle \overline{\langle f_j, \varphi_y \rangle} = \langle \varphi_y, \varphi_x \rangle.$$

This exchange is justified since, by the Cauchy–Schwarz inequality,

$$\sum_{j=1}^{\infty} |\langle f_j, \varphi_x \rangle \overline{\langle f_j, \varphi_y \rangle}| \leq \|\varphi_x\| \|\varphi_y\| \overset{\text{(ii)}}{\leq} c_1^2,$$

and therefore

$$\sum_{j=1}^{\infty} \int_{X \times X} |\langle f_j, \varphi_x \rangle| |\langle f_j, \varphi_y \rangle| |\langle \varphi_x, \varphi_y \rangle F(x) F(y)| \, d\mu(x) d\mu(y) \leq c_1^4 \|F\|_{L^1(X)}^2.$$

Proposition 3.2 *Assume, in addition to the hypotheses (i) and (ii), that the following Bessel type inequality holds for every $f \in \mathcal{H}$ and some $c_2 > 0$:*

$$\int_X |\langle f, \varphi_x \rangle|^2 \, d\mu(x) \le c_2^2 \|f\|^2. \tag{3.4}$$

Then, for $F \in L^2(X)$, L_F is a Hilbert–Schmidt operator and (3.3) holds true.

Proof We notice that the map $F \mapsto L_F$ is bounded from $L^2(X)$ into the space of linear bounded operators on $\mathcal{H}$ — which we will denote by $\mathcal{L}(\mathcal{H})$. Indeed, for every $g \in \mathcal{H}$ such that $\|g\| \le 1$ it holds

$$\begin{aligned} |\langle L_F f, g \rangle| &\le \int_X |F(x)||\langle f, \varphi_x \rangle||\langle \varphi_x, g \rangle| \, d\mu(x) \\ &\overset{\text{(ii)}}{\le} c_1 \|g\| \int_X |F(x)||\langle f, \varphi_x \rangle| \, d\mu(x) \\ &\le c_1 \|F\|_{L^2(X)} \left(\int_X |\langle f, \varphi_x \rangle|^2 \, d\mu(x) \right)^{1/2} \\ &\overset{(3.4)}{\le} c_1 c_2 \|F\|_{L^2(X)} \|f\|, \end{aligned}$$

and therefore

$$\|L_F f\| = \sup_{\|g\| \le 1} |\langle L_F f, g \rangle| \le c_1 c_2 \|F\|_{L^2(X)} \|f\|.$$

On the other hand, thanks to (3.4) we have

$$\operatorname{ess\,sup}_{y \in X} \int_X |\langle \varphi_x, \varphi_y \rangle|^2 \, d\mu(x) = \operatorname{ess\,sup}_{x \in X} \int_X |\langle \varphi_x, \varphi_y \rangle|^2 \, d\mu(y) \le c_1^2 c_2^2.$$

Hence, by Schur's test, we see that the right-hand side of (3.3) is a continuous quadratic form on $L^2(X)$. Therefore, if we take a sequence $F_n \in L^1(X) \cap L^2(X)$ that converges to F in $L^2(X)$, the sequence L_{F_n} is a Cauchy sequence in the space of Hilbert–Schmidt operators on $\mathcal{H}$, and therefore has a limit. Since $L_{F_n} \to L_F$ in $\mathscr{L}(\mathcal{H})$ due to the continuity of the map $F \mapsto L_F$, by the uniqueness of the limit we conclude that L_{F_n} converges to L_F also in the space of Hilbert–Schmidt operators on $\mathcal{H}$. Hence L_F is a Hilbert–Schmidt operator itself, for which (3.3) holds.

Remark 3.3 Assume, in place of (3.4), the stronger *resolution of the identity* formula

$$\int_X |\langle f, \varphi_x \rangle|^2 \, d\mu(x) = c \|f\|^2,$$

for some $c > 0$. Then the linear map $\mathcal{V} : \mathcal{H} \to L^2(X)$ given by

$$\mathcal{V} f(x) := \frac{1}{\sqrt{c}} \langle f, \varphi_x \rangle$$

is an isometry and its range is a reproducing kernel Hilbert space, i.e.:

$$\mathcal{V}f(x) = \frac{1}{\sqrt{c}}\langle \mathcal{V}f, \mathcal{V}\varphi_x\rangle_{L^2(X)} = \frac{1}{c}\int_X \mathcal{V}f(y)\langle \varphi_x, \varphi_y\rangle \, d\mu(y).$$

The above formula (3.3) was proved for particular reproducing kernel Hilbert spaces by several authors (see [3] and the references therein), in particular when F is the characteristic function of a subset $\Omega \subset X$ of finite measure. However, Proposition 3.1 shows that no reproducing property is in fact necessary in that case. Also, for Proposition 3.2 we only assumed the Bessel type inequality (3.4).

3.2 Time-Frequency Localization Operators

We are now switching our attention towards more classical time-frequency localization operators. Our measure space X is now $\mathbb{R}^d \times \mathbb{R}^d$ while the Hilbert space $\mathcal{H}$ is now $L^2(\mathbb{R}^d)$. Given the L^2-normalized Gaussian $\varphi(x) = 2^{d/4}e^{-\pi|x|^2}$, for any $z = (x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$ we consider the following function

$$\varphi_z(x) = e^{2\pi i \omega_0 \cdot x}\varphi(x - x_0), \quad x \in \mathbb{R}^d.$$

With this particular choice, given $f \in L^2(\mathbb{R}^d)$ the map $\mathbb{R}^{2d} \ni z \mapsto \langle f, \varphi_z\rangle$ is the usual short-time Fourier transform $\mathcal{V}f$ with Gaussian window as defined in (1.1), which is a continuous and therefore measurable function. Moreover $\|\varphi_z\|_{L^2} = 1$ for every $z \in \mathbb{R}^{2d}$ and $\mathcal{V} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is an isometry (see e.g. [27]). This means that the assumptions of Proposition 3.2 are satisfied and from a direct computation one can see that

$$|\langle \varphi_z, \varphi_w\rangle|^2 = e^{-\pi|z-w|^2}, \quad z, w \in \mathbb{R}^{2d}.$$

As a consequence of Propositions 3.1 and 3.2 we obtain, for $F \in L^1(\mathbb{R}^{2d}) + L^2(\mathbb{R}^{2d})$,

$$\|L_F\|^2_{\mathrm{HS}} = \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} F(z)e^{-\pi|z-w|^2}\overline{F(w)}\, dz dw. \tag{3.5}$$

We observe that the function $e^{-\pi t^2}$ for $t \geq 0$ is strictly decreasing.

