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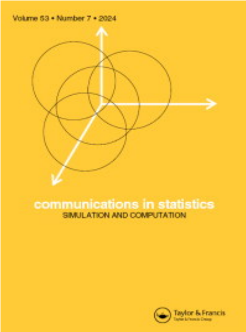

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Measures of extended fractional Deng entropy and extropy with applications

Nastaran Marzban Vaselabadi[a], Saeid Tahmasebi[a], Ahmad Keshavarz[b], and Francesco Buono[c]

[a]Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran; [b]Department of Electrical Engineering and the ICT Research Institute, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran; [c]Institute of Statistics, RWTH Aachen University, Aachen, Germany

ABSTRACT
Recently, Zhang and Shang introduced modifications to the concept of fractional entropy and proved some properties based on the inverse Mittag-Leffler function (MLF). The Deng entropy serves as a valuable measure in the Dempster-Shafer evidence theory (DST) to tackle uncertainty. In this study, we extend the fractional Deng entropy measure, introducing two distinct versions: $E^{\alpha}_{fd1}(m)$ and $E^{\alpha}_{fd2}(m)$. We call this new measure the extended fractional Deng entropy, EFDEn. Additionally, we apply a similar approach to the fractional Deng extropy measure, resulting in $EX^{\alpha}_{fd1}(m)$ and $EX^{\alpha}_{fd2}(m)$. We call this new measure the extended fractional Deng extropy, EFDEx. These two measures are complementary, leading to provide a deeper analysis of known and unknown information. Subsequently, we conduct a comparative analysis of these measures within the DST framework. We also propose the decomposable fractional Deng entropy, an extension of the decomposable entropy for Dempster–Shafer evidence theory, which effectively decomposes fractional Deng entropy. Finally, we delve into a pattern recognition classification problem to highlight the importance of these new measures.

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1. Introduction

The concept of entropy as a measure of uncertainty was first introduced by Shannon (Shannon 2001) and since then it has been used in several fields as information theory, signal and image processing, and economics. Let X be a discrete random variable with probability mass function vector $\underline{p} = (p_1, \ldots, p_m)$. The Shannon entropy of X is defined as

$$H(X) = H(\underline{p}) = -\sum_{i=1}^{m} p_i \log_2 p_i, \tag{1}$$

with the convention $0 \log 0 = 0$. The concept of Shannon entropy has been generalized to various fields with different applications. Zhang and Shang (2020) defined a new fractional entropy as follows

$$H_\alpha(X) = H_\alpha(\underline{p}) = \sum_{i=1}^{n} p_i [-Ln_\alpha p_i]^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1. \tag{2}$$

CONTACT Francesco Buono buono@isw.rwth-aachen.de Institute of Statistics, RWTH Aachen University, Aachen, Germany.

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where Ln_α is the inverse Mittag-Leffler function and is such that $Ln_\alpha 1 = 0$, $Ln_\alpha 0 = -\infty$, $0(Ln_\alpha 0)^{\frac{1}{\alpha}} = 1(Ln_\alpha 1)^{\frac{1}{\alpha}} = 0$, and $Ln_\alpha x < 0$ when $x < 1$. It is clear that $H_\alpha(X)$ is always non-negative. The parameter α of the fractional entropy is related to fractals, which are mathematical objects used to describe complex and irregular shapes or patterns. The fractal dimension is another measure of complexity. This fractional entropy concept has applications in physics, particularly in relation to Lesche and thermodynamic stability. In the context of fractional equations, Jumarie (2012) derived one of the most important expression in the theory of the inverse MLF which is given as

$$[Ln_\alpha pq]^{\frac{1}{\alpha}} = [Ln_\alpha p]^{\frac{1}{\alpha}} + [Ln_\alpha q]^{\frac{1}{\alpha}}. \tag{3}$$

In this framework, the inverse MLF is more suitable than the logarithmic function to calculate entropy and there is a well-known approximation, which will be useful in the following, given as $Ln_\alpha p \approx \log_2 p^{\alpha!}$.

