Representing Geometry Between Graphic Constructions and Mathematical Modeling

# Representing geometry between graphic constructions and mathematical modeling

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**Abstract.** Our study exposes a search for specific pedagogical practices between mathematical modeling and visual thinking to support first-year Architecture to acquire spatial visualization and representation skills and abilities by exploiting the connections between mathematics and architecture. For this purpose, in our Architectural Drawing and Survey Laboratory, activities were designed, discussed and validated by various critical reading tools across the disciplinary fields involved: designing and building objects that have certain properties and that satisfy certain specifications, such as the construction of a graphic representation 2D of a 3D object, has been described by mathematical education scholars as a particular type of mathematical modeling process and by architects as a process of developing spatial prefiguration skills and visual thinking. The analysed case study, the different possible plane sections of cones and cylinders, allows us to discuss the difficulties in learning the relationships between theoretical and/or analytical information, 2D and/or 3D representations, physical and/or digital modeling, as well as to introduce the concept of rigour from the mathematical and drawing point of view. Moments of assessment and interaction have led to highlighting not only the strong link between synthetic and analytical treatment, between object and representation, but also the need to formally structure the monitoring/tutoring activities to support each of the students, so that they can become part of the community in formation.

**Keywords:** Geometry, Graphic education, Mathematical modeling.

## 1 Introduction

There are many tools and methods for teaching Representation to first-year students in the Architecture degree course. This teaching unit, offered at the Politecnico di Torino to first-year students, includes the foundations and applications of descriptive geometry, drawing and survey. It lasts for the whole year and is an essential requirement before the students access the second year. The course structure, with the consequent temporal sequence of theoretical contents and applicative moments, is mediated by its educational urgencies: in fact, drawing appears not only as an operation for representation as a goal, but as a tool which is cognitive, analytical, and only lastly descriptive, to approach a shape, architectural or not, in an inevitable comparison and integration with other disciplinary approaches. As such, drawing relates to the other knowledge underlying the architect's training and is therefore included in what is defined as the year of instruction. On the one hand, the need for a graphic language defined by shared norms and codes to be univocal is indisputable while, on the other hand, it must be evident that considering drawing 'just a language' may be limiting, as it could prevent grasping the synergy between content and shape that becomes an expression of content. The problem is that newly arriving students have very variable backgrounds about geometrical and graphic language: in the last 4 years, since the orientation policies of our university management allowed access to reading and interpreting its data, the architecture entrance test highlights basic gaps in its drawing part, despite, for example, plane figures and nomenclature being included in the representation and geometry programs of lower secondary school teaching in Italy [Bocconcino et al. 2022; Pavignano and Zich 2022].

Our study exposes a search for specific pedagogical practices that can act as enablers of students' attempts to appropriate the process underlying 2D representation techniques. In this investigation, around 126 first-year students were engaged in lectures and exercises during the course and also in their assessment, tutoring, questionnaires, and interviews. All the activities had been structured within an interdisciplinary framework exploiting the connections between mathematics and architecture, to critically question procedures and techniques and suggest which aspects of mathematical thinking would be necessary for students to learn representation better.

## 2 Graphic & Geometrical Education Questions

At the Architectural Drawing and Survey Laboratory (hereinafter ADSLab), in the last 10 years of activity, it has emerged that almost 50% of the whole class always has very high difficulties passing descriptive geometry exercises and using basic geometry in architectural representation. A series of auxiliary tools were gradually introduced jointly with related disciplines, such as mathematics, actively involved in the definition of interdisciplinary activities and drafting of study support materials [Cumino et al. 2017, Cumino et al. 2019]. These auxiliary practices were initially satellites of the course; monitoring their outcomes led to making them an integral part of the ADSLab activity, by building a shared model of knowledge [Cumino et al. 2023b]. It is a fact that, in the training process of an architect, it is important to know abstract geometric shapes and their properties to recognize them in the built environment and acquire the mathematical tools that support the graphic ones. In this sense, the theory of mathematical modeling constitutes a theoretical framework, where we found a common ground to describe the graphic procedures leading to 2D representations and to highlight difficulties and problems underlying them [Niss 2015].

