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Interval Extropy and Weighted Interval Extropy

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Abstract

Recently, Extropy was introduced by Lad, Sanfilippo and Agrò as a complement dual of Shannon Entropy. In this paper, we propose dynamic versions of Extropy for doubly truncated random variables as measures of uncertainty called Interval Extropy and Weighted Interval Extropy. Some characterizations of random variables related to these new measures are given. Several examples are shown. These measures are evaluated under the effect of linear transformations and, finally, some bounds for them are presented.

Keywords: Uncertainty, Extropy, Weighted Extropy, Characterization

MSC Classification: 62N05 , 94A17

1 Introduction

The concept of Shannon entropy as a basic measure of uncertainty for a random variable was introduced by Shannon [20]. Let X be an absolutely continuous non-negative random variable having probability density function (pdf) f and cumulative distribution function (cdf) F. In Reliability Theory, X represents the lifetime of a system or a living organism. Shannon entropy for this kind of random variables is named differential entropy and is defined as follows:

$$H(X) = -\int_0^{+\infty} f(x) \log f(x)\, dx,$$

where log denotes the natural logarithm. Recently, another measure of uncertainty, known as extropy, was proposed by Lad et al. [13] as a dual measure of Shannon entropy. For a non-negative random variable X the extropy is defined as below:

$$J(X) = -\frac{1}{2}\int_0^{+\infty} f^2(x)\, dx. \tag{1}$$

The concept of extropy is useful in many fields: for instance, it is applied in automatic speech recognition [4]. In particular, the extropy of a network output with respect to the training set can be obtained in order to compute a kind of transformed cross entropy. Moreover, extropy is a measure better than entropy in some scenarios in statistical mechanics and thermodynamics [14]. More recently, some applications of extropy have been done in pattern recognition [3, 10].

Qiu et al. [19] defined the extropy for residual lifetime $X_t = (X - t \mid X > t)$ whose pdf is $f_{X_t}(x) = \frac{f(x+t)}{\overline{F}(t)}$ and survival function $\overline{F}_{X_t}(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}$, $x > 0$, called the residual extropy (REx) at time t and defined as

$$J(X_t) = -\frac{1}{2(\overline{F}(t))^2}\int_t^{+\infty} f^2(x)\, dx,$$

where $\overline{F}(t) = \mathbb{P}(X > t) = 1 - F(t)$ is the survival (reliability) function of X.

Krishnan et al. [12] and Kamari and Buono [9] studied the dual measure of residual extropy for past lifetime ${}_tX = (X \mid X \le t)$, whose pdf is $f_{{}_tX}(x) = \frac{f(x)}{F(t)}$ and cumulative distribution function $F_{{}_tX}(x) = \frac{F(x)}{F(t)}$, $0 < x < t$, called past extropy (PEx) and defined as follows:

$$J({}_tX) = -\frac{1}{2(F(t))^2}\int_0^{t} f^2(x)\, dx.$$

Recently, there has been growing attention to study uncertainty measures for doubly truncated random variable which is widely applied in many fields such as survival analysis and reliability engineering. In survival analysis, if the lifetime of the item falls in an interval (t_1, t_2), information about lifetime between these two points (also named doubly truncated failure time) is studied, see, for instance, Betensky and Martin [5], Khorashadizadeh et al. [11] and Poursaeed and Nematollahi [18]. Then, the random variable $(X \mid t_1 < X < t_2)$ is introduced with pdf $f_{X_{t_1,t_2}}(x) = \frac{f(x)}{F(t_2)-F(t_1)}$ and cdf $F_{X_{t_1,t_2}}(x) = \frac{F(x)-F(t_1)}{F(t_2)-F(t_1)}$, $t_1 < x < t_2$. With this motivation, Sunoj et.al. [21] introduced Interval Entropy to measure uncertainty in truncated random variable $(X \mid t_1 < X < t_2)$ that is defined as follows:

$$H(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)}\, dx, \tag{2}$$

which is an extension of Shannon Entropy. If $t_2 \to +\infty$, then $H(t_1, t_2)$ tends to residual entropy which was introduced by Ebrahimi [8]. Moreover, if $t_1 \to 0$, then $H(t_1, t_2)$ tends to the past entropy defined by Di Crescenzo and Longobardi [6]. Several other properties of the interval entropy were studied by Misagh and Yari [16].

