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Original
Grassmann semialgebras and the Cayley-Hamilton theorem / Gatto, L., Rowen, L.. - In: PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY. SERIES B. - ISSN 2330-1511. - ELETTRONICO. - 7:16(2020), pp. 183-201. [10.1090/bproc/53]

Availability:
This version is available at: 11583/2996250 since: 2025-01-06T06:50:49Z

Publisher:
American Mathematical Society

Published
DOI:10.1090/bproc/53

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26 November 2026

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 7, Pages 183–201 (November 25, 2020)
https://doi.org/10.1090/bproc/53

GRASSMANN SEMIALGEBRAS AND THE CAYLEY-HAMILTON THEOREM

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(Communicated by Jerzy Weyman)

ABSTRACT. We develop a theory of Grassmann semialgebra triples using Hasse-Schmidt derivations, which formally generalizes results such as the Cayley-Hamilton theorem in linear algebra, thereby providing a unified approach to classical linear algebra and tropical algebra.

1. INTRODUCTION

The main goal of this paper is to define and explore the semiring version of the theory [4–6] of the first author concerning the Grassmann exterior algebra, viewed more generally in terms of negation maps and systems, continuing the approach of [21]. This provides a robust structure which unifies and generalizes four major versions of Grassmann structures, cf. Note 1.1, and provides a generalization of the Cayley-Hamilton theorem in Theorem 3.15. In the process we investigate Hasse-Schmidt derivations on Grassmann systems. The version given in Theorem 2.4 (over a free module V over an arbitrary semifield) is the construction which seems to "work".

Generalizing negation in Definition 1.3 to the notion of a "negation map" $(-)$ satisfying all the usual properties of negation except $a(-)a = 0$ (studied in [21]), we find that the Grassmann semialgebra of a free module V, described in Theorem 2.4, has a natural negation map on all homogeneous vectors, with the ironic exception of V itself, obtained by switching two tensor components.

Theorem A (Theorem 2.4)**.** $\bigoplus_{n\geq 2} V^{\otimes(n)}$ *has a negation map* $(-)$ *satisfying*

$$b_{\pi(i_1)} \cdots \bar{b}_{\pi(i_t)} = (-)^{\pi} b_{i_1} \cdots \bar{b}_{i_t},$$

for $b_{i_j} \in V$.

Received by the editors May 14, 2018, and, in revised form, May 13, 2020.

2020 *Mathematics Subject Classification.* Primary 15A75, 16Y60, 15A18; Secondary 12K10, 14T10.

Key words and phrases. Cayley-Hamilton theorem, exterior semialgebras, Grassmann semialgebras, Hasse-Schmidt derivations, differentials, eigenvalues, eigenvectors Laurent series, Newton's formulas, power series, semifields, systems, semialgebras, tropical algebra, triples.

The first author was partially supported by INDAM-GNSAGA and by PRIN "Geometria sulle varietà algebriche" Progetto di Eccellenza Dipartimento di Scienze Matematiche, 2018–2022 no. E11G18000350001.

The second author was supported in part by the Israel Science Foundation, grant No. 1207/12 and his visit to Torino was supported by the "Finanziamento Diffuso della Ricerca", grant no. 53_RBA17GATLET, of Politecnico di Torino.

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This provides us "enough" negation, coupled with the relation $\preceq_\circ$ of Definition 1.6 (designed to replace equality), to adapt the methods of [6] to negation maps and systems of [21] to study $T(V)$, with $\mathcal{T}$ the nonzero simple tensors.

One obtains a Grassmann algebra by modding out all elements of the form $v \otimes v$, $v \in V$. A weaker version is obtained when we take a given base $\{b_0, b_1, \dots, b_{n-1}\}$ of V and mod only the $b_i \otimes b_i$, $0 \le i < n$. One can enhance these by means of a "symmetrizing" construction which provides a negation map and a triple for any Grassmann algebra:

Theorem B (Theorems 2.13, 2.14, 2.22)**.** *There is a symmetrized triple* $(\mathfrak{G}(V)^\natural, \widehat{\mathcal{T}}, (-)_{\mathrm{sw}})$*, together with an embedding of triples* $(\mathfrak{G}(V)_{\ge 2}, \mathcal{T}_{\mathfrak{G}(V)_{\ge 2}}, (-)) \to (\mathfrak{G}(V)^\natural, \widehat{\mathcal{T}}, (-)_{\mathrm{sw}})$ *given by* $c \mapsto (c, 0)$*. An analogous assertion holds for* $\mathfrak{G}(V)^\diamondsuit$*.*

Note 1.1. There are four major applications of this Grassmann triple:

(i) The classical situation is recovered, where V is a vector space and $(-)$ is the classical negation.
(ii) The tropical situation is treated in [8] to study matroids, taken over the max-plus algebra, cf. Remark 2.26 where $(-)$ is the identity map.
(iii) The supertropical situation, as described in Theorem 2.13, provides an alternate language for the theory of [8].
(iv) The symmetrized situation, as described in §2.1, provides a nontrivial negation map for any Grassmann semialgebra.

The rest of §2 is dedicated to laying the groundwork for further research in Grassmann semialgebras, and the reader only interested in the applications of this paper could go on to §3, where we study properties of derivations. There we learn how to associate to an endomorphism f of an $\mathcal{A}$ free module V_n a Hasse-Schmidt derivation $D\{z\} : \bigwedge V_n \to \bigwedge V_n[[z]]$, on the Grassmann semi-algebra, i.e $D\{z\}(u \wedge v) = D\{z\}u \wedge D\{z\}V$ and $D\{z\}_{|_{V_n}} = \sum_{j\ge 0} f_j z^j$. We suitably construct, starting from the data of $D\{z\}$, an operator valued polynomial $\overline{D}\{z\} : \bigwedge V_n \to \mathrm{End}(\bigwedge V_n)[z]$ showing in detail:

Theorem C (Theorem 3.12)**.** *The polynomial* $\overline{D}\{z\}$ *is a quasi-inverse of* $D\{z\}$*, in the sense that* $\overline{D}\{z\}D\{z\}u \succeq u$ *for all* $u \in \bigwedge V_n$*.*

Our main result, Theorem 3.14, describes the relation between a Hasse-Schmidt derivation $D\{z\}$ and its "quasi-inverse" $\overline{D}\{z\}$ defined in such a way to yield:

Theorem D (Theorem 3.14)**.** $\overline{D}\{z\}(D\{z\}u \wedge v) \succeq u \wedge \overline{D}\{z\}v$.

In the classical case, one gets equality as shown in Remark 3.16, so we recover [6]. Our main application in this paper is a generalization of the Cayley-Hamilton theorem to semi-algebras.

Theorem E (Theorem 3.15)**.**
(1.1)
$$((D_n u + e_1 D_{n-1}u + \cdots + e_n u) \wedge v)\,(-)\,((D_n u + e_1' D_{n-1}u + \cdots + e_n' u) \wedge v) \succeq 0$$
for all $u \in \bigwedge^{>0} V_n$*.*

We obtain true equality in these theorems by modding out all elements of the form $v \otimes v$. Again we recover [6] in the classical case.

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1.1. **Basic notions.** Much of this section is a review of [21], as summarized in [22], and also as in [15]. As customary, $\mathbb{N}$ denotes the natural numbers including 0, $\mathbb{N}^+$ denotes $\mathbb{N} \setminus \{0\}$, $\mathbb{Q}$ denotes the rational numbers, and $\mathbb{R}$ denotes the real numbers, all ordered monoids under addition.

A **semiring**†† $(\mathcal{A}, +, \cdot)$ is an additive abelian semigroup $(\mathcal{A}, +)$ and multiplicative semigroup $(\mathcal{A}, \cdot)$ satisfying the usual distributive laws. A **semiring**† $(\mathcal{A}, +, \cdot, \mathbb{1})$ is a semiring†† with a multiplicative unit $\mathbb{1}$. (Thus, an ideal of a semiring† is a semiring††.) A **semifield**† is a semiring† whose multiplicative monoid is a group. A **semiring** (resp. **semifield**) [9] is a semiring† (resp. semifield†) with an absorbing element $\mathbb{0}$ formally adjoined.

Definition 1.2. A $\mathcal{T}$**-module** over a set $\mathcal{T}$ is an additive monoid $(\mathcal{A}, +, \mathbb{0})$ with a scalar multiplication $\mathcal{T} \times \mathcal{A} \to \mathcal{A}$ satisfying the following axioms, $\forall k \in \mathbb{N}$, $a \in \mathcal{T}$, $b, b_j \in \mathcal{A}$:

(i) (Distributivity over $\mathcal{T}$): $a(\sum_{j=1}^k b_j) = \sum_{j=1}^k (ab_j)$.

(ii) $a\mathbb{0}_{\mathcal{A}} = \mathbb{0}_{\mathcal{A}}$.

A $\mathcal{T}$**-monoid module** over a multiplicative monoid $\mathcal{T}$ is a $\mathcal{T}$-module satisfying the extra conditions

$$\mathbb{1}_{\mathcal{T}} b = b, \qquad (a_1 a_2)b = a_1(a_2 b), \quad \forall a_i \in \mathcal{T},\ b \in \mathcal{A}.$$

A $\mathcal{T}$**-semiring** is a semiring that is also a $\mathcal{T}$-monoid module over a given multiplicative submonoid $\mathcal{T}$. This paper only concerns $\mathcal{T}$-semirings, which are closely related to blueprints in [19]. We put $\mathcal{T}_{\mathbb{0}} = \mathcal{T} \cup \{\mathbb{0}\}$.