Proposition 3.4 *Let $F \in L^1(\mathbb{R}^{2d}) + L^2(\mathbb{R}^{2d})$. Then*

$$\|L_F\|_{\mathrm{HS}} \leq \|L_{|F|^*}\|_{\mathrm{HS}},$$

where $|F|^*$ is the symmetric decreasing rearrangement of $|F|$. Equality occurs if and only if $F(z) = e^{i\theta}\rho(|z - z_0|)$ for a.e. $z \in \mathbb{R}^{2d}$ for some $\theta \in \mathbb{R}$, $z_0 \in \mathbb{R}^{2d}$ and some decreasing function $\rho\colon [0, +\infty) \to [0, +\infty)$.

Proof By (3.5) and Riesz's rearrangement inequality (Theorem 2.1) we have

$$\|L_F\|^2_{\mathrm{HS}} = \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} F(z)e^{-\pi|z-w|^2}\overline{F(w)}\, dz dw$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |F(z)|e^{-\pi|z-w|^2}|F(w)|\,dzdw \\
&\leq \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |F|^*(z)e^{-\pi|z-w|^2}|F|^*(w)\,dzdw.
\end{aligned}$$

The first inequality becomes an equality if and only if

$$F(z)\overline{F(w)} = |F(z)||F(w)|$$

for a.e. $z, w \in \mathbb{R}^{2d}$, which means that $F(z) = e^{i\theta}|F(z)|$ a.e. in $\mathbb{R}^{2d}$ for some $\theta \in \mathbb{R}$.

The second inequality is an equality if and only if $|F(z - z_0)|$ is symmetric decreasing for some $z_0 \in \mathbb{R}^{2d}$ (Theorem 2.1).

Given $\Omega \subset \mathbb{R}^{2d}$ of finite measure, we write L_Ω for L_{χ_Ω}. Since Ω has finite measure, we have $\chi_\Omega \in L^1(\mathbb{R}^{2d})$, hence we have the following corollary.

Corollary 3.5 *Let $\Omega \subset \mathbb{R}^{2d}$ be a subset of finite measure. Then*

$$\|L_\Omega\|_{\mathrm{HS}} \leq \|L_{\Omega^*}\|_{\mathrm{HS}},$$

where $\Omega^* \subset \mathbb{R}^{2d}$ is the open ball centered at 0 and measure $|\Omega^*| = |\Omega|$. Equality occurs if and only if Ω is (equivalent, up to a set of measure zero, to) a ball.

Remark 3.6 The quantity $\|L_{B_r}\|_{\mathrm{HS}}$ can be "explicitly" computed in terms of Bessel functions:

$$\begin{aligned}
\|L_{B_r}\|_{\mathrm{HS}}^2 &= \int_{\mathbb{R}^{2d}} \chi_{B_r}(z)\left(\int_{\mathbb{R}^{2d}} e^{-\pi|z-w|^2}\chi_{B_r}(w)\,dw\right)dz \\
&= \int_{\mathbb{R}^{2d}} \chi_{B_r}(z)(e^{-\pi|\cdot|^2} * \chi_{B_r})(z)\,dz \\
&\overset{\text{Parseval}}{=} \int_{\mathbb{R}^{2d}} |\widehat{\chi_{B_r}}(w)|^2 e^{-\pi|w|^2}\,dw
\end{aligned}$$

and $\widehat{\chi_{B_r}}(w)$ is given by (see [47, p. 324])

$$\widehat{\chi_{B_r}}(w) = 2\pi|w|^{-d+1}\int_0^r J_{d-1}(2\pi|w|R)R^d\,dR,$$

where J_{d-1} is the Bessel function of order $d-1$.

Remark 3.7

(a) Formula (3.5) can also be obtained by observing that L_F can be written as pseudodifferential operator with Weyl symbol $a(z)$, $z \in \mathbb{R}^{2d}$, given by

$$a = F * \Phi, \quad \Phi(z) = 2^d e^{-2\pi|z|^2},\ z \in \mathbb{R}^{2d},$$

see e.g. [48]. Hence,

$$\|L_F\|_{\mathrm{HS}}^2 = \|a\|_2^2 = \int_{\mathbb{R}^{2d}} e^{-\pi|w|^2} |\widehat{F}(w)|^2 \, dw = \int_{\mathbb{R}^{2d}} F(z) e^{-\pi|z-w|^2} \overline{F(w)} \, dz dw.$$

(b) Formula (3.5) can be written equivalently as

$$\|L_F\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}^{2d}} (F * e^{-\pi|\cdot|^2})(z) \overline{F(z)} \, dz.$$

Therefore, using the Cauchy–Schwarz inequality, Young's inequality and the fact that $\int_{\mathbb{R}^{2d}} e^{-\pi|z|^2} \, dz = 1$ it follows that

$$\|L_F\|_{\mathrm{HS}} \leq \|F\|_2,$$

which is a well known result (see e.g. [48]). However, this strategy also shows that equality can never occur if $F \neq 0$, because equality would imply $\hat{F}(w) = c\hat{F}(w)e^{-\pi|w|^2}$ for some $c \geq 0$. Also, observe that

$$\sup_{F \in L^2(\mathbb{R}^{2d}) \setminus \{0\}} \frac{\|L_F\|_{\mathrm{HS}}}{\|F\|_2} = 1,$$

as one sees by taking $F = \chi_{B_r}$ and letting $r \to +\infty$ (we leave the easy computation to the interested reader).

(c) Consider the so-called Schatten-von Neumann class $\mathcal{S}_p$, $1 \leq p < \infty$, constituted of the compact operators S on $L^2(\mathbb{R}^d)$ whose sequence of singular values σ_j, $j = 1, 2, \ldots$, belongs to ℓ^p, equipped with the norm $\|S\|_{\mathcal{S}_p} := \left(\sum_{j=1}^\infty \sigma_j^p\right)^{1/p}$. In particular, for $p = 2$ we have the class of Hilbert–Schmidt operators, with equal norm. For $p = \infty$ we set $\mathcal{S}_\infty = \mathcal{L}(L^2(\mathbb{R}^d))$, that is the set of all linear bounded operators on $L^2(\mathbb{R}^d)$. It is well known (see e.g. [48]) that

$$\|L_F\|_{\mathcal{S}_p} \leq \|F\|_p \tag{3.6}$$

for all $1 \leq p \leq \infty$. In fact, for $1 \leq p \leq \infty$ it holds

$$\sup_{F \in L^p(\mathbb{R}^{2d}) \setminus \{0\}} \frac{\|L_F\|_{\mathcal{S}_p}}{\|F\|_p} = 1.$$

Indeed, if the above supremum were strictly less than 1 for some $p_0 \in [1, 2)$, interpolating with the estimate (3.6) with $p = \infty$ would give

$$\|L_F\|_{\mathcal{S}_2} = \|L_F\|_{\mathrm{HS}} \leq C\|F\|_2,$$

for some $C < 1$, thus contradicting the previous remark. On the other hand, if the supremum were strictly less than 1 for some $p_0 \in (2, +\infty]$ one argues similarly by interpolating with (3.6) with $p = 1$.

