COMPRESSIVE SENSING USING EXTROPY MEASURES OF RANKED SET SAMPLING

SAEID TAHMASEBI*, MOHAMMAD REZA KAZEMI**, AHMAD KESHAVARZ***,
ALI AKBAR JAFARI****, AND FRANCESCO BUONO*****

ABSTRACT. The aim of this paper is to consider the extropy measure of uncertainty proposed by Lad, Sanfilippo and Agrò for the problem of compressive sensing. For this purpose, two sampling designs, i.e., simple random sampling (SRS) and a modified version of ranked set sampling, known as maximum ranked set sampling procedure with unequal samples (MRSSU) are utilized and some uncertainty measures such as extropy, cumulative extropy and residual extropy are obtained and compared for these sampling designs. Also, some results of extropy in record ranked set sampling data are developed. Then, a study on comparing the behavior of estimators of cumulative extropy in MRSSU and SRS using simulation method is obtained. As an example, two sampling methods MRSSU and SRS are utilized for compressive sensing technique and their performances are compared via signal to noise ratio (SNR), correlation coefficient of reconstructed and the original signal and cumulative extropy measure of uncertainty. The results show that the values of SNR and correlation coefficient for MRSSU are higher than those of SRS. Furthermore, it is shown that MRSSU scheme can efficiently reduce the uncertainty measure of cumulative extropy.

1. INTRODUCTION

Compressive sensing is a signal processing technique which is used to construct a signal using a small number of measurements (see [9]). This has received lots of attention in signal processing, statistics and computer science. The purpose of compressive sensing is to reconstruct a sparse signal using a linear random projection. A signal is considered sparse if it has very few non-zero elements. When a signal is not sparse in the time domain, its transform to another domain may be sparse. Common approaches for this purpose are discrete Fourier transform, discrete cosine transform (DCT) and wavelet transform. To achieve this goal, a signal of electrocardiography is utilized.

Ranked set sampling (RSS) design is a cost-effective sampling for situations where taking actual measurements on units is expensive but ranking of units is easy. McIntyre [21] indicated that RSS is a more efficient sampling method than SRS method for estimating the population mean. RSS and some of its variants are sampling designs that allow the experimenter to span the full range of values in the population and those have been applied widely in industrial, economics, environmental and ecological studies, biostatistics and statistical genetics.

Biradar and Santosha [5] proposed MRSSU to estimate the mean of the exponential distribution and indicated that an estimator based on MRSSU is better than an estimator based on SRS. In the MRSSU, we draw m simple random samples, where the size of the i-th sample is i, $i = 1, \dots, m$.

2010 *Mathematics Subject Classification.* Primary 62B10, 62D05, 94A17.
Key words and phrases. Compressive sensing, Cumulative extropy, Maximum ranked set sampling, Record Ranked set Sampling, Stochastic ordering.

The one-cycle MRSSU involves an initial ranking of m samples of size m as follows:

$$\begin{array}{lccccccc}
1: & \underline{\mathbf{X_{(1:1)1}}} & & & & \rightarrow & \tilde{X}_1 = X_{(1:1)1} \\
2: & X_{(1:2)2} & \underline{\mathbf{X_{(2:2)2}}} & & & \rightarrow & \tilde{X}_2 = X_{(2:2)2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
m: & X_{(1:m)m} & X_{(2:m)m} & \cdots & \underline{\mathbf{X_{(m:m)m}}} & \rightarrow & \tilde{X}_m = X_{(m:m)m}
\end{array}$$

where $X_{(i:l)j}$ denotes the i-th order statistic from the j-th SRS of size l. The resulting sample is called one-cycle MRSSU of size m and denoted by $\mathbf{X}^{(m)}_{MRSSU} = \{\tilde{X}_i : i = 1, \ldots, m\}$. Note that $\tilde{X}_i$ has the same distribution as $X_{(i:i)i}$ with probability density function (pdf) $f_{(i:i)i}(x) = if(x)[F(x)]^{i-1}$, where f and F are the pdf and cumulative distribution function (cdf) of the parent distribution, respectively. In reliability theory, $\tilde{X}_i$ measures the lifetime of a parallel system. Since the ranking may be done for example by using an easily measurable covariate, then it is not difficult to identify the maximum of ranked individuals in each subset. So, it can be stated that MRSSU is a very useful adjustment of RSS.

Salehi and Ahmadi [28] introduced a new sampling design for generating record data. Suppose we have m independent sequences of continuous random variables. The i-th sequence sampling is terminated when the i-th record is observed. Let us denote the last (upper or lower) record for the i-th sequence by $R_{i,i}$, then the record ranked set sampling (RRSS) of size m can be displayed as follows:

$$\begin{array}{lcccccc}
1: & \underline{\mathbf{R_{(1)1}}} & & & & \rightarrow & R_{1,1} = R_{(1)1} \\
2: & R_{(1)2} & \underline{\mathbf{R_{(2)2}}} & & & \rightarrow & R_{2,2} = R_{(2)2} \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
m: & R_{(1)m} & R_{(2)m} & \cdots & \underline{\mathbf{R_{(m)m}}} & \rightarrow & R_{m,m} = R_{(m)m}
\end{array}$$

where $R_{(i)j}$ is the i-th ordinary record in the j-th sequence. Note that $R_{i,i}$'s are independent random variables, but not necessarily ordered. Several authors have worked on measures of information for RSS and its variants. Jozani and Ahmadi [16] explored the notions of information content of RSS data and compared them with their counterparts in SRS data. Tahmasebi et al. [32] obtained some results of residual (past) entropy for ranked set samples. Eskandarzadeh et al. [11] considered information measures of MRSSU in terms of Shannon entropy, Rényi entropy and Kullback-Leibler information. Eskandarzadeh et al. [12] studied information measures for RRSS. Recently, Raqab and Qiu [27] considered the problems of uncertainty and information content of RSS data based on extropy measure and the related monotonic properties and stochastic comparisons. Qiu and Eftekharian [26] obtained some results of extropy for maximum and minimum ranked set sampling with unequal samples. Kazemi et al. [18] studied uncertainty measures of minimum ranked set sampling procedure with unequal samples in terms of cumulative residual extropy and its dynamic version.

This work develops some results of extropy measures for the MRSSU and RRSS designs.

For reconstructing the signal, the sampling methods SRS and MRSSU are used and their performances are compared using two concepts: signal to noise ratio (SNR) and the correlation coefficient of reconstructed and original signal. A design with higher values of SNR and the correlation coefficient outperforms another one. It is shown that using MRSSU sampling design can improve both SNR and the correlation coefficient of reconstructed and original signal instead of SRS one. Moreover, the performances of these sampling designs for compressive sensing in electrocardiography in term of extropy measure are investigated. Furthermore, we show how MRSSU scheme can efficiently reduce the uncertainty measure comparing with SRS design.

This paper is organized as follows: Section 2 deals with the results of extropy measure for MRSSU data by comparing with its counterpart under SRS data. Similar results related to cumulative extropy in MRSSU and SRS designs are presented in Section 3. Also, some properties in terms of residual and past extropy of MRSSU are developed. In Section 4 the extropy measure in RRSS data is discussed. In Section 5, an estimator for cumulative extropy in MRSSU design using empirical approach is proposed and a simulation study is performed to compare the cumulative extropy of MRSSU and SRS designs. In Section 6, a real example with the concept of compressive sensing is analyzed in details by comparing the effects of MRSSU and SRS schemes. Section 7 concludes the paper.

2. Extropy of MRSSU

Let X denote a continuous random variable with pdf f and cdf F. Lad et al. [19] introduced a new measure of uncertainty, known as extropy, associated to X and defined as

$$J(X) = -\frac{1}{2}\int_{-\infty}^{+\infty} [f(x)]^2 dx = -\frac{1}{2}\int_0^1 f(F^{-1}(u))du, \tag{2.1}$$

where $F^{-1}(\cdot)$ is the quantile function of X. The quantile function is an efficient alternative to the cdf in modelling and analysis of statistical data, see [14] for some applications, and it is defined by

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

The extropy was introduced as a measure of uncertainty dual to the well-known Shannon entropy [30]. Qiu [24] explored some characterization results, monotone properties, and lower bounds of extropy of order statistics and record values. The residual extropy of order statistics was studied in [25]. In this section, we compare the extropy of SRS data with its counterpart MRSSU of the same size. From (2.1), the extropy of $\mathbf{X}_{SRS}^{(m)}$ is given by, see [26]

$$J(\mathbf{X}_{SRS}^{(m)}) = -\frac{1}{2}[-2J(X)]^m. \tag{2.2}$$

In the sequel, under the MRSSU, it is easy to show that

$$\begin{aligned} J(\mathbf{X}_{MRSSU}^{(m)}) &= -\frac{1}{2}\prod_{i=1}^{m}[-2J(X_{(i:i)})] \\ &= -\frac{1}{2}\prod_{i=1}^{m} i^2 \int_0^1 f(F^{-1}(u))u^{2i-2}du \\ &= -\frac{1}{2}\prod_{i=1}^{m} i^2 \mathbb{E}\left[U^{2i-2} f(F^{-1}(U))\right], \end{aligned} \tag{2.3}$$

where $U \sim Uniform(0,1)$. In the following examples, we compare the $J(\mathbf{X}_{MRSSU}^{(m)})$ with $J(\mathbf{X}_{SRS}^{(m)})$ of size m.