Tensor products over semirings [17, 18, 23] are analogous to tensor products over rings. The tensor product $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$ of right and left $\mathcal{A}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ is $(\mathcal{F}_1 \oplus \mathcal{F}_2)/\Phi$, where $\mathcal{F}_i$ is the free module (respectively right or left) with base $\mathcal{M}_i$, and Φ is the congruence generated by all

$$\left((\sum_j x_{1,j}, \sum_k x_{2,k}), \sum_{j,k} (x_{1,j}, x_{2,k}) \right), \tag{1.2}$$

$$\Big((x_1 a, x_2), (x_1, a x_2) \Big) \quad \forall x_i, x_{i,j}, x_{i,k} \in \mathcal{M}_i,\ a \in \mathcal{A}.$$

1.2. **Negation maps, triples, and systems.**

Definition 1.3. A **negation map** on a $\mathcal{T}$-module M over a given set $\mathcal{T}$ is a semigroup isomorphism $(-) : M \to M$ of order ≤ 2, written $b \mapsto (-)b$, which also respects the $\mathcal{T}$-action in the sense that

$$(-)(ab) = a((-)b)$$

for $a \in \mathcal{T}$, $b \in M$.

A **semiring**†† **negation map** on a semiring†† $\mathcal{A}$ is a negation map which satisfies $(-)(ab) = a((-)b) = ((-)a)b$ for all $a, b \in \mathcal{A}$.

In the classical case the negative is a negation map. For tropical algebra, one could just take $(-)$ to be the identity map, but we want a less trivial example.

We write $a(-)b$ for $a + ((-)b)$, $(\pm)a$ for $\{a, (-)a\}$, and a° for $a(-)a$, called a **quasi-zero**. The set M° of quasi-zeroes is an important $\mathcal{T}$-submodule of M. When $\mathcal{A}$ is a semiring, $\mathcal{A}^\circ$ is an ideal.

We define $(-)^{\mathbb{0}}a$ to be a and, for $k \in \mathbb{N}$, we inductively define $(-)^k a$ to be $(-)((-)^{k-1}a)$.

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Lemma 1.4. $((-)^k a)((-)^{k'} a') = (-)^{k+k'}(aa')$ *for* $a, a' \in \mathcal{A}$.

Proof. $((-)^k a)((-)^{k'} a') = (-)((-)^{k-1} a)((-)^{k'} a') = (-)((-)^{k+k'-1}(aa'))$ $= (-)^{k+k'}(aa')$, by induction on k. □

Definition 1.5. A **pseudo-triple** is a collection $(\mathcal{A}, \mathcal{T}, (-))$, where $(-)$ is a negation map on both $\mathcal{T}$ and $\mathcal{A}$, and $\mathcal{A}$ is a $(\mathcal{T}, (-))$-module.

A $\mathcal{T}_{\mathcal{A}}$**-pseudo-triple** $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$ is a $\mathcal{T}$-module $\mathcal{A}$, with $\mathcal{T}_{\mathcal{A}}$ designated as a distinguished subset, and a negation map $(-)$ satisfying $(-)\mathcal{T}_{\mathcal{A}} = \mathcal{T}_{\mathcal{A}}$. A $\mathcal{T}_{\mathcal{A}}$**-triple**, called a **triple** when $\mathcal{T}$ is understood, is a $\mathcal{T}_{\mathcal{A}}$-pseudo-triple, in which $\mathcal{T}_{\mathcal{A}} \cap \mathcal{A}^\circ = \emptyset$ and $\mathcal{T}_{\mathcal{A}}$ generates $(\mathcal{A} \setminus \{\mathbb{0}\}, +)$.

Since $\mathcal{T}_{\mathcal{A}}$ is usually clear from the context, we abuse notation and write $\mathcal{T}$ for $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{A}$. A triple is **uniquely negated** when for any $a \in \mathcal{T}$, $a + b \in \mathcal{A}^\circ$ implies $b = (-)a$.

The structure is rounded out with the following relation.

Definition 1.6. Define the $\circ$**-surpassing relation** $\preceq_\circ$ on a module M with negation map by $a_0 \preceq_\circ a_1$ if $a_1 = a_0 + d$ for some $d \in M^\circ$.

A uniquely negated triple $(\mathcal{A}, \mathcal{T}, (-))$ together with a surpassing relation $\preceq$ is called a **system**.[1]

Remark 1.7. The relation $\preceq_\circ$ on a system restricts to equality on $\mathcal{T}$, by [21, Proposition 4.4]. In fact $\preceq_\circ$ is used to replace equality when we work with triples, and identities in classical algebra can often be replaced by relations expressed in terms of $\preceq_\circ$, by means of the transfer principle of [1], formulated for systems in [21, Theorem 6.17].

1.3. **Functions to** $\mathcal{A}$**.** The next construction, discussed in [16, §4.2], enables us to describe power series in a structural context. From now on, we suppose $(S, +)$ is a semigroup, often $(\mathbb{N}, +, 0)$. Given a triple $(\mathcal{A}, \mathcal{T}, (-))$, $\mathcal{A}^S$ denotes the maps from S to $\mathcal{A}$, and $\mathcal{T}^S$ denotes the nonzero maps of $\mathcal{A}^S$ sending S to $\mathcal{T}$. For example, for $c \in \mathcal{A}$, the **constant function** $\tilde{c}$ is given by $\tilde{c}(s) = c$ for all $s \in S$.

We modify the definition of support from [16, Definition 4.2].

Definition 1.8. Given $f \in \mathcal{A}^S$ we define its **support** $\operatorname{supp}(f) := \{s \in S : f(s) \neq \mathbb{0}\}$, and $\operatorname{supp}(\mathcal{A}^S)$ for $\cup\{\operatorname{supp}(f) : f \in \mathcal{A}^S\}$.

Definition 1.9. A set $\mathcal{A}\mathfrak{b}$ of maps $f : S \to \mathcal{A}$ is **convolution admissible** if for each $f, g \in \mathcal{A}\mathfrak{b}$ and $s \in S$ there are only finitely many $s' \in \operatorname{supp}(f)$, $s'' \in \operatorname{supp}(g)$, with $s' + s'' = s$.

Example 1.10. $\mathcal{A}\mathfrak{b}$ is convolution admissible whenever $S = \mathbb{N}^{(I)}$ (the direct sum) for some index set I, since the condition of Definition 1.9 already holds in S.

Definition 1.11. Suppose $\mathcal{A}\mathfrak{b}$ is a convolution admissible set. The **convolution product** $\mathcal{A}\mathfrak{b} \times \mathcal{A}\mathfrak{b} \to \mathcal{A}\mathfrak{b}$ is given by defining fg to be the function satisfying

$$fg(s) = \sum_{s'+s''=s} f(s')g(s'').$$

[1] In [16], in the systemic setting, a more general notion of surpassing map $\preceq$ is used, and $\mathcal{A}_{\mathrm{Null}}$ is introduced which equals $\mathcal{A}^\circ$ when $\preceq = \preceq_\circ$; here, using Lemma 2.18 as justification, we use $\mathcal{A}^\circ$ and $\preceq = \preceq_\circ$ to avoid complications.

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The intuitive way to receive a negation map on $\mathcal{A}\flat$ from a negation map on $\mathcal{A}$ would be to define $((-)f)(s) = (-)(f(s))$; these maps also are convolution admissible, so one would expand $\mathcal{A}\flat$ to include them.

1.4. **Graded semirings and modules.** We want to grade semirings and their modules. We define direct sums in the usual way.

Definition 1.12. An L-**graded** $\mathcal{T}$-semiring† is a $\mathcal{T}$-semiring† $\mathcal{R}$ which also is an L-graded $\mathcal{T}$-module $\mathcal{R} := \oplus_{\ell\in L}\mathcal{R}_\ell$ for semigroups $(\mathcal{R}_\ell, +)$ satisfying the following conditions, where $\mathcal{T}_\ell = \mathcal{T} \cap \mathcal{R}_\ell$:

(i) $\mathcal{T} = \cup_{\ell\in L}\mathcal{T}_\ell$;
(ii) $\mathcal{R}_\ell\mathcal{R}_{\ell'} \subseteq \mathcal{R}_{\ell+\ell'}$, $\forall \ell, \ell' \in L$.

Note that $\mathcal{R}_{\mathbb{0}}$ is a $\mathcal{T}_{\mathbb{0}}$-module, and also a semiring†, over which each $\mathcal{R}_\ell \cup \{\mathbb{0}\}$ is a module.

When we turn to Grassmann semialgebras, L will be ordered with a minimal element 0; one could take $L = \mathbb{N}$, for example. We write $\mathcal{M}^{>0} := \oplus_{\ell>0}\mathcal{M}_\ell$, a submodule of $\mathcal{M}$ lacking the constant component. Then $\mathcal{R}^{>0}$ is a sub-semiring†† of $\mathcal{R}$.

1.4.1. *Super-semialgebras.* Here is an interesting special case.

Definition 1.13. A **super-semialgebra** is a $\mathbb{Z}_2$-graded semialgebra $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1$, i.e., satisfying **twist multiplication**:

$$(a_0, a_1)(a_0', a_1') = (a_0a_0' + a_1a_1', a_0a_1' + a_1a_0'). \tag{1.3}$$

A natural way of getting a $\mathbb{Z}_2$-grade from an $\mathbb{N}$-graded semialgebra is to take the 0-grade to be the set of even indices and the 1-grade to be the set of odd indices.

1.4.2. *The power series semiring of a graded semiring*†. From now on, we take $S = \mathbb{N}$ as in Example 1.10, which is equivalent to the following.

Definition 1.14. Given an $\mathbb{N}$-graded $\mathcal{T}$-semiring $\mathcal{R}$ with respect to the semigroup $(\mathbb{N}, +)$, we define the power series semiring $\mathcal{R}[[z]]$ over a central indeterminate z, in the usual way as possibly infinite formal sums (with convolution product), and its sub-semiring $\mathcal{R}[z] = \sum_j \mathcal{R}_j z^j$.

