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THE UNIFIED EXTROPY AND ITS VERSIONS IN CLASSICAL AND DEMPSTER-SHAFER THEORIES

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Abstract

The measures of uncertainty are a topic of considerable and growing interest. Recently, the introduction of the extropy as a measure of uncertainty dual of Shannon entropy opened up interest in new aspects of the subject. Since there are many versions of entropy, a unified formulation for it has been introduced to work with all of them in an easy way. Here, we consider the possibility of defining a unified formulation for extropy by introducing a measure depending on two parameters. For particular choices of parameters, this measure will give back the well-known formulations of extropy. Moreover, the unified formulation of extropy is also analyzed in the context of Dempster-Shafer theory of evidence and an application to classification problems is given.

Keywords: Shannon entropy; Extropy; Dempster-Shafer Theory; Tsallis entropy; Fractional entropy.

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1. Introduction

Let X be a discrete random variable with support S of cardinality N and with probability vector $(p_1, \dots, p_N)$. In 1948, Shannon [16] introduced a measure of information or uncertainty about X, known as Shannon entropy, as

$$H(X) = -\sum_{i=1}^{N} p_i \log p_i, \tag{1}$$

where log denotes the natural logarithm. The paper of Shannon is considered a pioneering one since several studies have been devoted to the measures of information or discrimination, see, for instance, [7, 13, 14, 18]. Recently, a new measure, known as extropy, has been proposed by Lad et al. [11] as a measure dual of Shannon entropy and defined by

$$J(X) = -\sum_{i=1}^{N} (1-p_i) \log(1-p_i). \tag{2}$$

Another important and well-known generalizations of Shannon entropy, is Tsallis entropy defined in [19] as

$$S_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \sum_{i=1}^{N} p_i^\alpha\right), \tag{3}$$

where α is a parameter greater than 0 and different from 1. Tsallis entropy is a generalization of Shannon entropy since

$$\lim_{\alpha \to 1} S_\alpha(X) = H(X).$$

It is also possible to study the extropy-based version of Tsallis entropy, namely Tsallis extropy, as studied in details by Balakrishanan et al. [2]. In that paper, two equivalent expressions for Tsallis extropy are given by

$$\begin{aligned} JS_\alpha(X) &= \frac{1}{\alpha-1}\left(N-1-\sum_{i=1}^{N}(1-p_i)^\alpha\right) \qquad (4) \\ &= \frac{1}{\alpha-1}\sum_{i=1}^{N}(1-p_i)\left(1-(1-p_i)^{\alpha-1}\right), \end{aligned}$$

where $\alpha > 0$ and $\alpha \neq 1$. Of course, the definition of Tsallis extropy guarantees the preservation of the same relation which holds between Shannon entropy and extropy, in the sense that

$$\lim_{\alpha \to 1} JS_\alpha(X) = J(X). \tag{5}$$

The study of measures of uncertainty has been recently extended to the fractional calculus. In that context, Ubriaco [20] defined the fractional version of Shannon entropy, known as fractional entropy, by

$$H_q(X) = \sum_{i=1}^{N} p_i[-\log p_i]^q, \tag{6}$$

where $0 < q \leq 1$. Note that for $q = 0$ it simply reduces to 1 due to the normalization condition. Moreover, for $q = 1$ it reduces to the Shannon entropy, i.e., $H_1(X) = H(X)$. Up to now, the corresponding version based on extropy has not been studied in details. Of course, the fractional extropy can be easily defined as

$$J_q(X) = \sum_{i=1}^{N}(1 - p_i)[-\log(1 - p_i)]^q, \tag{7}$$

where $0 < q \leq 1$, and this is a particular case of the definition of fractional Deng extropy (see [10]) for the choice of a basic probability assignment that degenerates in a discrete probability function. Again, the case $q = 0$ is not of interest since it brings to a constant, whereas for $q = 1$ we obtain $J_1(X) = J(X)$.