4 Quantitative Estimate

In the previous section we proved an estimate for the Hilbert–Schmidt norm of time-frequency localization operators. In particular, for operators of the type L_Ω we proved that $\|L_\Omega\|_{\mathrm{HS}}$ is maximized, among all subsets Ω of a given finite measure, when Ω is a ball and the balls are the only maximizers (see Corollary 3.5). In this section we focus our attention on a quantitative version of Corollary 3.5. Roughly speaking, we want to prove that the difference $\|L_{\Omega^*}\|_{\mathrm{HS}}^2 - \|L_\Omega\|_{\mathrm{HS}}^2$ is bounded from below by some function of the set Ω which measures how much Ω differs from a ball, which implies that if the above "deficit" is small then Ω is "almost" a ball. The notion of Ω being close to a ball is made precise thanks to the *Fraenkel asymmetry index* $\alpha[\Omega]$ as defined in (1.5).

From (3.5) we have

$$\|L_\Omega\|_{\mathrm{HS}}^2 = \int_{\Omega\times\Omega} e^{-\pi|z-w|^2}\,dz\,dw, \tag{4.1}$$

and in the previous section we used Riesz's rearrangement inequality to prove that the right-hand side increases if Ω is replaced by Ω^*. To obtain a lower bound for $\|L_{\Omega^*}\|_{\mathrm{HS}}^2 - \|L_\Omega\|_{\mathrm{HS}}^2$ we will use a quantitative version of Riesz's rearrangement inequality, that was proved by Christ [10]; see also Frank and Lieb [19, Theorem 1] for a generalization to density functions.

Theorem 4.1 *Let $\delta \in (0, 1/2)$. Then, there exists a constant $c_{d,\delta}$ such that for all balls $B \subset \mathbb{R}^d$ centered at the origin, all $\Omega \subset \mathbb{R}^d$ such that*

$$\delta < \frac{|B|^{1/d}}{2|\Omega|^{1/d}} < 1-\delta$$

we have

$$\int_{\Omega\times\Omega} \chi_B(x-y)\,dxdy \le \int_{\Omega^*\times\Omega^*} \chi_B(x-y)\,dxdy - c_{d,\delta}|\Omega|^2\alpha[\Omega]^2. \tag{4.2}$$

We have therefore the following result.

Proposition 4.2 *For every subset $\Omega \subset \mathbb{R}^{2d}$ of positive finite measure it holds*

$$\|L_\Omega\|_{\mathrm{HS}}^2 \le \|L_{\Omega^*}\|_{\mathrm{HS}}^2 - c_1\beta(|\Omega|)\alpha[\Omega]^2, \tag{4.3}$$

where

$$\beta(t) = \begin{cases} t^{2+\frac{1}{d}}, & \text{for } 0 < t \le 1 \\ t^2 e^{-c_2 t^{1/d}}, & \text{for } t > 1 \end{cases}$$

for some constants $c_1, c_2 > 0$ depending only on d.

Proof We follow the same strategy as in Frank and Lieb [22, Theorem 4].

Using the formula (4.1) and the fact that

$$e^{-\pi t^2} = \int_0^{+\infty} \chi_{(0,R)}(t) 2\pi R e^{-\pi R^2}\, dR$$

we obtain

$$\begin{aligned}&\|L_{\Omega^*}\|_{\mathrm{HS}}^2 - \|L_\Omega\|_{\mathrm{HS}}^2 \\ &= \int_0^{+\infty} \left(\int_{\Omega^*\times\Omega^*} \chi_{B_R}(z-w)\, dz dw - \int_{\Omega\times\Omega} \chi_{B_R}(z-w)\, dz dw \right) 2\pi R e^{-\pi R^2}\, dR.\end{aligned}$$

Letting

$$I := \left\{ R > 0\colon \frac{1}{4} < \frac{|B_R|^{1/2d}}{2|\Omega|^{1/2d}} < \frac{3}{4} \right\}$$

we can use Christ's result (Theorem 4.1) with $\delta = 1/4$, thus obtaining

$$\begin{aligned}\|L_{\Omega^*}\|_{\mathrm{HS}}^2 - \|L_\Omega\|_{\mathrm{HS}}^2 &\geq c|\Omega|^2 \alpha[\Omega]^2 \int_I 2\pi R e^{-\pi R^2}\, dR \\ &= c|\Omega|^2 \alpha[\Omega]^2 \left(e^{-c_1'|\Omega|^{1/d}} - e^{-c_2'|\Omega|^{1/d}} \right),\end{aligned}$$

where c is the constant from Theorem 4.1, and $c_1' = \frac{\pi}{4|B_1|^{1/d}}$ and $c_2' = \frac{9\pi}{4|B_1|^{1/d}}$ are constants that depend only on d.

To highlight the behavior of the latter expression as $|\Omega| \to 0^+$ or $|\Omega| \to +\infty$, in the statement we introduced the function $\beta(t)$ which satisfies $C^{-1}\beta(t) \leq t^2(e^{-c_1' t^{1/d}} - e^{-c_2' t^{1/d}}) \leq C\beta(t)$ for some constant $C > 0$ depending in d.

5 Some Remarks on the Sharpness of Proposition 4.2

5.1 Sharpness of the Power $\alpha[\Omega]^2$

In this section we prove that the power $\alpha[\Omega]^2$ appearing in (4.3) is optimal, in the sense that we cannot take any exponent less than 2.

To this end, for $0 < \varepsilon < 1$ let

$$\Omega_\varepsilon := \{z \in \mathbb{R}^{2d} \colon |z| \leq 1 - \varepsilon \text{ or } 1 \leq |z| \leq 1 + \delta\},$$

where $\delta > 0$ is chosen so that $|\Omega_\varepsilon| = |B_1|$. This implies that $\delta = \delta(\varepsilon)$ depends on ε and from the implicit function theorem we see that

$$\delta(\varepsilon) = \varepsilon + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+.$$

Using the formula (4.1) and the fact that $\chi_{\Omega_\varepsilon} = \chi_{B_1} + \chi_{\Omega_\varepsilon} - \chi_{B_1}$ we obtain:

$$\begin{aligned}\|L_{B_1}\|_{\mathrm{HS}}^2 - \|L_{\Omega_\varepsilon}\|_{\mathrm{HS}}^2 = 2\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} &(\chi_{B_1} - \chi_{\Omega_\varepsilon})(z) e^{-\pi|z-w|^2} \chi_{B_1}(w)\, dz dw \\ - \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} &(\chi_{\Omega_\varepsilon} - \chi_{B_1})(z) e^{-\pi|z-w|^2} (\chi_{\Omega_\varepsilon} - \chi_{B_1})(w)\, dz dw.\end{aligned}$$