***Example* 1.** If $U \sim Uniform(0,1)$, then

$$J(\mathbf{U}_{MRSSU}^{(m)}) = -\frac{1}{2}\prod_{i=1}^{m}\frac{i^2}{2i-1} = -\frac{1}{2}\left(\frac{\sqrt{\pi}(m!)^2}{2^m\Gamma(m+\frac{1}{2})}\right) < -\frac{1}{2} = J(\mathbf{U}_{SRS}^{(m)}),$$

where $\Gamma(\cdot)$ is the Gamma function.

***Example* 2.** Let Z follows the exponential distribution with mean $\frac{1}{\theta}$. Then, $f(F^{-1}(u)) = \theta(1-u)$, $0 < u < 1$, and

$$\begin{aligned} J(\mathbf{Z}^{(m)}_{MRSSU}) &= -\frac{1}{2}\left(\frac{\theta}{2}\right)^m \prod_{i=1}^{m} \frac{i}{2i-1} = -\frac{1}{2}\left(\frac{\theta}{2}\right)^m \left(\frac{\sqrt{\pi}m!}{2^m\Gamma(m+\frac{1}{2})}\right) \\ &> -\frac{1}{2}\left(\frac{\theta}{2}\right)^m = J(\mathbf{Z}^{(m)}_{SRS}). \end{aligned}$$

Now, we state important properties of $J(\mathbf{X}^{(m)}_{MRSSU})$ using the stochastic ordering. For that we present the following definitions (for more details, one may refer to [29]).

Definition 1. Let X and Y be two non-negative random variables with pdf's f and g, cdf's F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, and hazard rate functions λ_X and λ_Y, respectively. Then

(1) X is said to be smaller than Y according to usual stochastically ordering (denoted by $X \leq_{st} Y$) if $P(X \geq x) \leq P(Y \geq x)$ for all $x \in \mathbb{R}$.
(2) X is smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\lambda_X(x) \geq \lambda_Y(x)$ for all x.
(3) X is smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if $f(F^{-1}(u)) \geq g(G^{-1}(u))$ for all $u \in (0,1)$, where F^{-1} and G^{-1} are right continuous inverses of F and G, respectively.
(4) A non-negative random variable X is said to have increasing (decreasing) failure rate [IFR (DFR)] if $\lambda_X(x)$ is increasing (decreasing) in x.
(5) X is smaller than Y in the convex transform order (denoted by $X \leq_c Y$) if $G^{-1}(F(x))$ is a convex function on the support of X.
(6) X is smaller than Y in the star order (denoted by $X \leq_* Y$) if $\frac{G^{-1}F(x)}{x}$ is increasing in $x \geq 0$.
(7) X is smaller than Y in the superadditive order (denoted by $X \leq_{su} Y$) if $G^{-1}(F(t+u)) \geq G^{-1}(F(t)) + G^{-1}(F(u))$ for $t \geq 0, u \geq 0$.
(8) X is said to have increasing failure rate average (IFRA) if $\int_0^x \frac{\lambda_X(t)}{x} dt$ is increasing in $x > 0$. Note that IFR distributions belong to the class IFRA.
(9) X is new better than used (NBU) if $\bar{F}(t+u) \leq \bar{F}(t)\bar{F}(u)$ for $t \geq 0$ and $u \geq 0$.

We recall the following result given in [26] in order to obtain some corollaries.

Theorem 2.1. *Let X and Y be two non-negative random variables. If $X \leq_{disp} Y$, then $J(\mathbf{X}^{(m)}_{MRSSU}) \leq J(\mathbf{Y}^{(m)}_{MRSSU})$ for $m > 1$.*

Corollary 2.1.1. *If $X \leq_{hr} Y$, and X or Y is DFR, then $J(\mathbf{X}^{(m)}_{MRSSU}) \leq J(\mathbf{Y}^{(m)}_{MRSSU})$ for $m > 1$.*

Proof. If $X \leq_{hr} Y$, and X or Y is DFR, then $X \leq_{disp} Y$, due to [4]. Thus, from Theorem 2.1, the desired result follows. □

Corollary 2.1.2. *Let X and Y be two non-negative random variables. If $X \leq_{su} Y$ ($X \leq_* Y$ or $X \leq_c Y$) and $f(0) \geq g(0) > 0$, then $J(\mathbf{X}^{(m)}_{MRSSU}) \leq J(\mathbf{Y}^{(m)}_{MRSSU})$ for $m > 1$.*

Proof. If $X \leq_{su} Y$ ($X \leq_* Y$ *or* $X \leq_c Y$) and $f(0) \geq g(0) > 0$, then $X \leq_{disp} Y$, due to [1]. So, from Theorem 2.1, the desired result follows. □

Corollary 2.1.3. *Let X be a non-negative random variable with decreasing pdf f such that $f(0) \leq 1$. Then $J(\mathbf{X}^{(m)}_{MRSSU}) \geq J(\mathbf{U}^{(m)}_{MRSSU})$, $m > 1$, where $J(\mathbf{U}^{(m)}_{MRSSU})$ is defined in Example 1.*

Proof. A non-negative random variable X has a decreasing pdf if, and only if, $U \leq_c X$, where $U \sim Uniform(0,1)$ (see [29]). Hence, from Theorem 2.1 and Corollary 2.1.2, the desired result follows. □

Theorem 2.2. *Let X and Y be two independent non-negative random variables. If X and Y have log-concave densities, then*

$$J(\mathbf{X}^{(m)}_{\text{MRSSU}} + \mathbf{Y}^{(m)}_{\text{MRSSU}}) \geq \max\{J(\mathbf{X}^{(m)}_{MRSSU}), J(\mathbf{Y}^{(m)}_{MRSSU})\}.$$

Proof. Suppose that X have a log-concave density, and be independent of random variable Y. From Theorem 3.B.7 of [29], we can conclude that $X \leq_{disp} X+Y$. Hence, from Theorem 2.1, $J(\mathbf{X}^{(m)}_{MRSSU}) \leq J(\mathbf{X}^{(m)}_{MRSSU} + \mathbf{Y}^{(m)}_{MRSSU})$. Similar result also holds when Y has a log-concave density, i.e., $J(\mathbf{Y}^{(m)}_{MRSSU}) \leq J(\mathbf{X}^{(m)}_{MRSSU} + \mathbf{Y}^{(m)}_{MRSSU})$. Therefore, the proof is completed. □

Remark 1. Suppose $X \sim Uniform(\alpha, \beta)$. Then

$$J(\mathbf{X}^{(m)}_{MRSSU}) = (\beta-\alpha)^{-m} J(\mathbf{U}^{(m)}_{MRSSU}), \quad J(\mathbf{X}^{(m)}_{SRS}) = (\beta-\alpha)^{-m} J(\mathbf{U}^{(m)}_{SRS}),$$

where $J(\mathbf{U}^{(m)}_{MRSSU})$ and $J(\mathbf{U}^{(m)}_{SRS})$ are as in the Example 1.

About the quantile function, note that $F(Q(u)) = u$ and differentiating it with respect to u yields $q(u)f(Q(u)) = 1$, where $q(u)$ is the derivative of $Q(\cdot)$ with respect to u. Let X be a non-negative random variable with pdf f and quantile function Q. Then $f(Q(u))$ is called the density quantile function and $q(u) = Q'(u)$ is known as the quantile density function of X (see [23]). Using (2.3), the corresponding quantile based extropy of $\mathbf{X}^{(m)}_{MRSSU}$ is obtained as

$$J(\mathbf{X}^{(m)}_{MRSSU}) = -\frac{1}{2} \prod_{i=1}^{m} i^2 \mathbb{E}\left[\frac{U^{2i-2}}{q(U)}\right]. \tag{2.4}$$

The hazard rate and reversed hazard rate functions are important concepts in reliability theory, see, for instance, [3, 6]. The quantile versions of these measures are defined by $R(u) = \frac{f(Q(u))}{1-F(Q(u))} = [(1-u)q(u)]^{-1}$ and $\Lambda(u) = \frac{f(Q(u))}{F(Q(u))} = [uq(u)]^{-1}$, respectively. Hence, by using (2.4), $J(\mathbf{X}^{(m)}_{MRSSU})$ can be expressed in terms of the hazard quantile function, $R(u)$, and the reversed hazard quantile function, $\Lambda(u)$, respectively, as

$$J(\mathbf{X}^{(m)}_{MRSSU}) = -\frac{1}{2} \prod_{i=1}^{m} i^2 \mathbb{E}\left[R(U)(U^{2i-2} - U^{2i-1})\right],$$
$$J(\mathbf{X}^{(m)}_{MRSSU}) = -\frac{1}{2} \prod_{i=1}^{m} i^2 \mathbb{E}\left[\Lambda(U) U^{2i-1}\right].$$