Furthermore, the study of measures of uncertainty has been extended to the Dempster-Shafer theory of evidence [4, 15]. This is a generalization of the classical probability theory which allows us to handle better uncertainty. More precisely, the discrete probability distributions are replaced by the mass functions which can give a weight, a sort of degree of belief, towards all the subsets of the space of the events. More details on this theory and on measures of uncertainty developed in its context will be recalled later on.

Recently, it was proposed by Balakrishnan et al. [1] a new definition given with the purpose of unifying the definitions of Shannon, Tsallis and fractional entropies. This was called fractional Tsallis entropy, or unified formulation of entropy, and it is given by

$$S_\alpha^q(X) = \frac{1}{\alpha - 1}\sum_{i=1}^{N} p_i(1 - p_i^{\alpha-1})(-\log p_i)^{q-1}, \tag{8}$$

where $\alpha > 0$, $\alpha \neq 1$ and $0 < q \leq 1$. We can readily observe that the fractional Tsallis entropy is always non-negative and that for $q = 1$ it reduces to the Tsallis entropy, that is $S^1_\alpha(X) = S_\alpha(X)$. Moreover, as α tends to 1, the fractional Tsallis entropy converges to the fractional entropy and if, in addition, $q = 1$ also to Shannon entropy. In [1] a unified formulation of entropy was proposed also in the context of Dempster-Shafer theory of evidence.

The purpose of this paper is to give the corresponding unified version of extropy, a definition which includes, as special cases, all previously defined extropies. Furthermore, a unified formulation for extropy is proposed also in the context of Dempster-Shafer theory of evidence.

2. Preliminaries on the Dempster-Shafer theory

Dempster [4] and Shafer [15] introduced a theory to study uncertainty. Their theory of evidence is a generalization of the classical probability theory. In Dempster-Shafer theory (DST) of evidence, an uncertain event with a finite number of alternatives is considered, and a mass function over the power set of the alternatives, that is a degree of confidence to all of its subsets, is defined. If we assign positive mass only to singletons, we recover a discrete probability distribution. By DST it is possible to describe situations in which there is less specific information.

Let X be a frame of discernment (FOD), i.e., a set of mutually exclusive and collectively exhaustive events indicated by $X = \{\theta_1, \theta_2, \ldots, \theta_{|X|}\}$. The power set of X is denoted by 2^X and it has cardinality $2^{|X|}$. A function $m : 2^X \to [0, 1]$ is called a mass function or a basic probability assignment (BPA) if

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in 2^X} m(A) = 1.$$

If $m(A) \neq 0$ implies $|A| = 1$ then m is also a probability mass function, i.e., BPAs generalize discrete random variables. Moreover, the elements A such that $m(A) > 0$ are called focal elements.

In DST, there are different indices to evaluate the degree of belief in a subset of the FOD. Among them, we recall here the definitions of belief function, plausibility function and pignistic probability transformation (PPT). The belief function and the plausibility function are defined as

$$Bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \quad Pl(A) = \sum_{B \cap A \neq \emptyset} m(B),$$

respectively. Note that the plausibility of A can be expressed also as 1 minus the sum of the masses of all sets whose intersection with A is empty. Moreover, both the belief and the plausibility vary from 0 to 1 and the belief is always less than or equal to the plausibility. Given a BPA, we can evaluate for each focal element the pignistic probability transformation (PPT) which represents a point estimate of belief and can be determined as

$$PPT(A) = \sum_{B: A \subseteq B} \frac{m(B)}{|B|}, \tag{9}$$

see [17].

