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Review Article

Soap films: from the Plateau problem to deformable boundaries

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Abstract

A review on the classical Plateau problem is presented. Then, the state of the art about the Kirchhoff-Plateau problem is illustrated as well as some possible future directions of research.

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AMS subject classification: 49Q05, 49Q20, 74K10, 74G65.

1. Introduction

Soap films arise as equilibrium interfaces between two fluids. Young and Laplace, at the beginning of the 19th century, gave an expression for the pressure difference p_c (the *capillary pressure*) over an interface between two fluids. Precisely, the relation, which is now called *Young-Laplace equation*, is given by

$$p_c = \sigma \left(\frac{1}{r_1} + \frac{1}{r_2} \right),$$

where $\sigma > 0$ is a constant called *surface tension* and r_1, r_2 are the principal radii of curvature of the interface S (see, for instance, [1] for a derivation of the Young-Laplace equation). The surface tension measures the amount of energy one needs to extend the surface S by one unit area. Taking into account the definition of mean curvature of S, denoted by H, we can rewrite the Young-Laplace equation as $p_c = \sigma H$. As a consequence, the interface is in equilibrium if and only if H is constant. From the physical point of view, two different kind of configurations are essentially possible. In one case, the interface S forms a closed surface, that is a compact surface without boundary; then, S must be a sphere, and this explains why soap bubbles are round. In the other case, the interface S is a surface with boundary. In this case, the Young-Laplace equation becomes $H = 0$. The best physical model for these kind of surfaces is represented by soap films: putting a rigid wire in a soap solution and extracting it, a thin soap film will remains attached to the wire. In the middle of the 19th century, the Belgian physicist Plateau devised many experiments putting rigid wires in a soap solution in order to understand the possible singular configurations of soap films. For this reason, still today we use the terminology *Plateau problem* to deal with the problem of finding the shape of soap films with some prescribed boundaries.

From a mathematical point of view, soap films turn out to be stable *minimal surfaces*. The connection between soap films and minimal surfaces dates back to Gauss who worked, in the 19th century, on capillarity problems. Actually, in the middle of the 18th century, Lagrange was the first who investigated minimal surfaces as critical points of the area functional. Indeed, at least in the smooth case, the *minimal surface equation* $H = 0$ is the Euler-Lagrange equation of the area functional. This suggests an interesting change of point of view: instead of solving directly the partial differential equation $H = 0$, one might look at minimizers of the area functional. This variational strategy permits to obtain directly a stable minimal surface, which should produce a corresponding soap film.

However, using some special solutions of the equation $H = 0$, one can construct many examples of minimal surfaces. The *catenoid*, discovered by Euler in 1744 and obtained as a revolution surface, is the first example of non planar minimal surface. The *helicoid* is another classical example as well as the *Enneper surface* and the *Scherk surface* [2]. Some of these examples of minimal surfaces can be seen as soap films: the catenoid appears also as a solution of the soap film bounded by two sufficiently close coaxial rings, while the helicoid spans a circular helix (see Figure 1).

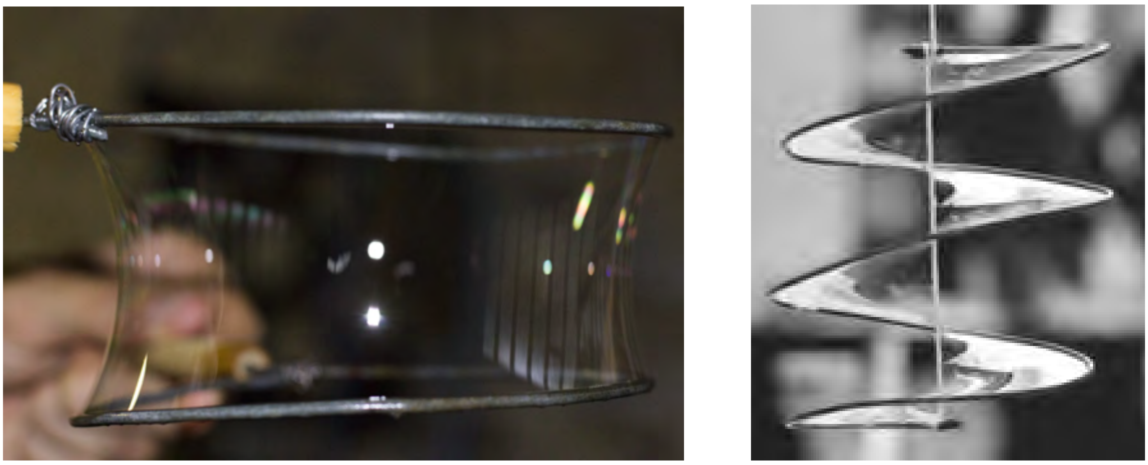

Figure 1. The catenoid (on the left) and the helicoid (on the right) as soap films.

The general formulation of the Plateau problem might be the following one: *given a closed curve* Γ *in the space find a surface with minimal area spanning* Γ. Since it seems that every closed wire spans some soap film, Plateau was convinced that every closed curve with no double points spans a surface which minimizes the area. Moreover, as far as experiments suggest, there are only two kind of singular configurations: the $\mathbb{Y}$-*configuration*, three plane sheets crossing on a line and forming a 120° angle, and the $\mathbb{T}$-*configuration*, four lines crossing in a point (called *tetrahedrical* point) and forming an angle of approximately 109, 47°. Pictures in Figure 2 show that these singularities may occur. The $\mathbb{Y}$ and the

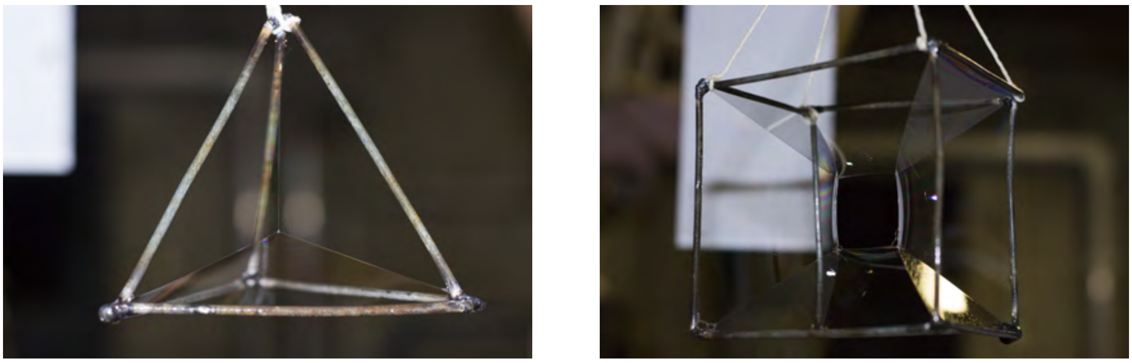

Figure 2. The soap film obtained by the edges of a tetrahedron (on the left) and the soap film realized by the edges of a cube (on the right).

$\mathbb{T}$ singularities are the only conjectured by Plateau. For this reason, the fact that a soap film can only produces $\mathbb{Y}$ and/or $\mathbb{T}$ singularities are known as *Plateau laws*.

In this paper, we will first of all review the main techniques for solving the Plateau problem. From the mathematical point of view, the problem is very difficult and a lot of possible formulations are available. Precisely, in Section 2 we will briefly mention how the classical solution by Douglas and Radó works, then we will pass to review more recent formulations of the problem in the context of Geometric Measure Theory: sets of finite perimeter, currents, and minimal sets. An important generalization of the

Plateau problem is presented in Section 3 where the so-called *Kirchhoff-Plateau problem* is introduced: the boundary is elastic and can sustain bending and twisting. We will present some recent results for the Kirchhoff-Plateau problem and some of its generalizations. Finally, we conclude in Section 4 with other results and some future directions of investigations.

2. Plateau problem: classical and non-classical tools

In order to rigorously state the Plateau problem, we need to clarify three things: what *surface* means, what *area of a surface* means, and the concept of *spanning* a prescribed boundary curve.

2.1. *Plateau problem for graphs*

The first attempt is to deal with graphs of functions. In this setting, the most elementary problem is to find a *global minimal graph*, namely a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ whose graph solves the equation $H = 0$. Any affine function is obviously a solution. Is it the unique solution? This is the celebrated *Bernstein problem*, that can be generalized to any dimension: if the graph of a smooth function $\mathbb{R}^n \to \mathbb{R}$ is a minimal surface in $\mathbb{R}^{n+1}$, does this imply that it is an affine function? Bernstein formulated the problem in 1914 and solved it, in the same year, only for $n = 2$. After that, many mathematicians tried to attack the problem in higher dimensions: we mention Simons who answered positively in $n = 6$ and gave an example of locally stable cones in $\mathbb{R}^8$ but without proving that these cones are minimal surfaces on the whole space $\mathbb{R}^8$. Finally, Bombieri, De Giorgi and Giusti showed that Simons cones are indeed minimal surfaces in $\mathbb{R}^n$ for $n \geq 8$. An example is the cone $\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$.