The second integral can be easily estimated as follows:

$$\left|\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} (\chi_{\Omega_\varepsilon} - \chi_{B_1})(z) e^{-\pi|z-w|^2} (\chi_{\Omega_\varepsilon} - \chi_{B_1})(w)\, dz dw\right| \le |\Omega_\varepsilon \triangle B_1|^2 = O(\varepsilon^2),$$

while the first integral can be written in the following way:

$$\begin{aligned}&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} (\chi_{B_1} - \chi_{\Omega_\varepsilon})(z) e^{-\pi|z-w|^2} \chi_{B_1}(w)\, dz dw \\ &\qquad = \int_{\mathbb{R}^{2d}} \left(\int_{B_1} e^{-\pi|z-w|^2}\, dw\right) (\chi_{B_1} - \chi_{\Omega_\varepsilon})(z)\, dz.\end{aligned}$$

We notice that the inner function is radial, so letting

$$f(|z|) := \int_{B_1} e^{-\pi|z-w|^2}\, dw$$

and using polar coordinates we see that this integral can be written (up to a multiplicative constant that depends only on d) as

$$\int_{1-\varepsilon}^{1} f(r) r^{2d-1}\, dr - \int_{1}^{1+\delta} f(r) r^{2d-1}\, dr.$$

Since f is smooth, we have that this difference is equal to

$$\varepsilon f(1) + O(\varepsilon^2) - \delta f(1) + O(\delta^2) \overset{\delta=\varepsilon+O(\varepsilon^2)}{=} O(\varepsilon^2).$$

Hence, by (4.3) we see that, for some $C > 0$,

$$C\varepsilon^2 \ge \|L_{B_1}\|_{\mathrm{HS}}^2 - \|L_{\Omega_\varepsilon}\|_{\mathrm{HS}}^2 \ge c_1 \alpha[\Omega_\varepsilon]^2.$$

Now, we claim that $\alpha[\Omega_\varepsilon] \ge c_d \varepsilon$ (where c_d is a constant that depends only d) and therefore the exponent of $\alpha[\Omega]$ in (4.3) cannot be replaced by any number smaller than 2.

To prove the claim, we notice that, up to rotations, it suffices to bound from below the quantity $|\Omega_\varepsilon \triangle B|$ when $B = B_1(xe_1)$, that is the ball of center xe_1 and radius 1, where e_1 is the first unit vector of the canonical basis and $x \ge 0$. Then, we have that $|\Omega_\varepsilon \triangle B_1(xe_1)| \ge |\Omega_\varepsilon \setminus B_1(xe_1)|$ and it is clear that $\Omega_\varepsilon \setminus B_1(xe_1)$ contains all the points of Ω_ε that are in the annulus $\{1 \leq |z| \leq 1+\delta\}$ and whose first component is negative. Since the measure of this set is half of the measure of the annulus, we have

$$\alpha[\Omega_\varepsilon] \geq \frac{1}{2}|B_1|[(1+\delta)^{2d}-1] = \frac{1}{2}|B_1|(2d\delta + o(\delta)) \overset{\delta=\varepsilon+O(\varepsilon^2)}{\geq} c_d\varepsilon.$$

5.2 Sharpness of the Power $t^{2+1/d}$ for $0 < t < 1$

Having proved that the exponent of $\alpha[\Omega]$ in (4.3) is optimal, we can prove that the behavior of the function $\beta(t)$ in (4.2) is optimal for $0 < t < 1$, i.e. the power $t^{2+1/d}$ is sharp.

To this end, we consider any subset $\Omega \subset \mathbb{R}^{2d}$ of positive finite measure, which is *not* (equivalent, up to set of measure zero, to) a ball. Then, if we consider the dilation of Ω by a factor $r > 0$, namely $\Omega_r = \{z \in \mathbb{R}^{2d} : \frac{z}{r} \in \Omega\}$, we have that the asymmetry index $\alpha[\Omega_r]$ is not zero and independent of r.

By (4.1), letting $g(|z|) = 1 - e^{-\pi|z|^2}$ and using the fact that $|\Omega_r^*| = |\Omega_r|$ we have:

$$\begin{aligned}
\|L_{\Omega_r^*}\|_{\mathrm{HS}}^2 - \|L_{\Omega_r}\|_{\mathrm{HS}}^2 &= \int_{\Omega_r^* \times \Omega_r^*} e^{-\pi|z-w|^2}\, dz dw - \int_{\Omega_r \times \Omega_r} e^{-\pi|z-w|^2}\, dz dw \\
&= \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} g(|z-w|) \left[\chi_{\Omega_r}(z)\chi_{\Omega_r}(w) - \chi_{\Omega_r^*}(z)\chi_{\Omega_r^*}(w)\right] dz dw \\
&\leq \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} g(|z-w|)\chi_{\Omega_r}(z)\chi_{\Omega_r}(w)\, dz dw. \\
&\leq cr^2|\Omega_r|^2 = c'|\Omega_r|^{2+1/d},
\end{aligned}$$

where we used the fact that $0 \leq g(r) \leq cr^2$.

Hence, if (4.3) holds true then

$$c\beta(|\Omega_r|)\alpha[\Omega_r]^2 \leq \|L_{\Omega_r^*}\|_{\mathrm{HS}}^2 - \|L_{\Omega_r}\|_{\mathrm{HS}}^2 \leq c'|\Omega_r|^{2+1/d},$$

which is possible if and only if $\beta(t) \leq c'' t^{2+1/d}$ for t small, since, as already observed, $\alpha[\Omega_r] > 0$ is independent of r.

5.3 The Behavior of $\beta(t)$ as $t \to +\infty$

In this section we are concerned with the behavior of the function $\beta(t)$ in Proposition 4.2 as $t \to +\infty$, which is probably not sharp.