The purpose of this paper is to extend Deng entropy by using the fractional entropy. Deng entropy (Deng 2016; Buono and Longobardi 2020) is a measure of uncertainty which is known in the context of Dempster-Shafer theory (DST) of evidence (Dempster 1967; Shafer 1976). DST is a generalization of the classical probability theory and it deals with uncertain events that have a finite number of alternatives. Moreover, in DST, a mass function is used to represent the degree of confidence or belief in different outcomes. DST allows us to handle situations where less specific information is available compared to classical probability theory. DST has several applications due to its advantages in dealing with uncertainty; for example, it is used in reliability analysis (Liu et al. 2017; Han and Deng 2018), in decision making (Yang and Xu 2013; Fu, Yang, and Yang 2015), and in several other fields (Liu et al. 2014; Kabir et al. 2015). Additionally, a novel failure mode and effects analysis, FMEA, a model based on the improved pignistic probability transformation (PPT) function in DST and grey relational projection method were proposed by Tang et al. (2024) to improve the accuracy and reliability in risk analysis with FMEA. Besides, although Dempster-Shafer theory is great for handling uncertain information, its fusion rule can lead to give odd results when encountering conflicting evidence. To this end (Tang et al. 2023), used a method which is inspired by complex networks. Actually, they treat each piece of evidence like a node, and measure the correlation to know how they are related. Then, the weights for each node based on its importance in the network are calculated and by adjusting the original evidence with these weights, they use Dempster's rule to fuze the information and get a better result.

The rest of the paper is organized as follows. In Sec. 2, we recall the basic notions of DST of evidence and some of the most important measures of uncertainty in this context. In Sec. 3, we define and study the extended fractional Deng entropy. The introduction of extended fractional Deng extropy, and several examples are given in Sec. 4. Section 5 presents the decomposable fractional Deng Entropy and it includes some related examples as well. We evaluate the efficiency of fractional Deng entropy on a problem of classification in Sec. 6.

2. Methodological background

Dempster-Shafer theory (Dempster 1967; Shafer 1976), alternatively known as the theory of belief functions or evidence theory, is widely applied in tackling uncertain scenarios. In this section, we introduce the fundamental concepts, principles, and background information.

2.1. Preliminaries in DST theory

In this subsection, we introduce key concepts in DST, such as the mass function, belief function, and PPT. These concepts play crucial roles in understanding the concept of DST theory.

Definition 1. Let Y be a set of mutually exclusive and collectively exhaustive events denoted as

$$Y = \{\theta_1, \theta_2, ..., \theta_i, ..., \theta_{|Y|}\},$$

where the set Y is named the frame of discernment (FOD). The power set of Y, denoted by 2^Y, consists of $2^{|Y|}$ elements given as

$$2^Y = \{\emptyset, \{\theta_1\}, ..., \{\theta_{|Y|}\}, \{\theta_1, \theta_2\}, ..., \{\theta_1, \theta_2, ..., \theta_i\}, ..., Y\}.$$

Definition 2. (Mass function) For a FOD $Y = \{\theta_1, \theta_2, ..., \theta_i, ..., \theta_{|Y|}\}$, the mass function is a mapping m from 2^Y to *[*0, 1*]*, defined as

$$m: \quad 2^Y \to [0, 1]$$

and satisfying the following conditions

$$m(\emptyset) = 0, \quad \sum_{A \in 2^Y} m(A) = 1. \tag{4}$$

The value $m(A)$ represents how strongly the evidence supports A and it measures the belief exactly assigned to A. If $m(A) > 0$, then A is called a focal element.

Within the framework of DST, the mass function is known as the basic probability assignment (BPA), which is the primary and initial step in DST and must be determined. Lately, new operations involving BPA have been introduced such as negation (Yin, Deng, and Deng 2019) and correlation (Jiang 2018). In several applications, a need arises to construct a new BPA by leveraging either independent BPAs or weight of evidence denoted by a coefficient $\beta \in (0, 1]$. To accomplish this, we can generate another BPA, m^β, following the method outlined in Shafer (1976)

$$m^\beta(A) = \begin{cases} \beta m(A), & \text{if } A \subset Y, \\ \beta m(Y) + (1-\beta), & \text{if } A = Y. \end{cases}$$

If we have two independent mass functions, denoted by m_1 and m_2, they can be combined with Dempster's rule of combination which is defined as (Dubois and Prade 1985; Yager 2008)

$$m(A) = (m_1 \oplus m_2)(A) = \frac{1}{1-K} \sum_{B \cap C = A} m_1(B) m_2(C),$$

where $K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C)$ is a normalization constant representing the degree of conflict between m_1 and m_2.

Within the realm of DST, there are different indices to evaluate the degree of belief in a subset of the FOD. Among them, we recall the explanations of belief function, plausibility function and PPT.