If $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a smooth graph, then the Plateau problem reads

$$\begin{cases} -\mathrm{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \text{ on } \Omega, \\ u = u_0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is open and bounded in $\mathbb{R}^n$. Concerning the solution, we mention Jenkin and Serrin [3]: if $\partial\Omega$ is of class $C^{2,\alpha}$ for some $\alpha \in (0, 1)$, and $u_0 \in C^{2,\alpha}(\overline{\Omega})$ then (1) has a solution if and only if the mean curvature of $\partial\Omega$ is everywhere non-negative. On the other hand, if $\partial\Omega$ is of class $C^{2,\alpha}$ for some $\alpha \in (0, 1)$, then there exists $\varepsilon > 0$ such that for every $u_0 \in C^{2,\alpha}(\overline{\Omega})$ with $\|u_0\|_{2,\alpha} \leq \varepsilon$ the problem (1) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ [4,5]. Moreover, (1) can be stated in a variational way [6]: it is the Euler-Lagrange equation of the area functional written for graphs like

$$u \mapsto \int_\Omega \sqrt{1 + |\nabla u|^2} dx. \tag{2}$$

In [6], the authors show that if Ω is convex and $u_0\colon \partial\Omega \to \mathbb{R}$ satisfies the bounded slope condition (that is u_0 satisfies a Lipschitz inequality), then the functional (2) has a unique minimizer among all Lipschitz functions with $u = u_0$ on $\partial\Omega$.

2.2. *Disc-type solutions*

The first rigorous solution to the Plateau problem is due to Douglas [7] and Radó [8], who independently developed an argument which works only for 2-dimensional surfaces in $\mathbb{R}^3$ and in codimension 1. We also mention simplifications in the proof of Courant, Tonelli and Dierkes [9]. The basic idea is to look at smooth parametrizations $\mathbf{X}\colon D \to \mathbb{R}^3$ where $D = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ is the disc and the trace of $\mathbf{X}$ on ∂D is a smooth parametrization of a prescribed Jordan curve Γ in $\mathbb{R}^3$. Thus, the Plateau problem reads

$$\mathbf{A}(\mathbf{X}) = \int_D |\partial_u \mathbf{X} \times \partial_v \mathbf{X}|\, dudv.$$

In order to apply the Direct Method of the Calculus of Variations, one immediately notices that $\mathbf{A}$ is weakly lower semicontinuous since the map $(u, v) \mapsto |\partial_u \mathbf{X}(u, v) \times \partial_v \mathbf{X}(u, v)|$ is a convex function of the determinants of the 2×2 minors of $\nabla \mathbf{X}$ (what is called *polyconvex* function [10]). Concerning compactness, the set $\{\mathbf{X} : \mathbf{A}(\mathbf{X}) \leq c\}$ is not bounded in any reasonable Sobolev norm since the area functional is invariant under reparametrization. However, considering conformal coordinates, it is easy to see that

$$\mathbf{A}(\mathbf{X}) \leq \frac{1}{2} \int_D |\nabla \mathbf{X}|^2 \, dudv = \mathbf{D}(\mathbf{X}),$$

and the equality holds true if and only if $\mathbf{X}$ is conformal, that is $|\mathbf{X}_u| = |\mathbf{X}_v|$ and $\mathbf{X}_u \cdot \mathbf{X}_v = 0$. This suggests to minimize directly the Dirichlet functional $\mathbf{D}$, which is not invariant under reparametrization, among all $\mathbf{X} \in W^{1,2}(D;\mathbb{R}^3)$ such that $\mathbf{X}_{|\partial D}$ is a reparametrization of Γ. Precisely, there exists a minimizer $\mathbf{X}_0 \in C^0(\overline{D};\mathbb{R}^3)$ which is harmonic on D and $\mathbf{X}_0$ is conformal, hence $\mathbf{D}(\mathbf{X}_0) = \mathbf{A}(\mathbf{X}_0)$. It is left to show that every minimizer $\mathbf{X}_0$ of $\mathbf{D}$ satisfying $\mathbf{D}(\mathbf{X}_0) = \mathbf{A}(\mathbf{X}_0)$ is a minimizer for the area functional. Obvioulsly, since for any admissible $\mathbf{X}$ it holds $\mathbf{A}(\mathbf{X}) \leq \mathbf{D}(\mathbf{X})$, then

$$\inf_{\mathbf{X}} \mathbf{A} \leq \inf_{\mathbf{X}} \mathbf{D}.$$

To have the equality one can apply the ε-conformal mappings Lemma due to Morrey [11]: if $\mathbf{X} \in C^0(\overline{D};\mathbb{R}^3) \cap W^{1,2}(D;\mathbb{R}^3)$ then for any $\varepsilon > 0$ there exists a homeomorphism $\tau_\varepsilon \colon \overline{D} \to \overline{D}$ of class $W^{1,2}$ such that $\mathbf{D}(\mathbf{X} \circ \tau_\varepsilon) \leq \mathbf{A}(\mathbf{X}) + \varepsilon$. Finally, $\mathbf{X}_0$ produces a regular surface, namely $\partial_u \mathbf{X} \times \partial_v \mathbf{X} \neq 0$ everywhere, indeed if Γ is an analytical Jordan curve and if its total curvature does not exceed 4π then any disc-type solution of Plateau problem is a regular minimal surface [12]. Moreover, disc-type minimal

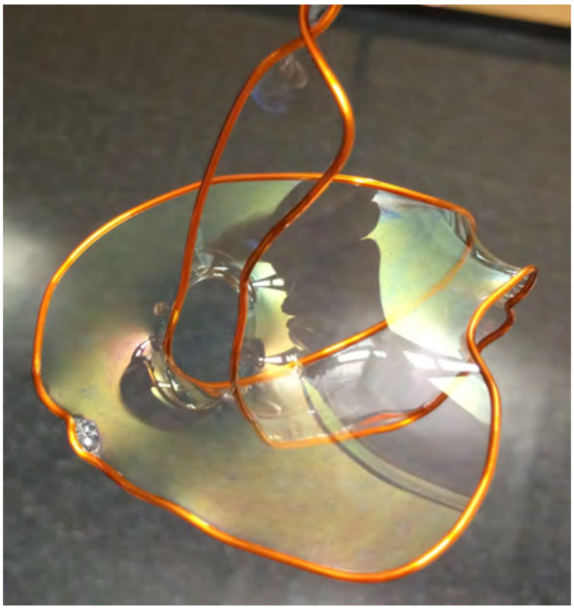

Figure 3. The area minimizing soap film spanning a disc wants to be embedded.

surfaces cannot produce singularities in the interior [13]: they are immersed surfaces and not embedded, without self-intersections (see for instance the embedded soap film solution in Figure 3), not providing a good model for soap films.

2.3. *Distributional approaches*

Concerning distributional approaches to solve the Plateau problem, we describe two approaches: sets of finite perimeter and currents.

We refer to the book by Ambrosio-Fusco-Pallara [14] or to the monograph by Maggi [15] for details on the theory of finite perimeter sets. Let $E \subset \mathbb{R}^n$ be a Borel set with finite Lebesgue measure and let

$\Omega \subseteq \mathbb{R}^n$ be open. We say that E is a *finite perimeter set in* Ω if

$$\mathscr{P}(E;\Omega) = \sup\left\{\int_{E\cap\Omega} \operatorname{div}\phi\, dx : \phi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\phi\|_\infty \le 1\right\} < +\infty.$$

The quantity $\mathscr{P}(E;\Omega)$ is the *perimeter of* E *in* Ω. If E is a bounded and open with smooth boundary, then E has finite perimeter in Ω and $\mathscr{P}(E;\Omega) = \mathscr{H}^{n-1}(\partial E \cap \Omega)$. By duality, we immediately can say that the map $E \mapsto \mathscr{P}(E;\Omega)$ is lower semicontinuous with respect to the L^1-convergence of characteristic function f sets. Moreover, if $\mathscr{P}(E_h)$ is bounded and E_h are contained in a given ball, then, up to a subsequence, $E_h \stackrel{L^1}{\to} E$ and E has finite perimeter.

Thus, the Plateau problem in this setting reads as follows:

$$\inf\left\{\mathscr{P}(E) : E \text{ has finite perimeter in } \Omega \text{ satisfying } \mathscr{L}^n((E\setminus\Omega)\triangle E_0) = 0\right\}, \tag{3}$$

where $E_0 \subset \mathbb{R}^n \setminus \Omega$ be such that $\partial E_0 \cap \partial\Omega = \Sigma_0 \subset \partial\Omega$ assigned. In particular, the Direct Method of the Calculus of Variations can be successfully applied and the problem (3) has a minimal solution.

Another distributional approach to the Plateau problem is the use of the theory of currents. The notion of current dates back to De Rham, while the variational and geometrical approach used today is mainly due to Federer and Fleming [16,17].

Let $\mathscr{D}^d(\mathbb{R}^n)$ be the set of d-forms on $\mathbb{R}^n$ with compact support. The space of *d-currents on* $\mathbb{R}^n$, denoted by $\mathscr{D}_d(\mathbb{R}^n)$, is the topological dual space of $\mathscr{D}^d(\mathbb{R}^n)$. Then, any d-dimensional smooth oriented surface S in $\mathbb{R}^n$ is an example of a current: $T_S \in \mathscr{D}_d(\mathbb{R}^n)$ is defined as follows

$$\langle T_S, \omega\rangle = \int_S \omega, \qquad \forall \omega \in \mathscr{D}^d(\mathbb{R}^n).$$

Also the boundary of a current can be defined via the Stokes' formula: if $T \in \mathscr{D}_d(\mathbb{R}^n)$, then $\partial T \in \mathscr{D}_{d-1}(\mathbb{R}^n)$ is the *boundary of* T and it is given by

$$\langle \partial T, \omega\rangle = \langle T, d\omega\rangle.$$

Moreover, by Stokes' formula and for smooth oriented surfaces, $\partial T_S = T_{\partial S}$.