Precisely, we claim that if the inequality

$$\|L_\Omega\|_{\mathrm{HS}}^2 \leq \|L_{\Omega^*}\|_{\mathrm{HS}}^2 - c_1\beta(|\Omega|)\alpha[\Omega]^2 \tag{5.1}$$

holds for some constant $c > 0$ and some function $\beta(t)$, and every subset $\Omega \subset \mathbb{R}^{2d}$ of finite large measure, then

$$\beta(t) \leq ct^{1-1/2d} \qquad \text{as } t \to +\infty.$$

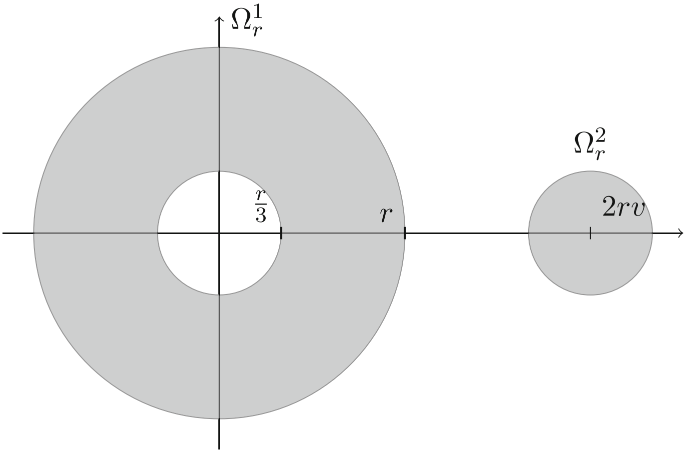

Fig. 1 Representation of Ω_r with $d = 1$ and $v = e_1$

Indeed, fix any $v \in \mathbb{R}^{2d}$ with $|v| = 1$ and let $\Omega_r = \Omega_r^1 \cup \Omega_r^2$, where

$$\Omega_r^1 = \left\{ z \in \mathbb{R}^{2d} : \frac{r}{3} \le |z| \le r \right\},$$
$$\Omega_r^2 = \left\{ z \in \mathbb{R}^{2d} : |z - 2rv| < \frac{r}{3} \right\}$$

(see Fig. 1).

For the sake of brevity, we let

$$I(f, g) := \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} f(z) e^{-\pi |z-w|^2} \overline{g(w)}\, dz dw.$$

Using (4.1) we have:

$$\|L_{\Omega_r^*}\|_{\text{HS}}^2 - \|L_{\Omega_r}\|_{\text{HS}}^2 = I(\chi_{\Omega_r^*}, \chi_{\Omega_r^*}) - I(\chi_{\Omega_r}, \chi_{\Omega_r}).$$

We notice that $\Omega_r^* = B_r$ so, using the fact that $\chi_{B_r} = \chi_{B_{r/3}} + \chi_{\Omega_r^1}$, $\chi_{\Omega_r} = \chi_{\Omega_r^1} + \chi_{\Omega_r^2}$ and that $I(\chi_{\Omega_r^2}, \chi_{\Omega_r^2}) = I(\chi_{B_{r/3}}, \chi_{B_{r/3}})$ we obtain

$$\|L_{\Omega_r^*}\|_{\text{HS}}^2 - \|L_{\Omega_r}\|_{\text{HS}}^2 = 2I(\chi_{\Omega_r^1}, \chi_{B_{r/3}}) - 2I(\chi_{\Omega_r^1}, \chi_{\Omega_r^2}) \le 2I(\chi_{\Omega_r^1}, \chi_{B_{r/3}}).$$

The function in the right-hand side can be written in the following way:

$$I(\chi_{\Omega_r^1}, \chi_{B_{r/3}}) = \int_{\mathbb{R}^{2d}} e^{-\pi |z|^2} \left(\chi_{\Omega_r^1} * \chi_{B_{r/3}} \right)(z)\, dz.$$

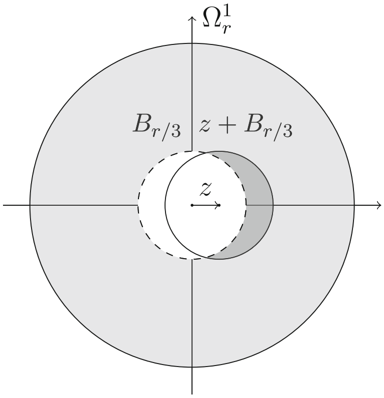

Fig. 2 The area of the darker gray region is $(\chi_{\Omega_r^1} * \chi_{B_{r/3}})(z)$

The function $\chi_{\Omega_r^1} * \chi_{B_{r/3}}$ is radial and it holds

$$(\chi_{\Omega_r^1} * \chi_{B_{r/3}})(z) \leq cr^{2d-1} \min\left\{|z|, \frac{2r}{3}\right\},$$

see Fig. 2 for a graphical intuition.

This implies that

$$I(\chi_{\Omega_r^1}, \chi_{B_{r/3}}) \leq cr^{2d-1} \int_{\mathbb{R}^{2d}} |z| e^{-\pi|z|^2}\, dz \leq c' r^{2d-1} \leq c'' |\Omega_r|^{1-1/2d}.$$

Since the Fraenkel index $\alpha[\Omega_r]$ is not zero and independent of r the above claim is proved.

One can see that the above example also "saturates" the analogous quantitative bound in $\mathbb{R}^d$ for the interaction kernel $|x|^{-\lambda}$, with $0 < \lambda < d$ [22, Theorem 4]. These facts and some further experimentation suggest the following conjecture.

Conjecture 5.1 For every subset $\Omega \subset \mathbb{R}^{2d}$ of finite positive measure it holds

$$\|L_\Omega\|_{\mathrm{HS}}^2 \leq \|L_{\Omega^*}\|_{\mathrm{HS}}^2 - c\tilde{\beta}(|\Omega|)\alpha[\Omega]^2, \tag{5.2}$$

where

$$\tilde{\beta}(t) = \begin{cases} t^{2+\frac{1}{d}}, & \text{for } 0 < t \leq 1 \\ t^{1-1/2d}, & \text{for } t > 1 \end{cases}$$

for some constant $c > 0$ depending only on d.

In view of the above connection with Christ's result (Theorem 4.1), this also suggests the following conjecture, of independent interest. We observe that Frank and Lieb arrived at the same conjecture when working on their papers [19, 22] (private communication).

Conjecture 5.2 Let $0 < \delta < 1$. There exists a constant $c_{d,\delta} > 0$ such that, for all balls $B \subset \mathbb{R}^d$ centered at the origin, and all subsets $\Omega \subset \mathbb{R}^d$ of finite measure satisfying

$$\frac{|B|^{1/d}}{2|\Omega|^{1/d}} \leq 1 - \delta$$

we have

$$\frac{1}{2}\int_{\Omega\times\Omega} \chi_B(x-y)\,dx\,dy \leq \frac{1}{2}\int_{\Omega^*\times\Omega^*} \chi_B(x-y)\,dx\,dy - c_{d,\delta}(|B|/|\Omega|)^{1+1/d}|\Omega|^2\,\alpha[\Omega]^2.$$

Indeed, arguing as in the proof of Proposition 4.2 it is easy to see that Conjecture 5.2 implies Conjecture 5.1. Also, in dimension $d = 1$ Conjecture 5.2 was positively solved by Christ [11, Theorem 2.4].