Recently, several measures of discrimination and uncertainty have been proposed in the literature and in the context of the Dempster-Shafer evidence theory (see, for instance, Zhou and Deng [21] where a belief entropy based on negation is proposed). For a detailed review we refer to Deng [6]. Among them, one of the most important is known as Deng entropy that was introduced in [5] for a BPA m as

$$ED(m) = -\sum_{A \subseteq X: m(A) > 0} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} \right). \tag{10}$$

This entropy is similar to Shannon entropy and they coincide if the BPA is also a probability mass function. The term $2^{|A|} - 1$ represents the potential number of states in A. For a fixed value of $m(A)$, $2^{|A|} - 1$ increases with the cardinality of A and then also Deng entropy does. Moreover, the fractional version of Deng entropy was proposed and studied by Kazemi et al. [10] whereas the Tsallis-Deng entropy was introduced by Liu et al. [12]. Based on these measures, Balakrishnan et al. [1] defined a unified formulation of entropy also in the context of Dempster-Shafer theory. This formulation of entropy is also called the fractional version of Tsallis-Deng entropy, and it is

$$SD_\alpha^q(m) = \frac{1}{\alpha - 1} \sum_{A \subseteq X: m(A) > 0} m(A) \left[1 - \left(\frac{m(A)}{2^{|A|} - 1} \right)^{\alpha - 1} \right] \left(-\log \frac{m(A)}{2^{|A|} - 1} \right)^{q-1}, \tag{11}$$

where $\alpha > 0$, $\alpha \neq 1$, $0 < q \leq 1$. It is a general expression of entropy as it includes several versions of entropy measure, both in the context of DST and in the classical probability theory.

In analogy with the relation between Shannon entropy and extropy, Buono and Longobardi [3] defined the Deng extropy as a measure of uncertainty dual of Deng entropy. The definition was given in order to satisfy the invariant property about the sum of entropy and extropy. For a BPA m over a FOD X, the Deng extropy is defined by

$$JD(m) = -\sum_{A\subset X: m(A)>0} (1-m(A)) \log\left(\frac{1-m(A)}{2^{|A^c|}-1}\right), \tag{12}$$

where A^c is the complementary set of A in X and $|A^c| = |X| - |A|$. In addition, in the context of fractional calculus, Kazemi et al. [10] defined the fractional version of Deng extropy as

$$JD^q(m) = \sum_{A\subset X: m(A)>0} (1-m(A)) \left[-\log\left(\frac{1-m(A)}{2^{|A^c|}-1}\right)\right]^q, \quad 0 < q \leq 1. \tag{13}$$

To the best of our knowledge, the corresponding measure related to Tsallis entropy has not been studied yet. This measure is called here Tsallis-Deng extropy and defined by

$$JD_\alpha(m) = \frac{1}{\alpha-1} \sum_{A\subset X: m(A)>0} (1-m(A)) \left[1 - \left(\frac{1-m(A)}{2^{|A^c|}-1}\right)^{\alpha-1}\right], \quad \alpha > 0, \alpha \neq 1. \tag{14}$$

3. Fractional Tsallis extropy

In this section, we introduce the unified formulation of extropy in the context of classical probability theory. We refer to this formulation as fractional Tsallis extropy and, for a discrete random variable X, it is

$$JS^q_\alpha(X) = \frac{1}{\alpha-1} \sum_{i=1}^N (1-p_i)[1-(1-p_i)^{\alpha-1}][-\log(1-p_i)]^{q-1}, \tag{15}$$

where $\alpha > 0$, $\alpha \neq 1$ and $0 < q \leq 1$. As mentioned above, the purpose of giving this definition is based on the fact that this expression includes the classical, Tsallis and fractional extropies as special cases.

Remark 1. The fractional Tsallis extropy is non-negative for any discrete random variable. In fact the term $1-(1-p_i)^{\alpha-1}$ is positive for $\alpha > 1$ and negative for $0 < \alpha < 1$, so that the sum in (15) has a definite sign and it is the same of $\alpha - 1$.

Remark 2. This extropy is non-additive. In fact, if we consider X with probability vector $(\frac{1}{3}, \frac{2}{3})$ and Y with probability vector $(\frac{1}{4}, \frac{3}{4})$ it is easy to show that $JS^q_\alpha(X*Y) \neq$

$JS^q_\alpha(X)+JS^q_\alpha(Y)$. For particular choices of α and q, we obtain $JS^{0.5}_2(X*Y)=1.2341$, whereas $JS^{0.5}_2(X)=0.5610$, $JS^{0.5}_2(Y)=0.5088$.