Definition 3. The belief function and plausibility function associated to a BPA and evaluated for A, subset of the FOD, are defined as

$$Bel(A) = \sum_{B|B \subseteq A} m(B), \quad Pl(A) = \sum_{B|B \cap A \neq \emptyset} m(B),$$

respectively.

Definition 4. Given a BPA m on a FOD Y, PPT of $A \subseteq Y$ is defined as

$$PPT(A) = \sum_{B: A \subseteq B} \frac{m(B)}{|B|}, \tag{5}$$

see (Smets 2000).

The PPT represents a point estimate of belief and it is particularly useful when comparing the elements of the FOD Y. Moreover, it is useful to note that, from the properties of the mass functions, we have

$$\sum_{i=1}^{|Y|} PPT(\{\theta_i\}) = 1.$$

2.2. Some uncertainty measures for DST theory

How to extend some measures of uncertainty for the classical probability theory to efficiently measure the uncertainty of a BPA is still an open issue. In the context of the DST, there are interesting measures of discrimination, as the Deng entropy, whose definition is recalled below, which has some advantages in some cases in comparison with other measures of uncertainty. This concept suggests us the introduction of a new extension. In Table 1, we present the definitions of some of the most important measures of uncertainty in DST.

Inspired by the fractional entropy in (2) and the property (3), we can define the fractional measures of uncertainty as the following. We believe that these measures can be good guide for future works about uncertainty measures.

Definition 5. Let m be a BPA on a FOD Y. We propose the first uncertainty measure of m for $0 < \alpha < 1$ as follows

$$E^{\alpha}_{Fract1}(m) = \sum_{A\subseteq Y:m(A)>0} m(A)[-Ln_\alpha(Bel(A)Pl(A))]^{\frac{1}{\alpha}} \tag{6}$$

$$= \sum_{A\subseteq Y:m(A)>0} m(A)[-Ln_\alpha Bel(A)]^{\frac{1}{\alpha}} + \sum_{A\subseteq Y:m(A)>0} m(A)[-Ln_\alpha Pl(A)]^{\frac{1}{\alpha}}$$
$$\approx [\alpha!]^{\frac{1}{\alpha}}\left[\sum_{A\subseteq Y:m(A)>0} m(A)\left[-\log_2 Bel(A)\right]^{\frac{1}{\alpha}}\right]$$
$$+[\alpha!]^{\frac{1}{\alpha}}\left[\sum_{A\subseteq Y:m(A)>0} m(A)\left[-\log_2 Pl(A)\right]^{\frac{1}{\alpha}}\right].$$

Definition 6. Let m be a BPA on a FOD Y. We propose the second uncertainty measure of m for $0 < \alpha < 1$ as follows

Table 1. Uncertainty measures in DST framework.

Uncertainty Measure	Definition
Hohle's confusion measure (Hohle 1982)	$C_H(m) = -\sum_{A\subseteq Y} m(A)\log_2 Bel(A)$
Yager's Dissonance Measure (Yager 2008)	$E_{Ya}(m) = -\sum_{A\subseteq Y} m(A)\log_2 Pl(A)$
Dubois and Prade's Weighted Hartley Entropy (Dubois and Prade 1985)	$E_{DP}(m) = -\sum_{A\subseteq Y} m(A)\log_2 \lvert A\rvert$
Klir and Ramer's discord measure (Klir and Ramer 1990)	$D_{KR}(m) = -\sum_{A\subseteq Y} m(A)\log_2 \sum_{B\subseteq Y} m(B)\frac{\lvert A\cap B\rvert}{\lvert B\rvert}$
Klir and Parviz's strife measure (Klir and Parviz 1992)	$S_{KP}(m) = -\sum_{A\subseteq Y} m(A)\log_2 \sum_{B\subseteq Y} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\rvert}$
George and Pal's total conflict measure (George and Pal 1996)	$TC_{GP}(m) = \sum_{A\subseteq Y} m(A)\sum_{B\subseteq Y} m(B)\left(1-\frac{\lvert A\cap B\rvert}{\lvert A\cup B\rvert}\right)$

$$E^{\alpha}_{Fract2}(m) = \sum_{A\subseteq Y:m(A)>0} m(A)[-Ln_{\alpha}(m(A)|A|)]^{\frac{1}{\alpha}} \tag{7}$$