Lemma 1.15. *$\mathcal{R}[[z]]$ and $\mathcal{R}[z]$ are indeed semirings. Both $\mathcal{R}[[z]]$ and $\mathcal{R}[z]$ are graded by the powers of z.*

Proof. $\mathcal{R}[[z]]$ satisfies the axioms of a semiring, by the customary verification, and its subset $\mathcal{R}[z]$ is closed under addition and multiplication, so both are semirings. The last assertion follows from the fact that $\mathcal{R}_j z^j \mathcal{R}_k z^k \subseteq \mathcal{R}_{j+k} z^{j+k}$. □

Definition 1.16. Suppose $\mathcal{A}$ is a semialgebra over a commutative base semiring†. We write $\mathrm{End}(\mathcal{A})$ for the set of module maps $\mathcal{A} \to \mathcal{A}$. Given $D \in \mathrm{End}(\mathcal{A})^S$, we write D_s for the map given by $s \mapsto D(s)$ and $s' \mapsto \mathbb{0}$, $\forall s' \neq s$. In the other direction, given $f_s :\in \mathrm{End}(\mathcal{A})$, $s \in S$, each of singleton support $\{s\}$ with the $\{s\}$ distinct, define $D^f \in \mathrm{End}(\mathcal{A})^S$ via $D^f(s) = f_s$.

Remark 1.17. $D^f(s)(ab) = D^f(s)(a)D^f(s)(b), \forall a, b \in \mathcal{A}$, under the usual product, seen by matching terms in the left side and the right side.

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1.5. **Higher derivations.** This discussion applies to any semialgebra $\mathcal{A}$, not necessarily associative and not necessarily a triple. A **derivation** $\delta : \mathcal{A} \to \mathcal{A}$ is a map in $\mathrm{End}(\mathcal{A})$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$. The following concepts were introduced by Hasse and Schmidt [10] and studied further by Heerema [11–13].

For S convolution admissible (not necessarily associative), a homogeneous map D in $(\mathrm{End}\,\mathcal{A})^S$ is called a **higher derivation** of A if it satisfies the conditions:

(i) $D_s(ab) = \sum_{s'+s''=s} D_{s'}(a)D_{s''}(b), \quad \forall s \in S, \quad \forall a, b \in \mathcal{A}.$
(ii) $D_0 = \mathbb{1}$ (the identity map on A).

Property (i) is called the **Leibniz rule**, obtained from the more familiar Leibniz rule for derivations for $S = \mathbb{N}$ by dividing by $k!$.

[10, pp. 190-191] indicates how to define a higher derivation D. We have a somewhat different take, along classical lines. We consider semialgebras over semifields containing $\mathbb{Q}_{>0}$, for the following definition to make sense. Given a map $f : \mathcal{A} \to \mathcal{A}[[z]]$ we define its **exponential** $\exp(f) = \sum_{j\geq 1} \frac{f^j}{j!} : \mathcal{A} \to \mathcal{A}[[z]]$. It is well-known that the exponential of a derivation is a homomorphism. In fact we have:

Lemma 1.18. *If $d_1, d_2, \dots$ is a sequence of derivations, then $\sum d_k z^k : \mathcal{A} \to \mathcal{A}[[z]]$ satisfies Leibniz' rule. Its exponential is a semialgebra homomorphism: $D := \sum D_r z^r := \exp(\sum_{k\geq 1} d_k z^k) : \mathcal{A} \to \mathcal{A}[[z]]$.*

Proof. Given in [24] and [4, Propositions 3.4.2 and 3.4.3]. The proof evidently extends to semialgebras over semifields containing $\mathbb{Q}_{>0}$. □

Matching coefficients in Lemma 1.18, one gets precisely the Schur polynomials associated to the sequence $d_1, d_2, \dots$. In particular:

$$D_1 = d_1, \quad D_2 = \frac{d_1^2}{2} + d_2, \quad D_3 = \frac{d_1^3}{3!} + d_1d_2 + d_3, \quad D_4 = \frac{d_1^4}{4!} + \frac{1}{2}d_1^2d_2 + \frac{1}{2}d_2^2 + d_1d_3 + d_4,$$

defines a higher derivation D.

When each $d_s = \delta$ for a given derivation δ, we call D the **higher derivation** of δ.

2. Grassmann semialgebras

Suppose $\mathcal{A}$ is a commutative semiring and V is an $\mathcal{A}$-module. $\mathfrak{H}$ will denote an $\mathcal{A}$-semialgebra generated by V (Often $\mathfrak{H}$ will be the tensor algebra $T(V)$ defined below). We write $\mathcal{T}_{\mathfrak{H}k}$ for the products of length k of elements of V, and $\mathfrak{H}_{\geq k}$ for the ideal $\sum_{j\geq k} \mathcal{T}_{\mathfrak{H}j}$ (with repetitions). Thus $\mathfrak{H} = \mathcal{A} + V + \mathfrak{H}_{\geq 2}$. The elements of $\mathcal{T}_{\mathfrak{H}k}$ and $\mathcal{T}_{\mathfrak{H}l}$ will satisfy $w_k w_l' = (-1)^{kl} w_l' w_k$, leading to the subject of our study.

Definition 2.1. A **Grassmann**, or **exterior**, semialgebra, over a semiring[†] $\mathcal{A}$ and an $\mathcal{A}$-module V, is a semialgebra $\mathfrak{H}$ generated by $\mathcal{A}$ and V, as above, together with a negation map on $\mathfrak{H}_{\geq 2}$ and an associative **wedge product** $\wedge : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ satisfying

$$v_1 \wedge v_2 = (-)v_2 \wedge v_1 \qquad \text{for} \qquad v_i \in V. \tag{2.1}$$

Thus $v_{\pi(1)} \wedge \cdots \wedge v_{\pi(t)} = (-)^\pi v_1 \wedge \cdots \wedge v_t$ for $t \geq 2$, where $(-)^\pi$ denotes the sign of the permutation. This ties in with the theory of triples since, taking $\mathcal{T}(V)$ to be the nonzero products of elements of V, then for $k \geq 2$, $(\mathfrak{H}_{\geq 2}, \mathcal{T}(V)_{\geq 2}, (-))$ often is a triple, for $(-)$ suitably defined (as in Theorem 2.4). Since $\mathcal{T}(V)_{\geq 2} \subset \mathfrak{H}_{\geq 2}$, it can

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be bypassed by restricting functions, but the negation map $(-)$ will play a crucial role.

Lemma 2.2. *If V is spanned by $\{b_i : i \in I\}$, then to verify the Grassmann relation* (2.1) *it is enough to check that*

$$b_i \wedge b_j = (-)b_j \wedge b_i, \qquad \forall i, j \in I.$$

Proof. Distributivity yields

$$\begin{aligned}\left(\sum \alpha_i b_i\right) \wedge \left(\sum \beta_j b_j\right) &= \sum \alpha_i \beta_j b_i \wedge b_j = (-) \sum \alpha_i \beta_j b_j \wedge b_i \\ &= (-) \left(\sum \beta_j b_j\right) \wedge \left(\sum \alpha_i b_i\right),\end{aligned}$$

yielding the assertion. □

We write v^k for $v \wedge \cdots \wedge v$ taken k times.

Lemma 2.3. $(\sum \alpha_i a_i)^2 \succeq_\circ \sum \alpha_i^2 a_i^2$ *for any Grassmann semialgebra.*

Proof. $(\sum \alpha_i a_i)^2 = \sum \alpha_i^2 a_i^2 + \sum_{i<j} \alpha_i \alpha_j (a_i \wedge a_j + a_j \wedge a_i)$. □

To obtain Grassmann semialgebras via Definition 2.10 below, we follow the familiar construction of the Grassmann algebra over a module V, but with modifications necessitated by working over semirings.

Accordingly, as in [21, Remark 6.35] and [16, Definition 6.10], we define the **tensor semialgebra** $T(V) = \bigoplus_n V^{\otimes(n)}$ (adjoining a copy of $\mathcal{A}$ if we want to have a unit element), with the usual multiplication $vv' := v \otimes v'$.

Theorem 2.4. *Write $T(V)_{\geq 2}$ for $\bigoplus_{n \geq 2} V^{\otimes(n)}$. Then $T(V)_{\geq 2}$ has a negation map $(-)$ satisfying*

$$b_{\pi(i_1)} \otimes \cdots \otimes b_{\pi(i_t)} = (-)^\pi b_{i_1} \otimes \cdots \otimes b_{i_t},$$

for $b_{i_j} \in V$.

Proof. By Lemma 2.2, we may take a generating set $\{b_i : i \in I\}$ of V, where I is an ordered index set. We define a negation on $V \otimes V$ by $(-)b_i \otimes b_j = b_j \otimes b_i$. (This is possible since it preserves the bilinear relations defining the tensor product.) Since this is homogeneous of degree 2, it defines a negation on $\mathfrak{G}(V)_2$ given by $(-)b_i \otimes b_j = b_j \otimes b_i$. When $i < j$ we thus rename $b_j \otimes b_i$ as $(-)b_i \otimes b_j$. It is easy to see that this is the same as defining a reduction procedure. Thus $b_{\pi(i_1)} \cdots b_{\pi(i_t)} \mapsto (-)^\pi b_{i_1} \cdots b_{i_t}$, where π is the permutation rearranging the indices $i_1 \ldots, i_t$ in ascending order. We get $(-)^\pi$ by writing π as a product of transpositions; since $(-)^\pi$ is independent of the way we write π in this manner, our reduction procedure is well-defined, cf. [20]. □

We continue to develop the Grassmann theory. We can eliminate many occurrences of $(-)$ in our formulas by switching two of the b_i. The tricky part is dealing with degree 1, i.e., in V itself, where we cannot perform this switch. But issues like determinants and linear independence of n vectors are trivial for $n = 1$, thereby enabling us to forego $(-)$ on elements of degree 1. In this manner, our way out in §3 is to focus on elements of degree > 1.