Proposition 1. *If $q=1$ the fractional Tsallis extropy is equal to Tsallis extropy.*

Proof. It follows from (15) for $q=1$. □

Proposition 2. *The fractional Tsallis extropy converges to the fractional extropy as α goes to 1.*

Proof. By taking the limit for α to 1 in (15) and by applying L'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{\alpha\to1} JS^q_\alpha(X) &= \lim_{\alpha\to1}\frac{1}{\alpha-1}\sum_{i=1}^N(1-p_i)[1-(1-p_i)^{\alpha-1}][-\log(1-p_i)]^{q-1}\\ &= \lim_{\alpha\to1}\sum_{i=1}^N(1-p_i)[-(1-p_i)^{\alpha-1}]\log(1-p_i)[-\log(1-p_i)]^{q-1}\\ &= \lim_{\alpha\to1}\sum_{i=1}^N(1-p_i)^\alpha[-\log(1-p_i)]^q=\sum_{i=1}^N(1-p_i)[-\log(1-p_i)]^q\\ &= J_q(X).\end{aligned}$$

□

Corollary 1. *If both parameters of the fractional Tsallis extropy go to 1, then*

$$\lim_{\alpha,q\to1} JS^q_\alpha(X)=J(X).$$

The results given in Propositions 1, 2 and Corollary 1 are summarized in Figure 1 in the form of a schematic diagram by displaying the relationships among different kinds of extropy.

In the following proposition, we show that the fractional Tsallis entropy and the fractional Tsallis extropy satisfy a classical property of entropy and extropy related to their sum.

Proposition 3. *Let X be a discrete random variable with finite support S and with corresponding probability vector $\mathbf{p}$. Then,*

$$S^q_\alpha(X)+JS^q_\alpha(X)=\sum_{i=1}^N S^q_\alpha(p_i,1-p_i)=\sum_{i=1}^N JS^q_\alpha(p_i,1-p_i), \tag{16}$$

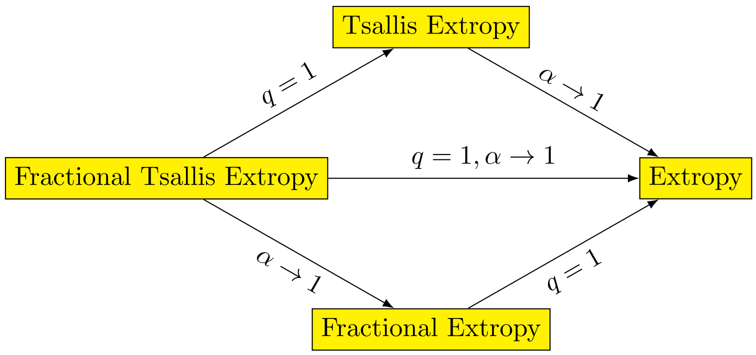

FIGURE 1: Relationships among different versions of extropy in classical probability theory.

where $S^q_\alpha(p_i, 1-p_i)$ *and* $JS^q_\alpha(p_i, 1-p_i)$ *are the fractional Tsallis entropy and extropy of a discrete random variable taking on two values with corresponding probabilities* $(p_i, 1-p_i)$*.*

Proof. The second equality readily follows by observing that, for a random variable with support of cardinality 2, the fractional Tsallis entropy and extropy coincide. In order to prove the first equality, by (8) and (15), note that,

$$S^q_\alpha(X) + JS^q_\alpha(X)$$
$$= \frac{1}{\alpha-1}\sum_{i=1}^{N}\left[p_i(1-p_i^{\alpha-1})(-\log p_i)^{q-1} + (1-p_i)[1-(1-p_i)^{\alpha-1}][-\log(1-p_i)]^{q-1}\right],$$

and then the statement follows. □

Theorem 1. *The supremum of the fractional Tsallis extropy as a function of* $q \in (0,1]$ *is attained in one of the extremes of the interval while the infimum is attained in one of the extremes of the interval or it is a minimum assumed in a unique point* $q_0 \in (0,1)$*.*