Di Crescenzo and Longobardi [7] defined weighted entropy, weighted residual entropy and weighted past entropy, which are respectively given by

$$\begin{aligned} H^w(X) &= -\int_0^{+\infty} x f(x) \log f(x)\, dx, \\ H^w(X_t) &= -\int_0^{+\infty} x \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)}\, dx, \\ H^w({}_tX) &= -\int_0^{+\infty} x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)}\, dx. \end{aligned}$$

Balakrishnan et al. [2] introduced weighted Extropy and its dynamic versions as Weighted Residual Extropy and Weighted Past Extropy for residual and past lifetime as below:

$$\begin{aligned} J^w(X) &= -\frac{1}{2} \int_0^{+\infty} x f^2(x)\, dx, \\ J^w(X_t) &= -\frac{1}{2(\overline{F}(t))^2} \int_t^{+\infty} x f^2(x)\, dx, \\ J^w({}_tX) &= -\frac{1}{2(F(t))^2} \int_0^{t} x f^2(x)\, dx. \end{aligned}$$

Weighted Interval Entropy was introduced by Misagh and Yari [15] for doubly truncated random variable $(X \mid t_1 < X < t_2)$ as follows:

$$IH^w(t_1, t_2) = -\int_{t_1}^{t_2} x \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)}\, dx.$$

In analogy with the novel measures of uncertainty (i.e., Interval Entropy and Weighted Interval Entropy), here we introduce the concepts of Interval Extropy and Weighted Interval Extropy for doubly truncated random variables.

2 Interval Extropy

Let us suppose that the random variable $(X \mid t_1 < X < t_2)$ represents the lifetime of a unit which fails between t_1 and t_2 where $(t_1, t_2) \in D = \{(u, v) \in R_+^2 : F(u) < F(v)\}$, the Extropy for the doubly truncated random variable is defined as follows:

$$IJ(t_1, t_2) = IJ(X \mid t_1 < X < t_2) = -\frac{1}{2(F(t_2) - F(t_1))^2} \int_{t_1}^{t_2} f^2(x)\, dx, \quad (3)$$

which is an extension of Extropy and is called Interval Extropy (IEx). In (3) we have omitted the dependence of X in the expression $IJ(t_1, t_2)$, but when it is necessary we denote by $IJ_X(t_1, t_2)$ the interval extropy of X to distinguish it from the interval extropy of another random variable.

Remark 1 It is clear that $IJ(0, t_2) = J({}_{t_2}X)$, $IJ(t_1, +\infty) = J(X_{t_1})$ and $IJ(0, +\infty) = J(X)$ are Past Extropy, Residual Extropy and Extropy, respectively.

Example 1 Let $X \sim Exp(\lambda)$, $\lambda > 0$ and support $(0, +\infty)$. Based on (3), we evaluate the interval extropy of X for $0 < t_1 < t_2 < +\infty$ and we obtain

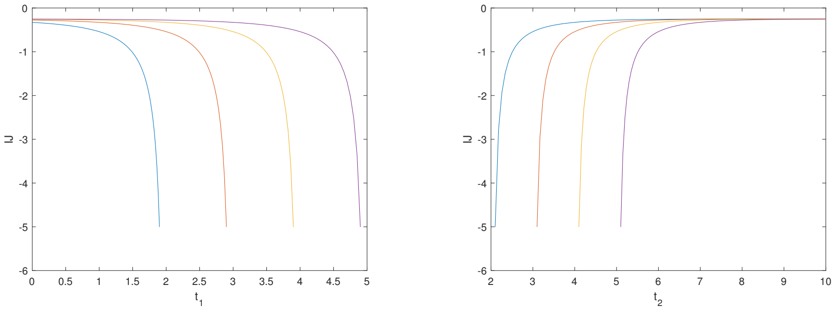
$$\begin{aligned} IJ(t_1, t_2) &= \frac{-1}{2(e^{-\lambda t_1} - e^{-\lambda t_2})^2} \int_{t_1}^{t_2} \lambda^2 e^{-2\lambda x} dx \\ &= -\frac{\lambda}{4} \cdot \frac{e^{-\lambda t_2} + e^{-\lambda t_1}}{e^{-\lambda t_1} - e^{-\lambda t_2}}. \end{aligned}$$

In Figure 1, we plot the interval extropy as a function of t_1 for fixed t_2 (Figure 1(a)) and vice versa (Figure 1(b)) for $\lambda = 1$.

Example 2 Let X follow the Weibull distribution, $W2(\alpha, \lambda)$, with parameters $\alpha = \lambda = 2$, $X \sim W2(2, 2)$. The cdf and the pdf of X are expressed as

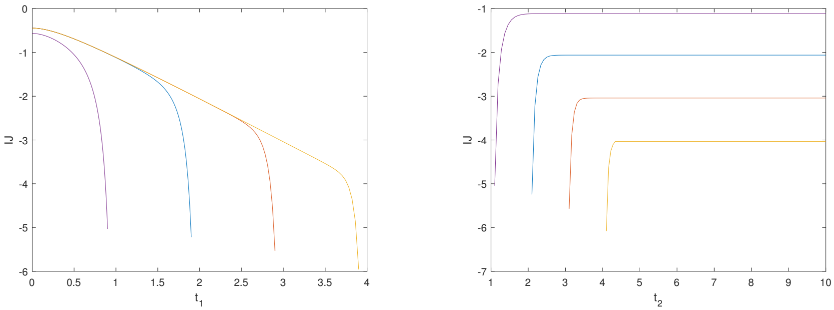
$$F(x) = 1 - \exp(-2x^2), \quad f(x) = 4x \exp(-2x^2), \quad x \in (0, +\infty).$$