Definition 2.5. $\mathcal{T}_{\text{even}}^{\geq 2}$ is the set of all even products of elements of V, not including the constants $\mathcal{A}$, $\mathfrak{G}_{\text{even}}^{\geq 2}$ is the submodule of $\mathfrak{G}$ generated by $\mathcal{T}_{\text{even}}^{\geq 2}$, $\mathcal{T}_{\text{odd}}$ is the set of all odd products of elements of V, and $\mathfrak{G}_{\text{odd}}$ is the submodule of $\mathfrak{G}$ generated by $\mathcal{T}_{\text{odd}}$.

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Lemma 2.6. *If $v \in \mathfrak{G}_i$ and $v' \in \mathfrak{G}_j$ for $i, j \geq 1$ then*

$$v \wedge v' = (-)^{i+j} v' \wedge v, \tag{2.2}$$

where $(-)$ is given as in Theorem 2.4.

Proof. Easy induction on i and j. □

Definition 2.7. $T(V)^{\circ\sharp}$ is the ideal of $T(V)$ generated by $T(V)^\circ$ and all elements $v \otimes v$, $v \in V$.

(This is just $T(V)^\circ$ when $\frac{1}{2} \in \mathcal{A}$ since then $v \otimes v = (\frac{1}{2} v \otimes v)^\circ$). We now weaken $\preceq_\circ$.

Definition 2.8. (Supplanting Definition 1.6) $a_0 \preceq a_1$ in $T(V)$ if $a_1 = a_0 + d$ for some $d \in T(V)^{\circ\sharp}$.

2.0.1. *The standard Grassmann semialgebra.* Recall that the way to define factor structures in universal algebra (in particular, for semirings† or modules over semirings†) is to mod out by a congruence.

Theorem 2.9. *If $v^2 = \mathbb{0}$ for all v in V, then for any permutation π and all $v_i \in V$,*

(i) $v \wedge v_1 \wedge \cdots \wedge v_n \wedge v = \mathbb{0}$.
(ii) $v_{\pi(1)} \wedge \cdots \wedge v_{\pi(n)} = v_1 \wedge \cdots \wedge v_n$ *if π is even;*
(iii) $v_1 \wedge \cdots \wedge v_n + v_{\pi(1)} \wedge \cdots \wedge v_{\pi(n)} = \mathbb{0}$ *if π is odd.*

Thus the only quasi-zeros are 0.

Proof. Linearizing yields

$$\mathbb{0} = (v_1 + v_2)^2 = v_1^2 + v_2^2 + v_1 \wedge v_2 + v_2 \wedge v_1 = \mathbb{0} + \mathbb{0} + v_1 \wedge v_2 + v_2 \wedge v_1,$$

so $v_1 \wedge v_2 + v_2 \wedge v_1 = \mathbb{0}$. Now (i) is by induction on n, since $v \wedge v_1 \wedge \cdots \wedge v_n \wedge v = v \wedge v_1 \wedge \cdots \wedge v_n \wedge +\mathbb{0} = (v \wedge v_1 + v_1 \wedge v) \wedge \cdots \wedge v_n \wedge v = \mathbb{0}$.

To get (ii) and (iii) we write π as a product of transpositions $\pi_1 \cdots \pi_k$ of the form $(i, i+1)$. If $k = 2$ then $v_1 \wedge v_2 \wedge v_3 = v_1 \wedge v_2 \wedge v_3 + (v_2 \wedge (v_1 \wedge v_3 + v_3 \wedge v_1)) = (v_1 \wedge v_2 + v_2 \wedge v_1) \wedge v_3 + v_2 \wedge v_3 \wedge v_1 = v_2 \wedge v_3 \wedge v_1$, and then we have (ii) for all even k.

For k odd, we use (ii) to reduce to $v_1 \wedge v_2 \wedge \cdots \wedge v_n + v_2 \wedge v_1 \wedge \cdots \wedge v_n = \mathbb{0}$.

The last assertion follows by using these equalities to reduce every quasi-zero until reaching $\mathbb{0}$. □

Definition 2.10. The **standard Grassmann semialgebra** $\bigwedge V$ with respect to a given generating set $\{b_i : i \in I\}$ of V, also denoted $\mathfrak{G}(V)$, is $T(V)/\Phi$, where Φ is the congruence generated by $(v \wedge v, \mathbb{0})$, $\forall v \in V$. Accordingly $\mathfrak{G}(V)_k$ is $T(V)_k/\Phi$ and $\mathfrak{G}(V)_{\geq 2} = T(V)_{\geq 2}/\Phi$.

The **standard Grassmann triple** is $(\mathfrak{G}(V)_{\geq 2}, \mathcal{T}_{\mathfrak{G}(V)_{\geq 2}}, (-))$, where $\mathcal{T}_{\mathfrak{G}(V)_{\geq 2}}$ is the product of elements of V of length ≥ 2, and $(-)$ is as in Theorem 2.4.

2.1. **Symmetrization and the twist action.** There is a general way to provide a negation map for arbitrary Grassmann semialgebras. Although $\mathcal{T}$-modules initially may lack negation, one can obtain negation maps for them through the next main idea, the symmetrization process, which although a special case of supersemialgebras and their modules, provides a crucial method of creating a triple. Tropical symmetrization dates back to [7], and we recall the treatment for systems from [16].

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Definition 2.11. Given any $\mathcal{T}$-monoid module $\mathcal{M}$, define its $\mathbb{Z}_2$-graded **symmetrization** $\widehat{\mathcal{M}} = \mathcal{M} \times \mathcal{M}$, with componentwise addition.

Also define $\widehat{\mathcal{T}} = (\mathcal{T} \times \{\mathbb{0}\}) \cup (\{\mathbb{0}\} \times \mathcal{T})$ with the **twist action** of $\widehat{\mathcal{T}}$ on $\widehat{\mathcal{M}}$ given by the super-action, namely

$$(a_0, a_1) \cdot_{\mathrm{tw}} (c_0, c_1) = (a_0c_0 + a_1c_1, a_0c_1 + a_1c_0), \quad a_i \in \mathcal{T}, c_i \in \mathcal{M}. \tag{2.3}$$

Definition 2.12. The **switch map** $(-)_{\mathrm{sw}}$ on the symmetrized module $\widehat{\mathcal{M}}$ is given by $(-)_{\mathrm{sw}}(c_0, c_1) = (c_1, c_0)$.

If $\mathcal{A}$ is a semiring containing $\mathcal{T}$, then we define the **twist action** as in (2.3), but this time with $a_i, c_i \in \mathcal{A}$.

Theorem 2.13 ([15, Theorems 2.41, 2.43])**.** *For any $\mathcal{T}$-module $\mathcal{A}$, we can embed $\mathcal{A}$ into $\widehat{\mathcal{A}}$ via*

$$b \mapsto (b, \mathbb{0}),$$

thereby obtaining a faithful functor from the category of semirings into the category of semirings with a negation map (and preserving additive idempotence). This also yields a faithful functor from ordered semigroups to signed $(-)_{\mathrm{sw}}$-bipotent systems. Any $\mathcal{A}$-module $\mathcal{M}$ yields a $\widehat{\mathcal{A}}$-module $\widehat{\mathcal{M}} = \mathcal{M} \oplus \mathcal{M}$, which has a signed decomposition where $\mathcal{M}^+$ is the first component.

This applies to the Grassmann semialgebra:

Theorem 2.14. *Define $\mathfrak{G}(V)^\natural$ to be $\widehat{\mathfrak{G}(V)}$ modded out by the congruence generated by $(v \wedge v', 0) \cong (0, v' \wedge v)$ for all $v, v' \in V$, and $\overline{\mathcal{T}}$ to be the corresponding image of $\widehat{\mathcal{T}}$. There is a triple $(\mathfrak{G}(V)^\natural, \widehat{\mathcal{T}}, (-)_{\mathrm{sw}})$, together with an embedding of triples*

$$(\mathfrak{G}(V)_{\geq 2}, \mathcal{T}_{\mathfrak{G}(V)_{\geq 2}}, (-)) \to (\mathfrak{G}(V)^\natural, \overline{\mathcal{T}}, (-)_{\mathrm{sw}})$$

given by $c \mapsto (c, 0)$.

Proof. Take $\mathcal{A} = \mathfrak{G}(V)$ in Theorem 2.13.

$$(-)(v \wedge v') = v' \wedge v \mapsto (v' \wedge v, 0) = (-)_{\mathrm{sw}}(0, v' \wedge v) = (-)_{\mathrm{sw}}(v \wedge v', 0).$$

Thus $(-)$ matches by (2.13), so the Grassmann relations match. □

2.2. **The partially reduced Grassmann system (when V is free).** The main results of this paper involve the **free module** V with base $\mathcal{B} = \{b_0, \dots, b_{n-1}\}$, in the sense that any element of V can be written uniquely as an $\mathcal{A}$-linear combination of the b_i. Let $V_n := \mathcal{A}^{(n)}$ be the free module over the semiring $\mathcal{A}$ with basis $\mathbf{b} := \{b_0, \dots, b_{n-1}\}$ of n elements. When $V = V_n$, this includes the definition in [8, Definition 3.1.2], in which $(-)$ is the identity map. In this work we have two candidates for the Grassmann algebra, given respectively in Theorems 2.14 and 2.22.