Proposition 2.3. *Let the quantile density function $q(w) = \frac{1}{M} < +\infty$, where $w = \sup\{u : q(u) \geq \frac{1}{M}\}$ is the distribution mode of a random variable X. Then,*

$$J(\mathbf{X}^{(m)}_{MRSSU}) \geq -\frac{M^m}{2} \prod_{i=1}^{m} \frac{i^2}{2i-1}, \quad m > 1. \tag{2.5}$$

Proof. The proof is straightforward and hence it is omitted. □

3. Cumulative extropy of MRSSU

Let X denote the lifetime of a system with cdf F and finite support (α, β), where $0 \leq \alpha < \beta < +\infty$. Recently, a new measure of information was proposed in the literature by replacing the pdf by F in the definition of extropy (2.1). This new measure is called cumulative extropy (CEX) and defined as

$$\mathcal{CJ}(X) = -\frac{1}{2}\int_{\alpha}^{\beta} F^2(x)dx, \tag{3.1}$$

see [22] for further details. Random variables with finite support have finite CEX and it is always non-positive. If the CEX of X is less than that of another random variable, say Y, that is $\mathcal{CJ}(X) \leq \mathcal{CJ}(Y)$, then X has more uncertainty than Y. For the MRSSU and SRS designs, we have

$$\begin{aligned}\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) &= -\frac{1}{2}\prod_{i=1}^{m}[-2\mathcal{CJ}(X_{(i:i)})] = -\frac{1}{2}\prod_{i=1}^{m}\int_0^1 \frac{u^{2i}}{f(F^{-1}(u))}du \\ &= -\frac{1}{2}\prod_{i=1}^{m}\mathbb{E}\left[\frac{U^{2i}}{f(F^{-1}(U))}\right],\end{aligned} \tag{3.2}$$

where $U \sim Uniform(0,1)$, and

$$\mathcal{CJ}(\mathbf{X}^{(m)}_{SRS}) = -\frac{1}{2}[-2\mathcal{CJ}(X)]^m. \tag{3.3}$$

To compare the above measures, let us consider the following examples.

***Example* 3.** If $U \sim Uniform(0,1)$, then

$$\mathcal{CJ}(\mathbf{U}^{(m)}_{MRSSU}) = -\frac{1}{2}\prod_{i=1}^{m}\frac{1}{2i+1} = -\frac{1}{2}\left(\frac{\sqrt{\pi}}{2^{m+1}\Gamma(m+\frac{3}{2})}\right) > \mathcal{CJ}(\mathbf{U}^{(m)}_{SRS}) = -\frac{1}{2}\left(\frac{1}{3}\right)^m. \tag{3.4}$$

***Example* 4.** Let Z be a random variable with the cdf $F(z) = z^a,\ 0 < z < 1,\ a > 1$. Then, $f(F^{-1}(u)) = au^{1-\frac{1}{a}}$, $0 < u < 1$, and we have

$$\mathcal{CJ}(\mathbf{Z}^{(m)}_{MRSSU}) = -\frac{1}{2}\prod_{i=1}^{m}\frac{1}{2ia+1} > \mathcal{CJ}(\mathbf{Z}^{(m)}_{SRS}) = -\frac{1}{2}(2a+1)^{-m}. \tag{3.5}$$

***Example* 5.** Let X be a random variable with the pdf $f(x) = 5(1-x)^4,\ 0 < x < 1$. Then, we have

$$\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) = -\frac{1}{2}\left(\frac{1}{5}\Gamma\left(\frac{1}{5}\right)\right)^m\prod_{i=1}^{m}\frac{\Gamma(1+2i)}{\Gamma(\frac{6}{5}+2i)} > \mathcal{CJ}(\mathbf{X}^{(m)}_{SRS}) = -0.5\left(\frac{25}{33}\right)^m. \tag{3.6}$$

Theorem 3.1. *Let $\mathbf{X}^{(m)}_{MRSSU}$ be the MRSSU from population X with cdf F and finite support (α, β). Then, $\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) \geq \mathcal{CJ}(\mathbf{X}^{(m)}_{SRS})$ for $m > 1$.*

Proof. Since $F^2(x) \geq F^{2i}(x)$ for $i \geq 1$, we have

$$\left(\int_{\alpha}^{\beta} F^2(x)dx\right)^m \geq \prod_{i=1}^{m}\int_{\alpha}^{\beta}F^{2i}(x)dx.$$

The proof follows from Equations (3.2) and (3.3). □

In the following, we provide some results on the cumulative extropy of $\mathbf{X}^{(m)}_{MRSSU}$ in terms of stochastic ordering properties.

Theorem 3.2. *If $X \leq_{st} Y$, then $\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) \leq \mathcal{CJ}(\mathbf{Y}^{(m)}_{MRSSU})$, $m > 1$.*

Proof. By the assumption on the stochastically ordering, $F^{2i}(x) \geq G^{2i}(x)$ for all $x \geq 0$. Now using (3.2), for $m > 1$, we get the desired result. □

Theorem 3.3. *Let X and Y be two non-negative random variables. If $X \leq_{disp} Y$, then $\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) \geq \mathcal{CJ}(\mathbf{Y}^{(m)}_{MRSSU})$ for $m > 1$.*

Proof. The proof is similar to the one of Theorem 2.1 and based on the characterization of dispersive ordering in terms of cdf's and hence it is omitted. □

Proposition 3.4. *If $f(F^{-1}(u)) \geq 1$, $0 < u < 1$, then $\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU})$ is increasing in $m \geq 1$.*

Proof. From (3.2), we get

$$\frac{\mathcal{CJ}(\mathbf{X}^{(m+1)}_{MRSSU})}{\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU})} = \int_0^1 \frac{u^{2m+2}}{f(F^{-1}(u))} du \leq \frac{1}{2m+3} \leq 1,$$

and the result follows readily, since the extropy is negative. □

Theorem 3.5. *If $X \leq_{hr} Y$, and X or Y is DFR, then $\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) \geq \mathcal{CJ}(\mathbf{Y}^{(m)}_{MRSSU})$ for $m > 1$.*

Proof. The proof is similar to Corollary 2.1.1 and based on the characterization of hazard rate ordering in terms of cdf's. □

Theorem 3.6. *Let X and Y be two non-negative random variables with pdf's f and g, respectively. If $X \leq_{su} Y (X \leq_* Y$ or $X \leq_c Y)$ and $f(0) \geq g(0) \geq 0$, then $\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) \geq \mathcal{CJ}(\mathbf{Y}^{(m)}_{MRSSU})$ for $m > 1$.*

Proof. The proof is similar to Corollary 2.1.2 and based on the characterization of superadditive ordering in terms of cdf's. □

In the following propositions, the cumulative extropy of MRSSU is studied for linear transformation and it is shown a connection with the cumulative residual extropy for symmetric distributions. The proof of the results are straightforward and hence they are omitted.

Proposition 3.7. *Let $Y = aX + b$ with $a > 0$ and $b \geq 0$. Then,*

$$\mathcal{CJ}(\mathbf{Y}^{(m)}_{MRSSU}) = a^m \mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}).$$

Proposition 3.8. *Let X be a symmetric random variable with respect to the finite mean $\mu = \mathbb{E}(X)$. Then*

$$\mathcal{CJ}(\mathbf{X}^{(m)}_{MRSSU}) = \mathcal{EJ}(\mathbf{X}^{(m)}_{MRSSU}),$$

where $\mathcal{EJ}(X) = \frac{-1}{2}\int_0^{+\infty} [\bar{F}_X(x)]^2 dx$ is the cumulative residual extropy and $\bar{F}_X(x) = 1 - F_X(x)$ (see [15]).