Remark 5.3 Rupert Frank pointed out to us that Conjecture 5.2 easily implies the sharp isoperimetric inequality in quantitative form (first proved by Fusco et al. [23] by rearrangement techniques), that is

$$P(\Omega) \geq P(\Omega^*) + c_d|\Omega|^{\frac{d-1}{d}}\alpha[\Omega]^2$$

for $0 < |\Omega| < \infty$, where $P(\Omega)$ is the (distributional) perimeter of Ω.

Indeed, it follows from [14] that for every bounded measurable set $\Omega \subset \mathbb{R}^d$ of finite perimeter we have

$$\lim_{s\to 1^-}(1-s)P_s(\Omega) = K_d P(\Omega) \tag{5.3}$$

where, for $s \in (0, 1)$,

$$P_s(\Omega) := \int_\Omega\int_{\Omega^c} \frac{1}{|x-y|^{d+s}}\,dx\,dy$$

is the fractional s-perimeter of Ω and $K_d > 0$ is a constant depending only on the dimension. Hence it is sufficient to obtain a suitable quantitative isoperimetric inequality for P_s, that is uniform in s as $s \to 1^-$. To this end, one can argue as in the proof of Proposition 4.2. Precisely, using the conjecture one arrives at integrating R^{-s} on the interval $(0, c'_d|\Omega|^{1/d})$, which gives

$$P_s(\Omega) \geq P_s(\Omega^*) + \frac{c_d}{1-s}|\Omega|^{\frac{d-s}{d}}\alpha[\Omega]^2$$

uniformly with respect to $s \in (0, 1)$, namely with an (explicit) constant $c_d > 0$ independent of s, which is precisely what we need.

We observe that the latter inequality was proved by Figalli, Fusco, Maggi, Millot and Morini [18] for $s \in (s_0, 1)$, for any $s_0 \in (0, 1)$, with a (non-explicit) constant $c(d, s_0)$ depending also on s_0 (in place of c_d above).

6 The Hilbert–Schmidt Norm of Wavelet Localization Operators

In this section we are going to prove that a result analogous to Corollary 3.5 holds also for wavelet localization operators. Just for this section, we consider the following normalization for the Fourier transform (which is a common choice in the wavelet literature):

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t}\, dt.$$

The role that was previously played by the Gaussian window is now taken by the so-called Cauchy wavelet ψ_β, which for $\beta > 0$ is defined by

$$\widehat{\psi_\beta}(\omega) = \frac{1}{c_\beta} \chi_{[0,+\infty)}(\omega) \omega^\beta e^{-\omega}, \tag{6.1}$$

where $c_\beta > 0$ is given by $c_\beta^2 = 2\pi 2^{-2\beta}\Gamma(2\beta)$ and is chosen so that $\|\widehat{\psi_\beta}\|^2_{L^2(\mathbb{R}_+, d\omega/\omega)} = 1/(2\pi)$ (where $\mathbb{R}_+ = (0, +\infty)$). In this context, the Hilbert space $\mathcal{H}$ is given by the Hardy space $H^2(\mathbb{R})$, that is the space of functions of $L^2(\mathbb{R})$ whose Fourier transform is supported in $[0, +\infty)$, endowed with the L^2-norm. In particular, it holds that $\psi_\beta \in H^2(\mathbb{R})$ for every $\beta > 0$. The "coherent states" here are given by

$$\varphi_z(t) := \pi(z)\psi_\beta(t) := \frac{1}{\sqrt{s}} \psi_\beta\left(\frac{t-x}{s}\right), \quad t \in \mathbb{R},$$

with $z = (x, s) \in \mathbb{R} \times \mathbb{R}_+$, regarded as the hyperbolic space introduced in Sect. 2. We remark that $\pi(z)$ is a unitary representation on $H^2(\mathbb{R})$ of the affine ("$b + ax$") group. The transform related with these coherent states is the *wavelet transform* $\mathcal{W}_{\psi_\beta}$, which for $f \in H^2(\mathbb{R})$ is given by

$$\mathcal{W}_{\psi_\beta} f(z) := \frac{1}{\sqrt{s}} \int_{\mathbb{R}} f(t) \overline{\psi_\beta\left(\frac{t-x}{s}\right)}\, dt, \quad z = (x, s) \in \mathbb{R} \times \mathbb{R}_+.$$

With the above normalization of ψ_β, $\mathcal{W}_{\psi_\beta} : H^2(\mathbb{R}) \to L^2(\mathbb{R} \times \mathbb{R}_+, d\nu)$ is an isometry, with $d\nu = dx\, ds/s^2$.

Now, in this context, the kernel appearing in (3.3) is given by

$$|\langle \varphi_z, \varphi_w \rangle|^2 = |\langle \psi_\beta, \pi(z^{-1} \circ w)\psi_\beta \rangle|^2, \quad z, w \in \mathbb{R} \times \mathbb{R}_+,$$

where $\circ$ is the product in the affine group ($z \circ w = (x, s) \circ (y, t) = (x + ys, st)$) while z^{-1} is the inverse of z in the group ($z^{-1} = (-x/s, 1/s)$).

An essential property for the proof of Proposition 3.4 was that the integral kernel in (3.5) depends only on the distance $|z - w|$ and the function $e^{-\pi t^2}$ appearing in it is strictly decreasing. A similar fact holds true in this context, namely $|\langle \psi_\beta, \pi(z)\psi_\beta \rangle|^2 = |\mathcal{W}_{\psi_\beta} \psi_\beta(z)|^2$ is strictly symmetric decreasing around $(0, 1)$ (unit of the group) with respect to the Poincaré metric of $\mathbb{R} \times \mathbb{R}_+$.

Indeed, we have:

$$\begin{aligned}
\mathcal{W}_{\psi_\beta}\psi_\beta(z) &= \frac{1}{\sqrt{s}}\int_{\mathbb{R}} \psi_\beta(t)\overline{\psi_\beta\left(\frac{t-x}{s}\right)}\,dt \\
&= \sqrt{s}\int_{\mathbb{R}} \widehat{\psi_\beta}(\omega)\overline{e^{-ix\omega}\widehat{\psi_\beta}(s\omega)}\,d\omega \\
&= \frac{s^{\beta+\frac{1}{2}}}{c_\beta^2}\int_0^{+\infty} \omega^{2\beta}e^{-[(1+s)-ix]\omega}\,d\omega \\
&= \frac{s^{\beta+\frac{1}{2}}}{c_\beta^2}[(1+s)-ix]^{-2\beta-1}\Gamma(2\beta+1),
\end{aligned}$$

hence, for some $C > 0$, it holds

$$|\mathcal{W}_{\psi_\beta}\psi_\beta(z)|^2 = Cs^{2\beta+1}\left[(1+s)^2+x^2\right]^{-2\beta-1}.$$

On the other hand, identifying $\mathbb{R}\times\mathbb{R}_+$ with $\mathbb{C}_+$ via $z = x+is$, we have

$$\frac{4s}{(1+s)^2+x^2} = 1-\left|\frac{z-i}{z+i}\right|^2,$$

therefore

$$|\mathcal{W}_{\psi_\beta}\psi_\beta(z)|^2 = \varrho(d_H(z,i)),$$

where $\varrho(t) = C[1-\tanh(t/2)]$ is a strictly decreasing function $[0,+\infty)\to[0,+\infty)$.