### 2.1 Plane figures in orthographic projections: polygons and circumferences

Within the ADSLab, the teaching of the fundamentals of geometry starts from the concept of projection, therefore also involving subjects who have not had specific training in secondary schools. Simple exercises of orthographic projections of plane figures assigned in space with respect to the reference planes, belonging or parallel or orthogonal to them, have shown the difficulty of correlating information between object and representation, highlighting the need to reconstruct the logical sequence underlying the representation itself. While representation of polygons is achieved immediately by almost 70% of the subjects involved, representation of a circumference is achieved by less than 35% and the difficulty emerges in identifying the minimum requirements necessary to proceed. For example, in an exercise that requires the representation of a circular element belonging to a building facade arranged with vertical walls not parallel to the projection planes, going through the practice of overlapping is particularly problematic: leaving aside the outcomes where the object description is not understood, the variety of errors ranges between complete absence of construction, inappropriate direct construction in projection with subsequent transformation on the plane into true size and correct construction on the auxiliary plane which does not find correlation with the projections on the reference planes.

### 2.2 Plane sections of solids: representing conics graphically

The activity of representing different plane sections of cones and cylinders allowed us to discuss the difficulties in the relationships between theoretical and/or analytical information with 2D and/or 3D representations, as well as between physical and/or digital modelling. This topic is present in the curricular paths common to the entire student population, because it belongs to compulsory training; it is proposed and explored in depth by distributing it over the ADSLab time with different tools and contextualization: orthographic projections, axonometries, perspectives, modelling AutoCAD 3D, various dynamical geometric software; therefore, we need to define how and what to represent, according to the purposes; so, the exercise about conics here proposed is a clear demonstration of this.

The activity on conics starts almost immediately, talking about ruled surfaces, showing examples of rotation and translation surfaces, introducing the concept of graphic construction and ‘rigour’ in representation with an awareness that any outcome of a graphic construction is affected by the chosen tools. To address the concept of rigour in a graphic representation, the possibility of comparing representations with ruler and compass (henceforth R&C) with those with CAD (2D and 3D) was fundamental, as well as the comparison between the operational sequences and their intersection with mathematics.

In the first training moment, the construction of cone sections was only graphically discussed. As visible in Fig. 1, the cone, always represented on 3 related projection planes, is cut by a series of projecting planes whose description can always be traced back to geometric properties of the cone which become a reference system for the entire construction. The activity, described verbally with geometric vocabulary, is initially carried out graphically on the blackboard as a sequence of section planes useful for identifying points deemed necessary for the graphic construction of the identified sections. It follows that what was obtained is the result of an operation that is only theoretically rigorous because any tracing on the blackboard cannot be so, but it is equally clear that any tracing performed with R&C cannot be so. The same process is then proposed with CAD, explaining when certain tools can be used for tracing the sections obtained. The graphic sequences for R&C / CAD construction may therefore differ. For points, lines and circles in both modes, every element necessary for tracing is easily identified with the available tools, but what may affect their rigour is how they are used. For hyperbolas and parabolas, it is necessary to proceed, both in the R&C environment and in 2D CAD, with a sequence of section planes to identify the points useful for tracing the curve in question: the outcome changes, depending on how many and which planes are used, on the detail with which the tracing is carried out, and, in the case of CAD, also on which tool is chosen [Cumino et al. 2020].