$$\begin{aligned}
&= \sum_{A\subseteq Y:m(A)>0} m(A)[-Ln_{\alpha}m(A)]^{\frac{1}{\alpha}} + \sum_{A\subseteq Y:m(A)>0} m(A)[-Ln_{\alpha}|A|]^{\frac{1}{\alpha}} \\
&\approx [\alpha!]^{\frac{1}{\alpha}}\left[\sum_{A\subseteq Y:m(A)>0} m(A)\left[-\log_2 m(A)\right]^{\frac{1}{\alpha}}\right] \\
&+[\alpha!]^{\frac{1}{\alpha}}\left[\sum_{A\subseteq Y:m(A)>0} m(A)\left[-\log_2|A|\right]^{\frac{1}{\alpha}}\right].
\end{aligned}$$

Nonetheless, these defined measures are not satisfactory for entropy measures of mass functions. Additionally, they suffer from complex calculations. Consequently, Deng (2016) introduced a novel uncertainty measure for mass functions that has yielded to relatively promising experimental outcomes.

Definition 7. (Deng entropy) The Deng entropy was introduced in Deng (2016) for a BPA m as

$$E_d(m) = -\sum_{A\subseteq Y:m(A)>0} m(A)\log_2\left(\frac{m(A)}{2^{|A|}-1}\right), \tag{8}$$

where $|A|$ denotes the cardinality of the focal element A. The mass of each focal element in the Deng entropy is divided by $2^{|A|}-1$ which represents the potential number of states in A.

Deng entropy degenerates to the Shannon entropy if, and only if, a positive mass function value is assigned only to singleton elements, which is $E_d(m) = -\sum_{i=1}^{|Y|} m(\{\theta_i\})\log_2 m(\{\theta_i\})$. Deng entropy has attracted the interest of researchers and several of its generalizations were studied. In Table 2 we present some modified versions of Deng entropy.

3. Extended fractional Deng entropies

In this section, we propose the concepts of extended fractional Deng entropy in the following definitions.

Definition 8. Let m be a BPA on a FOD Y. In order to obtain an analogue of (18), we introduce an extended fractional Deng entropy (EFDEn) of type 1 of m as

$$E^{\alpha}_{fd1}(m) = \sum_{A\subseteq Y:m(A)>0} m(A)\left[-Ln_{\alpha}\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^{\frac{1}{\alpha}} \tag{9}$$

Table 2. Modified deng entropy in DST framework.

Uncertainty Measure	Definition
Zhou et al.'s Entropy (Zhou, Tang, and Jiang 2017)	$E_{Md}(m) = -\sum_{A\subseteq Y} m(A)\log_2\left(\frac{m(A)}{2^{|A|}-1}e^{\frac{|A|-1}{|Y|}}\right)$
Pan et al.'s Entropy (Pan and Deng 2018)	$P_{Bel}(m) = -\sum_{A\subseteq Y}\frac{Bel(A)+Pl(A)}{2}\log_2\left(\frac{Pl(A)+Bel(A)}{2(2^{|A|}-1)}\right)$
Cui et al.'s Entropy (Cui et al. 2019)	$E(m) = -\sum_{A\subseteq Y} m(A)\log_2\left(\frac{m(A)}{2^{|A|}-1}e^{\sum_{B\subseteq X, B\neq A}\frac{\lvert A\cap B\rvert}{2^{\lvert Y\rvert}-1}}\right)$
Kazemi et al.'s Entropy (Kazemi et al. 2021)	$E^q_{Md}(m) = \sum_{A\subseteq Y} m(A)\left[-\log_2\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^q$

$$\approx [\alpha!]^{\frac{1}{\alpha}} \sum_{A\subseteq Y:m(A)>0} m(A)\left[-\log_2\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^{\frac{1}{\alpha}}, \quad 0<\alpha<1.$$

Definition 9. Let m be a BPA on a FOD Y. We define the EFDEn of type 2 of m as

$$E^{\alpha}_{fd2}(m) = -\sum_{A\subseteq Y:m(A)>0} m(A)Ln_{\alpha}\left(\frac{m(A)}{2^{|A|}-1}\right) \approx \alpha! E_d(m), \quad 0<\alpha<1, \tag{10}$$

where $E_d(m)$ is the Deng entropy.

Remark 1. It is clear that $E^{\alpha}_{fd2}(m) < E_d(m)$. Hence, the information content is lower than $E_d(m)$.