Remark 2.15.

(i) By Theorem 2.9, $b_{\pi(i_1)} \wedge \cdots \wedge b_{\pi(i_t)} = (-)^\pi b_{i_1} \wedge \cdots \wedge b_{i_t}$ for any permutation π. Also, every simple tensor in which some b_i repeats is $\mathbb{0}$.

(ii) $\mathfrak{G}_k$ is free with a base of $2\binom{n}{k}$ elements. For instance $\mathfrak{G}_2(V_3)$ has base $b_1 \wedge b_2$, $b_1 \wedge b_3$, $b_2 \wedge b_3$ and their "negations" $b_2 \wedge b_1$, $b_3 \wedge b_1$, $b_3 \wedge b_2$. This phenomenon gives rise to the "eigenvalue pair" of §3.2 below.

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Lemma 2.16. *For the free Grassmann semialgebra, $\mathfrak{G} = \mathfrak{G}_{\rm even} \oplus \mathfrak{G}_{\rm odd}$ is a super-semialgebra, and its ideal $\mathfrak{G}^{\geq 2} = \mathfrak{G}^{\geq 2}_{\rm even} \oplus \mathfrak{G}^{\geq 2}_{\rm odd}$ has the negation map from Theorem 2.4.*

Proof. By linearity, we need only check products of the b_i. □

Lemma 2.17. $(-)$ *is well-defined, and*

$$\bar{b}_{\pi(i_1)} \wedge \cdots \wedge \bar{b}_{\pi(i_t)} = (-)^{\pi} \bar{b}_{i_1} \wedge \cdots \wedge \bar{b}_{i_t}, \quad \forall t \geq 2.$$

Proof. $(-)$ is well-defined by Theorem 2.4. The formula follows from writing a permutation as the product of transpositions, noting that the sign of a permutation is well-defined, and counting the number of times $(-)$ occurs. □

Lemma 2.18. *Suppose $\sum_{\mathbf{i}} \alpha_{\mathbf{i}} b_{i_1} \otimes \cdots \otimes b_{i_k} + d \preceq \sum_{\mathbf{i}} \alpha_{\mathbf{i}'} b'_{i_1} \otimes \cdots \otimes b'_{i_k} + d'$, where $i_1 < \cdots < i_k$, $\alpha_{\mathbf{i}}, \alpha_{\mathbf{i}'} \in \mathcal{A}$, $d, d' \in \overline{\mathfrak{G}^{\circ}}$. Then $\sum_{\mathbf{i}} \alpha_{\mathbf{i}} b_{i_1} \otimes \cdots \otimes b_{i_k} \preceq_{\circ} \sum_{\mathbf{i}} \alpha_{\mathbf{i}'} b'_{i_1} \otimes \cdots \otimes b'_{i_k}$, where $i_1 < \cdots < i_k$.*

Proof. Match components, eliminating those components in which some b_i repeats or some of the b_i descend. □

Definition 2.19. $T(V)_{\rm doub}$ is the ideal of $T(V)$ generated by all elements $b_i \otimes b_i$ for all i.

Lemma 2.20. *Any nonzero element of $T(V)$ is a sum of terms $(\pm)\alpha\, b_{i_1} \cdots \otimes b_{i_k} + d$, where $i_1 < \cdots < i_k$, $\alpha \in \mathcal{A}$, and $d \in T(V)_{\rm doub}$.*

Proof. We rearrange the b_i appearing in the summands, noting that any time a b_i repeats, the product is in $T(V)_{\rm doub}$. □

We use Lemma 2.18 to avoid $T(V)_{\rm doub}$ in our computations. We will need the following nondegeneracy result.

Proposition 2.21. *Suppose $V = \mathcal{A}^{(n)}$ and $u, u' \in \mathfrak{G}(V)_k$ for $2 \leq k < n$.*

(i) *If $u \wedge v = u' \wedge v$ for all $v \in \mathfrak{G}(V)_{n-k}$, then $u = u'$.*

(ii) *If $u \notin T(V)^{\circ}_k$ then there is some $v \in T(V)_{n-k}$ for which $u \wedge v \notin T(V)^{\circ}_k$.*

Proof. Using Lemma 2.20, write

$$u = \sum_{i_1 < \cdots < i_k} \alpha_{i_1, \ldots i_k} b_{i_1} \wedge \cdots \wedge b_{i_k}, u' = \sum_{i_1 < \ldots i_k} \alpha'_{i_1, \ldots i_k} b_{i'_1} \wedge \cdots \wedge b_{i'_k}.$$

(i) For any $\alpha_{1,\ldots k} \neq \mathbb{0}$, $u \wedge b_{i_{k+1}} \wedge \cdots \wedge b_{i_n} = (\pm)\alpha_{1,\ldots k} b_1 \wedge \cdots \wedge b_n$, which must be $(\pm)\alpha'_{1,\ldots k} b_1 \wedge \cdots \wedge b_n$, with the base elements matching up.

(ii) Adjusting notation, we may assume that $\alpha_{1,\ldots k} \neq 0$. But then

$$u \wedge b_{k+1} \wedge \cdots \wedge b_n = \alpha_{1,\ldots k} b_1 \wedge \cdots \wedge b_n \notin T(V)^{\circ \sharp}_n.$$

□

Theorem 2.22. *More in line with* [8]*, define the* ***partially reduced Grassmann algebra*** *$\mathfrak{G}(V)^{\diamondsuit}_{\geq 2}$ to be $T(V)_{\geq 2}$ modded out by the congruence generated by $(b_i \wedge b_i, 0)$ for all i, $\mathfrak{G}(V)^{\sharp}$ to be $\widehat{T(V)}$ modded out by the congruence generated by $(b_i \wedge b_j, 0) \cong (0, b_j \wedge b_i)$ for all i, j, and $\widehat{\mathcal{T}}^{\sharp}$ to be the corresponding image of $\widehat{\mathcal{T}}$.*

There is a triple $(\mathfrak{G}(V)^{\sharp}, \widehat{\mathcal{T}}^{\sharp}, (-)_{\rm sw})$, together with an embedding of triples

$$(\mathfrak{G}(V)^{\diamondsuit}_{\geq 2}, \mathcal{T}_{\mathfrak{G}(V)_{\geq 2^{\diamondsuit}}}, (-)) \to (\mathfrak{G}(V)^{\sharp}, \mathcal{T}^{\sharp}, (-)_{\rm sw})$$

given by $c \mapsto (c, 0)$.

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Note 2.23. $\mathfrak{G}(V)_{\geq 2}$ is clearly a homomorphic image of $\mathfrak{G}(V)^{\diamondsuit}_{\geq 2}$, since it has more relations. (All $v \otimes v$ are sent to $\mathbb{0}$, not just the $b_i \otimes b_i$.) Although $\mathfrak{G}(V)^{\diamondsuit}_{\geq 2}$ and $\mathfrak{G}(V)_{\geq 2}$ both coincide with the Grassmann algebra in the classical setting where V is a vector space over a field, they differ in the semiring setting, since $b_i \otimes b_j + b_j \otimes b_i$ is not $\mathbb{0}$ in $\mathfrak{G}(V)^{\sharp}_{\geq 2}$.

2.3. **Digression: Related notions.** For the remainder of this section we examine algebraic notions related to this paper, even though one can bypass them for the proofs of Theorems 3.14 and 3.15.

2.3.1. *The case when V already has a negation map.* We have seen that V itself need not have a negation map, for us "almost" to define a negation map on $T(V)$. In case V does have a negation map $(-)$,[2] we need a slight modification. We define a negation map on the tensor product $V \otimes W$ by $(-)(v \otimes w) = ((-)v) \otimes w$. When W also has a negation map $(-)$ we define a **negated tensor product** $V \otimes_{(-)} W$ by imposing the extra axiom

$$((-)v) \otimes_{(-)} w = v \otimes_{(-)} ((-)w), \qquad v \in V, w \in W.$$

(One mods out the tensor product by the congruence generated by all elements $((-)v \otimes w,\ v \otimes (-)w)$.)

Remark 2.24. The appropriate triple is $(\mathfrak{G}, \mathcal{T}_{\mathfrak{G}}, (-))$, where $\mathcal{T}_{\mathfrak{G}} = \{v_1 \wedge \cdots \wedge v_t : v_i \in V,\ t \in \mathbb{N}\}$, the submonoid generated by $\mathcal{T}$, with $(-)(v_1 \wedge \cdots \wedge v_t) = ((-)v_1) \wedge \cdots \wedge v_t$.

2.3.2. *Comparison with* [8]. The arguments of [8], whose objective is to obtain a Grassmann algebra point of view for Plücker relations, can be adapted to this situation. We use the systemic version which enables us to replace the "bend relation" $f \sim g$ of [8] for $f, g \in \mathrm{Hom}(V, \mathcal{A})$ by $f + g$ being a quasi-zero in the sense that $(f + g)(v) \in \mathcal{A}^{\circ}$ for every $v \in V$. Then the Plücker relations in [8, Proposition 4.1.2] become the conditions that $\sum_{i \in A \setminus B} v_{A-\{i\}} v_{B+\{i\}}$ is a quasi-zero.

Note 2.25. The flavor of the Grassmann algebra might be better preserved by taking the negation map $(-)$ not to be the identity map, but rather as defined here, which also could be obtained using symmetrization. Note also that $\mathfrak{G}$ is commutative in [8, Definition 3.1.2]. So why does [8, Definition 3.1.2] work? The answer is that $\mathfrak{G}$ is largely a book-keeping devise to keep track of sets of vectors without repetition, and application of its theory to matroids does not require much of multiplication other than $e_i \wedge e_i = 0$. [8, Proposition 3.1.4] is formal. One needs a cancelation result parallel to [8, Lemma 3.2.2], and [8, Proposition 4.2.1] requires the ability to switch vectors e_i and e_j.