Proof. The fractional Tsallis extropy is a convex function of q. Then there are three possible cases to consider. In the first one, it is strictly increasing in q, so that the infimum is attained at 0 and the maximum at $q=1$. In the second one, it is strictly decreasing in q and then the minimum is reached at $q=1$ while the supremum at 0. In the last case, it is decreasing up to $q_0 \in (0,1)$ and then increasing, so that q_0 is the point of minimum and the supremum is reached at one of the extremes of the interval $(0,1)$. □

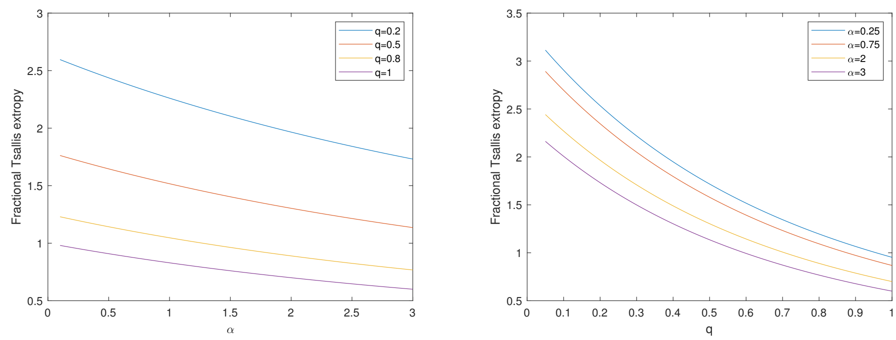

FIGURE 2: The fractional Tsallis extropy in Example 1 as a function of α for different choices of q (left) and as a function of q for different choices of α (right).

Example 1. Consider a discrete random variable X with support of cardinality 4 and probability vector $(0.1, 0.2, 0.3, 0.4)$. It is possible to study the fractional Tsallis extropy as a function of either α or q by fixing the other one as shown in Figure 2. There it is possible to observe a decreasing monotonicity for all the choices of α and q.

Example 2. Consider X_N as a discrete random variable uniformly distributed over a support of cardinality N. Then the components of the probability vector are $p_i = \frac{1}{N}$, $i = 1, \dots, N$, and the fractional Tsallis extropy is obtained as

$$JS_\alpha^q(X_N) = \frac{N-1}{\alpha-1}\left[1-\left(1-\frac{1}{N}\right)^{\alpha-1}\right]\left[-\log\left(1-\frac{1}{N}\right)\right]^{q-1}, \tag{17}$$

for $\alpha > 0$, $\alpha \neq 1$, and $0 < q \leq 1$. In Table 1, the values of $JS_\alpha^q(X_N)$ are given in function of N for different choices of the parameters α and q. Moreover, these values are plotted in Figure 3 showing an increasing behavior in terms of N. By considering the Taylor expansion of Equation (17), we note that as N goes to infinity, $JS_\alpha^q(X_N)$ is asymptotically N^{1-q}. So it diverges and this explains why in Figure 3 the cases related to the same value of q seem to be paired.

Table 1: Values of the fractional Tsallis extropy for the discrete uniform distribution as a function of N, for different choices of q and α.

N	$\alpha = 0.5,\ q = 0.2$	$\alpha = 0.5,\ q = 0.5$	$\alpha = 0.5,\ q = 0.75$	$\alpha = 5,\ q = 0.2$	$\alpha = 5,\ q = 0.5$	$\alpha = 5,\ q = 0.75$
5	3.1349	1.9990	1.3739	1.9601	1.2498	0.8590
10	5.8921	2.9997	1.7090	4.6824	2.3838	1.3581
15	8.3445	3.7415	1.9175	7.1670	3.2135	1.6470
20	10.6254	4.3588	2.0743	9.4836	3.8904	1.8514
25	12.7888	4.8989	2.2020	11.6792	4.4739	2.0110
30	14.8637	5.3851	2.3107	13.7825	4.9934	2.1426
35	16.8683	5.8309	2.4060	15.8120	5.4657	2.2553
40	18.8149	6.2450	2.4911	17.7805	5.9016	2.3541
45	20.7122	6.6332	2.5683	19.6975	6.3082	2.4424
50	22.5670	7.0000	2.6391	21.5699	6.6907	2.5225