Let X be the random lifetime of a system, then we recall that $X_{[t]} = [t - X \mid X \leq t]$ describes the inactivity time of the system. For all $t \geq 0$ the mean inactivity time is given by [13]

$$\tilde{\mu}(t) = \mathbb{E}[t - X \mid X \leq t] = \frac{1}{F(t)} \int_0^t F(x) dx.$$

Now, we can define a generalized measure of cumulative residual extropy called the dynamic failure extropy as, see [22] for details,

$$\mathcal{CJ}(X; t) = -\frac{1}{2} \int_0^t \left[\frac{F(x)}{F(t)}\right]^2 dx. \tag{3.7}$$

Note that $\mathcal{CJ}(X;t) \geq \frac{-\tilde{\mu}(t)}{2F(t)}$. Moreover

$$\mathcal{CJ}(\mathbf{X}_{SRS}^{(m)};t) = -\frac{1}{2}[-2\mathcal{CJ}(X,t)]^m. \tag{3.8}$$

Under the MRSSU design, one can show that

$$\begin{aligned}
\mathcal{CJ}(\mathbf{X}_{MRSSU}^{(m)};t) &= -\frac{1}{2}\prod_{i=1}^{m}[-2\mathcal{CJ}(X_{(i:i)};t)] \\
&= -\frac{1}{2}\prod_{i=1}^{m}\int_0^t \left[\frac{F(x)}{F(t)}\right]^{2i} dx \\
&= -\frac{1}{2}\prod_{i=1}^{m}\mathbb{E}\left[\frac{U^{2i}F(t)}{f(F^{-1}(UF(t)))}\right],
\end{aligned} \tag{3.9}$$

where $U \sim Uniform(0,1)$.

Theorem 3.9. *Let X be a non-negative random variable. Then, for $m > 1$*

$$\mathcal{CJ}(\mathbf{X}_{MRSSU}^{(m)};t) \geq \mathcal{CJ}(\mathbf{X}_{SRS}^{(m)};t). \tag{3.10}$$

Proof. The proof is similar to Theorem 3.1. □

3.1. **Residual extropy of MRSSU.** In reliability theory, the data are generally truncated and in such situations $J(X)$ is not applicable to a system which has survived for some unit of time. Ebrahimi [10] defined uncertainty of the residual lifetime of the random variable $X_t = [X - t \mid X > t]$ as

$$H(X;t) = -\int_t^{+\infty} \frac{f(x)}{\bar{F}(t)} \log\left(\frac{f(x)}{\bar{F}(t)}\right) dx. \tag{3.11}$$

Analogous to the residual entropy, the entropy of $[X \mid X \leq t]$, called the past entropy at time t, has also drawn attention in the literatures. The past entropy is given in [8] as

$$\bar{H}(X;t) = -\int_0^t \frac{f(x)}{F(t)} \log\left(\frac{f(x)}{F(t)}\right) dx. \tag{3.12}$$

By analogy to (3.11) and (3.12), the residual extropy and its past form can be written in this way as

$$\mathcal{RJ}(X;t) = -\frac{1}{2}\int_t^{+\infty}\left(\frac{f(x)}{\bar{F}(t)}\right)^2 dx,$$
$$\overline{\mathcal{RJ}}(X;t) = -\frac{1}{2}\int_0^t\left(\frac{f(x)}{F(t)}\right)^2 dx,$$

one may refer to [25] for the residual extropy and to [17] for the past extropy and related properties. Moreover, those measure of uncertainty are connected with another interesting one known as interval extropy [7]. For mutually independent random variables $X_1, X_2, \ldots, X_m$, with cdf F_i for $i = 1, \ldots, m$, we have

$$\mathcal{RJ}(X_1,\ldots,X_m;t) = -\frac{1}{2}\prod_{i=1}^{m}\left[-2\mathcal{RJ}(X_i;t)\right].$$

Therefore, the residual extropy of $\mathbf{X}_{SRS}^{(m)}$ is

$$\mathcal{RJ}(\mathbf{X}_{SRS}^{(m)};t) = -\frac{1}{2}\left[-2\mathcal{RJ}(X;t)\right]^m. \tag{3.13}$$

The residual extropy associated with the i-th order statistic in a sample of size i has the form

$$\mathcal{RJ}(X_{(i:i)};t) = -\frac{1}{2\bar{B}^2_{F(t)}(i,1)}\int_t^{+\infty} f^2(x)F^{2i-2}(x)dx,$$

where $\bar{B}_t(a,b) = \int_t^1 u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function. So, the residual extropy of $\mathbf{X}^{(m)}_{MRSSU}$ can be written as

$$\mathcal{RJ}(\mathbf{X}^{(m)}_{MRSSU};t) = -\frac{1}{2}\prod_{i=1}^{m}\left[-2\mathcal{RJ}(X_{(i:i)};t)\right]. \tag{3.14}$$

Proposition 3.10. *Let X be a random variable with pdf f and cdf F. If $f(F^{-1}(u)) \geq 1$, $0 < u < 1$, then $\mathcal{RJ}(\mathbf{X}^{(m)}_{MRSSU};t)$ is decreasing in $m \geq 1$.*

Proof. From (3.14), we get

$$\begin{aligned}\frac{\mathcal{RJ}(\mathbf{X}^{(m+1)}_{MRSSU};t)}{\mathcal{RJ}(\mathbf{X}^{(m)}_{MRSSU};t)} &= \frac{(m+1)^2}{(1-[F(t)]^{m+1})^2}\int_{F(t)}^1 f(F^{-1}(u))u^{2m}du \\ &\geq \frac{(m+1)^2}{2m+1}\times\frac{1-[F(t)]^{2m+1}}{(1-[F(t)]^{m+1})^2}.\end{aligned}$$

Since $1-[F(t)]^{2m+1} > 1-[F(t)]^{m+1} > \left(1-[F(t)]^{m+1}\right)^2$ and $\frac{(m+1)^2}{2m+1} \geq 1$, then the proof follows by the fact that the extropy is negative. □

Similarly, the past extropy for $\mathbf{X}^{(m)}_{MRSSU}$ and $\mathbf{X}^{(m)}_{SRS}$ are obtained as follows:

$$\overline{\mathcal{RJ}}(\mathbf{X}^{(m)}_{MRSSU};t) = -\frac{1}{2}\prod_{i=1}^{m}\int_0^{F(t)}\frac{i^2 f(F^{-1}(u))u^{2i-2}}{[F(t)]^{2i}}du,$$

$$\overline{\mathcal{RJ}}(\mathbf{X}^{(m)}_{SRS};t) = -\frac{1}{2}\left[-2\overline{\mathcal{RJ}}(X;t)\right]^m.$$

In the following proposition, it is shown a sufficient condition to have a decreasing past extropy of MRSSU as a function of m. The proof is similar to the that of Proposition 3.10 and hence it is omitted.

Proposition 3.11. *Let X be a non-negative random variable with pdf f and cdf F. If $f(F^{-1}(u)) \geq 1$, $0 < u < 1$, then $\overline{\mathcal{RJ}}(\mathbf{X}^{(m)}_{MRSSU};t)$ is decreasing in $m \geq 1$.*

In the following, we evaluate and compare the residual and past extropies for MRSSU and SRS using two examples.

***Example* 6.** If $U \sim Uniform(0,1)$, then

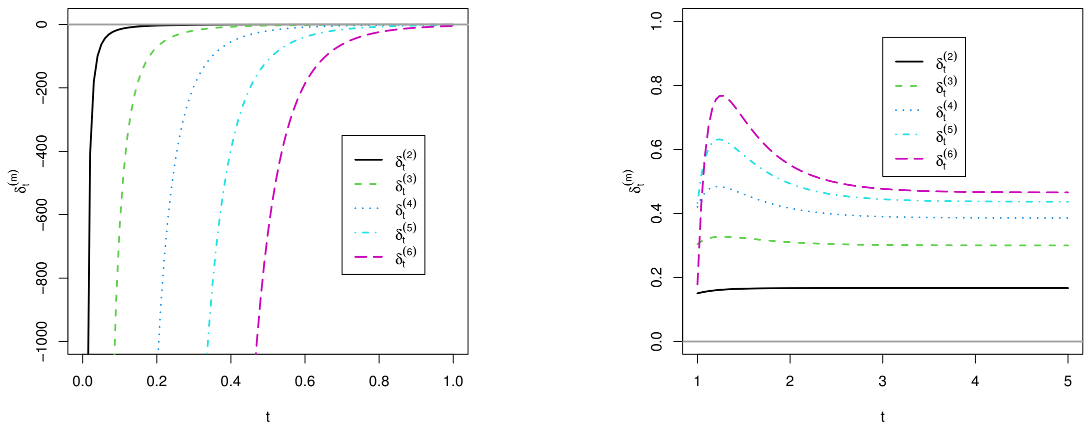
$$\mathcal{RJ}(\mathbf{U}^{(m)}_{MRSSU};t) = -\frac{1}{2}\prod_{i=1}^{m}\frac{i^2}{2i-1}\times\frac{1-t^{2i-1}}{(1-t^i)^2},\quad \mathcal{RJ}(\mathbf{U}^{(m)}_{SRS};t) = -\frac{1}{2}\left(\frac{1}{1-t}\right)^m,$$

$$\overline{\mathcal{RJ}}(\mathbf{U}^{(m)}_{MRSSU};t) = -\frac{1}{2}\prod_{i=1}^{m}\frac{i^2}{(2i-1)t} = -\frac{1}{2}\left(\frac{\sqrt{\pi}(m!)^2}{2^m\Gamma(m+\frac{1}{2})}\right)t^{-m},$$

$$\overline{\mathcal{RJ}}(\mathbf{U}^{(m)}_{SRS};t) = -\frac{1}{2}t^{-m}.$$

Also, let

$$\delta_t^{(m)} = \overline{\mathcal{RJ}}(\mathbf{U}^{(m)}_{MRSSU};t) - \overline{\mathcal{RJ}}(\mathbf{U}^{(m)}_{SRS};t) = \frac{1}{2t^m}\left(1-\frac{\sqrt{\pi}(m!)^2}{2^m\Gamma(m+\frac{1}{2})}\right).$$



(A) (B)

FIGURE 1. The Values of $\delta_t^{(m)}, m = 2, 3, \ldots, 6$ for exponential distribution in Example 7: (a) $\theta < 2$ and $0 < t < 1$ and (b) $\theta > 2$ and $t > 1$.