Remark 2.26. Suppose that $\mathcal{A}$ is "zero sum free" in the sense that $a_1 + a_2 = \mathbb{0}$ implies $a_1 = a_2 = \mathbb{0}$. Then the base $\mathcal{B}$ of a free module V is unique up to multiplication of invertible elements of $\mathcal{A}$. (Otherwise some b_i does not appear in the new base, and we cannot recover b_i since we cannot zero out extraneous coefficients.)

[2]For example V could be the free $\mathcal{A}$-module with a negation map, with base $\{b_i, (-)b_i : i \in I\}$.

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2.3.3. *Digression: The Grassmann envelope.*

Remark 2.27. Just as with classical algebra, one can use $\mathfrak{G}$ to study a super-semialgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ using its **Grassmann envelope** $\mathcal{A}_0 \otimes \mathfrak{G}_0 + \mathcal{A}_1 \otimes \mathfrak{G}_1 \subset \mathcal{A} \otimes \mathfrak{G}$. Following Zelmanov, we say that a super-semialgebra $\mathcal{A}$ is **super-P** if its Grassmann envelope is P. For example, $\mathcal{A}$ is super-commutative if its Grassmann envelope is commutative. In particular, $\mathfrak{G}$ itself is super-commutative.

Then one can study linear algebra over super-commutative super-semialgebras, super-anticommutative super-semialgebras, and so forth, as indicated in [21, §8.2.2].

3. Hasse-Schmidt derivations on Grassmann semi-algebras

Having set out the general framework, let us turn to the situation at hand. We review our set-up, in the special case of power series over endomorphisms of the Grassmann algebra. As before, $V_n := \mathcal{A}^{(n)}$ is the free module with basis $\mathbf{b} := \{b_0, \dots, b_{n-1}\}$. (We start our subscripts with 0 in consonance with the notation for projective space.) Let $T_0(V_n) = \mathcal{A}$, and $T_k(V_n) := V_n \otimes V_n \otimes \cdots \otimes V_n$ be its k tensor power. Define a negation $(-) : T_2(V_n) \to T_2(V_n)$ by mapping $u \otimes v$ to $v \otimes u$. In particular $(-)(u \otimes u) = u \otimes u$. We extend this to $(-) : T_k(V_n) \to T_k(V_n)$ by means of Theorem 2.4 and Lemma 2.17. Let

$$T_{\geq 2}(V_n) = \{\mathbb{0}\} \cup \bigoplus_{k \geq 2} T_k(V_n),$$

a semiring[††] with multiplication given by tensoring. We consider two variants of the Grassmann algebra:

(1) $\mathfrak{G}(V)^{\sharp}_{\geq 2}$, modding out by the congruence $\mathcal{I}$ of $T_{\geq 2}(V_n)$ generated by all $\{(b_i \otimes b_i, 0) : 0 \leq i < n\}$,
(2) $\mathfrak{G}(V)_{\geq 2}$, modding out by the congruence $\mathcal{I}$ of $T_{\geq 2}(V_n)$ generated by all $\{(u \otimes u, 0) : u \in V_n\}$.

In either case we will work with a graded Grassmann semialgebra $\mathfrak{G}$, which now we denote as

$$\bigwedge V_n = \bigoplus_{r \geq 0} \overset{r}{\bigwedge} V_n, \qquad \text{where}$$

$$\overset{0}{\bigwedge} V_n = \mathcal{A}, \qquad \overset{1}{\bigwedge} V_n = V_n, \qquad \text{and} \qquad \overset{r}{\bigwedge} V_n := \frac{T_r(V_n)}{\mathcal{I} \cap T_r(V_n)} \quad \textit{for } r \geq 2.$$

Thus $u \wedge v$ denotes the image of $u \otimes v$ through the natural map $T(V_n) \to \bigwedge V_n$. Here $\preceq$ is $\preceq_\circ$.

Remark 3.1. By Theorem 2.4, each submodule $\bigwedge^r V_n$, $r \geq 2$, inherits a negation map by putting

$$(-)(u_1 \wedge u_2 \cdots \wedge u_r) = u_2 \wedge u_1 \wedge \cdots \wedge u_r.$$

Remark 3.2.

(i) For each $r \geq 2$, $\bigwedge^r V_n$ is spanned by words $b_{i_0} \wedge b_{i_1} \wedge \cdots \wedge b_{i_{r-1}}$ of length r. In particular, $\bigwedge^r V_n$ is a free $\mathcal{A}$ module spanned by $[\mathbf{b}]^r_{\boldsymbol{\lambda}}$, where

$$\boldsymbol{\lambda} := (\lambda_1 \geq \cdots \geq \lambda_r), \qquad [\mathbf{b}]^r_{\boldsymbol{\lambda}} := b_{\lambda_r} \wedge b_{1+\lambda_{r-1}} \wedge \cdots \wedge b_{r-1+\lambda_1}.$$

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We are interested in the $\mathbb{N}$-graded power series semiring $(\bigwedge V_n)[[z]] := \oplus_{r\geq 0} \bigwedge V_n z^r$ of Definition 1.14 (and later its super-version), and its endomorphisms.

Since the congruences are homogeneous, we define

(3.1)
$$\bigwedge^{\geq 1} V_n := \bigoplus_{r\geq 1} \bigwedge^r V_n, \qquad \bigwedge^{\geq 2} V_n := \bigoplus_{r\geq 2} \bigwedge^r V_n, \qquad \text{and} \qquad \bigwedge^{\neq 1} V_n := \bigoplus_{r\neq 1} \bigwedge^r V_n$$

Definition 3.3. Let $D\{z\} := \sum_{i\geq 0} D_i z^i \in \mathrm{End}(\bigwedge V_n)[[z]]$ be homogeneous of degree 0 (i.e. $D_i(\bigwedge^r V_n) \subseteq \bigwedge^r V_n$ and in particular $D_i(V_n) \subseteq V_n$)). If

$$D\{z\}(u \wedge v) = D\{z\}u \wedge D\{z\}v \tag{3.2}$$

we say that it is a *Hasse-Schmidt (HS) derivation* on $\bigwedge V_n$.

To simplify notation let us simply denote the identity map on V_n as "$\mathbb{1}_V$," also identified with $D\{z\}$ where $D_0 = \mathbb{1}$ and all other $D_i = \mathbb{0}$.

Equation (3.2) is equivalent to:

$$D_k(u \wedge v) = \sum_{i+j=k} D_i u \wedge D_j v, \quad \forall k \geq 0, \quad \forall u, v \in \bigwedge V_n. \tag{3.3}$$

For $r \geq 2$, any element of $\bigwedge^r V_n$ is a linear combination of monomials $v_1 \wedge \cdots \wedge v_r$ of length r. The definition shows that $D\{z\}$ is uniquely determined by the values it takes on elements of V.

In the following we shall restrict to a special class of HS derivations, useful for the applications.

Proposition 3.4. For any $f \in \mathrm{End}_{\mathcal{A}}(V_n)$, there exists a unique HS-derivation $D^f\{z\}$ on $\bigwedge V_n$ such that $D^f\{z\}_{|V_n} = \sum_{i\geq 0} f^i z^i$.

Proof. For the chosen $\mathcal{A}$-basis of the module V we necessarily have $D^f\{z\}(b_j) = \sum_{i\geq 0} f^i(b_j) z^i$. Write $f(z)$ for $\sum_{i\geq 0} f^i z^i$. One defines $D^f\{z\}$ on $\bigwedge V$ by setting for each degree:

$$D^f\{z\}(b_{i_1} \wedge \cdots \wedge b_{i_j}) = f(z)b_{i_1} \wedge \cdots \wedge f(z)b_{i_j}, \quad 1 \leq j \leq r. \tag{3.4}$$

If D were another derivation satisfying the same initial condition, it would coincide on all the basis elements of $\bigwedge V_n$, which generate all elements of $\bigwedge V_n$. □

Example 3.5. Let us compute $D_2^f(b_1 \wedge b_2)$ where $f(b_i) = b_{i+1}$. Then

$$\begin{aligned} D_2^f(b_1 \wedge b_2) &= D_2^f(b_1) \wedge b_2 + D_1^f b_1 \wedge D_1^f b_2 + b_1 \wedge D_2^f b_2 \\ &= f^2(b_1) \wedge b_2 + f(b_1) \wedge f(b_2) + b_1 \wedge f^2(b_2) \\ &= b_3 \wedge b_2 + b_2 \wedge b_3 + b_1 \wedge b_4 \succeq b_1 \wedge b_4, \end{aligned}$$

since $b_3 \wedge b_2 + b_2 \wedge b_3$ is a quasi-zero.

From now on we shall fix the endomorphism f once and for all, and write $D\{z\} := D^f(z)$ and $D := D_{1|V} := f$. Also we write $D_i v$ for $D_i(v)$ and $D\{z\}v$ for $\sum D_i v \cdot z^i$. In particular, for each $v \in V_n$ the equality $D_i v = D_1^i v = f^i(v)$ holds.

Lemma 3.6. *For $u, v \in V_n$,*

(i) $D\{z\}v = v + D\{z\}(D_1 v)z$.

(ii) $D\{z\}(u \wedge v) = u \wedge D\{z\}v + zD\{z\}(D_1 u \wedge v)$.

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Proof.