Since the expression of the interval extropy is not given in terms of elementary functions, in Figure 2, we plot the interval extropy as a function of t_1 for fixed t_2 (Figure 2(a)) and vice versa (Figure 2(b)). From Figure 2(b) we observe an asymptotic behavior of the interval extropy as $t_2 \to +\infty$ towards $-t_1$, i.e., when the interval extropy



(a) $IJ(t_1, t_2)$ as a function of t_1 (b) $IJ(t_1, t_2)$ as a function of t_2

Fig. 1: Plot of IJ in Example 1 as a function of t_1 or t_2 fixing the other one with $t_i = 2$ (blue), 3 (red), 4 (yellow) and 5 (violet), $i = 1, 2$.



(a) $IJ(t_1, t_2)$ as a function of t_1 (b) $IJ(t_1, t_2)$ as a function of t_2

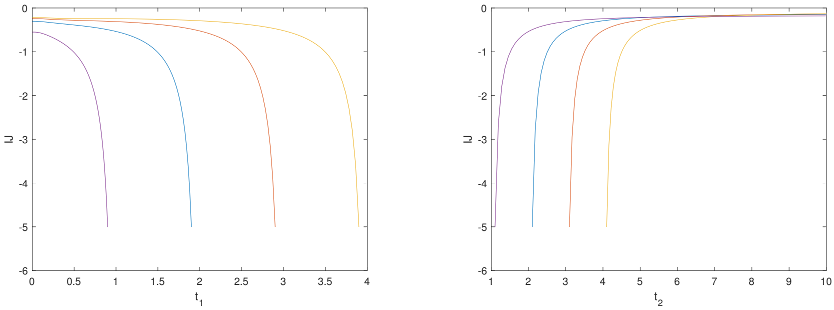
Fig. 2: Plot of IJ in Example 2 as a function of t_1 or t_2 fixing the other one with $t_i = 1$ (violet), 2 (blue), 3 (red) and 4 (yellow), $i = 1, 2$.

$IJ(t_1, t_2)$ reduces to the residual extropy $J(X_{t_1})$. In fact, the residual extropy of X in t can be derived as

$$\begin{aligned} J(X_t) &= -\frac{1}{2\exp(-4t^2)} \int_t^{+\infty} 16x^2 \exp(-4x^2) dx \\ &= -t - \frac{1}{\exp(-4t^2)} \int_t^{+\infty} \exp(-4x^2) dx \\ &= -t - \frac{1}{2\sqrt{2}\exp(-4t^2)} \int_{2\sqrt{2}t}^{+\infty} \exp\left(-\frac{y^2}{2}\right) dy = -t - \frac{\sqrt{\pi}}{2} \frac{\overline{F}_Z(2\sqrt{2}t)}{\exp(-4t^2)}, \end{aligned}$$

where $\overline{F}_Z(\cdot)$ is the survival function of $Z \sim N(0, 1)$.

Example 3 Let X follow the Lognormal distribution, $Lognormal(\mu, \sigma^2)$, with parameters $\mu = 0$, $\sigma^2 = 1$, $X \sim Lognormal(0, 1)$. The pdf of X is expressed



(a) $IJ(t_1, t_2)$ as a function of t_1 (b) $IJ(t_1, t_2)$ as a function of t_2

Fig. 3: Plot of IJ in Example 3 as a function of t_1 or t_2 fixing the other one with $t_i = 1$ (violet), 2 (blue), 3 (red) and 4 (yellow), $i = 1, 2$.

as

$$f(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{\log^2 x}{2}\right), \qquad x \in (0, +\infty).$$

Since the expression of the interval extropy is not given in terms of elementary functions, in Figure 3, we plot the interval extropy as a function of t_1 for fixed t_2 (Figure 3(a)) and vice versa (Figure 3(b)).

Based on the above examples, it could seem that the interval extropy is always decreasing with respect to t_1 and always increasing with respect to t_2. In the following, we provide two counterexamples to show that the interval extropy can be non monotonous with respect to t_1 and t_2.