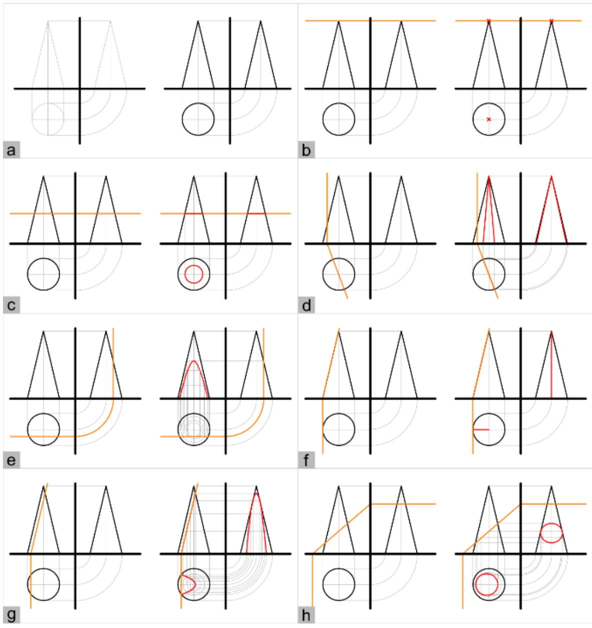


**Fig. 1.** Cone and its plane sections. 2D AutoCAD graphic construction and elaboration by UZ.

A focus on the ellipse allows us to introduce a further step: in addition to the dynamics explained for the construction by points, it is also possible, with 2D CAD, to compare the outcome with the tool used for tracing the ellipse, leading the students to consider the rigour of the CAD tool itself. It is clear that to use the ‘ellipse’ tool proposed by CAD, certain information is needed and therefore the geometric sequence mentioned above can be oriented towards direct identification of points necessary to use the dedicated tool, thus making the different approaches converge.

The subsequent possibility of working in a 3D CAD environment and obtaining not only ellipses but also hyperbolas and parabolas, as a direct result of intersection of the modelled surface by insertion of a section plane, leads further to compare and integrate tools useful for create the critical context in which to educate on the representation of architecture complexity. To assess learned skills, we proposed a test (from now on "the skewer") giving students a sheet with data in orthographic projections without defined thicknesses; students had to complete it with R&C constructions to show their ability in visualizing the simple solids co-present in the same environment and not intersecting each other; each student had to insert a straight line intersecting only two solids of the assigned set, following a textual description [Zich 2017]. A further request was to recognize which conic section is identified by the auxiliary plane used to solve the first question. "The skewer" emphasized the difficulties of the students in identifying the assigned solids, which relied on an oral description and a multitude of mistakes in the conic sections they had determined. Beyond the problems relating to graphic constructions, the lack of awareness of geometric objects has become evident: the most widespread error arises from plane sections of the cone which, being sometimes only an arc of an ellipse, are mistakenly considered as arcs of parabolas or hyperbolas. Hence, the interdisciplinary intervention, already planned within the ADSLab, has been further supported to explore the question in depth with other tools, not just graphics.

### 2.3 Plane sections of solids: other descriptions of conics sections

During regular drawing lessons on conics and surfaces, we made an interdisciplinary intervention that joined professors of architectural drawing and mathematics, and we presented conics as plane sections of circular right cones, and different types of graphical representation of an ellipse (as section of a circular right cylinder or cone, in the plane as a foreshortened circumference, as the gardener's ellipse, as tangents envelope). The choice of language for both disciplines was basic, accessible to all, also due to the presence of high school students in virtual class, according to a practice consolidated in previous years, in the context of an orientation project [Pavignano and Zich 2022]. To fill the gaps mentioned above and to integrate what has been done from the point of view of drawing, we provided different types of graphical representation for any geometric object we introduced, using a mathematical point of view. An analytical study was added to the traditional gardener's ellipse; then, we proposed not only to write the equation of the ellipse but also to give a geometric motivation for the construction of the ellipse as a geometric locus, by using GeoGebra, and as an envelope of its tangents; this last topic was outside the topics related to drawing, however it provided the theoretical support to create a folding procedure of origami type to construct by hand an ellipse [Cumino et al. 2018]; this material experience has involved students and favoured discussions. Moreover, we proposed GeoGebra applets with animations to show conics in the space and ellipses in the plane as draggable objects to highlight shape variations and invariant properties with respect to movements of the foci, pointing out a "tangible" link between synthetic and analytic treatment.

# 3 Graphic & Geometrical Education Questions

Designing and constructing objects to have certain properties and to meet certain specifications, like the construction of a 2D graphic representation of a 3D object, has been described by scholars of mathematical education as a particular type of mathematical modelling process (that of prescriptive type) [Niss 2015; Cumino et al. 2023a]. Generally, the mathematical modelling process is schematized by a so-called modelling cycle like the following Fig. 2:

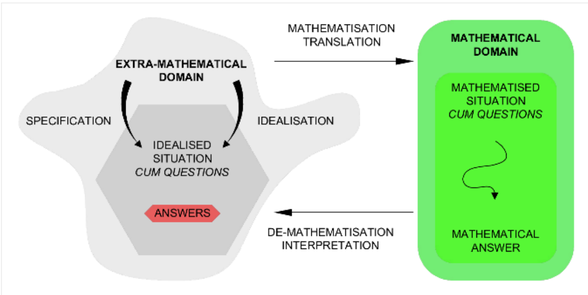


**Fig. 2.** Mathematical modeling cycle [Authors' graphic elaboration after Niss 2010, p. 44].

## 3.1 Mathematical modelling versus representation

Here is an example of mathematical modelling as an investigation tool applied to "the skewer" test. To highlight the points where students might have difficulties in the mathematical thinking supporting the graphic procedure, we examined the second question of the self-assessment test (which conic section is identified by the plane used to find the line/cone intersection points?), considering the theory of mathematical modelling and its cycle. This exercise aimed to construct a 2D representation to faithfully reflect one's spatial vision of a conic curve as a plane section of a right circular cone.

The modelling cycle for this type of modelling is very rudimentary. Initially there is no existing extra-mathematical domain to model. Mathematization amounts to translating the design requirements into mathematical requirements amenable to mathematical treatment. For example, independently of the graphical procedure, in regards with the choice of the plane, a mathematical question is how one can predict which conic is determined by a given plane sectioning the cone. The mathematical treatment is made up by a series of straightforward orthographic projections; but it is the crucial step, because students are engaged in identifying and analysing the hidden or explicit assumptions and the prerequisites that underlie the procedure undertaken; for example, the choice of points on the trace circumference of the cone, and the choice of their number, to identify generating lines on the cone and points of the conic: how many points to identify a conic? Is there some symmetry to be respected? The de-mathematization step is trivial, since it consists of nothing but inserting details to the obtained drawing, such as thickness and type of lines, symbols, following the drawing standards [Cumino et al. 2023a]. As for validation, we engage students in a critical evaluation of their outcomes with respect to the issues addressed and the enterprise purpose, proposing them the possible iteration of the proof, until they obtain an acceptable result.

Research has shown that there is a common underlying obstacle at different stages of the modelling cycle: the need in several places, for the modeler, to know in advance what can be done in subsequent stages, after the current stage has been completed [Niss 2010]. This need, called “implemented anticipation”, is central to completing the process of modelling successfully and involves teachers reflecting on students’ thinking as intermediary to the problem itself (see also the concept of “anticipatory metacognition” by [Geiger 2022]).

With this aim, after interdisciplinary supporting activities and production of specific materials, we proposed a questionnaire about conics (103 participating students), where we posed questions about their representation and synthetic 2D/3D geometry, with dichotomous and multiple-choice questions, using different approaches and languages, to stimulate the awareness of the students and investigate aspects of their thinking before they face a new test with “the skewer.” The critical analysis of students’ answers was done synergically by drawing and mathematics professors. As a consequence, we decided to offer a questionnaire review ex cathedra to the entire class, before the further test, but we wondered whether or not the errors found in the results were independent of the high schools of origin; this thought pushed us to choose a few students (20% of the entire class) for an interview following the authors’ prediction of the questions that could be particularly confusing and/or contradictory for the students to answer, Our aim was both a targeted action to see how to plan the entire subsequent activity, and a check whether it was useful for them to be individually sensitised before reviewing the questionnaire in class. The interview essentially consisted of two parts: at first a request to think about the errors made in the questionnaire and above all about the reasons behind them, then a request to explain one’s method in answering and checking the accuracy of given answers. The questionnaire highlights among other things the effectiveness of different languages for asking questions (such as Rhino, Dynamic Geometric Software like GeoGebra) and the power of visualization compared to word questions in quizzes on conic sections. Here is an example of a conceptual error attributable to poor mastery of languages: in the questionnaire, almost 20% of the students admitted the existence, on a sphere, of an elliptic section with axes of different length, showing an identification between object and its representation; even if, during the interview, this error was mainly attributed to haste or distraction. Also, the questionnaire highlights misconceptions about the concept of rigour [Dedò 2012] in graphic (30%) and analytic (40%) representations of an ellipse; about this, from the sample of students interviewed it emerges among other things that the concept of rigour, both graphic and mathematical, is not easily acquired; interestingly it turned out that the activity of constructing the ellipse with the origami technique favoured its achievement [Cumino et al. 2018].