Remark 2. Given a FOD $Y = \{a_1, a_2, \ldots, a_n\}$. For a mass function $m(a_1) = m(a_2) = \ldots = m(a_n) = \frac{1}{n}$, we have $E^{\alpha}_{fd1}(m) \approx [\alpha! \log_2 n]^{\frac{1}{\alpha}} \approx [E^{\alpha}_{fd2}(m)]^{\frac{1}{\alpha}}$.

Theorem 3.1. *If m is a BPA on a FOD Y, it holds that*

$$E^{\alpha}_{fd1}(m) \geq \left[E^{\alpha}_{fd2}(m)\right]^{\frac{1}{\alpha}}.$$

Proof. From (9) we have

$$\begin{aligned} E^{\alpha}_{fd1}(m) &= \sum_{A\subseteq Y:m(A)>0} m(A)\left[-Ln_{\alpha}\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^{\frac{1}{\alpha}} \\ &\geq \sum_{A\subseteq Y:m(A)>0} \left[-m(A)Ln_{\alpha}\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^{\frac{1}{\alpha}}. \end{aligned} \tag{11}$$

Since $h(x) = x^{\frac{1}{\alpha}}, \ 0<\alpha<1$ is a convex function, the Jensen inequality gives

$$\sum_{A\subseteq Y:m(A)>0} \left[-m(A)Ln_{\alpha}\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^{\frac{1}{\alpha}} \geq \left[\sum_{A\subseteq Y:m(A)>0} -m(A)Ln_{\alpha}\left(\frac{m(A)}{2^{|A|}-1}\right)\right]^{\frac{1}{\alpha}},$$

and the result follows. ■

4. Extended fractional Deng extropy

Buono and Longobardi (2020) proposed the Deng extropy as a measure of uncertainty dual to the Deng entropy.

Definition 10. (Deng Extropy) The Deng extropy was proposed in Buono and Longobardi (2020) for a BPA m on a FOD Y as

$$EX_d(m) = -\sum_{A\subset Y:m(A)>0} (1-m(A))\log_2\left(\frac{1-m(A)}{2^{|A^c|}-1}\right),$$

where A^c is the complementary of A in Y and $|A^c| = |Y| - |A|$.

Now, in analogy with EFDEn, we introduce the fractional versions of the Deng extropy.

Definition 11. Let m be a BPA on a FOD Y. We define the extended fractional Deng extropy (EFDEx) of type 1 of m as

$$EX_{fd1}^{\alpha}(m) = \sum_{A\subset Y:m(A)>0} (1-m(A))\left[-Ln_{\alpha}\left(\frac{1-m(A)}{2^{|A^c|}-1}\right)\right]^{\frac{1}{\alpha}} \tag{12}$$

$$\approx \sum_{A\subset Y:m(A)>0} (1-m(A))\left[-\alpha! \log_2\left(\frac{1-m(A)}{2^{|A^c|}-1}\right)\right]^{\frac{1}{\alpha}}.$$

Definition 12. Let m be a BPA on a FOD Y. We introduce the EFDEx of type 2 of m as

$$EX_{fd2}^{\alpha}(m) = -\sum_{A\subseteq Y:m(A)>0} (1-m(A))Ln_{\alpha}\left(\frac{1-m(A)}{2^{|A^c|}-1}\right) \approx \alpha! EX_d(m), \quad 0<\alpha<1, \tag{13}$$

where $EX_d(m)$ is the Deng extropy.

Remark 3. Note that $EX_{fd2}^{\alpha}(m) < EX_d(m)$. So, the information content is lower than $EX_d(m)$.

Remark 4. Let m be a BPA on a FOD Y, then it holds that

$$EX_{fd1}^{\alpha}(m) \geq \left[EX_{fd2}^{\alpha}(m)\right]^{\frac{1}{\alpha}}.$$

Next, we give some examples of evaluation of the extended fractional Deng entropy and extended fractional Deng extropy in different situations.

Example 1.

i. Assume that the FOD is $Y=\{a,b,c\}$. For a mass function $m(a)=m(b)=m(c)=\frac{1}{3}$, the associated EFDEn and EFDEx are obtained in Table 3. In Figure 1, the plot of $EX_{fd1}^{\alpha}(m) - E_{fd1}^{\alpha}(m)$ is given.
ii. Assume there exist $a\in Y$ such that $m(a)=1$ then EFDEn and EFDEx of type one and two are equal and are given as

$$E_{fd1}^{\alpha}(m) = E_{fd1}^{\alpha}(m) = EX_{fd1}^{\alpha}(m) = EX_{fd2}^{\alpha}(m) = 0$$

Example 2. Let us consider a FOD $Y=\{a,b,c\}$. For a mass function $m(a)=m(b)=m(c)=m(a,b)=m(a,c)=m(b,c)=m(a,b,c)=\frac{1}{7}$, we obtain, the associated EFDEn and EFDEx in Table 4.