To state the Plateau problem, the concept of the *mass* of a current $T \in \mathscr{D}_d(\mathbb{R}^n)$ is defined as

$$\mathbb{M}(T) = \sup_{||\omega(x)||\le 1} \langle T, \omega\rangle,$$

where $\|\omega(x)\|$ is a suitable notion of norm for d-forms. It turns out that $\mathbb{M}$ is lower semicontinuous with respect to the weak convergence of currents. For smooth oriented surfaces, it holds $\mathbb{M}(T_S) = \mathscr{H}^d(S)$.

Since the space of currents is too large, a subspace must be introduced: $T \in \mathscr{D}_d(\mathbb{R}^n)$ is a *d-rectifiable current with integer multiplicity* if there exist:

(a) a d-rectifiable set E in $\mathbb{R}^n$,
(b) an *orientation τ on E*, namely a Borel map that to $\mathscr{H}^d$-a.e. $x \in E$ assigns a unit simple d-vector $\tau(x)$ which spans T_xE,
(c) a *multiplicity function*, that is a $\mathscr{H}^d$-summable function $m\colon E \to \mathbb{N}$,

such that T can be defined as follows

$$\langle T, \omega\rangle = \int_E \langle \omega(x), \tau(x)\rangle\, m(x)\, d\mathscr{H}^d(x), \qquad \forall \omega \in \mathscr{D}^d(\mathbb{R}^n).$$

A current of this type is denoted by $[E,\tau,m]$. If S is a smooth d-dimensional surface oriented by τ then $T_S = [S,\tau,1]$.

Finally, a current $T \in \mathscr{D}_d(\mathbb{R}^n)$ is said to be a *d-integral current* if both T and ∂T are rectifiable currents with integer multiplicity. In this final class, a compactness theorem holds true (Federer-Fleming

Compactness Theorem): if (T_h) is a sequence of integral d-currents with $\mathbb{M}(T_h)+\mathbb{M}(\partial T_h)$ bounded then, up to a subsequence, $T_h \to T$ in $\mathscr{D}_d(\mathbb{R}^n)$, $\partial T_h \to \partial T$ where T is an integral current. Moreover,

$$\mathbb{M}(T) \leq \liminf_h \mathbb{M}(T_h), \qquad \mathbb{M}(\partial T) \leq \liminf_h \mathbb{M}(\partial T_h).$$

The Plateau problem in terms of integral currents can be stated as follows: let T_0 be a given integral d-current on $\mathbb{R}^n$; find a minimizer of $\mathbb{M}(T)$ among all currents d integral currents T with $\partial T = \partial T_0$.

Notice that this approach has some limitations:

1. the current solution to the Plateau problem can have multiplicity different from 1. Indeed, the right object to minimize should be the *size* of a current defined as

$$\mathbb{S}\left([E,\tau,m]\right) = \mathscr{H}^d\left(\{x \in E : m(x) \neq 0\}\right).$$

   However, for $\mathbb{S}$ a compactness theorem does not hold true.
2. any discontinuity on the orientation produces an nonphysical boundary. A possibility to overcome this issue is to produce non-orientable soap films (see Figure 4) and mathematically to deal with rectifiable currents *modulo* ν, where $\nu \geq 2$ is an integer or using the theory of varifold, for definitions and details see [18,19].

Figure 4. A Möbius strip-like soap film.

Finally, we would like to say that in both cases a minimizer produces actually a soap film. Thus, a regularity theory has been developed. For the set of finite perimeter the regularity is obtained for a set E in $\mathbb{R}^n$ which minimizes the perimeter with respect to all possible compactly supported perturbations. In this case it is possible to prove that $\partial E \setminus S$ is smooth, where S is the closed set of singularities and

(a) if $2 \leq n \leq 7$ then S is empty and ∂E is analytical;
(b) if $n = 8$ then S has no accumulation points in E;
(c) if $n \geq 9$ then $\mathscr{H}^d(S) = 0$ for every $d > n - 8$.

A similar regularity result holds for mass-minimizing currents: if T is a mass-minimizing 1-integral current in $\mathbb{R}^2$ then the "interior part" of T (the part of T which is not in the boundary of T) is made of disjoint line segments. Moreover, similar to the set of finite perimeter case, if $2 \leq n \leq 7$ then the interior part of any mass-minimizing $(n-1)$-integral current in $\mathbb{R}^n$ is a smooth embedded hypersurface: in Figure 3 the soap film solution corresponds to an embedded solution, which actually should be the mass-minimizing integral current. When $n > 7$, the Simons cone $C = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$ is an area-minimizer current developing a singularity in the origin [20]. Unfortunately, since in lower dimension, especially the physical one $n = 3$, both the set of finite perimeter approach and the current one do not develop singularities, they do not provide a good model to study soap films.

2.4. *Almgren minimal sets approach and Taylor regularity*

Minimal sets, introduced by Almgren in [21], represent the best model for soap films.

Let us recall the definition of minimal set. Let $S \subset \mathbb{R}^n$ be a closed set and $A \subset \mathbb{R}^n$ be an open set. We say that S is a *d-dimensional minimal set in A*, briefly *minimal set*, if for any closed ball $B \subset A$ and for every Lipschitz function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi_{|_{\mathbb{R}^n \setminus C}} = id$ and with $\varphi(C) \subset C$ we have $\mathscr{H}^d(S) \leq \mathscr{H}^d(\varphi(S))$. In 1976, Taylor [22] proved that 2-dimensional minimal sets in $\mathbb{R}^3$ may have singularities and these are exactly the ones produced by soap films and observed by Plateau in his experiments. More precisely, there are two kind of singularities (see Figure 2):

(a) the so-called $\mathbb{Y}$-*configuration*: three sheets crossing on a line and forming a $120°$ angle;
(b) the so-called $\mathbb{T}$-*configuration*: four lines crossing in a point forming a $109,47°$ angle.

In order to state a Plateau problem in this framework the main difficulty stems from the notion of boundary. Recently, a suitable theory has been developed and some existence results have been proved. The main idea has been introduced by Harrison [23,24]. The approach by Harrison and Pugh is based on differential chains and it permits to represent all types of observed soap films as well as immersed surfaces of various genus types, both orientable and nonorientable, see Figure 5.

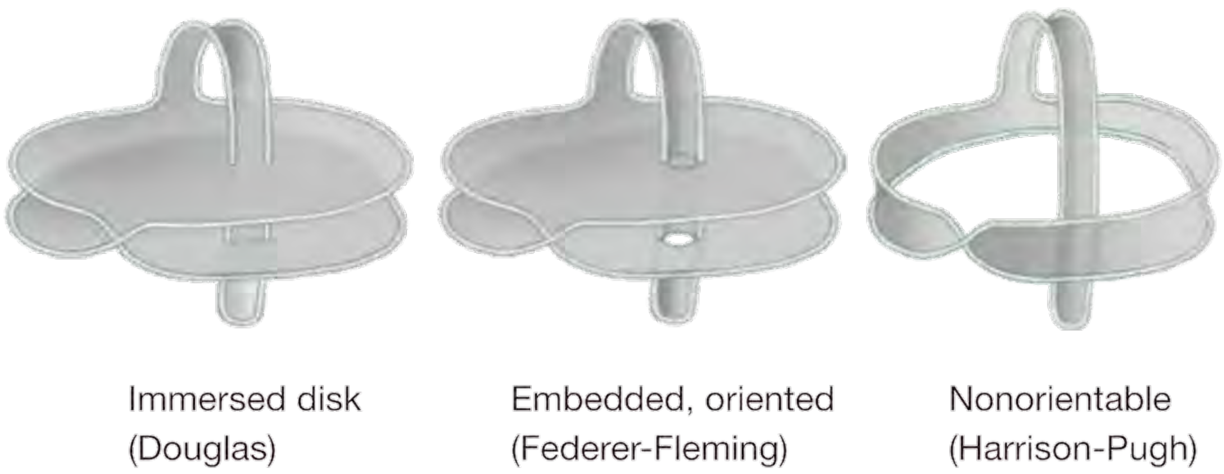

Figure 5. Three different solutions for the same wire (courtesy of J. Harrison [19]).

Later on, De Lellis, Ghiraldin, and Maggi reformulated the concept of spanning in a more Geometric Measure Theory setting [25]: let $n \geq 3$ and let H be a closed subset of $\mathbb{R}^n$. Let

$$\mathscr{C}_H = \{\gamma\colon \mathbb{S}^1 \to \mathbb{R}^n \setminus H \text{ smooth embedding of } \mathbb{S}^1 \text{ into } \mathbb{R}^n\}.$$

Fix $\mathscr{C} \subset \mathscr{C}_H$ be a closed subset by homotopy and let K be a relatively closed set in $\mathbb{R}^n \setminus H$. The set K is said to be a *$\mathscr{C}$-spanning set of H* if

$$K \cap \gamma(\mathbb{S}^1) \neq \emptyset, \qquad \forall \gamma \in \mathscr{C}. \tag{4}$$

Let us denote by $\mathscr{F}(H, \mathscr{C})$ the class of all relatively closed sets in $\mathbb{R}^n \setminus H$ which are $\mathscr{C}$-spanning sets of H. If there exists $K \in \mathscr{F}(H, \mathscr{C})$ such that $\mathscr{H}^{n-1}(K) < +\infty$, then the problem

$$\min_{K \in \mathscr{F}(H,\mathscr{C})} \mathscr{H}^{n-1}(K)$$

has a solution which is a $(n-1)$-dimensional minimal set in $\mathbb{R}^n \setminus H$.