Example 4 Let X be a random variable with support $S = (a, +\infty)$, $a > 0$, whose cdf is defined as $F(x) = 1 - \left(\frac{a}{x}\right)^b$, $b > 0$. The interval extropy of X can be obtained as follows

$$\begin{aligned} IJ(t_1, t_2) &= -\frac{1}{2\left[\left(\frac{a}{t_1}\right)^b - \left(\frac{a}{t_2}\right)^b\right]^2} \int_{t_1}^{t_2} \frac{b^2 a^{2b}}{x^{2b+2}} dx \\ &= \frac{1}{2\left[\frac{1}{t_1^b} - \frac{1}{t_2^b}\right]^2} \frac{b^2}{2b+1} \left[\frac{1}{t_2^{2b+1}} - \frac{1}{t_1^{2b+1}}\right] \\ &= \frac{b^2\left(t_1^{2b+1} - t_2^{2b+1}\right)}{2(2b+1)t_1 t_2 \left(t_2^b - t_1^b\right)^2}. \end{aligned}$$

Let us focus on the case $a = 1$ and $b = 10$. In Figure 4 we have plotted the interval extropy of X as a function of t_1 for fixed different values of t_2 and we can observe that it is initially increasing and then decreasing with respect to t_1.

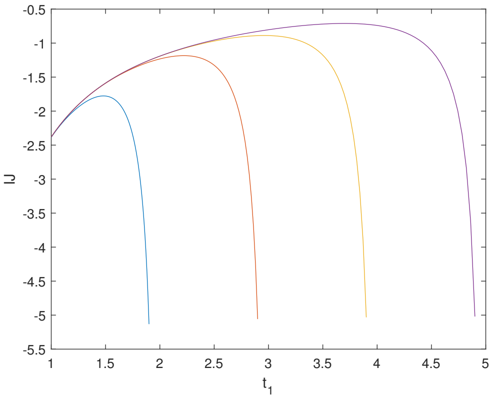


Fig. 4: Plot of IJ in Example 4 as a function of t_1 fixing $t_2 = 2$ (blue), 3 (red), 4 (yellow) and 5 (violet).

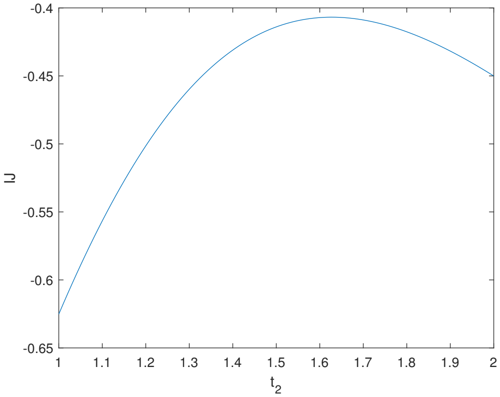


Fig. 5: Plot of IJ in Example 5 as a function of $t_2 \in (1, 2)$ fixing $t_1 = 0.1$.

Example 5 Let X be a random variable with cdf and pdf respectively defined as

$$F(x) = \begin{cases} \exp\left(-\frac{1}{2} - \frac{1}{x}\right), & \text{if} \quad x \in (0, 1] \\ \exp\left(-2 + \frac{x^2}{2}\right), & \text{if} \quad x \in [1, 2) \end{cases}$$

$$f(x) = \begin{cases} \exp\left(-\frac{1}{2} - \frac{1}{x}\right) \frac{1}{x^2}, & \text{if} \quad x \in (0, 1] \\ \exp\left(-2 + \frac{x^2}{2}\right) x, & \text{if} \quad x \in [1, 2). \end{cases}$$

In Figure 5 we have plotted the interval extropy as a function of $t_2 \in (1, 2)$ with fixed $t_1 = 0.1$ and we can observe a non monotonic behavior.

In the following theorem, we show the connection among Extropy and its dynamic versions.

Theorem 1 *Let X be a random variable denoting the lifetime of a component. For all $0 < t_1 < t_2 < +\infty$ the extropy can be decomposed as follows:*

$$J(X) = F^2(t_1)J({}_{t_1}X) + (F(t_2) - F(t_1))^2 IJ(t_1, t_2) + \overline{F}^2(t_2)J(X_{t_2}), \tag{4}$$

i.e., the Extropy is a function of PEx, REx and IEx.

Proof From the definition of extropy (1), we can write

$$\begin{aligned} J(X) &= -\frac{1}{2}\int_0^{+\infty} f^2(x)\,dx \\ &= -\frac{1}{2}\int_0^{t_1} f^2(x)\,dx - \frac{1}{2}\int_{t_1}^{t_2} f^2(x)\,dx - \frac{1}{2}\int_{t_2}^{+\infty} f^2(x)\,dx. \end{aligned} \tag{5}$$

Now, we observe that the terms in the RHS of (5) are related to past extropy, interval extropy and residual extropy as

$$\begin{aligned} -\frac{1}{2}\int_0^{t_1} f^2(x)\,dx &= F^2(t_1)J({}_{t_1}X), \\ -\frac{1}{2}\int_{t_1}^{t_2} f^2(x)\,dx &= (F(t_2) - F(t_1))^2 IJ(t_1, t_2), \\ -\frac{1}{2}\int_{t_2}^{+\infty} f^2(x)\,dx &= \overline{F}^2(t_2)J(X_{t_2}), \end{aligned}$$

and then we obtain the stated result. □

Relation (4) shows that the uncertainty about the failure time of a component consists of 3 parts:

1^{st}. The uncertainty of the failure time in $(0, t_1)$;
2^{nd}. The uncertainty of the failure time in (t_1, t_2);
3^{rd}. The uncertainty about the failure time in $(t_2, +\infty)$.