It is easily found that $\delta_t^{(m)} < 0$, i.e., $\overline{\mathcal{RJ}}(\mathbf{U}_{MRSSU}^{(m)}; t) < \overline{\mathcal{RJ}}(\mathbf{U}_{SRS}^{(m)}; t)$.

***Example* 7.** Let Z follows exponential distribution with mean $\frac{1}{\theta}$. Then

$$\begin{aligned}
\mathcal{RJ}(\mathbf{Z}_{MRSSU}^{(m)}; t) &= \frac{-1}{2}\left(\frac{\theta}{2}\right)^m \prod_{i=1}^{m} \frac{i^2 \bar{B}_{1-e^{-\theta t}}(2i-1, 2)}{(1-(1-e^{-\theta t})^i)^2}, \\
\mathcal{RJ}(\mathbf{Z}_{SRS}^{(m)}; t) &= \frac{-1}{2}\left(\frac{\theta}{2}\right)^m, \\
\overline{\mathcal{RJ}}(\mathbf{Z}_{MRSSU}^{(m)}; t) &= \frac{-1}{2}\left(\frac{\theta}{2}\right)^m \prod_{i=1}^{m} \frac{i[1-(e^{-\theta t})(1-2i)]}{(2i-1)(1-e^{-\theta t})}, \\
\overline{\mathcal{RJ}}(\mathbf{Z}_{SRS}^{(m)}; t) &= \frac{-1}{2}\left(\frac{\theta}{2}\right)^m \left(\frac{1-e^{-2\theta t}}{(1-e^{-\theta t})^2}\right)^m.
\end{aligned}$$

Define $\delta_t^{(m)} = \overline{\mathcal{RJ}}(\mathbf{Z}_{MRSSU}^{(m)}; t) - \overline{\mathcal{RJ}}(\mathbf{Z}_{SRS}^{(m)}; t)$. Figure 1 shows the the behavior of $\delta_t^{(m)}$ for different values of t. Our findings beside this figure show that for $\theta < 2$ and $0 < t < 1$, the $\overline{\mathcal{RJ}}$ for MRSSU is smaller than that of for SRS and also, for $\theta > 2$ and $t > 1$, this measure for MRSSU is bigger than that of for SRS design.

4. Extropy for RRSS

Let $\mathbf{L}_{RRSS}^{(m)} = (L_{1,1}, L_{2,2}, \ldots, L_{m,m})$ and $\mathbf{U}_{RRSS}^{(m)} = (U_{1,1}, U_{2,2}, \ldots, U_{m,m})$ be the lower and upper RRSS, respectively. The joint pdf's of $\mathbf{L}_{RRSS}^{(m)}$ and $\mathbf{U}_{RRSS}^{(m)}$ respectively are given by

$$\begin{aligned} f_{\mathbf{L}_{RRSS}^{(m)}}(\mathbf{l_r}) &= \prod_{i=1}^{m} \frac{\{-\log F(l_{i,i})\}^{i-1}}{(i-1)!} f(l_{i,i}), \\ f_{\mathbf{U}_{RRSS}^{(m)}}(\mathbf{u_r}) &= \prod_{i=1}^{m} \frac{\{-\log \bar{F}(u_{i,i})\}^{i-1}}{(i-1)!} f(u_{i,i}), \end{aligned}$$

respectively, where $\mathbf{l_r} = (l_{1,1}, l_{2,2}, \ldots, l_{m,m})$ and $\mathbf{u_r} = (u_{1,1}, u_{2,2}, \ldots, u_{m,m})$ (see [2]). For $\mathbf{L}_{RRSS}^{(m)}$ and $\mathbf{U}_{RRSS}^{(m)}$ designs, we have

$$\begin{aligned} J(\mathbf{L}_{RRSS}^{(m)}) &= -\frac{1}{2}\prod_{i=1}^{m}[-2J(L_{i,i})] = -\frac{1}{2}\prod_{i=1}^{m}\frac{1}{[\Gamma(i)]^2}\int_0^1 f(F^{-1}(u))[-\log u]^{2i-2}du \\ &= -\frac{1}{2}\prod_{i=1}^{m}\tau_i \mathbb{E}\left[f(F^{-1}(e^{-V}))\right], \qquad (4.1) \\ J(\mathbf{U}_{RRSS}^{(m)}) &= -\frac{1}{2}\prod_{i=1}^{m}\tau_i \mathbb{E}\left[f(F^{-1}(1-e^{-V}))\right], \qquad (4.2) \end{aligned}$$

where $\tau_i = \binom{2i-2}{i-1}$ and $V \sim Gamma(2i-1, 1)$ with pdf $f_V(v) = \frac{1}{(2i-2)!}v^{2i-2}e^{-v}$, $v \in (0, +\infty)$.

***Example* 8.** If $U \sim Uniform(0, 1)$, then

$$J(\mathbf{L}_{RRSS}^{(m)}) = -\frac{1}{2}\prod_{i=1}^{m}\tau_i = J(\mathbf{U}_{RRSS}^{(m)}).$$

Comparing with the results of Example 1, it is found that the uncertainty measure based on $\mathbf{L}_{RRSS}^{(m)}$ is more than its SRS counterpart. Also, the extropy measures of lower and upper RRSS are equal.

***Example* 9.** Let Z follows exponential distribution with mean $\frac{1}{\theta}$. Therefore, $f(F^{-1}(u)) = \theta(1-u)$, and

$$\begin{aligned} \mathbb{E}\left[f(F^{-1}(e^{-V}))\right] &= \theta\left(1 - \phi_V(-1)\right) = \theta\left(1 - 2^{-(2i-1)}\right), \\ \mathbb{E}\left[f(F^{-1}(1-e^{-V}))\right] &= \theta\left(\phi_V(-1)\right) = \theta 2^{-(2i-1)}, \end{aligned}$$

where $\phi_V(\cdot)$ is the moment generating function of the random variable V. For $\mathbf{L}_{RRSS}^{(m)}$ and $\mathbf{U}_{RRSS}^{(m)}$ designs, we obtain

$$\begin{aligned} J(\mathbf{L}_{RRSS}^{(m)}) &= -\frac{\theta^m}{2}\prod_{i=1}^{m}\tau_i\left(1 - 2^{-(2i-1)}\right), \\ J(\mathbf{U}_{RRSS}^{(m)}) &= -\frac{\theta^m}{2}\prod_{i=1}^{m}\tau_i 2^{-(2i-1)}. \end{aligned}$$

It is obvious that $1-2^{-(2i-1)} > 2^{-(2i-1)}$, and therefore $J(\mathbf{L}_{RRSS}^{(m)}) < J(\mathbf{U}_{RRSS}^{(m)})$. Straightforward simplifications yields

$$J(\mathbf{U}_{RRSS}^{(m)}) = -\frac{1}{2}\left(\frac{\theta}{2}\right)^m \delta_m,$$

where $\delta_m = \frac{\prod_{i=1}^{m} \tau_i}{2^{m(m-1)}}$. It can be shown that δ_m is decreasing in m and $\delta_m < 1$. So, $J(\mathbf{U}_{RRSS}^{(m)}) > J(\mathbf{Z}_{SRS}^{(m)})$, where $J(\mathbf{Z}_{SRS}^{(m)})$ is the extropy measure of SRS samples from exponential distribution computed in Example 2. Similar discussions show that $J(\mathbf{L}_{RRSS}^{(m)}) < J(\mathbf{Z}_{SRS}^{(m)})$. So, in general we can say that

$$J(\mathbf{L}_{RRSS}^{(m)}) < J(\mathbf{Z}_{SRS}^{(m)}) < J(\mathbf{U}_{RRSS}^{(m)}).$$

Define

$$\Lambda_m^{(1)} = \frac{J(\mathbf{Z}_{MRSSU}^{(m)})}{J(\mathbf{L}_{RRSS}^{(m)})},\ \Lambda_m^{(2)} = \frac{J(\mathbf{U}_{RRSS}^{(m)})}{J(\mathbf{L}_{RRSS}^{(m)})},\ \Lambda_m^{(3)} = \frac{J(\mathbf{U}_{RRSS}^{(m)})}{J(\mathbf{Z}_{MRSSU}^{(m)})}. \tag{4.3}$$

Figure 2 shows the values of $\Lambda_m^{(\alpha)}, \alpha = 1, 2, 3$. It can be found that all $\Lambda_m^{(\alpha)}$ are less than 1. Since all of the extropy measures are negative then, $J(\mathbf{L}_{RRSS}^{(m)}) < J(\mathbf{Z}_{MRSSU}^{(m)}) < J(\mathbf{U}_{RRSS}^{(m)})$.