In Figure 2, we depict all four measures to see how they change in terms of α. $E_{fd1}^{\alpha}(m)$ is decreasing in α while $E_{fd2}^{\alpha}(m)$ is parabola plane curves and has minimum at $(0.462, 3.443)$.

Table 3. The value of EFDEn and EFDEx when α changes in Example 1.

Measure	Expression (approx)	$\alpha=0.3$	$\alpha=0.6$	$\alpha=0.9$
$E_{fd1}^{\alpha}(m)$	$[\alpha!\log_2 3]^{\frac{1}{\alpha}}$	3.237	1.786	1.597
$E_{fd2}^{\alpha}(m)$	$\alpha!\log_2 3$	1.422	1.416	1.524
$EX_{fd1}^{\alpha}(m)$	$2\left[\alpha!\log_2\frac{9}{2}\right]^{\frac{1}{\alpha}}$	18.446	6.029	4.529
$EX_{fd2}^{\alpha}(m)$	$2\alpha!\log_2\frac{9}{2}$	3.895	3.878	4.174

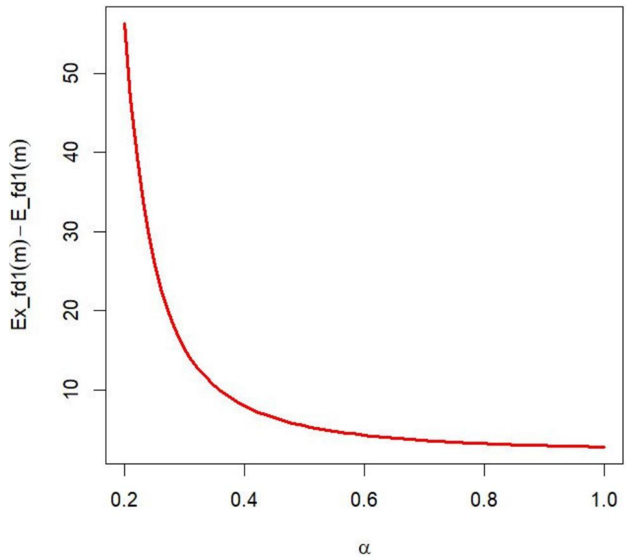


Figure 1. Plot of $EX^{\alpha}_{fd1}(m) - E^{\alpha}_{fd1}(m)$ in Example 1 as a function of α.

Table 4. The value of EFDEn and EFDEx when α changes in Example 2.

Measure	Expression (approx)	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
$E^{\alpha}_{fd1}(m_1)$	$(\alpha!)^{\frac{1}{\alpha}}\left[\frac{3}{7}\left([\log_2 7]^{\frac{1}{\alpha}} + [\log_2 21]^{\frac{1}{\alpha}}\right) + \frac{1}{7}[\log_2 49]^{\frac{1}{\alpha}}\right]$	83.134	8.270	4.347
$E^{\alpha}_{fd2}(m_1)$	$\alpha!\left[\frac{3}{7}(\log_2 7 + \log_2 21) + \frac{1}{7}\log_2 49\right]$	3.490	3.474	3.739
$EX^{\alpha}_{fd1}(m)$	$\frac{18}{7}\left([\alpha!(\log_2 7 - 1)]^{\frac{1}{\alpha}} + [\alpha!(\log_2 7 - \log_2 6)]^{\frac{1}{\alpha}}\right)$	12.906	5.890	5.216
$EX^{\alpha}_{fd2}(m)$	$\alpha!\frac{18}{7}[2\log_2 7 - 1 - \log_2 6]$	4.684	4.663	5.02