Hence, for the kernel in (3.3) we have

$$|\langle\varphi_z,\varphi_w\rangle|^2 = |\langle\psi_\beta,\pi(z^{-1}\circ w)\psi_\beta\rangle|^2 = |\mathcal{W}_{\psi_\beta}(z^{-1}\circ w)|^2 = \varrho(d_H(z,w)).$$

We can therefore state the analog of Proposition 3.4 for wavelet localization operators. The proof is exactly the same, with the only difference that one has to use Theorem 2.2 in place of Theorem 2.1.

Proposition 6.1 *Let $F\in L^1(\mathbb{C}_+,\nu)+L^2(\mathbb{C}_+,\nu)$. Then*

$$\|L_F\|_{\mathrm{HS}} \le \|L_{|F|^*}\|_{\mathrm{HS}},$$

where $|F|^*$ is the symmetric decreasing rearrangement of $|F|$. Equality occurs if and only if $F(z) = e^{i\theta}|F|^*(az+b)$ for almost every $z\in\mathbb{C}_+$ and some $a>0$ and $b\in\mathbb{R}$.

Appendix A Remarks on the Uniqueness of the Extremizers in Riesz's Rearrangement Inequality

In Theorem 2.1 the cases of equality in Riesz's rearrangement inequality were characterized under the hypothesis that g is *strictly* decreasing (see also [6] for a complete

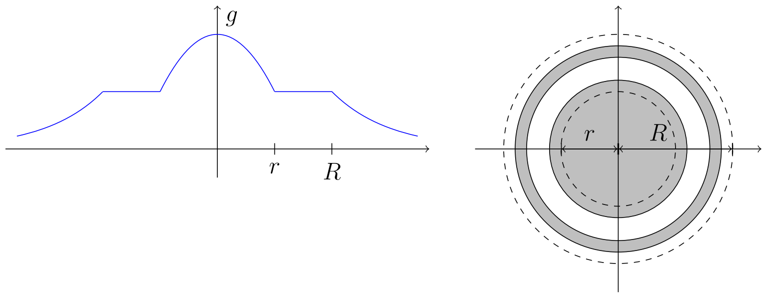

Fig. 3 Example of g that is symmetric decreasing but not strictly (left, where $g(x_1, 0)$ is represented) and the corresponding set Ω in dimension $d = 2$ (right)

study of the cases of equality). We also stressed this fact in the proof of Proposition 3.4.

In this appendix, for the benefit of the non-expert reader, we would like to clarify the necessity of this hypothesis by showing that if g is not strictly decreasing we can *always* find characteristic functions f and h that achieve equality in (2.1) even though f (say) is not, up to a translation, symmetric decreasing.

If we assume that g is decreasing but not strictly then there exist $0 \leq r < R$ such that g is constant on the annulus $B_R \setminus B_r$ (where $B_r = \emptyset$ if $r = 0$). Then, we consider the following set

$$\Omega = \{x \in \mathbb{R}^d \colon |x| < r + 2\delta \text{ or } R - 4\delta < |x| < R - 2\delta\},$$

where δ is small enough to have $r + 2\delta < R - 4\delta$ and we set

$$f = \chi_\Omega, \qquad h = \chi_{B_\delta}.$$

Since 3 $h(-y) = h(y)$, the left-hand side of (2.1) can be written as follows:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(x-y) h(y)\, dx dy = \int_{\mathbb{R}^d} g(x) (f * h)(x)\, dx.$$

Now we observe that (obviously) $h^* = h$ and that the functions $(f * h)(x)$ and $(f^* * h^*)(x)$

- Both vanish on $\mathbb{R}^d \setminus B_R$;
- Coincide on B_r;
- Have the same integral: $\int_{\mathbb{R}^d} (f * h)(x)\, dx = \int_{\mathbb{R}^d} (f^* * h^*)(x)\, dx = |B_\delta| \cdot |\Omega|$.

As a consequence, we have

$$\int_{B_R \setminus B_r} (f * h)(x)\, dx = \int_{B_R \setminus B_r} (f^* * h^*)(x)\, dx.$$

Hence, since $g(x) = c$ (constant) for $x \in B_R \setminus B_r$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)\,dxdy \\
&\quad = \int_{B_r} g(x)(f*h)(x)\,dx + \int_{B_R\setminus B_r} g(x)(f*h)(x)\,dx \\
&\quad = \int_{B_r} g(x)(f*h)(x)\,dx + c\int_{B_R\setminus B_r} (f*h)(x)\,dx \\
&\quad = \int_{B_r} g(x)(f*h)(x)\,dx + c\int_{B_R\setminus B_r} (f^**h^*)(x)\,dx \\
&\quad = \int_{B_r} g(x)(f*h)(x)\,dx + \int_{B_R\setminus B_r} g(x)(f^**h^*)(x)\,dx \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(x)g(x-y)h^*(y)\,dx\,dy,
\end{aligned}$$

where in the last step we used the fact that $f*h = f^* * h^*$ on B_r.

Acknowledgements We would like to thank Rupert Frank for interesting discussions and for his observations about Conjecture 5.2 in Remark 5.3. F. N. is a Fellow of the *Accademia delle Scienze di Torino.*

Funding Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement.

References

1. Abreu, L.D., Dörfler, M.: An inverse problem for localization operators. Inverse Prob. **28**(11), 115001 (2012)
2. Abreu, L.D., Speckbacher, M.: Donoho-Logan large sieve principles for modulation and polyanalytic Fock spaces. Bull. Sci. Math. **171**(25), 103032 (2021)
3. Abreu, L.D., Gröchenig, K., Romero, J.L.: On accumulated spectrograms. Trans. Am. Math. Soc. **368**(5), 3629–3649 (2016)
4. Baernstein, A., II., Drasin, D., Laugesen, R.: Symmetrization in Analysis, vol. 36. Cambridge University Press, Cambridge (2019)
5. Berezin, F.A.: Wick and anti-Wick symbols of operators. Mat. Sb. **86**(128), 578–610 (1971)
6. Burchard, A.: Cases of equality in the Riesz rearrangement inequality. Ann. of Math. **143**(3), 499–527 (1996)
7. Burchard, A., Chambers, G.R.: Geometric stability of the Coulomb energy. Calc. Var. Partial Differential Equ. **54**(3), 3241–3250 (2015)
8. Burchard, A., Choksi, R., Topaloglu, I.: Nonlocal shape optimization via interactions of attractive and repulsive potentials. Indiana Univ. Math. J. **67**(1), 375–395 (2018)