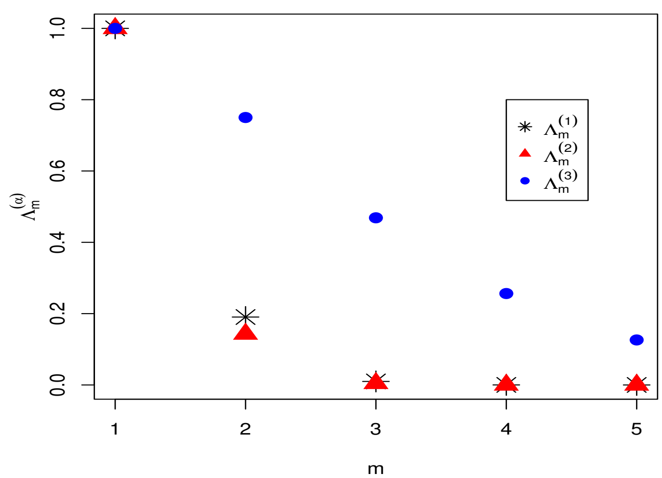


FIGURE 2. The Values of $\Lambda_m^{(\alpha)}, \alpha = 1, 2, 3$ in (4.3).

In the sequel, we provide some results on the extropy of $\mathbf{L}_{RRSS}^{(m)}$ and $\mathbf{U}_{RRSS}^{(m)}$ in terms of ordering properties.

Proposition 4.1. *Let $\mathbf{L}_{RRSS}^{(m)}$ and $\tilde{\mathbf{L}}_{RRSS}^{(m)}$ be the lower RRSS's from populations X and Y with pdf's f and g, respectively. If $X \leq_{disp} Y$, then $J(\mathbf{L}_{RRSS}^{(m)}) \leq J(\tilde{\mathbf{L}}_{RRSS}^{(m)})$.*

Proof. The proof is similar to Theorem 2.1. □

Proposition 4.2. *Suppose that X is a random variable with cdf F and pdf f. If $f(F^{-1}(e^{-V})) \geq 1$, then $J(\mathbf{L}_{RRSS}^{(m)})$ is decreasing in $m > 1$.*

Proof. From (4.1), we have

$$\frac{J(\mathbf{L}_{RRSS}^{(m+1)})}{J(\mathbf{L}_{RRSS}^{(m)})} = \binom{2m}{m} \mathbb{E}\left[f(F^{-1}(e^{-V}))\right] \geq 1.$$

Thus, the result follows readily. □

Remark 2. For a random variable X with cdf F and pdf f, $J(\mathbf{U}_{RRSS}^{(m)})$ is decreasing in $m > 1$, if $f(F^{-1}(1-e^{-V})) \geq 1$.

In the following propositions, some properties of extropy of upper RRSS are studied in connection with stochastic orders. The proof of the results are straightforward and hence they are omitted.

Proposition 4.3. *Let $\mathbf{U}_{RRSS}^{(m)}$ and $\tilde{\mathbf{U}}_{RRSS}^{(m)}$ be the upper RRSS's from populations X and Y with pdf's f and g, respectively. If $X \leq_{hr} Y$, and X or Y is DFR, then $J(\mathbf{U}_{RRSS}^{(m)}) \leq J(\tilde{\mathbf{U}}_{RRSS}^{(m)})$.*

Proposition 4.4. *Let $\mathbf{U}_{RRSS}^{(m)}$ and $\tilde{\mathbf{U}}_{RRSS}^{(m)}$ be the upper RRSS's from populations X and Y with pdf's f and g, respectively. If $X \leq_{su} Y (X \leq_{*} Y$ or $X \leq_{c} Y)$, and $f(0) \geq g(0) \geq 0$, then $J(\mathbf{U}_{RRSS}^{(m)}) \leq J(\tilde{\mathbf{U}}_{RRSS}^{(m)})$ for $m > 1$.*

5. Empirical measure of $\mathcal{CJ}(X)$

Let $X_1, X_2, ..., X_m$ be a random sample of size m from cdf F. If $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ represent the order statistics of the sample $X_1, X_2, ..., X_m$, then the empirical measure of F is defined as

$$\hat{F}_m(x) = \begin{cases} 0, & x < X_{(1)}, \\ \frac{k}{m}, & X_{(k)} \leq x \leq X_{(k+1)}, \qquad k = 1, 2, \ldots, m-1, \\ 1, & x > X_{(m)}. \end{cases}$$

Thus, the empirical measure of $\mathcal{CJ}(X)$ is obtained as

$$\begin{aligned} \mathcal{CJ}(\hat{F}_m) &= -\frac{1}{2}\int_{X_{(1)}}^{X_{(m)}} \hat{F}_m^2(x)dx \\ &= -\frac{1}{2}\sum_{k=1}^{m-1}\int_{X_{(k)}}^{X_{(k+1)}} \left(\frac{k}{m}\right)^2 dx \\ &= -\frac{1}{2}\sum_{k=1}^{m-1} U_{k+1}\left(\frac{k}{m}\right)^2, \end{aligned} \tag{5.1}$$

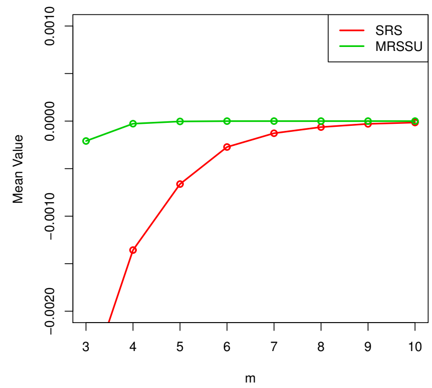
where $U_k = X_{(k)} - X_{(k-1)}, \;\; k = 1, 2, \ldots, m$ and $X_{(0)} = 0$. It can be shown that $\mathcal{CJ}(\hat{F}_m)$ almost surely converges to the CEX of X (3.1):

$$\mathcal{CJ}(\hat{F}_m) \overset{a.s.}{\to} \mathcal{CJ}(X) \quad as \quad m \to +\infty,$$

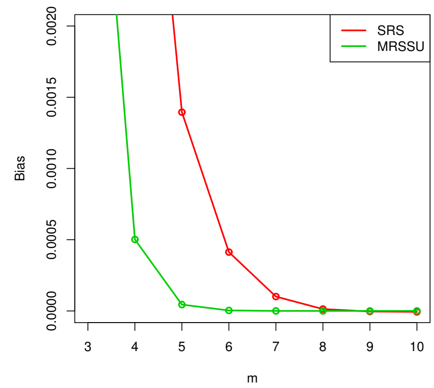
(see, for example, [15] and [20]). A simulation study is performed to evaluate the performance of the proposed $\mathcal{CJ}(\hat{F}_m)$ in both SRS and MRSSU schemes. We denote by $\widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)})$ the empirical versions of (3.3) and (3.2), respectively. Based on (3.2), (3.3) and (5.1), we obtain

$$\begin{aligned} \widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)}) &= -\frac{1}{2}\left(\sum_{k=1}^{m-1} U_{k+1}\left(\frac{k}{m}\right)^2\right)^m, \\ \widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)}) &= -\frac{1}{2}\prod_{i=1}^{m}\sum_{k=1}^{m-1} Z_{k+1}\left(\frac{k}{m}\right)^{2i}, \end{aligned}$$

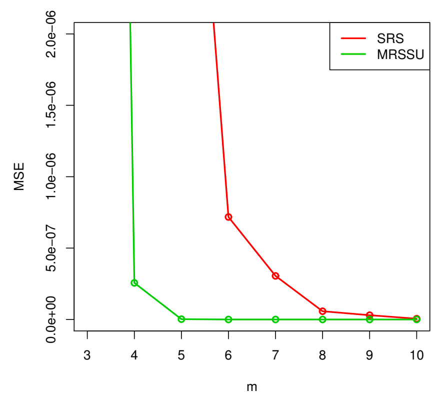
where $Z_k = Y_{(k)} - Y_{(k-1)}$ and $Y_{(k)}$'s are the ordered values of the MRSSU sample. To verify the validity of $\widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)})$, we obtain the mean value, bias and the root of mean square error (RMSE) of them for three examples 3–5 presented in Section 3. We consider the sample sizes $m = 3, 4, \ldots, 10$. For each case, the simulation has been performed with 5000 replications.