This approach furnishes a good answer to the Plateau problem: when H is a Jordan curve in $\mathbb{R}^3$ the spanning condition corresponds to the fact that the soap film K wets entirely the curve H. There exists a minimal set K in $\mathbb{R}^3 \setminus H$ that spans H. Therefore, "the boundary of K is H" and, by Taylor's result, K can develop Plateau's type singularities.

3. The Kirchhoff-Plateau problem

A recent generalization of the Plateau problem consists in the situation in which the boundary is not rigid but it is given by a flexible manifold. The first results were given by Bernatzky [26] and Bernatzky and Ye [27] who proved existence in the framework of currents. The first formulation of the Kirchhoff-Plateau problem was given by Giusteri, Franceschini and Fried [28], where the boundary of the soap film lies on a 3-dimensional elastic rod; more precisely, stability of equilibrium configurations is analyzed. We also mention [29–32]. The first rigorous existence result for the Kirchhoff-Plateau problem has been provided by Fried, Giusteri and Lussardi [33] where the energy functional to be minimized is composed by the elastic energy of the rod, the weight of the rod and the area of the soap film spanned by the rod. Further details and relative bibliography can be found therein; we refer also to [34–38].

3.1. *The bounding loop*

To model the boundary manifold, in [33] the theory of Kirchhoff rods has been implemented (see for instance the book of Antman [39]). A $3D$-rod is completely described by its midline curve and a family of two-dimensional *material cross-section* attached to each point of the midline. Moreover, in order to encode how the cross-sections are "appended" to the midline, a family of *material frames* completes the framework. Here, it is also assumed that the material cross-section lies in the plane orthogonal to the midline at any point of the midline, namely that the rod is *unshearable*, and that its midline is *inextensible*, denoting wit $L > 0$ its length. Under these assumptions, the final shape of the rod is uniquely determined

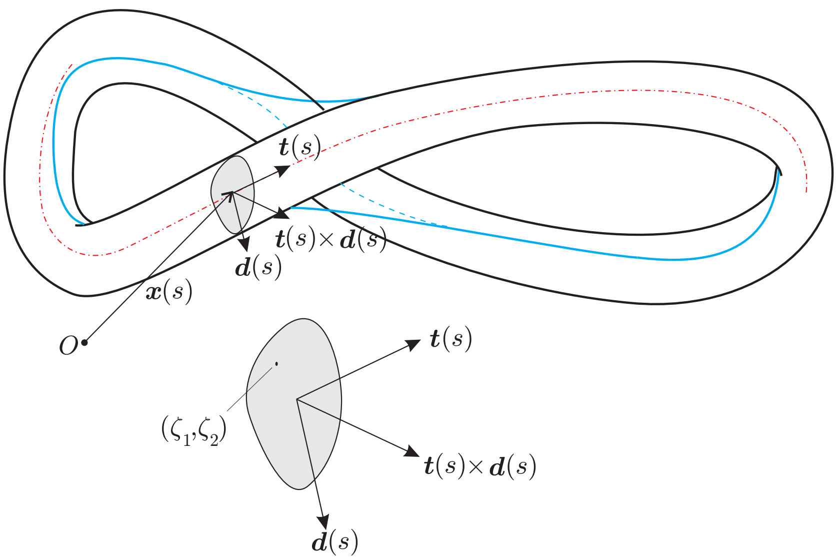

Figure 6. The shape of the rod constructed by a moving frame.

by assigning the cross-sections and three scalar fields: the *flexural densities* κ_1 and κ_2 and a *twist density* ω as we illustrate. Fix the clamping point $\boldsymbol{x}_0 \in \mathbb{R}^3$ and fix $\boldsymbol{t}_0, \boldsymbol{d}_0 \in \mathbb{R}^3$ unit orthogonal vectors. Let $p > 1$ and $V = L^p(0, L) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, then $\boldsymbol{w} \in V$ and it is given by

$$\boldsymbol{w} = ((\kappa_1, \kappa_2, \omega), \boldsymbol{x}_0, \boldsymbol{t}_0, \boldsymbol{d}_0).$$

Starting from $\boldsymbol{w}$, we can reconstruct the midline $\boldsymbol{x}$ and a director field $\boldsymbol{d}$ as the unique solutions of the following system of ordinary differential equations (for a graphical representation see Figure 6)

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{t}, \\ \boldsymbol{t}' = \kappa_1 \boldsymbol{d} + \kappa_2 \boldsymbol{t} \times \boldsymbol{d}, \\ \boldsymbol{d}' = \omega \boldsymbol{t} \times \boldsymbol{d} - \kappa_1 \boldsymbol{t}, \end{cases} \tag{5}$$

supplemented by the initial conditions

$$\begin{cases} \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{t}(0) = \boldsymbol{t}_0, \\ \boldsymbol{d}(0) = \boldsymbol{d}_0. \end{cases} \tag{6}$$

A classical result of Carathéodory [40] ensures that (5)–(6) has a unique solution:

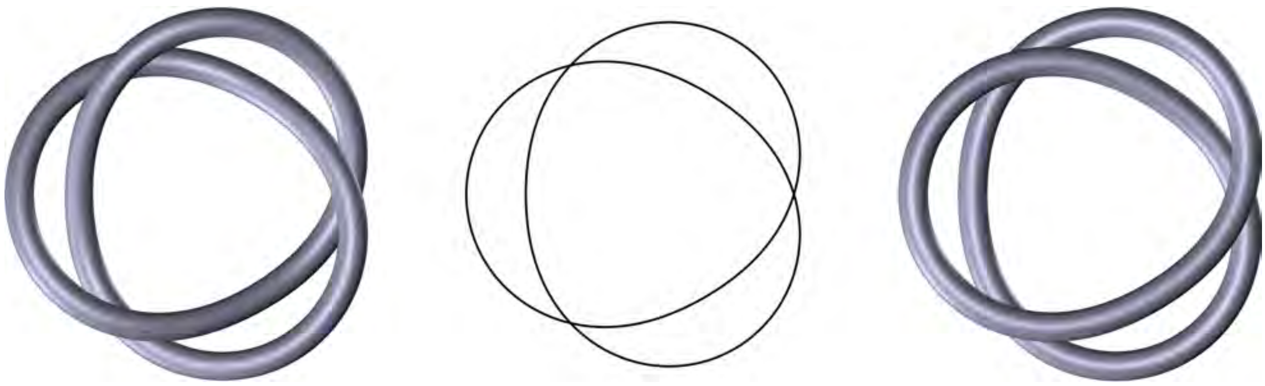

Figure 7. The trefoil knot on the left and the unknot structure on the right are distinct objects when the cross-sections have a non-vanishing thickness, even in the presence of self-contact. The distinction between a knot and an unknot is lost in the vanishing-thickness limit.

$$\boldsymbol{x} \in W^{2,p}((0,L);\mathbb{R}^3) \qquad \text{and} \qquad \boldsymbol{d} \in W^{1,p}((0,L);\mathbb{R}^3).$$

Moreover, since $\boldsymbol{t}_0, \boldsymbol{d}_0$ are unit orthogonal vectors, we can reconstruct the moving material frame given by $\{(\boldsymbol{t}(s), \boldsymbol{d}(s), \boldsymbol{t}(s) \times \boldsymbol{d}(s)) : s \in [0,L]\}$. In particular, the midline $\boldsymbol{x}$ is parametrized by the arc-length. In addition, the problem is equipped with some constraints. First of all, in order to have a closed and smooth midline we require that

$$\boldsymbol{x}(L) = \boldsymbol{x}(0), \qquad \text{and} \qquad \boldsymbol{t}(L) = \boldsymbol{t}(0). \tag{7}$$

We want also to encode the knot type of midline. To do that we simply fix a continuous map $\ell\colon [0,L] \to \mathbb{R}^3$ with $\ell(L) = \ell(0)$ and we ask that

$$\boldsymbol{x} \simeq \ell, \tag{8}$$

where $\simeq$ is the isotopy equivalence relation in the sense of the theory of knots. We point out that a non-vanishing cross-sectional thickness is crucial for distinguishing knot types in the presence of self-contact, see Figure 7. Next, we discuss how to reconstruct the shape of the rod. The material cross-section at each $s \in [0,L]$ is given by a compact and simply connected set $A(s) \subset \mathbb{R}^2$ which contains the origin. The corresponding rod can then be described as the set $\boldsymbol{p}[\boldsymbol{w}](\Omega)$, where

$$\Omega = \big\{(s, \zeta_1, \zeta_2) : s \in [0,L] \text{ and } (\zeta_1, \zeta_2) \in A(s)\big\}$$

and $\boldsymbol{p}[\boldsymbol{w}]$ is given by

$$\boldsymbol{p}[\boldsymbol{w}](s, \zeta_1, \zeta_2) = \boldsymbol{x}(s) + \zeta_1 \boldsymbol{d}(s) + \zeta_2 \boldsymbol{t}(s) \times \boldsymbol{d}(s).$$

In what follows, $\Lambda[\boldsymbol{w}]$ will stand for the set $\boldsymbol{p}[\boldsymbol{w}](\Omega)$. We define the elastic energy of the rod as

$$E_{\text{sh}}(\boldsymbol{w}) = \int_0^L f(w_1(s), s)\, ds$$

where $f\colon \mathbb{R}^3 \times [0,L] \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:

(a) $f(\cdot, s)$ is continuous and convex for any s in $[0, L]$;
(b) $f(a, s)$ is bounded from below;
(c) $f(a, \cdot)$ is measurable for any $a \in \mathbb{R}^3$.
(d) $f(a, s) \geq c_1|a|^p + c_2$ for some $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Under these assumptions the functional $E_{\rm sh}$ is coercive and lower semicontinuous with respect to the weak topology of V. We now discuss other necessary physical constraints. First of all, we have to specify how we glue the last cross-section to the first one when we close the rod. More precisely, we have to prescribe how many times the ends of the rod are twisted before being glued together. Thus, for a small parameter $\varepsilon > 0$, the curve $\boldsymbol{x} + \varepsilon \boldsymbol{d}$ remains inside the rod $\Lambda[\boldsymbol{w}]$. Up to add a straigth line-segment we can assume that the curve $\boldsymbol{x} + \varepsilon \boldsymbol{d}$ is closed. We ask that the linking number between the midline $\boldsymbol{x}$ and the curve $\boldsymbol{x} + \varepsilon \boldsymbol{d}$ is a prescribed integer number z, i.e.

$$\text{Link}(\boldsymbol{x}, \boldsymbol{x} + \varepsilon \boldsymbol{d}) = z. \tag{9}$$

To complete the global gluing condition, see Figure 8, we also fix the angle between $\boldsymbol{d}_0$ and $\boldsymbol{d}(L)$ as

$$\text{angle}\,(\boldsymbol{d}_0, \boldsymbol{d}(L)) \text{ is fixed.} \tag{10}$$

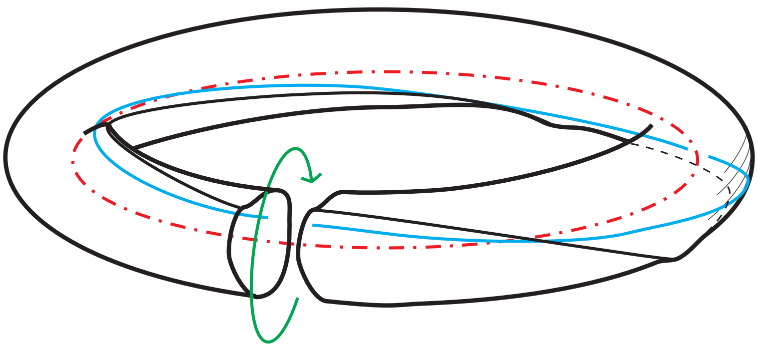

Figure 8. The gluing of the rod: the curve in blue is close to the midline in red.

We finally have to discuss the non-interpenetration of matter, see Figure 9. In order to guarantee that

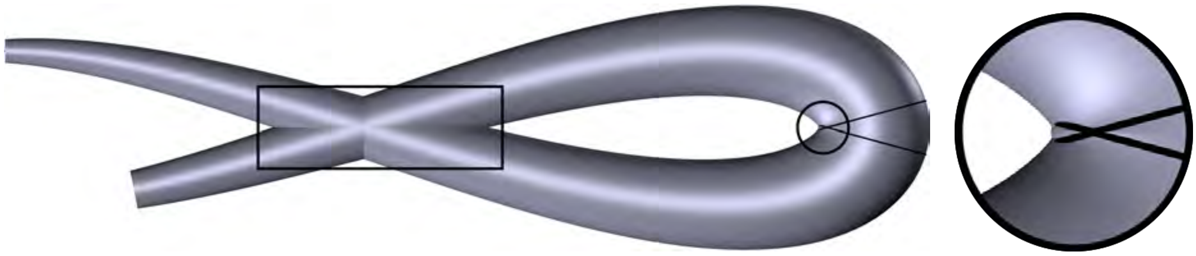

Figure 9. Lost of global and local injectivity.

the map $\boldsymbol{p}$ is globally injective on the interior part of Ω we have to assume two conditions. The first one is the *local non-interpenetration constraint*, which we employ adding to the energy of the loop the term

$$E_{\rm ni}(\boldsymbol{w}) = \begin{cases} 0 & \text{if } \boldsymbol{w} \in N\,, \\ +\infty & \text{if } \boldsymbol{w} \in V \setminus N \end{cases}$$

where

$$N = \left\{ \boldsymbol{w} \in V : \max_{(\zeta_1, \zeta_2) \in A(s)} \big(\zeta_1 \kappa_2(s) - \zeta_2 \kappa_1(s)\big) \leq 1, \text{ a.e. } s \in (0, L) \right\}.$$

The penalization of E_{ni} can be seen as a sort of relaxation of the orientation-preservation of $\boldsymbol{p}[\boldsymbol{w}]$. Besides, the global injectivity follows from the Ciarlet-Nečas condition

$$\int_{\Omega} \det D\boldsymbol{p}[\boldsymbol{w}](s,\zeta_1,\zeta_2)\, ds d\zeta_1 d\zeta_2 \leq \mathscr{L}^3(\Lambda[\boldsymbol{w}]). \tag{11}$$

It remains to add the effects of the weight of the rod: we consider the potential energy

$$E_g(\boldsymbol{w}) = -\int_{\Omega} \rho(s,\zeta_1,\zeta_2)\, \boldsymbol{g}\cdot\boldsymbol{p}(s,\zeta_1,\zeta_2)\, ds d\zeta_1 d\zeta_2,$$

where $\rho > 0$ is the mass density and $\boldsymbol{g}$ is the constant acceleration of gravity.

The final form of the loop energy reads as

$$E_{\text{loop}} = E_{\text{sh}} + E_{\text{ni}} + E_g.$$

3.2. *The Kirchhoff–Plateau problem*

As in the classical Plateau problem, we model the liquid film by a two-dimensional object K, but we want to keep track of the fact that it is reminiscent of two adhering surfactant leaflets. Then, we define the energy of the liquid film as

$$E_{\text{film}}(K) = 2\sigma\mathscr{H}^2(K)$$

where $\sigma > 0$ is the surface tension.

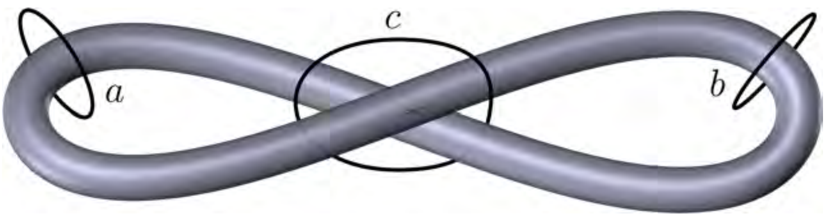

Figure 10. For this loop, if we look for a spanning set relative to the homotopy class of the loops a or b, spanning surfaces covering only the hole on the left or on the right one will be allowed, respectively. If, instead, we consider the homotopy class of the loop c, both holes must be covered by the spanning set.

Using the framework and the notation presented in Section 2.4, we impose the spanning condition choosing a suitable class of loops closed by homotopy. Indeed, an appropriate choice of homotopy classes determines which holes of a bounding loop with points of self-contact are covered, see Figure 10. Precisely, we use the subset $\mathscr{D}_{\Lambda[\boldsymbol{w}]} \subset \mathscr{C}_{\Lambda[\boldsymbol{w}]}$ containing all γ that have linking number 1 or -1 with the midline $\boldsymbol{x}$. Then, we seek a surface $K \in \mathscr{F}(\Lambda[\boldsymbol{w}], \mathscr{D}_{\Lambda[\boldsymbol{w}]})$ that is a $\mathscr{D}_{\Lambda[\boldsymbol{w}]}$-spanning set of the bounding loop $\Lambda[\boldsymbol{w}]$ in the sense of (4) where $\mathcal{C} = \mathscr{D}_{\Lambda[\boldsymbol{w}]}$, see Figure 11.

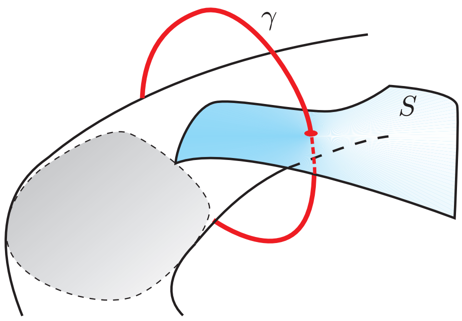

Figure 11. The surface S must intersect the loop γ.

The Kirchhoff-Plateau problem concerns the minimization of the energy functional

$$E_{\mathrm{KP}}(\boldsymbol{w}) = E_{\mathrm{loop}}(\boldsymbol{w}) + \inf\left\{E_{\mathrm{film}}(K) : K \in \mathscr{F}(\Lambda[\boldsymbol{w}], \mathscr{D}_{\Lambda[\boldsymbol{w}]})\right\}$$

under the constraints (7)–(11). The main Theorem proved in [33] is the following one.

Theorem 3.1. *Assume that there exists $\tilde{\boldsymbol{w}} \in V$ satisfying (7)–(11) with $E_{\mathrm{KP}}(\tilde{\boldsymbol{w}}) < +\infty$. Then there exists a minimizer of E_{KP} satisfying (7)–(11). Moreover, there is a relatively closed subset $K[\boldsymbol{w}]$ of $\mathbb{R}^3 \setminus \Lambda[\boldsymbol{w}]$ such that*

$$E_{\mathrm{film}}(K[\boldsymbol{w}]) = \inf\left\{E_{\mathrm{film}}(K) : K \in \mathscr{F}(\Lambda[\boldsymbol{w}], \mathscr{D}_{\Lambda[\boldsymbol{w}]})\right\}.$$

Finally, $K[\boldsymbol{w}]$ is a minimal set in $\mathbb{R}^3 \setminus \Lambda[\boldsymbol{w}]$.