The subsequent self-assessment test made it possible to evaluate the impact of the questionnaire and the individual interviews on the students. It was possible to remark that the questionnaire stimulated students to ask themselves questions (as we expected), but there was an unexpected outcome: only the interviewed subjects did the test better and concretely showed a critical approach to the procedure chosen for the representation process, both in evaluating the information necessary to proceed with the individual steps (questioning for example: are 8 points enough for drawing the curve?), and in the evaluation step of the final result.

### 3.2 Cutting 3D objects: issues from the point of view of the science of representation

As already recalled within the frame of mathematical modelling (see preceding paragraph) and according to Niss (2015), the construction of a 2D graphic representation of a 3D object may be described as a mathematical modelling process of prescriptive type.

Here we discuss what other abilities contribute to this elaboration from the point of view of the science of representation (besides the ability in drawing and with its tools). In this sense, we can highlight the contribution of spatial visualization, which mainly deals with the theoretical framework of spatial thinking (Fig. 3 left). The term spatial thinking refers to different concepts such as spatial perception, spatial ability, visual perception, and spatial intelligence [Maresch and Sorby 2021, p. 274] (Fig. 3 right).

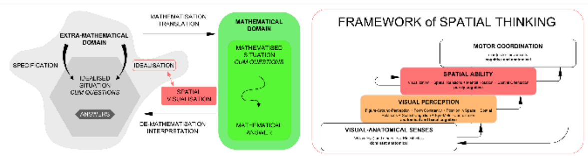


**Fig. 3.** Spatial visualization and spatial thinking within the framework of mathematical modelling [MP's graphic elaboration after Niss 2015] with a focus on the successive stages of the process of spatial thinking. We highlight the phases of our interest [MP's graphic elaboration after Maresch and Sorby 2021, p. 275].

For our study we focus on the concepts of spatial perception, ability, and intelligence. It is not possible to provide unique definitions of the recalled declination of spatial thinking [Yilmaz 2009], nonetheless: spatial abilities may be referred to as "skills in representing, transforming, generating and recalling symbolic, non-linguistic information" [Linn and Petersen 1985, p. 1482], hence a complex system [Nagy-Kondor and Esmailnia 2022, p. 3752]; visual perception may be defined as the awareness of the external world essentially through sensorial relationships [Kesner 2014], or, as R. Arnheim argues, «any line drawn on a sheet of paper, the simplest form modelled from a piece of clay, it is like a rock thrown into the pond. It upsets repose, it mobilizes space. Seeing is the perception of action» [Arnheim 1974, p. 16]; spatial intelligence can be defined as the skill of visualizing with the mind's eye and is of great importance for STEAM students, so also for those of architecture [Schaik 2008; Buckley et al. 2018; El Bedewy et al. 2022].

So, most of the questions of the described questionnaire are intended to stimulate students' spatial thinking. We now focus our attention on question n. 3, asking: "Which of the ellipses A, B, C, D in the figure can be obtained as a planar section from a right circular cylinder generated by the translation of the circumference E? You can provide one or more answers." From the point of view of spatial thinking, students had two different ways to solve the question: 1) visualize the 3D shape of the cylinder, then visualize the possible cutting planes defining the correct sections (A, C); 2) compare the diameter of the circular section of the cylinder with the minor diameter of the elliptical sections and identify the correct couple of answers.

The first hypothesis clearly required a strong set of spatial thinking abilities, while the second hypothesis required a good set of theoretical knowledge, connecting different aspects of geometry (basic knowledge of ellipses and their properties, basic knowledge of the right circular cylinder and its properties. To answer the question in the first way students must visualize the problem as in Fig. 4. A similar approach can be used to solve questions about sections of cones. In fact, question n. 5 asked students to name four conics on the base of the isometric representation of four cones and different sectioning planes highlighting the results of each intersection (upper part of Fig. 5). In this very case, the application of spatial ability, specifically mental rotation, allows students to rotate intersection curves on the horizontal plane, hence allowing them to visualize curves and their geometries.

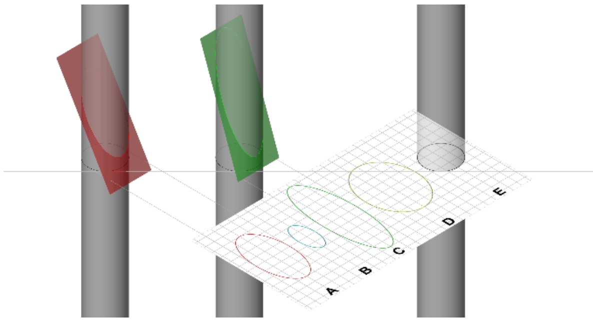

**Fig. 4.** Possible answer to question 3 conducted via spatial thinking abilities (visualization and spatial intelligence). Modeling and graphic elaboration by MP.