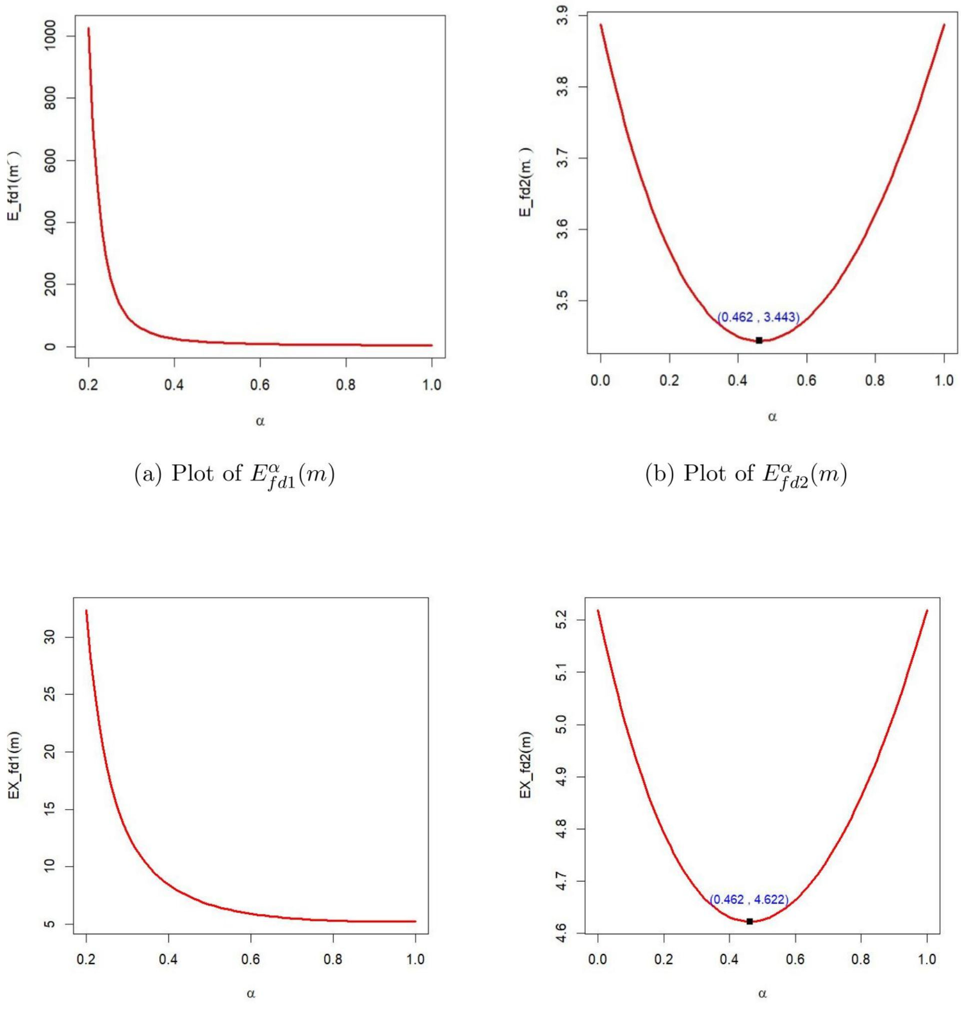
Table 5. The value of EFDEn and EFDEx when α changes in Example 3.

Measure	Expression (approx)	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
$E^{\alpha}_{fd1}(m)$	$0.4\left[-\alpha!\log_2\left(\frac{0.4}{2^{10}-1}\right)\right]^{\frac{1}{\alpha}} + 0.6\left[-\alpha!\log_2\left(\frac{0.6}{2^{10}-1}\right)\right]^{\frac{1}{\alpha}}$	2050.465	44.907	13.708
$E^{\alpha}_{fd2}(m)$	$(-\alpha!)\left[0.4\log_2\left(\frac{0.4}{2^{10}-1}\right) + 0.6\log_2\left(\frac{0.6}{2^{10}-1}\right)\right]$	9.845	9.801	10.550
$EX^{\alpha}_{fd1}(m)$	$0.6\left[-\alpha!\log_2\left(\frac{0.6}{2^{10}-1}\right)\right]^{\frac{1}{\alpha}} + 0.4\left[-\alpha!\log_2\left(\frac{0.4}{2^{10}-1}\right)\right]^{\frac{1}{\alpha}}$	2050.465	44.907	13.708
$EX^{\alpha}_{fd2}(m)$	$\alpha!\left[-0.6\log_2\left(\frac{0.6}{2^{10}-1}\right) - 0.4\log_2\left(\frac{0.4}{2^{10}-1}\right)\right]$	9.845	9.801	10.550

Besides, $EX^{\alpha}_{fd1}(m)$ is decreasing in α and $EX^{\alpha}_{fd2}(m)$ is convex function and has minimum at $(0.462, 4.622)$.