Proof. We just give the main ideas. Consider a minimizing sequence $(\boldsymbol{w}_h)$ for E_{KP} such that $E_{\mathrm{KP}}(\boldsymbol{w}_h) \le M$ for some $M \ge 0$. It is possible to extract a weakly converging subsequence, not relabeled, $\boldsymbol{w}_h \rightharpoonup \boldsymbol{w}$ where $\boldsymbol{w}$ satisfies the constraints (7)–(11). The key point is to prove that if $K_h \in \mathscr{F}(\Lambda[\boldsymbol{w}_h], \mathcal{D}_{\Lambda[\boldsymbol{w}_h]})$ and a loop γ in $\mathcal{D}_{\Lambda[\boldsymbol{w}]}$ then for any $\varepsilon > 0$ such that the tubular neighborhood $U_{2\varepsilon}(\gamma)$ of radius 2ε around γ is contained in $\mathbb{R}^3 \setminus \Lambda[\boldsymbol{w}_h]$, there exists $M = M(\varepsilon) > 0$ such that, for any h large enough,

$$\mathscr{H}^2(K_h \cap U_\varepsilon(\gamma)) \ge M. \tag{12}$$

Indeed, take K_h with

$$\mathscr{H}^2(K_h) = \inf\left\{E_{\mathrm{film}}(K) : K_h \in \mathscr{F}(\Lambda[\boldsymbol{w}_h], \mathcal{D}_{\Lambda[\boldsymbol{w}_h]})\right\}.$$

This is always possible essentially thanks to [25, Thm. 2]. The measures $\mu_h := \mathscr{H}^2 \llcorner K_h$ constitute a bounded sequence, $\mu_h \overset{*}{\rightharpoonup} \mu$ up to the extraction of a subsequence, and the limit measure satisfies

$$\mu \ge \mathscr{H}^2 \llcorner K_\infty$$

where $K_\infty := \mathrm{spt}(\mu) \setminus \Lambda[\boldsymbol{w}]$ is a countably $\mathscr{H}^2$-rectifiable set. Assume by contradiction that there exists $\gamma \in \mathcal{D}_{\Lambda[\boldsymbol{w}]}$ with $\gamma \cap K_\infty = \emptyset$ and take ε as before. We therefore find that $\mu(U_{2\varepsilon}(\gamma)) = 0$ and then

$$\lim_{h \to +\infty} \mathscr{H}^2(K_h \cap U_\varepsilon(\gamma)) = 0$$

which contradicts (12). This means that $K_\infty \in \mathscr{F}(\Lambda[\boldsymbol{w}], \mathcal{D}_{\Lambda[\boldsymbol{w}]})$.

We also get

$$\begin{aligned}
\liminf_{h \to +\infty} \inf\{\mathscr{H}^2(K) : K_h \in \mathscr{F}(\Lambda[\boldsymbol{w}_h], \mathcal{D}_{\Lambda[\boldsymbol{w}_h]})\} \\
&\ge \liminf_{h \to +\infty} \mathscr{H}^2(K_h) \\
&= \liminf_{h \to +\infty} \mu_h(\mathbb{R}^3) \\
&\ge \mu(\mathbb{R}^3) \\
&\ge \mathscr{H}^2(K_\infty) \\
&\ge \inf\{\mathscr{H}^2(K) : K \in \mathscr{F}(\Lambda[\boldsymbol{w}], \mathcal{D}_{\Lambda[\boldsymbol{w}]})\}
\end{aligned}$$

which establishes the lower semicontinuity of the functional E_{KP} (the lower semicontinuity of the loop energy is a standard).

Finally, the fact that there exists a minimal set $K[\boldsymbol{w}]$ in $\mathbb{R}^3 \setminus \Lambda[\boldsymbol{w}]$ with

$$E_{\mathrm{film}}(K[\boldsymbol{w}]) = \inf\left\{E_{\mathrm{film}}(K) : K \in \mathscr{F}(\Lambda[\boldsymbol{w}], \mathcal{D}_{\Lambda[\boldsymbol{w}]})\right\}$$

follows from [25] and this yields the conclusion. □

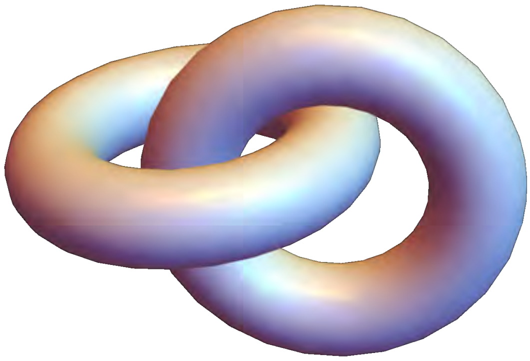

Figure 12. A possible geometry of two linked rods.

3.3. *Linked rods*

The first generalization of the Kirchhoff-Plateau problem has been investigated in [35] where a more complex configuration of the bounding loop is considered. Precisely, the loop consists in a finite number of rods linked in an arbitrary way, as for instance in Figure 12. Following the same notation as before, and limiting to the case of two rods, two vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ are introduced to describe the two midlines:

$$\boldsymbol{w}^{(1)} = ((\kappa_1^{(1)}, \kappa_2^{(1)}, \omega^{(1)}), \boldsymbol{x}_0^{(1)}, \boldsymbol{t}_0^{(1)}, \boldsymbol{d}_0^{(1)}), \quad \boldsymbol{w}^{(2)} = ((\kappa_1^{(2)}, \kappa_2^{(2)}, \omega^{(2)}), \boldsymbol{x}_0^{(2)}, \boldsymbol{t}_0^{(2)}, \boldsymbol{d}_0^{(2)}).$$

In particular, it is assumed that only the midline generated by $\boldsymbol{w}^{(1)}$ is clamped, that means that $(\boldsymbol{x}_0^{(1)}, \boldsymbol{t}_0^{(1)}, \boldsymbol{d}_0^{(1)})$ is prescribed. Concerning the second rod, we do not assume a priori its position in space, namely the vector $(\boldsymbol{x}_0^{(2)}, \boldsymbol{t}_0^{(2)}, \boldsymbol{d}_0^{(2)})$ is an unknown of the problem. For each rod we assume the correspond-

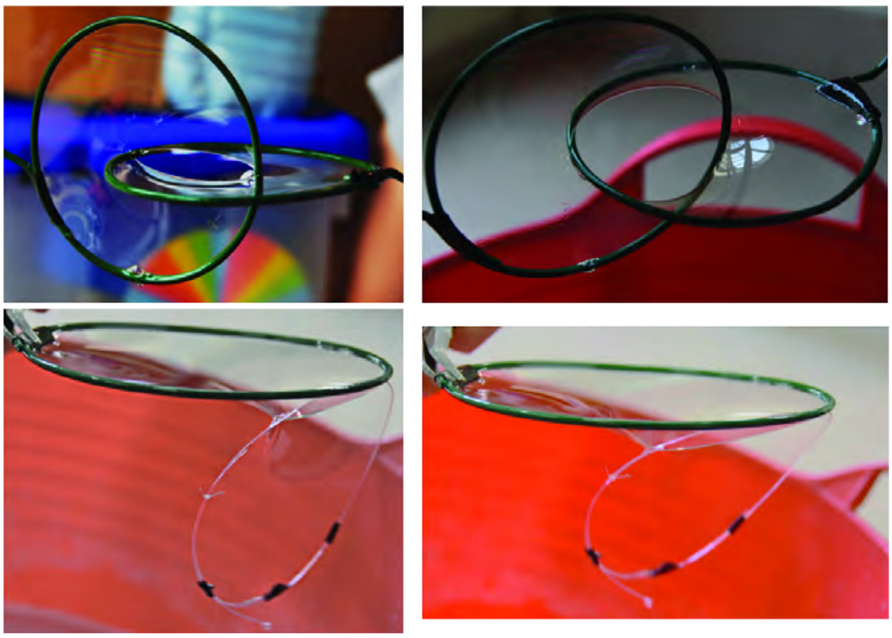

Figure 13. Above: two fixed linked rigid metallic wires in a soap solution. Below: one rod is more flexible than the other one.

ing analogous constraints (7)–(11). Moreover, we have also to ask that the linking number between the two midlines is prescribed:

$$\text{Link}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = \eta \tag{13}$$

for some $\eta \in \mathbb{Z}$. Finally, since we need to have a non-interpenetration between the two rods we are going to assume that

$$\mathscr{L}^3(\Lambda[\boldsymbol{w}^{(1)}] \cap \Lambda[\boldsymbol{w}^{(2)}]) = 0. \tag{14}$$

Concerning the spanning conditions, in this case we choose the loops which are not homotopic to a constant and such that the sum of the linking numbers between the loop and the two rods is always one: this means that a loop cannot link at the same time both rods. Thus, the Kirchhoff-Plateau problem concerns the minimization of the following energy functional

$$(\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}) \mapsto E_{\text{loop}}(\boldsymbol{w}^{(1)}) + E_{\text{loop}}(\boldsymbol{w}^{(2)}) + \inf\left\{E_{\text{film}}(K) : K \text{ spans } \Lambda[\boldsymbol{w}^{(1)}] \cup \Lambda[\boldsymbol{w}^{(2)}]\right\},$$

under all of the constraints described above. In [35], we provide the existence of a minimizer and we perform some experiments, see Figure 13.

3.4. *Soap films spanning repulsive links*

As a second generalization, in [34] the case of knotted proteins is treated. To consider processes like the adsorption of a protein by a biomembrane, in [34] we introduce an additional repulsional energy between the two linked rods; see Figure 14.