(A)



(B)



(C)

FIGURE 3. The Mean Value (a), Bias (b) and RMSE (c) of $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{SRS})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{MRSSU})$ in Example 3.

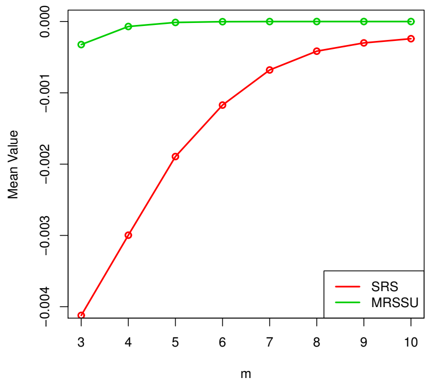
The results are summarized in Figures 3–5, based on the Examples 3–5. From these results we can conclude that:

(i) As we see, the result of Theorem 3.1 is also true for experimental values $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{SRS})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{MRSSU})$ and one can find that the results of MRSSU are less uncertainty than SRS counterpart in the sense that $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{MRSSU}) > \widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{SRS})$.

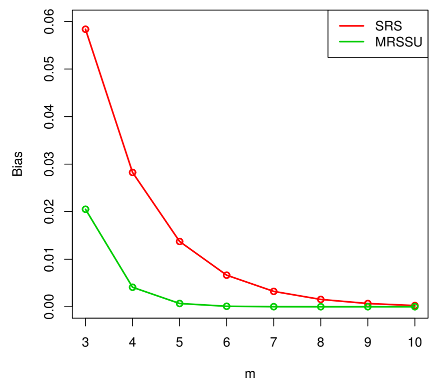
(ii) The bias values of $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{MRSSU})$ are less than that of $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{SRS})$ in all three examples.

(iii) The RMSE values of $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{MRSSU})$ are less than that of $\widehat{\mathcal{CJ}}(\mathbf{X}^{(m)}_{SRS})$ in all three examples.

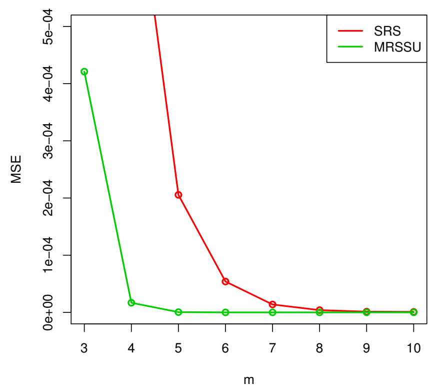
(iv) In all cases, the bias and RMSE tend to zero as m increases.



(A)



(B)



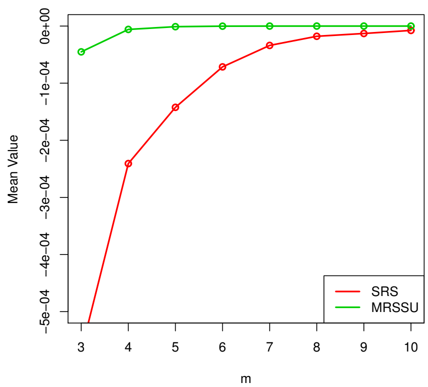
(C)

FIGURE 4. The Mean Value (a), Bias (b) and RMSE (c) of $\widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)})$ in Example 4.

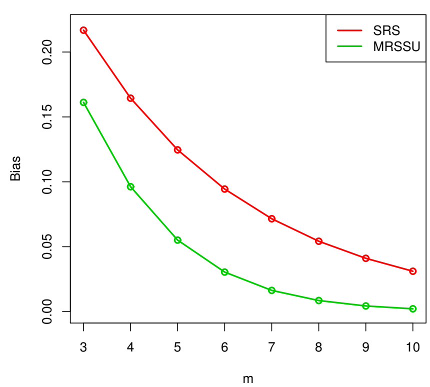
6. Real Example of Compressive sensing using MRSSU

Here, DCT is used for this example when a signal is not sparse in the time domain. The signal reconstruction is carried out using measurements by optimization techniques. Least absolute shrinkage and selection operator (LASSO) is used in this example as the optimization approach for signal reconstruction (see [31]). Here, a segment of an electrocardiogram (ECG) is shown in Figure 6. This signal is divided to some segments with 32 samples length. Then each segment is measured and finally it is reconstructed using LASSO. The signal is formed using the concatenating of reconstructed segments.

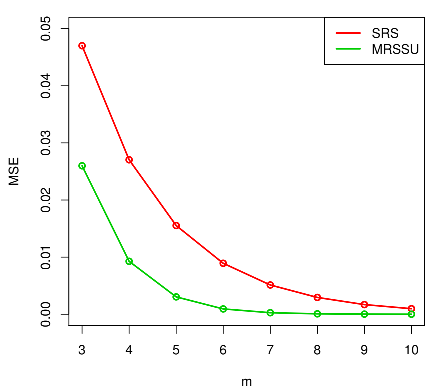
The SRS and MRSSU designs are used for exploiting of random measurements of DCT coefficients of signal. To investigate the performance of the SRS and MRSSU approaches, signal to noise ratio (SNR) and correlation between the reconstructed and original signals are used in this example. The results are shown in Figure 7. Figure 7 shows the correlation and SNR increase



(A)



(B)



(C)

FIGURE 5. The Mean Value (a), Bias (b) and RMSE (c) of $\widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)})$ in Example 5.

as the number of measurements (m) increases. Furthermore, it can be seen the performance of MRSSU measurements is much better than SRS ones. Also the results show the SNR difference of SRS and MRSSU approaches increases with an increase of the number of measurements for m less than 8. For correlation difference, this phenomenon is true for m less than 6. Indeed, MRSSU measurements cause to better performance than SRS ones for small measurements, and there is no significant difference between them for large measurements. For more investigation, cumulative extropy of SRS and MRSSU measurements are calculated using $\widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)})$ and $\widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)})$, respectively, and shown in Figure 8. For each case, a simulation with 1000 replications is performed. It can be seen that the difference between cumulative extropy of SRS and MRSSU measurements decreases with an increasing of measurements number. Moreover, one can find that MRSSU scheme can efficiently reduce the uncertainty measure comparing with SRS design in the sense

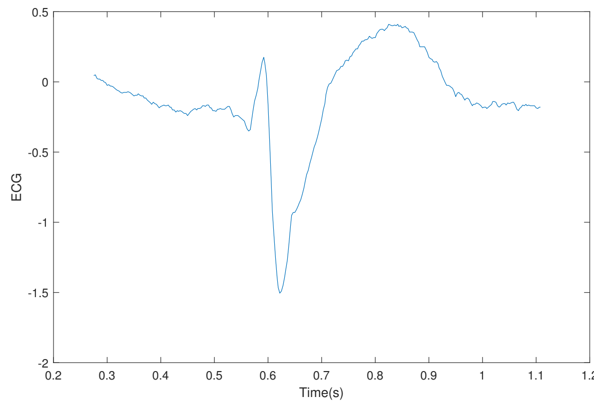
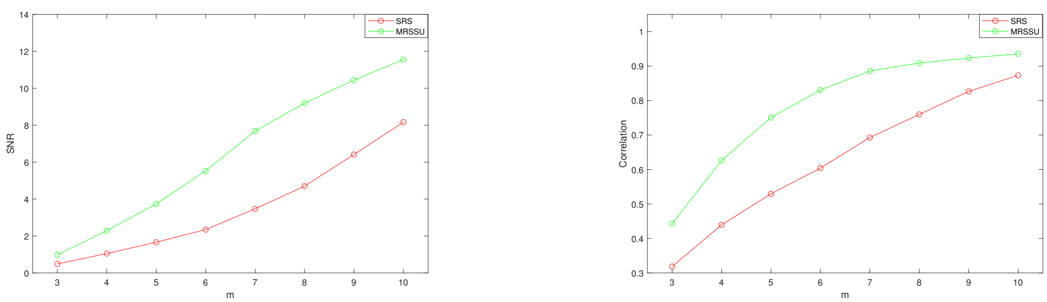


FIGURE 6. A segment of ECG signal.



(A)

(B)

FIGURE 7. (a) SNR of SRS and MRSSU measurements, (b) correlation of SRS and MRSSU measurements based on 1000 replications.

that $\widehat{\mathcal{CJ}}(\mathbf{X}_{MRSSU}^{(m)}) \geq \widehat{\mathcal{CJ}}(\mathbf{X}_{SRS}^{(m)})$. So, it is expected for small number of measurements, MRSSU causes to better SNR and correlation, as it is shown in Figure 7.