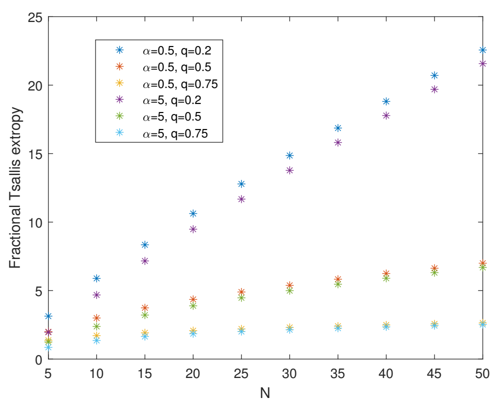

Figure 3: The fractional Tsallis extropy in Example 2 as a function of N with different choices of the parameters α and q.

4. A unified formulation of extropy

Now, we introduce a unified formulation of extropy in the context of Dempster-Shafer theory of evidence as

$$JD_\alpha^q(m) = \frac{1}{\alpha-1} \sum_{A\subset X: m(A)>0} (1-m(A)) \left[1 - \left(\frac{1-m(A)}{2^{|A^c|}-1}\right)^{\alpha-1}\right] \left(-\log \frac{1-m(A)}{2^{|A^c|}-1}\right)^{q-1}, \quad (18)$$

where $\alpha > 0$, $\alpha \neq 1$, $0 < q \leq 1$. As well as the fractional Tsallis extropy, this is a general formulation, since it includes several versions of extropy measure both in the context of DST and in the classical probability theory.

Remark 3. The unified formulation of extropy (18) is non-negative too.

Proposition 4. *If $q=1$, the unified formulation of extropy in* (18) *is equal to Tsallis-Deng extropy in* (14).

Proof. By taking $q=1$ in (18), we have

$$JD_\alpha^1(m)=\frac{1}{\alpha-1}\sum_{A\subset X:m(A)>0}(1-m(A))\left[1-\left(\frac{1-m(A)}{2^{|A^c|}-1}\right)^{\alpha-1}\right]=JD_\alpha(m).$$

□

Proposition 5. *The unified formulation of extropy in* (18) *converges to the fractional Deng extropy in* (13), *as α goes to* 1.

Proof. By taking the limit for α going to 1 in (18), and by using L'Hôpital's rule, it follows

$$\begin{aligned}\lim_{\alpha\to1}JD_\alpha^q(m) &= \lim_{\alpha\to1}\sum_{A\subset X:m(A)>0}(1-m(A))\left[\left(\frac{1-m(A)}{2^{|A^c|}-1}\right)^{\alpha-1}\right]\left(-\log\frac{1-m(A)}{2^{|A^c|}-1}\right)^q\\ &= \sum_{A\subset X:m(A)>0}(1-m(A))\left(-\log\frac{1-m(A)}{2^{|A^c|}-1}\right)^q=JD^q(m),\end{aligned}$$

and the proof is complete. □

Corollary 2. *When both parameters α and q in* (18) *tend to* 1, *the unified formulation of extropy in* (18) *converges to Deng extropy in* (12), *i.e.,*

$$\lim_{\alpha,q\to1}JD_\alpha^q(m)=JD(m).$$

Remark 4. If the BPA m is such that all focal elements have cardinality $n-1$, then in the expression of the unified formulation of extropy $|A^c|=1$ for each addend in the sum. Hence, the unified formulation of extropy reduces to

$$\begin{aligned}JD_\alpha^q(m) &= \frac{1}{\alpha-1}\sum_{A\subset X:m(A)>0}(1-m(A))\left[1-(1-m(A))^{\alpha-1}\right][-\log(1-m(A))]^{q-1}\\ &= JS_\alpha^q(Y),\end{aligned}$$

where Y is a discrete random variable with support $X=\{\theta_1,\dots,\theta_n\}$ such that $p_i=\mathbb{P}(\theta_i)=m(X\setminus\{\theta_i\})$.