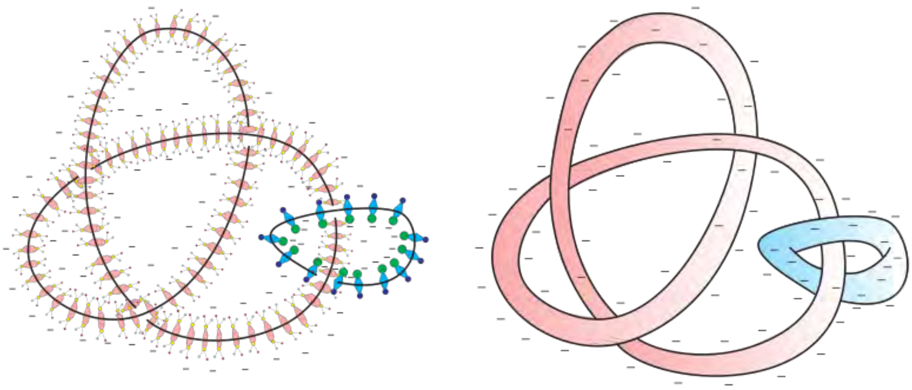

Figure 14. Knotted protein linked to another one.

The general setting is the same as in the case previously considered of two linked rods: $\boldsymbol{w}^{(1)}$ and $\boldsymbol{w}^{(2)}$ generate two midlines $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(1)}$ respectively. We assume all the usual constraints (7)–(11) on $\boldsymbol{x}^{(i)}$, as well as (13). We substitute (14) with electrical potential energy term which, physically, encodes the repulsion between the two rods. Precisely, the repulsion is modeled by

$$\int_0^{L_1} \int_0^{L_2} \frac{1}{h(\|\boldsymbol{x}^{(1)}(s_1) - \boldsymbol{x}^{(2)}(s_2)\|)}\, ds_1 ds_2, \tag{15}$$

where h is a suitable increasing, nonnegative and continuous function. With this choice, we are introducing a positively unbounded energy, that may be infinite if the midlines are sufficiently close. A possible choice for h is represented in Figure 15: a function which is 0 until some positive and small parameter ε and then grows linearly. Therefore, the energy functional becomes

$$\begin{aligned}(\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}) \mapsto E_{\text{loop}}(\boldsymbol{w}^{(1)}) + E_{\text{loop}}(\boldsymbol{w}^{(2)}) + \int_0^{L_1} \int_0^{L_2} \frac{1}{h(\|\boldsymbol{x}^{(1)}(s_1) - \boldsymbol{x}^{(2)}(s_2)\|)}\, ds_1 ds_2 \\ + \inf\left\{E_{\text{film}}(K) : K \text{ spans } \Lambda[\boldsymbol{w}^{(1)}] \cup \Lambda[\boldsymbol{w}^{(2)}]\right\}\end{aligned}$$

which is minimized under all of the described constraints.

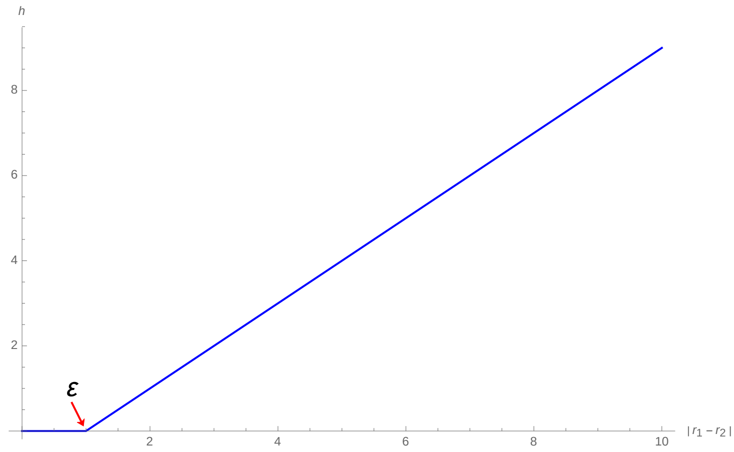

Figure 15. Example of the "repulsive" function h.

3.5. *Dimensional reduction*

The paper [36] deals with a sort of dimensional reduction of the Kirchhoff-Plateau problem. The aim of that work is then to perform a formal dimensional reduction of the classical Kirchhoff-Plateau problem where the limiting curve is the midline of the rod. We require that $\boldsymbol{w}$ satisfies the assumptions (7)–(11). Moreover, we assume that the so-called *global radius of curvature* $\Delta(\boldsymbol{x}[\boldsymbol{w}])]$ is bounded from below by a constant $\Delta_0 > 0$; this assumption prevents self-intersection for a sufficiently small cross section (see [41]). In this context the cross section $A(s)$ is replaced by a rescaled cross section $\varepsilon A(s)$, where $\varepsilon > 0$ is a positive and vanishing parameter. In addition, we can define the map $\boldsymbol{p}^\varepsilon[\boldsymbol{w}]$ and the corresponding rod $\Lambda^\varepsilon[\boldsymbol{w}] = \boldsymbol{p}^\varepsilon[\boldsymbol{w}](\Omega^\varepsilon)$. The rescaled energy functional reads as

$$E^\varepsilon_{\mathrm{KP}}(\boldsymbol{w}) = E^\varepsilon_{\mathrm{loop}}(\boldsymbol{w}) + \inf\left\{E_{\mathrm{film}}(K) : K \in \mathscr{F}(\Lambda^\varepsilon[\boldsymbol{w}], \mathscr{D}_{\Lambda^\varepsilon[\boldsymbol{w}]})\right\}$$

where

$$E^\varepsilon_{\mathrm{loop}}(\boldsymbol{w}) = E_{\mathrm{sh}}(\boldsymbol{w}) - \frac{1}{\varepsilon^2}\int_{\Omega^\varepsilon} \rho(s,\zeta_1,\zeta_2)\boldsymbol{g}\cdot\boldsymbol{p}^\varepsilon[\boldsymbol{w}](s,\zeta_1,\zeta_2)\, ds d\zeta_1 d\zeta_2.$$

As $\varepsilon \to 0^+$ we obtain a limit functional (in the sense of Γ-convergence) which is given by

$$E^0(\boldsymbol{w}) = E_{\mathrm{sh}}(\boldsymbol{w}) - \int_0^L |A(s)|\rho_0(s)\boldsymbol{g}\cdot\boldsymbol{x}[\boldsymbol{w}](s)\, ds + \inf\left\{E_{\mathrm{film}}(K) : K \text{ spans } \boldsymbol{x}[\boldsymbol{w}]([0,L])\right\}$$

being

$$\rho_0(s) = \lim_{(\xi_1,\xi_2)\to(0,0)} \rho(s,\xi_1,\xi_2).$$

The approximating problems have minima which converge weakly to the minimum energy solution of the limit problem, as well as the corresponding value of the energy. This also shows that the Plateau solution with elastic line boundary may be approximated by solutions of the problems with a rod boundary.

4. Further results and work in progress

Once the existence of minimizers has been obtained, it is useful to characterize them deriving, for instance, the Euler-Lagrange equations. Unfortunately, since the definition of the bounding loop requires a high number of constraints, a first simplification is to consider an elastic curve instead of the Kirchhoff-rod as the boundary wire. Thus, a first simplification can be considering the Elastic-Plateau problem: we are interested in performing the variational analysis of energy functionals of the type

$$\mathcal{E}[\gamma, \boldsymbol{X}] = \int_\gamma f(\kappa,\tau)\, d\ell + \mathrm{Area}(S)$$

where $\boldsymbol{\gamma}$ is a closed curve in $\mathbb{R}^3$ with curvature κ and torsion τ and Area(S) is the area of the spanning surface spanning the elastic curve $\boldsymbol{\gamma}$. The derivation of minimizers and their characterization leads to several difficulties like getting compactness in the disc-type approach or dealing with Plateau singularities. It seems that one should use the framework of Lytchak and Wenger [42] and Creutz [43] to set the problem in Sobolev spaces for disc-type surfaces spanning a curve with possible self-intersections, while we expect to deal with Geometric Measure Theory to treat general surfaces.

A first attempt to investigate the mentioned problem has been done in [44–46], where in order to deal with elastic curves, we minimize $\mathcal{E}$ among all disc-type maps $\boldsymbol{X}: D \to \mathbb{R}^3$ with trace pointwise equal to the elastic curve $\boldsymbol{\gamma}$ (this condition is different from the classical Plateau problem rigorously solved by Douglas and Radò [7,8] where the trace of $\boldsymbol{X}$ is a suitable reparametrization $\sigma(s)$ with $s \in [0,1]$ of the curve γ). Moreover, the area functional is substituted with $\int_D \Psi(\nabla \boldsymbol{X})\, dudv$, where D is the unit disc in $\mathbb{R}^2$ and $\boldsymbol{X}: D \to \mathbb{R}^3$ is a parametrization of a membrane spanning the elastic curve $\boldsymbol{\gamma}$, modeled through its deformation gradient $\nabla \boldsymbol{X} \in \mathbb{R}^{3\times 2}$. We adopt two different approaches to model the line integral: *the parametrized curves approach and the framed curves approach.* For the first one, the curve $\boldsymbol{\gamma}$ is modeled as the Euler-Bernoulli elastica, while only linear elastic membranes are taken into account (For details we refer to [46, Theorem 2.2 - Theorem 2.3 - Theorem 2.7]). Concerning the second one, it is introduced to deal with more general energies, both for the boundary curve and for the membrane. Precisely, we introduce a moving orthonormal frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\} \in W^{1,p}((0, 2\pi); SO(3))$ with $p > 1$ which generates a curve $\boldsymbol{r}$ by integration. On this basis, we impose suitable constraints to get a closed curve (For details we refer to [44, Theorem 3.1] and [46, Theorem 3.3 - Theorem 3.5]).