9. Carlen, E.A.: Some integral identities and inequalities for entire functions and their application to the coherent state transform. J. Funct. Anal. **97**(1), 231–249 (1991)
10. Christ, M.: A sharpened Riesz–Sobolev inequality. Preprint at http://arxiv.org/abs/1706.02007 (2017)
11. Christ, M.: Near equality in the Riesz-Sobolev inequality. Acta Math. Sin. (Engl. Ser.) **35**(6), 783–814 (2019)
12. Cordero, E., Gröchenig, K.: Time-frequency analysis of localization operators. J. Funct. Anal. **205**(1), 107–131 (2003)
13. Daubechies, I.: Time-frequency localization operators: a geometric phase space approach. IEEE Trans. Inform. Theory **34**(4), 605–612 (1988)
14. Dávila, J.: On an open question about functions of bounded variation. Calc. Var. Partial Differential Equ. **15**(4), 519–527 (2002)
15. Dias, N.C., Luef, F., Prata, J.N.: Uncertainty principle via variational calculus on modulation spaces. J. Funct. Anal. **283**(8), 109605 (2022)
16. Fernández, C., Galbis, A.: Compactness of time-frequency localization operators on $L^2(\mathbb{R}^d)$. J. Funct. Anal. **233**(2), 335–350 (2006)
17. Figalli, A.: Stability in geometric and functional inequalities. In: European Congress of Mathematics, pp. 585–599. European Mathematical Society, Zürich (2013)
18. Figalli, A., Fusco, N., Maggi, F., Millot, V., Morini, M.: Isoperimetry and stability properties of balls with respect to nonlocal energies. Comm. Math. Phys. **336**(1), 441–507 (2015)
19. Frank, R.L., Lieb, E.H.: A note on a theorem of M. Christ. Preprint at http://arxiv.org/abs/1909.04598 (2019)
20. Frank, R.L., Nicola, F., Tilli, P.: The generalized Wehrl entropy bound in quantitative form. J. Eur. Math. Soc., to appear. Preprint at arXiv:2307.14089 (2023)
21. Frank, R.L.: Sharp inequalities for coherent states and their optimizers. Adv. Nonlinear Stud. **23**(1), 28 (2023)
22. Frank, R.L., Lieb, E.H.: Proof of spherical flocking based on quantitative rearrangement inequalities. Ann. Sc. Norm. Super. Pisa Cl. Sci. **22**(3), 1241–1263 (2021)
23. Fusco, N., Maggi, F., Pratelli, A.: The sharp quantitative isoperimetric inequality. Ann. of Math. **168**(3), 941–980 (2008)
24. Galbis, A.: Norm estimates for selfadjoint Toeplitz operators on the Fock space. Complex Anal. Oper. Theory **16**(1), 15 (2022)
25. Gómez, J., Guerra, A., Ramos, J.P.G., Tilli, P.: Stability of the Faber-Krahn inequality for the short-time Fourier transform. Invent. Math. **236**(2), 779–836 (2024)
26. Gröchenig, K.: Aspects of Gabor analysis on locally compact abelian groups. In: Gabor Analysis and Algorithms, pp. 211–231. Birkhäuser Boston, Boston (1998)
27. Gröchenig, K.: Foundations of time-frequency analysis. In: Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc, Boston, MA (2001)
28. Gröchenig, K.: Uncertainty principles for time-frequency representations. In: Advances in Gabor Analysis, pp. 11–30. Birkhäuser Boston, Boston (2003)
29. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
30. Kalaj, D.: Contraction property of differential operator on Fock space. Preprint at http://arxiv.org/abs/2207.13606 (2022)
31. Kalaj, D.: Contraction property of certain classes of log-$\mathcal{M}$-subharmonic functions in the unit ball. J. Funct. Anal. **286**(1), 110203 (2024)
32. Kulikov, A., Nicola, F., Ortega-Cerdà, J., Tilli, P.: A monotonicity theorem for subharmonic functions on manifolds. Preprint at http://arxiv.org/abs/2212.14008 (2023)
33. Kulikov, A.: Functionals with extrema at reproducing kernels. Geom. Funct. Anal. **32**(4), 938–949 (2022)
34. Lieb, E.H.: Proof of an entropy conjecture of Wehrl. Comm. Math. Phys. **62**(1), 35–41 (1978)
35. Lieb, E.H.: Integral bounds for radar ambiguity functions and Wigner distributions. J. Math. Phys. **31**(3), 594–599 (1990)
36. Lieb, E.H., Loss, M.: Analysis, vol. 14, 2nd edn. American Mathematical Society, Providence, RI (2001)
37. Luef, F., Skrettingland, E.: Convolutions for localization operators. J. Math. Pures Appl. **9**(118), 288–316 (2018)

38. Nicola, F.: The uncertainty principle for the short-time Fourier transform on finite cyclic groups: cases of equality. J. Funct. Anal. **284**(12), 109924 (2023)
39. Nicola, F.: Maximally localized Gabor orthonormal bases on locally compact Abelian groups. Adv. Math. **451**, 109786 (2024)
40. Nicola, F., Tilli, P.: The Faber-Krahn inequality for the short-time Fourier transform. Invent. Math. **230**(1), 1–30 (2022)
41. Nicola, F., Tilli, P.: The norm of time-frequency and wavelet localization operators. Trans. Amer. Math. Soc. **376**(10), 7353–7375 (2023)
42. Nicola, F., Romero, J.L., Trapasso, S.I.: On the existence of optimizers for time-frequency concentration problems. Calc. Var. Partial Differential Equ. **62**(1), 21 (2023)
43. Poincaré, H.: Sur un théoréme de M. Liapunoff relatif à l'équilibre d'une masse fluide. Comptes Rendus de L'Academie des Sciences **104**, 622–625 (1887)
44. Ramos, J.P.G., Tilli, P.: A Faber-Krahn inequality for wavelet transforms. Bull. Lond. Math. Soc. **55**(4), 2018–2034 (2023)
45. Riccardi, F.: A new optimal estimate for the norm of time-frequency localization operators. J. Funct. Anal. **287**(6), 110523 (2024)
46. Seip, K.: Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. SIAM J. Math. Anal. **22**(3), 856–876 (1991)
47. Stein, E.M., Shakarchi, R.: Functional Analysis: Introduction to Further Topics in Analysis, vol. 4. Princeton University Press, Princeton, NJ (2011)
48. Wong, M.W.: Wavelet Transforms and Localization Operators, vol. 136. Birkhäuser Verlag, Basel (2002)