To summarize the results given in Propositions 4 and 5, Corollary 2 and Remark 4, the relationships among different formulations of extropies are depicted in Figure 4 in the form of a schematic diagram.

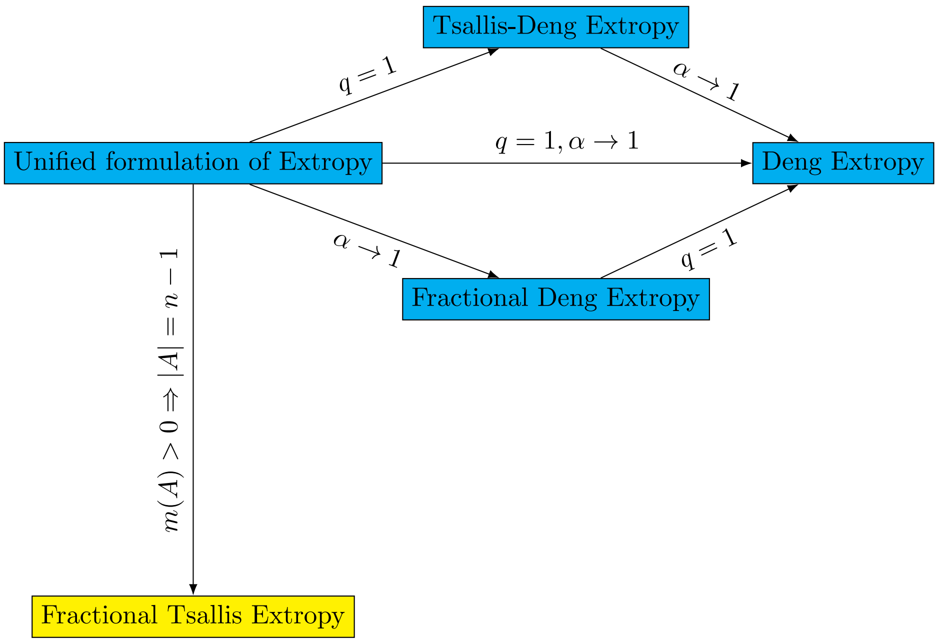

FIGURE 4: Relationships among different entropies in DST theory (blue) and in classical probability theory (yellow).

Theorem 2. *The supremum of the unified formulation of extropy in* (18)*, as a function of* $q \in (0, 1]$*, is attained at one of the extremes of the interval, and the infimum either is attained at one of the extremes of the interval or it is a minimum at a unique* $q_0 \in (0, 1)$*.*

Proof. The proof is similar to that of Theorem 1. □

Example 3. Let X be a frame of discernment with cardinality 4 and consider the BPA m^* such that

$$m^*(A) = \frac{2^{|A|} - 1}{\sum_{B \subseteq X}(2^{|B|} - 1)}, \quad A \subseteq X.$$

It is a well-known BPA which gives the same mass to all the subsets with the same cardinality. More precisely, with $|X| = 4$, we have four subsets with cardinality 1 and mass $1/65$, six subsets with cardinality 2 and mass $3/65$, four subsets with cardinality 3 and mass $7/65$ and one subset with cardinality 4 and mass $15/65$. Remember that the last one, that is the entire frame of discernment X, is not involved in the evaluation of the unified formulation of extropy. In Figure 5, the values of the unified formulation