The corresponding aging classes are defined as follows.

Definition 1 The random variable X is said to have decreasing IJ property if and only if for any fixed t_2, $IJ(t_1, t_2)$ is decreasing respect to t_1.

Definition 2 The random variable X is said to have increasing IJ property if and only if for any fixed t_1, $IJ(t_1, t_2)$ is increasing respect to t_2.

Remark 2 As seen in Example 1, the exponential distribution satisfies both the conditions in Definitions 1 and 2.

Definition 3 Let X be a non-negative and absolutely continuous random variable with cdf F and pdf f. The Generalized Failure Rate (GFR) functions of X in t_1 and t_2 (with $F(t_2) - F(t_1) > 0$) are defined in [17] as

$$h_i(t_1, t_2) = \frac{f(t_i)}{F(t_2) - F(t_1)}, \qquad i = 1, 2. \tag{6}$$

An upper bound in terms of GFR is obtained for Interval Extropy in the following theorem.

Theorem 2 *Let X be an absolutely continuous non-negative random variable. If IJ is increasing in t_2, then*

$$IJ(t_1, t_2) \leq -\frac{h_2(t_1, t_2)}{4}. \tag{7}$$

Proof By differentiating IEx with respect to t_2, we have

$$\frac{\partial IJ(t_1, t_2)}{\partial t_2} = -\frac{h_2^2(t_1, t_2)}{2} - 2h_2(t_1, t_2) IJ(t_1, t_2). \tag{8}$$

If $IJ(t_1, t_2)$ is increasing in t_2 then (8) implies (7). □

In the following proposition, we analyze the effect of a linear transformation on the interval extropy.

Proposition 3 *Let X be a non negative and absolutely continuous random variable and let $Y = aX + b$ where $a > 0$ and $b \geq 0$. The interval extropy of Y is given in terms of the interval extropy of X as*

$$IJ_Y(t_1, t_2) = \frac{1}{a} IJ_X\left(\frac{t_1 - b}{a}, \frac{t_2 - b}{a}\right), \tag{9}$$

where $t_1, t_2 \in S_Y$.

Proof The cdf and the pdf of Y can be expressed in terms of F_X and f_X as

$$F_Y(x) = F_X\left(\frac{x - b}{a}\right), \quad f_Y(x) = \frac{1}{a} f_X\left(\frac{x - b}{a}\right). \tag{10}$$

Hence, the interval extropy of Y can be expressed as

$$\begin{aligned}
IJ_Y(t_1, t_2) &= -\frac{1}{2\left(F_X\left(\frac{t_2-b}{a}\right) - F_X\left(\frac{t_1-b}{a}\right)\right)^2} \int_{t_1}^{t_2} \frac{1}{a^2} f_X^2\left(\frac{x - b}{a}\right) dx \\
&= -\frac{1}{2\left(F_X\left(\frac{t_2-b}{a}\right) - F_X\left(\frac{t_1-b}{a}\right)\right)^2} \int_{\frac{t_1-b}{a}}^{\frac{t_2-b}{a}} \frac{1}{a} f_X^2(x)\, dx \\
&= \frac{1}{a} IJ_X\left(\frac{t_1 - b}{a}, \frac{t_2 - b}{a}\right),
\end{aligned}$$

which completes the proof. □

In the following theorem, we give a characterization of the exponential distribution based on the interval extropy.

Theorem 4 *Let X be a random variable with support $(0,+\infty)$, differentiable and strictly positive pdf f and cdf F. Then, X is exponentially distributed if, and only if, for all (t_1,t_2) such that $0<t_1<t_2<+\infty$, the following relation holds*

$$IJ(t_1,t_2) = -\frac{1}{4}\left[h_1(t_1,t_2)+h_2(t_1,t_2)\right]. \tag{11}$$

Proof Let us suppose $X \sim Exp(\lambda)$. In Example 1, we have evaluated the interval extropy that is given by

$$IJ(t_1,t_2) = -\frac{\lambda}{4}\cdot\frac{e^{-\lambda t_2}+e^{-\lambda t_1}}{e^{-\lambda t_1}-e^{-\lambda t_2}}.$$

Moreover, about GFR functions of X, we have

$$\begin{aligned} h_1(t_1,t_2) &= \frac{\lambda e^{-\lambda t_1}}{e^{-\lambda t_1}-e^{-\lambda t_2}}, \\ h_2(t_1,t_2) &= \frac{\lambda e^{-\lambda t_2}}{e^{-\lambda t_1}-e^{-\lambda t_2}}, \end{aligned}$$

and then the first part of the proof is completed.