Moreover, another interesting direction of investigation would be to perform a numerical study in order to visualize minimizers and their behaviour. A first attempt has been proposed in [46, Section 4] where *ad hoc* method has been introduced to test some simple configurations in the membrane case. Precisely, developing a numerical approach is quite challenging due to the large number of constraints in the formulation of the problem, for instance we mention the pointwise length preserving constraint which is ill-suited in the application of a finite element method, or the choice of energy functionals, non linear terms are hard to be treat numerically.

Finally, in its classical formulation, the Plateau problem is an optimization problem: looking for the surface with minimal area spanning the assigned boundary. However, it would be interesting to characterize the dynamical process since, especially from the physical viewpoint, Plateau devised many experiments putting a wire frame into a soap solution to a soap film. In particular, a first step can be to formulate and solve the dynamical Plateau problem in its quasi-static approximation: the idea is to prescribe the motion of the elastic curve and, at each time step $t \in [0,1]$, a minimal surface spanning the assigned curve must be determined. This approach generalizes the machinery introduced by Dal Maso and co-authors for studying fractures [47].

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References

1. L. M. Siqveland and S. M. Skjæveland, Derivations of the Young-Laplace equation, *Capillarity*, vol. 4, no. 2, pp. 23–30, 2021.
2. M. P. Do Carmo, *Differential geometry of curves and surfaces: revised and updated second edition.* Courier Dover Publications, 2016.
3. H. Jenkins and J. Serrin, Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation, *Archive for Rational Mechanics and Analysis*, vol. 21, no. 4, pp. 321–342, 1966.
4. A. Korn, *Über Minimalflächen, deren Randkurven wenig von ebenen Kurven abweichen*, vol. II. Verlag der Königl. Akad. der Wiss., 1909.
5. C. Müntz, Zum Randwertproblem der partiellen Differentialgleichung der Minimalflächen, *Journal für die Reine und Angewandte Mathematik*, vol. 139, pp. 52–79, 1911.
6. P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Mathematica*, vol. 115, no. 1, pp. 271–310, 1966.
7. J. Douglas, Solution of the problem of Plateau, *Transactions of the American Mathematical Society*, vol. 33, no. 1, pp. 263–321, 1931.
8. T. Radó, The problem of the least area and the problem of Plateau, *Mathematische Zeitschrift*, vol. 32, no. 1, pp. 763–796, 1930.
9. U. Dierkes, S. Hildebrandt, and F. Sauvigny, *Minimal surfaces.* Springer, 2010.
10. B. Dacorogna, *Direct methods in the calculus of variations*, vol. 78 of *Applied Mathematical Sciences.* Springer, New York, second ed., 2008.
11. C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Transactions of the American Mathematical Society*, vol. 43, no. 1, pp. 126–166, 1938.
12. J. C. Nitsche, *Lecture on minimal surfaces.* Cambridge university press, 1989.
13. R. D. Gulliver, Regularity of minimizing surfaces of prescribed mean curvature, *Annals of Mathematics*, vol. 97, no. 2, pp. 275–305, 1973.
14. L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
15. F. Maggi, *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory.* No. 135 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2012.
16. H. Federer, *Geometric measure theory*, vol. Band 153 of *Die Grundlehren der mathematischen Wissenschaften.* Springer-Verlag New York, Inc., New York, 1969.
17. L. Simon, *Lectures on geometric measure theory*, vol. 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University.* Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
18. F. Almgren, *The theory of varifolds.* Mimeographed notes, Princeton, 1965.
19. J. E. Hutchinson, Second fundamental form for varifolds and the existence of surfaces minimising curvature, *Indiana Univ. Math. J.*, vol. 35, no. 1, pp. 45–71, 1986.
20. E. Bombieri, E. De Giorgi, and E. Giusti, Minimal cones and the Bernstein problem, *Invent. Math.*, vol. 7, pp. 243–268, 1969.
21. F. J. Almgren, Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, *Mem. Amer. Math. Soc.*, vol. 4, no. 165, pp. viii+199, 1976.
22. J. E. Taylor, The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces,

Ann. of Math., vol. 103, no. 3, pp. 489–539, 1976.

23. J. Harrison, Soap film solutions to Plateau's problem, *J. Geom. Anal.*, vol. 24, no. 1, pp. 271–297, 2014.
24. J. Harrison and H. Pugh, Existence and soap film regularity of solutions to Plateau's problem, *Adv. Calc. Var.*, vol. 9, no. 4, pp. 357–394, 2016.
25. C. De Lellis, F. Ghiraldin, and F. Maggi, A direct approach to Plateau's problem, *J. Eur. Math. Soc.*, vol. 19, no. 8, pp. 2219–2240, 2017.
26. F. Bernatzki, On the existence and regularity of mass-minimizing currents with an elastic boundary, *Annals of Global Analysis and Geometry*, vol. 15, pp. 379–399, 1997.
27. F. Bernatzki and R. Ye, Minimal surfaces with an elastic boundary, *Annals of Global Analysis and Geometry*, vol. 19, no. 1, pp. 1–9, 2001.
28. G. G. Giusteri, P. Franceschini, and E. Fried, Instability paths in the Kirchhoff–Plateau problem, *Journal of Nonlinear Science*, vol. 26, pp. 1097–1132, 2016.
29. Y.-c. Chen and E. Fried, Stability and bifurcation of a soap film spanning a flexible loop, *Journal of Elasticity*, vol. 116, pp. 75–100, 2014.
30. A. Biria and E. Fried, Buckling of a soap film spanning a flexible loop resistant to bending and twisting, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 470, no. 2172, p. 20140368, 2014.
31. A. Biria and E. Fried, Theoretical and experimental study of the stability of a soap film spanning a flexible loop, *International Journal of Engineering Science*, vol. 94, pp. 86–102, 2015.
32. T. Hoang and E. Fried, Influence of a spanning liquid film on the stability and buckling of a circular loop with intrinsic curvature or intrinsic twist density, *Mathematics and Mechanics of Solids*, vol. 23, no. 1, pp. 43–66, 2018.
33. G. G. Giusteri, L. Lussardi, and E. Fried, Solution of the Kirchhoff-Plateau problem, *J. Nonlinear Sci.*, vol. 27, no. 3, pp. 1043–1063, 2017.
34. G. Bevilacqua, L. Lussardi, and A. Marzocchi, Soap film spanning electrically repulsive elastic protein links, *Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur.*, vol. 96, pp. A1, 13, 2018.
35. G. Bevilacqua, L. Lussardi, and A. Marzocchi, Soap film spanning an elastic link, *Quart. Appl. Math.*, vol. 77, no. 3, pp. 507–523, 2019.
36. G. Bevilacqua, L. Lussardi, and A. Marzocchi, Dimensional reduction of the Kirchhoff-Plateau problem, *J. Elasticity*, vol. 140, no. 1, pp. 135–148, 2020.
37. A. De Rosa and L. Lussardi, On the anisotropic Kirchhoff-Plateau problem, *Math. Eng.*, vol. 4, no. 2, pp. Paper No. 011, 13, 2022.
38. G. Bevilacqua and C. Lonati, Effects of surface tension and elasticity on critical points of the Kirchhoff–Plateau problem, *Bollettino dell'Unione Matematica Italiana*, vol. 17, no. 2, pp. 221–240, 2024.
39. S. S. Antman, *Nonlinear problems of elasticity*, vol. 107 of *Applied Mathematical Sciences*. Springer, New York, second ed., 2005.
40. P. Hartman, *Ordinary differential equations*. Birkhäuser, Boston, MA, second ed., 1982.
41. O. Gonzalez, J. H. Maddocks, F. Schuricht, and H. von der Mosel, Global curvature and self-contact of nonlinearly elastic curves and rods, *Calc. Var. Partial Differential Equations*, vol. 14, no. 1, pp. 29–68, 2002.
42. A. Lytchak and S. Wenger, Area minimizing discs in metric spaces, *Archive for Rational Mechanics and Analysis*, vol. 223, pp. 1123–1182, 2017.

43. P. Creutz, Plateau's problem for singular curves, *Communications in Analysis and Geometry*, vol. 30, no. 8, pp. 1779–1792, 2022.
44. G. Bevilacqua, L. Lussardi, and A. Marzocchi, Variational analysis of inextensible elastic curves, *Proceedings of the Royal Society A*, vol. 478, no. 2260, p. 20210741, 2022.
45. G. Bevilacqua, L. Lussardi, and A. Marzocchi, Geometric invariants of non-smooth framed curves, *Accepted in INdAM-Series, "Anisotropic Isoperimetric Problems & Related Topics", arXiv preprint arXiv:2301.03525*, 2023.
46. F. Ballarin, G. Bevilacqua, L. Lussardi, and A. Marzocchi, Elastic membranes spanning deformable curves, *ZAMM-Zeitschrift für Angewandte Mathematik und Mechanik*, p. e202300890, 2024.
47. G. Dal Maso and R. Toader, A model for the quasi-static growth of brittle fractures: Existence and approximation results, *Archive for Rational Mechanics and Analysis*, vol. 162, pp. 101–135, 2002.