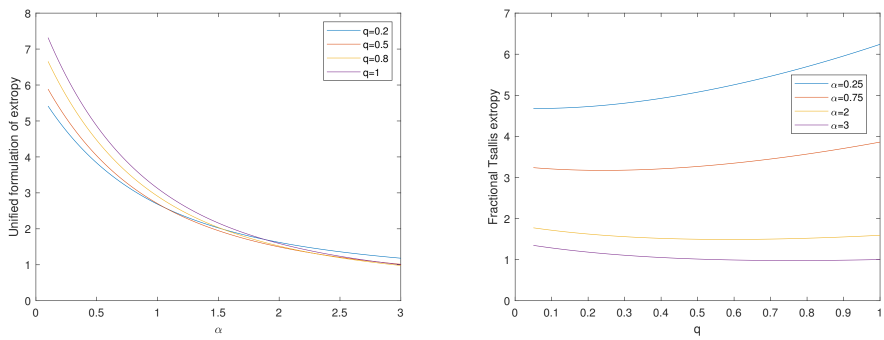

FIGURE 5: The unified formulation of extropy in Example 3 as a function of α for different choices of q (left) and as a function of q for different choices of α (right).

of extropy of m^* are plotted as a function of α or q.

4.1. Application to classification problems

The measures of uncertainty are an efficient tool in the classification problems, see, for instance, [1, 2, 3]. Here, the unified formulation of extropy is applied to the classification problem based on the Iris dataset given in [8]. It is composed of 150 instances equally divided into three classes, Iris Setosa (Se), Iris Versicolor (Ve) and Iris Virginica (Vi), and, for each of them, four attributed are known, i.e., the sepal length in cm (SL), the sepal width in cm (SW), the petal length in cm (PL) and the petal width in cm (PW). By using the method of max–min values, the model of interval numbers is obtained and is presented in Table 10 in [1]. Suppose the selected instance is $(6.3, 2.7, 4.9, 1.8)$. It belongs to the class Iris Virginica and our purpose is to classify it correctly. Four BPAs, one for each attribute, are generated by using the similarity of interval numbers proposed by Kang et al. [9]. Without any additional information, the final BPA is determined by giving the same weight to each attribute. In order to discriminate among classes, we evaluate the PPT (9) of singleton classes for the final BPA obtaining $PPT(Se) = 0.1826$, $PPT(Ve) = 0.4131$, $PPT(Vi) = 0.4043$. Thus, the focal element with the highest PPT is the class Iris Versicolor, which would be our final decision, which is not the correct one. Hence, we try to improve the method by using the unified formulation of extropy in (18), by choosing as parameters $q = 0.5$ and $\alpha = 5$. The unified formulation of extropy of BPAs obtained by using the similarity of interval numbers is evaluated and the corresponding results are given in Table 2.

TABLE 2: The unified formulation of extropy of BPAs based on similarity of interval numbers.

Attribute	**Alcohol**	**Malic Acid**	**Ash**	**OD**
$JD_5^{0.5}$	1.5733	1.5855	0.6664	0.7108

A greater value of the unified formulation of extropy represents a higher uncertainty, then it is reasonable to give more weight to the attributes with lower value of that measure. Here, we define the weights by normalizing to 1 the reciprocal values of the unified formulation of extropy, and the results are given in Table 3.

TABLE 3: The weights of attributes based on the unified formulation of extropy.

Attribute	**SL**	**SW**	**PL**	**PW**
Weight	0.1523	0.1511	0.3595	0.3371

Based on the weights in Table 3, a weighted version of the final BPA is obtained. Then, the PPT of the singleton classes are computed as $PPT(Se) = 0.1156$, $PPT(Ve) = 0.4360$, $PPT(Vi) = 0.4485$. Hence, the focal element with the highest PPT is the type Iris Virginica, and would therefore be our final decision, which is the correct one. By using the method based on the unified formulation of extropy, a gain in terms of recognition rate is obtained going from 94% of the non-weighted method to 94.67% of the method explained here.

5. Conclusions

The recent study of the extropy, as a dual measure of uncertainty, has inspired us to introduce a unified formulation for it. We have obtained well-known concepts of extropy for particular choices of the two parameters involved. This unified formulation has also been analyzed in the context of Dempster-Shafer theory of evidence. Further, an application to classification problems of the proposed measure is described.

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