Conversely, let us suppose (11) holds. Then, by making explicit the interval extropy and GFR functions, we obtain

$$-\frac{1}{2(F(t_2)-F(t_1))^2}\int_{t_1}^{t_2} f^2(x)\,dx = -\frac{f(t_1)+f(t_2)}{4(F(t_2)-F(t_1))}.$$

From the above equation, we have

$$\int_{t_1}^{t_2} f^2(x)\,dx = \frac{1}{2}(F(t_2)-F(t_1))(f(t_1)+f(t_2)). \tag{12}$$

By differentiating both sides of (12) with respect to t_1, we get

$$-f^2(t_1) = -\frac{1}{2}f(t_1)(f(t_1)+f(t_2)) + \frac{1}{2}f'(t_1)(F(t_2)-F(t_1)),$$

which reduces to

$$-f^2(t_1)+f(t_1)f(t_2) = f'(t_1)(F(t_2)-F(t_1)). \tag{13}$$

By differentiating both sides of (13) with respect to t_2, we get

$$f(t_1)f'(t_2) = f'(t_1)f(t_2),$$

which is equivalent to

$$\frac{f'(t_1)}{f(t_1)} = \frac{f'(t_2)}{f(t_2)},$$

i.e., the ratio is constant for $x>0$,

$$\frac{f'(x)}{f(x)} = A. \tag{14}$$

Hence, by integrating both sides of (14) from 0 to t, we get

$$f(t) = f(0)\,e^{At},$$

and in order to satisfy the condition of normalization for the pdf f, we need $A = -f(0)$, i.e., f is the pdf of an exponential distribution. $\square$

Remark 3 The equilibrium random variable Y associated to a renewal process with lifetime X is a random variable of primary interest in the context of reliability theory, as pointed out in Barlow and Proschan [1]. The survival function and the probability density function of Y are expressed as

$$\overline{F}_Y(t) = \frac{1}{\mathbb{E}(X)} \int_t^{+\infty} \overline{F}_X(x)\, dx, \quad f_Y(t) = \frac{\overline{F}_X(t)}{\mathbb{E}(X)}$$

where $\mathbb{E}(X)$ is the expectation of X. We can define Extropy and its interval version of Y as follows:

$$J(Y) = -\frac{1}{2\mathbb{E}^2(X)} \int_0^{\infty} \overline{F}_X^2(x)\, dx,$$

$$IJ_Y(t_1, t_2) = -\frac{1}{2} \frac{\int_{t_1}^{t_2} \overline{F}_X^2(x)\, dx}{\left(\int_{t_1}^{t_2} \overline{F}_X(x)\, dx\right)^2}.$$

3 Weighted Interval Extropy

In order to give importance to the value assumed by the random variable, it is significant to introduce weighted versions of the measures of uncertainty. In fact, most of the well-known measure of discrimination are position-free, in the sense that they assume the same value for X and $X + b$ for any $b \in \mathbb{R}$. In Proposition 3, we have showed that the interval extropy does not change under translations since, for $Y = X + b$, we have $IJ_Y(t_1 + b, t_2 + b) = IJ_X(t_1, t_2)$. In this section, we will introduce and study the weighted version of the interval extropy and we will show that it is not invariant under translations.

Suppose X be a non-negative absolutely continuous random variable. For all t_1 and t_2 such that $(t_1, t_2) \in D = \{(u, v) \in R_+^2 : F(u) < F(v)\}$ we define the Weighted Interval Extropy (WIEx) of X as

$$IJ^w(t_1, t_2) = -\frac{1}{2(F(t_2) - F(t_1))^2} \int_{t_1}^{t_2} x f^2(x)\, dx, \tag{15}$$

in the same way in which in [15] the weighted interval entropy has been defined.

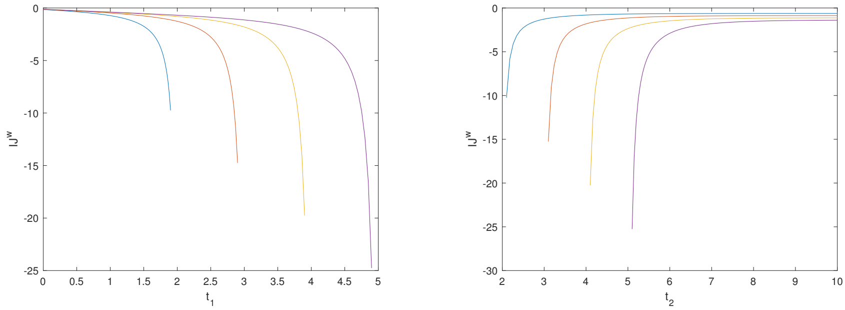
Remark 4 We notice that

$$\lim_{t_1 \to 0} IJ^w(t_1, t_2) = IJ^w({}_{t_2}X) \text{ and } \lim_{t_2 \to +\infty} IJ^w(t_1, t_2) = IJ^w(X_{t_1}),$$

where $IJ^w({}_{t_2}X)$ and $IJ^w(X_{t_1})$ are Weighted Past Extropy at time t_2 and Weighted Residual Extropy at time t_1, respectively.