7. Conclusion

In this paper are studied the uncertainty measures of MRSSU, RRSS and SRS data using the extropy, cumulative extropy and residual extropy. Several results on the above extropy measures including monotonic properties, stochastic orders and quantile based extropy were obtained for MRSSU and RRSS compared to SRS data. Also, we provided two empirical cumulative extropy measures for both SRS and MRSSU data and compared them according to the mean value, bias and RMSE criteria. It can be seen that the bias and RMSE of MRSSU data are less than that of SRS data for all configurations. Also, two sampling methods MRSSU and SRS were utilized for compressive sensing technique and their performances were compared via signal to noise ratio (SNR), correlation coefficient of reconstructed and the original signal and cumulative extropy measure of uncertainty. Our results show that the values of SNR and correlation of coefficient of MRSSU are higher than that of SRS. Furthermore, it was shown that MRSSU scheme can efficiently reduce the uncertainty measure of cumulative extropy.

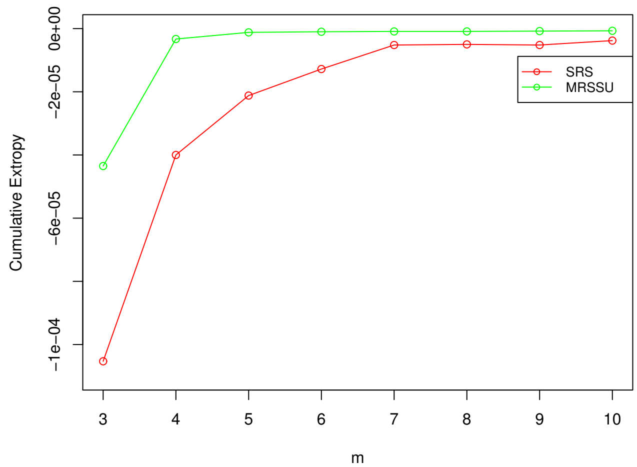


FIGURE 8. Cumulative extropy of SRS and MRSSU measurements based on 1000 replications .

Acknowledgement. Francesco Buono is member of the research group GNAMPA of INdAM (Istituto Nazionale di Alta Matematica) and is partially supported by MIUR-PRIN 2017, project "Stochastic Models for Complex Systems", no. 2017 JFFHSH. The present work was developed within the activities of the project 000009_ALTRI_CDA_75_2021_FRA_LINEA_B funded by "Programma per il finanziamento della ricerca di Ateneo - Linea B" of the University of Naples Federico II.

REFERENCES

[1] AHMED, A. N.— ALZAID, A.— BARTOSZEWICZ, J. — KOCHAR, S. C. (1986). *Dispersive and Superadditive Ordering.* Advances in Applied Probability, **18**, 1019–1022.

[2] ARNOLD, B. C. — BALAKRISHNAN, N. — NAGARAJA, H. N. (2008). *A First Course in Order Statistics*; Society for Industrial and Applied Mathematics.

[3] BARLOW, R. E. — PROSCHAN, F. J. (1996). *Mathematical Theory of Reliability.* Philadelphia: Society for Industrial and Applied Mathematics.

[4] BAGAI, I.— KOCHAR, S. C. (1986). *On tail-ordering and comparison of failure rates.* Communications in Statistics - Theory and Methods, **15**, 1377–1388.

[5] BIRADAR, B.S. — SANTOSHA, C.D. (2014). *Estimation of the Mean of the Exponential Distribution using Maximum Ranked Set Sampling with unequal Samples.* Open Journal of Statistics, **4**, 641–649.

[6] BLOCK, H.W.— SAVITS, T.H. (1998). *The Reversed Hazard Rate Function.* Probability in the Engineering and Informational Sciences, **12** 69–90.

[7] BUONO, F.— KAMARI, O.— LONGOBARDI, M. (2021). *Interval extropy and weighted interval extropy.* Ricerche di Matematica, https://doi.org/10.1007/s11587-021-00678-x.

[8] DI CRESCENZO, A.— LONGOBARDI, M. (2002). *Entropy-based measure of uncertainty in past lifetime distributions.* Journal of Applied Probability, **39**, 434–440.

[9] DONOHO, D. L. (2006). *Compressed sensing.* IEEE Transactions on Information Theory, **52**, 1289–1306.

[10] EBRAHIMI, N. (1996). *How to Measure Uncertainty in the Residual Life Time Distribution.* Sankhyā: The Indian Journal of Statistics, Series A (1961-2002), **58**, 48–56.

[11] ESKANDARZADEH, M.— DI CRESCENZO, A.— TAHMASEBI, S. (2018). *Measures of information for maximum ranked set sampling with unequal samples.* Communications in Statistics - Theory and Methods, **47**, 4692–4709.

[12] ESKANDARZADEH, M.— TAHMASEBI, S.— AFSHARI, M. (2016). *Information measures for record ranked set samples.* Ciência e Natura, **39**, 554.
[13] FINKELSTEIN, M.S. (2002). *On the reversed hazard rate.* Reliability Engineering and System Safety, **78**, 71–75.
[14] GILCHRIST, W. (2000). *Statistical Modelling with Quantile Functions.* Champman & Hall: New York, NY, USA.
[15] JAHANSHAHI, S. M. A. — ZAREI, H. — KHAMMAR A. H. (2019). *On cumulative residual extropy.* Probability in the Engineering and Informational Sciences, **34**, 605–625.
[16] JOZANI, M. J. — AHMADI, J. (2014). *On uncertainty and information properties of ranked set samples.* Information Sciences, **264**, 291–301.
[17] KAMARI, O.— BUONO, F. (2021). *On extropy of past lifetime distribution.* Ricerche di Matematica, **70**, 505–515, DOI: 10.1007/s11587-020-00488-7.
[18] KAZEMI, M. R.— TAHMASEBI, S.— CALÌ, C.— LONGOBARDI, M. (2021). *Cumulative residual extropy of minimum ranked set sampling with unequal samples.* Results in Applied Mathematics, DOI:10.1016/j.rinam.2021.100143.
[19] LAD, F.— SANFILIPPO, G.— AGRÒ, G. (2015). *Extropy: complementary dual of entropy.* Statistical Science, **30**, 40–58.
[20] LONGOBARDI, M. (2014). *Cumulative measures of information and stochastic orders.* Ricerche di Matematica, **63**, 209–223.
[21] MCINTYRE, G. A. (1952). *A method for unbiased selective sampling, using ranked sets.* Australian Journal of Agricultural Research, **3**, 385–390.
[22] NAIR, R. D.— SATHAR, A. E. I. (2020). *On dynamic failure extropy.* Journal of the Indian Society for Probability and Statistics, **21**, 287–313.
[23] PARZEN, E. (1979). *Nonparametric Statistical Data Modeling.* Journal of the American Statistical Association, **74**, 105–121.
[24] QIU, G. (2017). *The extropy of order statistics and record values.* Statistics & Probability Letters **120**, 52-–60.
[25] QIU, G.— JIA, K. (2018). *The residual extropy of order statistics.* Statistics & Probability Letters, **133**, 15–22.
[26] QIU, G.— EFTEKHARIAN, A. (2020). *Extropy information of maximum and minimum ranked set sampling with unequal samples.* Communications in Statistics–Theory and Methods, DOI:10.1080/03610926.2019.1678640.
[27] RAQAB, M. Z.— QIU, G. (2019). *On extropy properties of ranked set sampling.* Statistics, **53**, 210–226.
[28] SALEHI, M.— AHMADI, J. (2014). *Record ranked set sampling scheme.* METRON, **72**, 351–365.
[29] SHAKED, M.— SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*; Springer: New York, NY, USA.
[30] SHANNON, C. E. (1948). *A mathematical theory of communication.* Bell System Technical Journal, **27**, 379–423.
[31] SHEN, X.— CHEN, L.— GU, Y.— SO, H. C. (2016). *Square-Root Lasso With Nonconvex Regularization: An ADMM Approach.* IEEE Signal Processing Letters, **23**, 934–938.
[32] TAHMASEBI, S.— JAFARI, A. A.— ESKANDARZADEH, M. (2016). *Some results on residual entropy of ranked set samples.* Statistics & Probability Letters, **112**, 137–145.

* Department of Statistics
Persian Gulf University
Bushehr
IRAN
Email address: tahmasebi@pgu.ac.ir

** Department of Statistics, Faculty of Science
Fasa University
Fass
IRAN
Email address: kazemi@fasau.ac.ir

*** Department of Electrical Engineering and the ICT Research Institute
Persian Gulf University
Bushehr
IRAN
Email address: a.keshavarz@pgu.ac.ir

**** Department of Statistics
Yazd University
Yazd
IRAN
Email address: aajafari@yazd.ac.ir

***** Dipartimento di Matematica e Applicazioni "Renato Caccioppoli"
Università degli Studi di Napoli Federico II
Naples
ITALY
Email address: francesco.buono3@unina.it