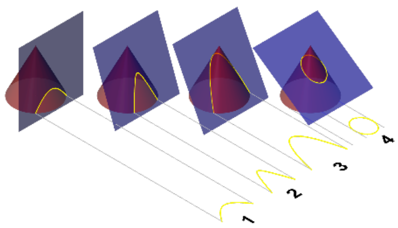

**Fig. 5.** Possible answers to question n. 5 conducted via spatial thinking abilities (visualization by mental rotation and spatial intelligence). Modeling and graphic elaboration by MP.

### 3.3 Text versus 3D description

Within the questionnaire, the question about recognition of plane sections of a cone was proposed by presenting the same case in two versions, to verify how students correlate geometric language with spatial visualization: the first one starting from textual description only (hereinafter referred to as TD) and the second one starting from the representation of a digital modelling (hereinafter referred to as 3DD).

The graphic question was: which of the assigned planes (one or more) identifies the proposed curves on the cone; it was possible to indicate one or more answers for each case. Subsequently, the question was presented by following textual geometric description: “an infinite double-sided rotational cone is given, with opening angle 2θ at the vertex, with the following planes α, β, γ, δ, where: α (alpha) is parallel to the axis of the cone, β (beta) is inclined at an angle θ with respect to the axis of the cone, γ (gamma) is inclined at an angle smaller than θ with respect to the axis of the cone, δ (delta) is inclined at an angle greater than θ with respect to the axis of the cone”.

In both descriptive methods, the 4 cases proposed led to the recognition of a parabola, an ellipse and two hyperbola descriptions (see Fig. 5); these latter caused the greatest number of errors. The results were evaluated as follows: considered correct when both solutions were identified - TD25/103 - 3DD 43/103; incomplete if only one of the 2 cases answering the question has been recognized - TD 53/103 - 3DD 39/103; partially incorrect if one or more incorrect answers have been added to a correct answer - TD17/103 - 3DD 13/103; completely incorrect if the answer (one or more) was entirely incorrect - TD 8/103 - 3DD 8/103. 19 participants answered correctly in both TD and 3DD, only one student got both completely wrong; however, many subjects made a mistake in one mode of communication and not in the other. Furthermore, the difference between completely correct TD and 3DD responses is very high, but by adding these numbers with those of incomplete answers, the numbers get very close. In the second test “the skewer” – proposed in orthographic projections as an exercise to be completed by drawing –, the question of recognizing the section identified on the cone by the auxiliary plane was still object of many errors; comparing these ones with the questionnaire results, we could deduce that: recognizing a plane section when this is the only object under analysis is simpler than when the context is a space shared with other objects; choosing from a list of terms the correct one is more accessible than having to know it directly; having to produce the representation by themselves was no help in this regard.

## 4 Conclusions

We provided students with different approaches to architectural representation, allowing them to combine their background to train their graphic intelligence [Cicalò 2017] and better grasp the geometrical properties of theoretical and built shapes. In our context, enhancing students’ competency to independently and successfully conduct mathematical modelling processes, which underlies their drawing activities, has not to be perceived as a vehicle for learning mathematics but as an educational objective of architect training. On the other hand, solutions to the ‘conics’ issues might seem trivial but highlight quite an important perspective within the STEAM studies. Previous studies have already demonstrated the fundamental role of spatial abilities for STEAM students [for example Gerber 2020; Nagy-Kondor, Esmailnia 2022], for example when they are asked to produce drawings of specific intersections between given surfaces. Nonetheless, our study frames how the mathematical modelling processes can support students' learning geometry with an interdisciplinary approach.

## Credits and acknowledgements

Although the contribution was conceived jointly, paragraphs were written by: C. Cumino 3, 3.1; M. Pavignano 3.2; U. Zich 2, 2.1, 2.2, 2.3, 3.3; all Authors: 1 and 4. Authors would like to thank the Reviewers for their helpful suggestions.

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