Example 6 Let $X \sim Exp(\lambda)$, $\lambda > 0$. Based on (15), we evaluate the weighted interval extropy of X for $0 < t_1 < t_2 < +\infty$ and we obtain

$$IJ^w(t_1, t_2) = \frac{-1}{2(e^{-\lambda t_1} - e^{-\lambda t_2})^2} \int_{t_1}^{t_2} x\lambda^2 e^{-2\lambda x} dx$$



(a) $IJ^w(t_1, t_2)$ as a function of t_1 (b) $IJ^w(t_1, t_2)$ as a function of t_2

Fig. 6: Plot of IJ^w in Example 6 as a function of t_1 or t_2 fixing the other one with $t_i = 2$ (blue), 3 (red), 4 (yellow) and 5 (violet), $i = 1, 2$.

$$= -\frac{\lambda}{4} \cdot \frac{t_1 e^{-2\lambda t_1} - t_2 e^{-2\lambda t_2}}{(e^{-\lambda t_1} - e^{-\lambda t_2})^2} - \frac{1}{8} \cdot \frac{e^{-\lambda t_2} + e^{-\lambda t_1}}{e^{-\lambda t_1} - e^{-\lambda t_2}}.$$

In Figure 6, we plot the interval extropy as a function of t_1 for fixed t_2 (Figure 6(a)) and vice versa (Figure 6(b)) for $\lambda = 1$.

Example 7 Let X follow the Weibull distribution with parameters $\alpha = \lambda = 2$, $X \sim W2(2, 2)$. Based on (15), we evaluate the weighted interval extropy of X for $0 < t_1 < t_2 < +\infty$ and we obtain

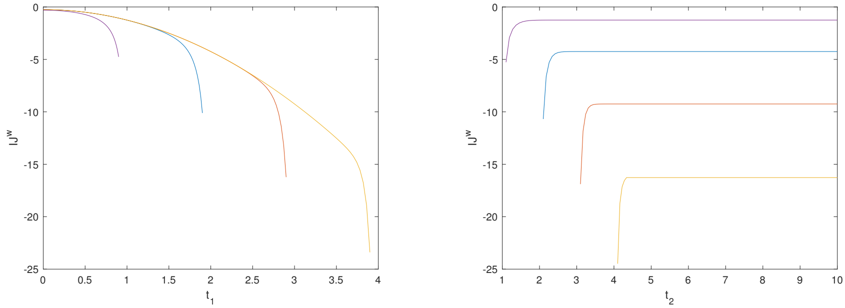
$$\begin{aligned} IJ^w(t_1, t_2) &= \frac{-1}{2(\exp(-2t_1^2) - \exp(-2t_2^2))^2} \int_{t_1}^{t_2} 16x^3 \exp(-4x^2) dx \\ &= \frac{t_2^2 \exp(-4t_2^2) - t_1^2 \exp(-4t_1^2)}{(\exp(-2t_1^2) - \exp(-2t_2^2))^2} - \frac{1}{4} \cdot \frac{\exp(-2t_1^2) + \exp(-2t_2^2)}{\exp(-2t_1^2) - \exp(-2t_2^2)}. \end{aligned}$$

In Figure 7, we plot the interval extropy as a function of t_1 for fixed t_2 (Figure 7(a)) and vice versa (Figure 7(b)). From Figure 7(b) we observe an asymptotic behavior of the weighted interval extropy as $t_2 \to +\infty$, i.e., when the weighted interval extropy $IJ^w(t_1, t_2)$ reduces to the weighted residual extropy $J^w(X_{t_1})$. In fact, the weighted residual extropy of X in t can be expressed as

$$J(X_t) = -t^2 - \frac{1}{4}.$$

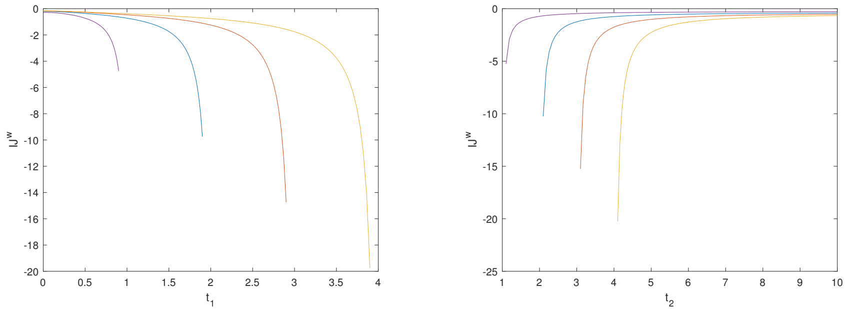
Example 8 Let X follow the Lognormal distribution with parameters $\mu = 0$, $\sigma^2 = 1$, $X \sim Lognormal(0, 1)$. Since the expression of the weighted interval extropy is not given in terms of elementary functions, in Figure 8, we plot the weighted interval extropy as a function of t_1 for fixed t_2 (Figure 8(a)) and vice versa (Figure 8(b)).