(i) $D\{z\}v = v + \sum_{i\geq 1} D_i v\, z^i = v + \sum_{i\geq 1}(D_{i-1}D_1 v\, z^{i-1})z = v + D\{z\}(D_1 v)z.$

(ii) $D\{z\}(u\wedge v) = D\{z\}u \wedge D\{z\}v = (u + zD\{z\}D_1 u)\wedge D\{z\}v)$
$= u\wedge D\{z\}v + zD\{z\}D_1 u \wedge D\{z\}v = u\wedge D\{z\}v + zD\{z\}(D_1 u\wedge v).$

□

3.1. **The canonical quasi-inverse of $D\{z\}$.**

Definition 3.7. $\overline{D}\{z\} := \sum_{i\geq 0}\overline{D}_i z^i \in \mathrm{End}(\bigwedge^{\neq 1} V_n)[[z]]$ is a **(left) quasi-inverse** of $D\{z\}$ if

$$\overline{D}\{z\}D\{z\}u \succeq u, \quad \forall u \in \bigwedge^{\neq 1} V_n. \tag{3.5}$$

Our next task consists in constructing a quasi-inverse $\overline{D}\{z\}$ of the HS derivation $D\{z\}$, that we will achieve through a number of steps necessary to cope with the difficulty of not having a natural negation map on V_n. This can be done in two ways: First do it in the classical case, and then apply the "transfer principle" of Remark 1.7. However, one gets more precise information by taking the direct analog.

Construction. Towards this purpose we first consider the map:

$$\begin{cases} \overline{D} & : & V_n & \longrightarrow & \mathrm{End}_{\mathcal{A}}(\bigwedge V_n) \\ & & u & \longmapsto & \overline{D}u \end{cases} \tag{3.6}$$

such that $\overline{D}u(v) := v\wedge D_1 u$ for all $v\in V_n$. If $v\in \bigwedge^{\geq 2} V_n$, we may assume it is of the form $v = v_1\wedge v_2$ with $v_1\in V_n$. In this case we define

$$\overline{D}u(v) = \overline{D}u(v_1\wedge v_2) = \overline{D}u(v_1)\wedge v_2 = v_1\wedge D_1 u\wedge v_2. \tag{3.7}$$

It is easily seen that (3.7) suffices to define $\overline{D}u$ on all the Grassmann semi-algebras and also that the definition does not depend on the representation of the same element $v\in\bigwedge V_n$. In fact any one such is a finite linear combination of tensors of the form $v := v_1\wedge\cdots\wedge v_k$, and the definition of $\overline{D}u(v)$ does not change if we replace the given expression of v with an equivalent one after an even permutation of the factors. It will be useful to identify V_n as a subset of $\mathrm{End}_{\mathcal{A}}(\bigwedge V_n)$, by viewing each of its elements as a (wedge) multiplication operator, under the map $v\mapsto v\wedge_$, i.e. $v(w) = v\wedge w$ for all $w\in\bigwedge V_n$.

Definition 3.8. Let $\overline{D}\{z\} := 1 + \overline{D}_1 z + \overline{D}_2 z^2 + \cdots + \overline{D}_n z^n : V_n \to \mathrm{End}_{\mathcal{A}}(\bigwedge V_n)[z]$ defined as follows. If $u_1\wedge\cdots\wedge u_k\in\bigwedge^k V_n$, then

$$\overline{D}\{z\}(u_1\wedge\cdots\wedge u_k) = \overline{D}\{z\}(u_1)\circ\cdots\circ\overline{D}\{z\}(u_k) \tag{3.8}$$

where $\circ$ is the composition in $\mathrm{End}_{\mathcal{A}}(\bigwedge V_n)$ and where for all $u\in V_n = \bigwedge^1 V_n$ we set

$$(\overline{D}\{z\}u)(v) = u\wedge v + z\overline{D}u(v) \in \mathrm{End}_{\mathcal{A}}(\bigwedge V_n)[z] \tag{3.9}$$

acting on $v\in\bigwedge V_n$ as $\overline{D}\{z\}u(v) = u\wedge v + (\overline{D}u)v$. Now we extend $\overline{D}\{z\}$ to a map $\bigwedge V_n \to \mathrm{End}_{\mathcal{A}}(\bigwedge V_n)$ defining it on monomials of degree $k\geq 2$ via

$$(1+\overline{D}_1 z+\cdots+\overline{D}_k z^k)(u_1\wedge\cdots\wedge u_k) = \overline{D}\{z\}(u_1\wedge\cdots\wedge u_k) = \overline{D}\{z\}(u_1)\circ\cdots\circ\overline{D}\{z\}(u_k)$$

Remark 3.9. By definition it follows that if $u\in\bigwedge^i V_n$, then $\overline{D}_j u = 0$ for all $j > i$.

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Example 3.10. In this example we compute $\overline{D}_1(u_1 \wedge u_2)$, $\overline{D}_2(u_1 \wedge u_2)$ and $\overline{D}_2(u_1 \wedge u_2 \wedge u_3)$ to illustrate how the definition works. By definition $\overline{D}_1(u_1 \wedge u_2)$ and $\overline{D}_2(u_1 \wedge u_2)$ are the coefficients of z and z^2 in the expansion $(\overline{D}u_1) \circ (\overline{D}_1 u_2)$ that we apply to a test element $w \in \bigwedge V_n$. By definition one may assume $w \in V_n$. One has:

$$\begin{aligned}
& (u_1 \wedge u_2 + z(\overline{D}u_1 \circ u_2 + u_1 \circ \overline{D}u_2) + z^2 \overline{D}u_1 \circ \overline{D}u_2)w \\
= \;& u_1 \wedge u_2 \wedge w + z\overline{D}u_1(u_2 \wedge w) + u_1 \wedge \overline{D}u_2(w) + z^2 \overline{D}u_1(\overline{D}u_2(w)) \\
= \;& u_1 \wedge u_2 \wedge w + z(u_2 \wedge D_1 u_1 \wedge w + u_1 \wedge (w \wedge D_1 u_2)) + z^2 \overline{D}u_1(w \wedge D_1 u_2) \\
= \;& u_1 \wedge u_2 \wedge w + z(u_2 \wedge D_1 u_1 + D_1 u_2 \wedge u_1) \wedge w + z^2 (w \wedge D_1 u_1 \wedge D_1 u_2) \\
= \;& [u_1 \wedge u_2 + (D_1 u_2 \wedge u_1 + u_2 \wedge D_1 u_1)z + (D_1 u_1 \wedge D_1 u_2)z^2] \wedge w.
\end{aligned}$$

We have obtained:

$$\overline{D}_1(u_1 \wedge u_2) = (D_1 u_2 \wedge u_1 + u_2 \wedge D_1 u_1) \wedge _ \; : \bigwedge V_n \to \bigwedge V_n,$$

$$\overline{D}_2(u_1 \wedge u_2) = (D_1 u_1 \wedge D_1 u_2) \wedge _ \; : \bigwedge V_n \to \bigwedge V_n.$$

Similarly we can find that

$$\overline{D}_2(u_1 \wedge u_2 \wedge u_3) = \overline{D}_2(u_1 \wedge u_2) \circ u_3 + \overline{D}_1(u_1 \wedge u_2) \circ \overline{D}_1 u_3,$$

i.e., more explicitly

$$\overline{D}_2(u_1 \wedge u_2 \wedge u_3) \wedge w = D_1 u_1 \wedge D_1 u_2 \wedge u_3 \wedge w + (D_1 u_2 \wedge u_1 + u_2 \wedge D_1 u_1) \wedge w \wedge D_1 u_3$$

from which

$$\begin{aligned}
& \overline{D}_2(u_1 \wedge u_2 \wedge u_3) \wedge w \\
& \quad = (D_1 u_1 \wedge D_1 u_2 \wedge u_3 + D_1 u_2 \wedge D_1 u_3 \wedge u_1 + u_2 \wedge D_1 u_3 \wedge D_1 u_1) \wedge w \\
& \quad = (D_1 u_1 \wedge D_1 u_2 \wedge u_3 + u_1 \wedge D_1 u_2 \wedge D_1 u_3 + D_1 u_1 \wedge u_2 \wedge D_1 u_3) \wedge w.
\end{aligned}$$

We could say that the operators $\overline{D}_1(u_1 \wedge u_2)$, $\overline{D}_2(u_1 \wedge u_2)$, and $\overline{D}_2(u_1 \wedge u_2 \wedge u_3)$ are "represented" respectively by the following elements of $\bigwedge V_n$:

$$D_1 u_2 \wedge u_1 + u_2 \wedge D_1 u_1 = D_1(u_2 \wedge u_1), \qquad D_1 u_1 \wedge D_1 u_2,$$

and

$$D_1 u_1 \wedge D_1 u_2 \wedge u_3 + u_1 \wedge D_1 u_2 \wedge D_1 u_3 + D_1 u_1 \wedge u_2 \wedge D_1 u_3.$$

Remark 3.11. Let us check that $D_1 u \wedge v + (\overline{D}_1 u)(v) \succeq 0$, which is the sense we want to give to the expression $D_1 + \overline{D}_1 \succeq 0$. For all $u, v \in \bigwedge V_n$. If $u \in V_n$ and $v = v_1 \wedge v_2$ with $v_1 \in V_n$:

$$\begin{aligned}
D_1 u \wedge v + (\overline{D}_1 u)(v) &= (D_1 u \wedge v_1) \wedge v_2 + (\overline{D}u)(v_1) \wedge v_2 \\
&= D_1 u \wedge v_1 \wedge v_2 + v_1 \wedge D_1 u \wedge v_2 \\
&= (D_1 u \wedge v_1 + v_1 \wedge D_1 u) \wedge v_2 \succeq 0.
\end{aligned}$$

More in general we have the following crucial:

Theorem 3.12. *The polynomial $\overline{D}\{z\}$ is a quasi-inverse of $D\{z\}$, in the sense that for all $u \in \bigwedge V_n$*

$$\overline{D}\{z\} D\{z\} u \succeq u. \tag{3.10}$$

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Proof. We first check that the property holds for all $u \in V_n$. Then, for all $w \in \bigwedge V_n$

$$\begin{aligned}(\overline{D}\{z\}D\{z\}u)(w) &= (D\{z\}u + z\overline{D}D\{z\}u)(w)\\ &= D\{z\}u \wedge w + zw \wedge D_1 D\{z\}u\\ &= (u + zD_1 D\{z\}u) \wedge w + z \cdot w \wedge D_1 D\{z\}u\\ &= u \wedge w + z\big(D_1 D\{z\}u \wedge w + w \wedge D_1 D\{z\}u\big) \succeq u \wedge w\end{aligned}$$

for all $w \in \bigwedge V_n$. Thus we have proved that $\overline{D}\{z\}D\{z\}u \wedge _ \ \succeq \ u \wedge _$, and the property is checked for all $u \in \bigwedge^1 V_n$. Now, we argue by induction, by supposing the property holds true for all $u \in \bigwedge^{\leq k-1} V_n$. Let us prove it for all $u \in \bigwedge^k V_n$. In this case we can assume u of the form $u_1 \wedge v$, with $u_1 \in V_n$. Then

$$\overline{D}\{z\}(D\{z\}(u_1 \wedge v)) = \overline{D}\{z\}(D\{z\}u_1) \circ \overline{D}\{z\}(\overline{D}\{z\}v) \succeq (u_1 \wedge v) \wedge _$$

having used induction and the first step. □

Counterexample 3.13. Quite surprisingly, while $\overline{D}\{z\}$ is a quasi-inverse of $D\{z\}$, the reverse is not true. For instance, for all $u, v \in V_n$ one has: $D\{z\}(\overline{D}\{z\}u)(v) \succeq u \wedge D\{z\}v = u + D\{z\}D_1 u$. Let us check it, recalling that if $u \in V_n$ then $D\{z\}u = u + \sum_{i\geq 0} D_1^i u \cdot z^i$.

$$\begin{aligned}&D\{z\}(\overline{D}\{z\}u)(v)\\ &\quad = D\{z\}(u \wedge v + z \cdot v \wedge D_1 u) && \text{(definition (3.7) of}(D\{z\}u)(v))\\ &\quad = D\{z\}u \wedge D\{z\}v + zD\{z\}v \wedge D\{z\}D_1 u && (D\{z\}\text{is a HS derivation})\\ &\quad = u \wedge D\{z\}v + zD\{z\}D_1 u \wedge D\{z\}v\\ &\quad + zD\{z\}v \wedge D\{z\}D_1 u && \text{(using } D\{z\}u = u + D\{z\}D_1 u)\\ &\quad \succeq u \wedge D\{z\}v.\end{aligned}$$

As an additional check, notice that if $D\{z\}$ were a quasi-inverse of $\overline{D}\{z\}$, then

$$(1 + D_1 z + D_2 z^2 + \cdot)(1 + \overline{D}_1 + \overline{D}_2 z^2 + \cdots) \succeq 0$$

In particular, considering the coefficient of z^2 in both sides, the following surpassing relation should hold:

$$D_2 + D_1\overline{D}_1 + \overline{D}_2 \succeq 0. \tag{3.11}$$

But (3.11) already fails for $u \in V_n = \bigwedge^1 V_n$. Indeed, for all $v \in V_n$, and noting that $\overline{D}_2 u = 0$, by Remark 3.9:

$$\begin{aligned}&(D_2 u + D_1\overline{D}_1 u + \overline{D}_2 u)(v)\\ &\quad = D_2 u \wedge v + D_1(v \wedge D_1 u) && \text{(applying to a test vector } v)\\ &\quad = D_2 u \wedge v + D_1 v \wedge D_1 u + v \wedge D_2 u && \text{(Leibniz rule enjoyed by } D_1)\\ &\quad = D_1 v \wedge D_1 u + D_2 u \wedge v + v \wedge D_2 u\\ &\quad \succeq D_1 v \wedge D_1 u\end{aligned}$$

Theorem 3.14. $\overline{D}\{z\}(D\{z\}u \wedge v) \succeq u \wedge \overline{D}\{z\}v, \qquad \forall u, v \in \bigwedge V_n.$

Proof. Suppose u, v are homogeneous, say $u \in \bigwedge^k V_n$ and $v \in \bigwedge^\ell V_n$. Then, by Definition 3.8 of $\overline{D}\{z\}$,

$$\overline{D}\{z\}(D\{z\}u \wedge v) = \overline{D}\{z\}(D\{z\}u) \circ \overline{D}\{z\}v$$

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By Theorem 3.12, $\overline{D}\{z\}(D\{z\}u\wedge v)(w) \succeq u\circ(\overline{D}\{z\}v)(w) = u\wedge(\overline{D}\{z\}v)(w)$, because u acts as an endomorphism on vectors of $\bigwedge V_n$ as $u\wedge _$, for all $w\in\bigwedge V_n$. □

3.2. **The Cayley-Hamilton formulas for semialgebras.** Formally define $\zeta = b_0\wedge b_1\wedge\cdots\wedge b_{n-1}$ and $\zeta' = b_1\wedge b_0\wedge\cdots\wedge b_{n-1}$. Thus $\zeta' = (-)\zeta$, and

$$\overline{D}_i\zeta = e_i\zeta + e_i'\zeta', \quad e_i, e_i'\in\mathcal{A}. \tag{3.12}$$

In other words, (e_i, e_i') could be called the **eigenvalue pair** of $\overline{D}_i$ restricted to $\bigwedge^n V_n$ (where in some sense e_i' is the negated part). Let $E_n(z)$ be the eigenvalue polynomial of $\overline{D}\{z\}$, i.e.

$$E_n(z)\zeta := \overline{D}\{z\}\zeta + \overline{D}\{z\}\zeta' = (1+e_1z+\cdots+e_nz^n)\zeta + (1+e_1'z+\cdots+e_n'z^n)\zeta'.$$

In particular if one sets $D_i\zeta = h_i\zeta + h_i'\zeta'$, the relations $\overline{D}\{z\}D\{z\}\zeta \succeq \zeta$ and $\overline{D}\{z\}D\{z\}\zeta'\succeq\zeta'$ yield the relation

$$(h_n + e_1h_{n-1}+\cdots+e_n) + (h_n' + e_1'h_{n-1}'+\cdots+e_n') \succeq \mathbb{0}. \tag{3.13}$$

Theorem 3.15. *The Cayley-Hamilton formulas*

$$((D_nu + e_1D_{n-1}u+\cdots+e_nu)\wedge v)\,(-)\,((D_nu+e_1'D_{n-1}u+\cdots+e_n'u)\wedge v)\succeq\mathbb{0} \tag{3.14}$$

hold for all $u\in\bigwedge^{>0}V_n$, i.e., the left side is a quasi-zero.

Proof. If $u=\zeta$ the theorem is true, due to (3.13). Then assume that $u\in\bigwedge^{n-i}V_n$, for some $1\le i\le n-1$. This follows from the transfer principle of Remark 1.7, since the assertion was proved (with equality) for classical algebras in [6], and all the extra quasi-zeros appear in the right. But we also would like to give a direct proof. For all $v\in\bigwedge^i V_n$ we have the surpassing relation (3.14). Matching degrees yields the surpassing relation between the n-th degree coefficient of the left side and the n-th degree coefficient of the right side of (3.14) which is:

$$D_nu\wedge v + \overline{D}_1(D_{n-1}u\wedge v) + \cdots + \overline{D}_n(u\wedge v) \succeq u\wedge\overline{D}_nv.$$

Since $\overline{D}\{z\}v$ is a polynomial of degree at most $i<n$, it follows that $\overline{D}_kv\succeq\mathbb{0}$ for all $k>i$. On the other hand $\overline{D}_i(D_{n-i}u\wedge v) = e_i(D_{n-i}u\wedge v)(-)e_i'(D_{n-i}u\wedge v)$ because (e_i,e_i') is the eigenvalue pair of $\overline{D}_i$ against any element of $\bigwedge^n V_n\cong(\mathcal{A}\zeta+\mathcal{A}\zeta')$. Thus we have proved (3.14) for all $v\in\bigwedge V_n$. □

Remark 3.16. In the classical case where the only quasi-zero is $\{\mathbb{0}\}$, we get

$$((D_nu + e_1D_{n-1}u+\cdots+e_nu)\wedge v)\,(-)\,((D_nu+e_1'D_{n-1}u+\cdots+e_n'u)\wedge v) = \mathbb{0}. \tag{3.15}$$

Corollary 3.17. $(D_1^n + (e_1(-)e_1')D_1^{n-1}+\cdots+(e_n(-)e_n'))u\succeq\mathbb{0}$ *for all* $u\in\bigwedge^{>0}V_n$, *where we interpret* $(e_i(-)e_i')D_i^{n-i}(u)$ *to be* $e_iD_i^{n-i}u\,(-)\,e_i'D_i^{n-i}u$.

Proof. By Theorem 3.15,

$$((D_nu + e_1D_{n-1}u+\cdots+e_nu)\wedge v)\,(-)\,((D_nu+e_1'D_{n-1}u+\cdots+e_n'u)\wedge v)\succeq\mathbb{0}.$$

But $D\{z\}$ is by hypothesis the unique HS-derivation on $\bigwedge V_n$ associated to the endomorphism D_1 (see Proposition 3.4). In particular $D_iu = D_1^iu$. □

Note 3.18. When working with $\mathfrak{G}(V)_{\ge2}$, we obtain equality in Theorem 3.15 and Corollary 3.17 since the only quasi-zeros are $\mathbb{0}$, by Theorem 2.9.

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