In the following theorem, we prove that the expression of the weighted extropy is composed of three terms in function of the weighted past, residual and interval extropies.



(a) $IJ^w(t_1, t_2)$ as a function of t_1 (b) $IJ^w(t_1, t_2)$ as a function of t_2

Fig. 7: Plot of IJ^w in Example 7 as a function of t_1 or t_2 fixing the other one with $t_i = 1$ (violet), 2 (blue), 3 (red) and 4 (yellow), $i = 1, 2$.



(a) $IJ^w(t_1, t_2)$ as a function of t_1 (b) $IJ^w(t_1, t_2)$ as a function of t_2

Fig. 8: Plot of IJ^w in Example 8 as a function of t_1 or t_2 fixing the other one with $t_i = 1$ (violet), 2 (blue), 3 (red) and 4 (yellow), $i = 1, 2$.

Theorem 5 *Let X be a random variable denoting the lifetime of a component. For all $0 < t_1 < t_2 < +\infty$ the weighted extropy can be decomposed as follows:*

$$J^w(t_1, t_2) = F^2(t_1) J^w({}_{t_1}X) + (F(t_2) - F(t_1))^2 IJ^w(t_1, t_2) + \overline{F}^2(t_2) J^w(X_t),$$

i.e., Weighted Extropy is a function of WPEx, WREx and WIEx.

Proof The proof is similar to the one of Theorem 1 and hence it is omitted. □

In the following proposition, we analyze the effect of a linear transformation on the weighted interval extropy.

Proposition 6 *Let X be a non negative and absolutely continuous random variable and let $Y = aX + b$ where $a > 0$ and $b \geq 0$. The weighted interval extropy of Y is given in terms of the interval extropy and weighted interval extropy of X as*

$$IJ_Y^w(t_1,t_2) = IJ_X^w\left(\frac{t_1-b}{a},\frac{t_2-b}{a}\right) + \frac{b}{a} IJ_X\left(\frac{t_1-b}{a},\frac{t_2-b}{a}\right), \tag{16}$$

where $t_1, t_2 \in S_Y$.

Proof By using the expressions of the cdf and the pdf of Y in terms of F_X and f_X obtained in (10), the weighted interval extropy of Y can be expressed as

$$\begin{aligned} IJ_Y^w(t_1,t_2) &= -\frac{1}{2\left(F_X\left(\frac{t_2-b}{a}\right)-F_X\left(\frac{t_1-b}{a}\right)\right)^2}\int_{t_1}^{t_2}\frac{x}{a^2}f_X^2\left(\frac{x-b}{a}\right)dx \\ &= -\frac{1}{2\left(F_X\left(\frac{t_2-b}{a}\right)-F_X\left(\frac{t_1-b}{a}\right)\right)^2}\int_{\frac{t_1-b}{a}}^{\frac{t_2-b}{a}} x f_X^2(x)\,dx \\ &\quad -\frac{1}{2\left(F_X\left(\frac{t_2-b}{a}\right)-F_X\left(\frac{t_1-b}{a}\right)\right)^2}\int_{\frac{t_1-b}{a}}^{\frac{t_2-b}{a}} \frac{b}{a} f_X^2(x)\,dx \\ &= IJ_X^w\left(\frac{t_1-b}{a},\frac{t_2-b}{a}\right) + \frac{b}{a} IJ_X\left(\frac{t_1-b}{a},\frac{t_2-b}{a}\right), \end{aligned}$$

which completes the proof. □

Remark 5 The results given in Propositions 3 and 6 about linear transformations could be generalized to monotonic transformations but, in these cases, we do not obtain a formula of interest, in the sense that the interval extropy and the weighted interval extropy of the transformed random variable are not expressed in terms of the ones of the original random variable.

In the following theorem, we present an upper bound for the weighted interval extropy given in terms of the generalized failure rate function.

Theorem 7 *For an absolutely continuous non-negative random variable X, if the WIEx is increasing in t_2, then we have*

$$IJ^w(t_1,t_2) \leq -\frac{t_2 h_2(t_1,t_2)}{4}. \tag{17}$$

Proof The proof follows in analogy with the one of Theorem 2 and hence it is omitted. □

4 Conclusion

In this paper dynamic versions of extropy for double truncated random variables have been presented. Several examples are given. The behavior under linear transformations of these new measures has been studied. Some bounds for them have been found in relation with the Generalized Failure Rate.

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Conflict of interest

The authors declare that they have no conflict of interest.

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