Example 3. Assume that the FOD is $Y = \{a_1, a_2, \ldots, a_{20}\}$. For a mass function $m(\{a_1, a_2, \ldots, a_{10}\}) = 0.4$, $m(\{a_{11}, a_{12}, \ldots, a_{20}\}) = 0.6$, we obtain the results presented in Table 5 and Figure 3.

Example 4. Given a FOD $Y = \{a, b, c\}$ and a BPA m such that $m(a) = 0.9$, $m(a, b) = 0.01$ and $m(Y) = 0.09$, we have the results in Table 6. Besides, Figure 4 shows the plot of all four measures.



(a) Plot of $E^{\alpha}_{fd1}(m)$

(b) Plot of $E^{\alpha}_{fd2}(m)$

(c) Plot of $EX^{\alpha}_{fd1}(m)$

(d) Plot of $EX^{\alpha}_{fd2}(m)$

Figure 2. Plot of EFDEn and EFDEx in Example 2.

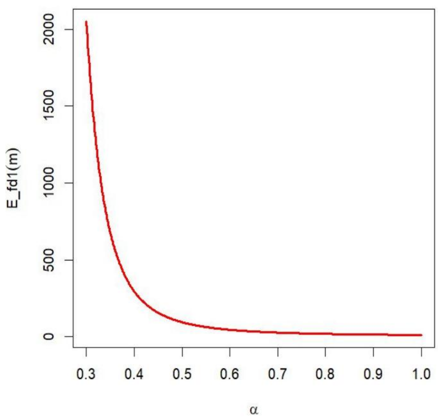
5. Decomposable fractional Deng entropy in DST

In this section, we extend the fractional versions of decomposable Deng entropy based on the decomposable entropy. In the following, we define a concept of commonality function in DST.

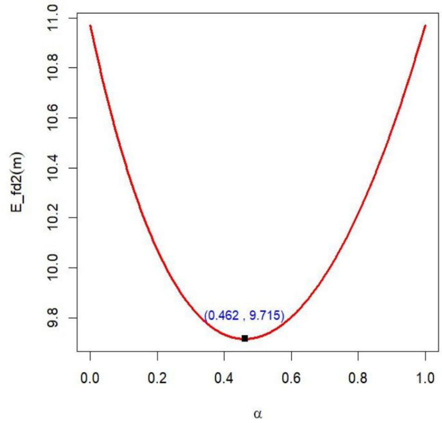
Definition 13. (Commonality Function) The information in a BPA m for Y can also be represented by a corresponding commonality function (CF) Q for Y that is defined as follows

$$Q(E) = \sum_{C \in 2^Y,\ C \supseteq E} m(C), \ \ \forall E \in 2^Y. \tag{14}$$

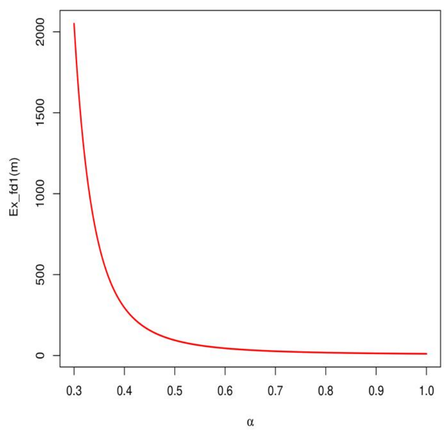
Note that the commonality function can be obtained through mass function, indicating that there is a strong connection between them.



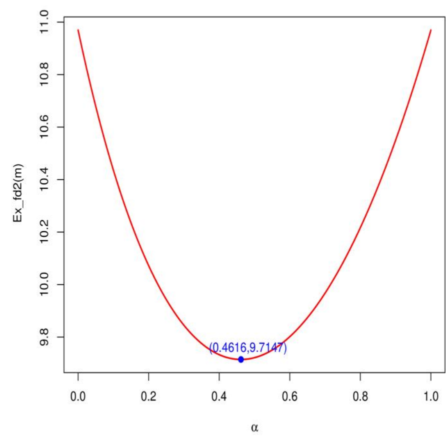
(a) Plot of $E^{\alpha}_{fd1}(m)$



(b) Plot of $E^{\alpha}_{fd2}(m)$



(c) Plot of $EX^{\alpha}_{fd1}(m)$



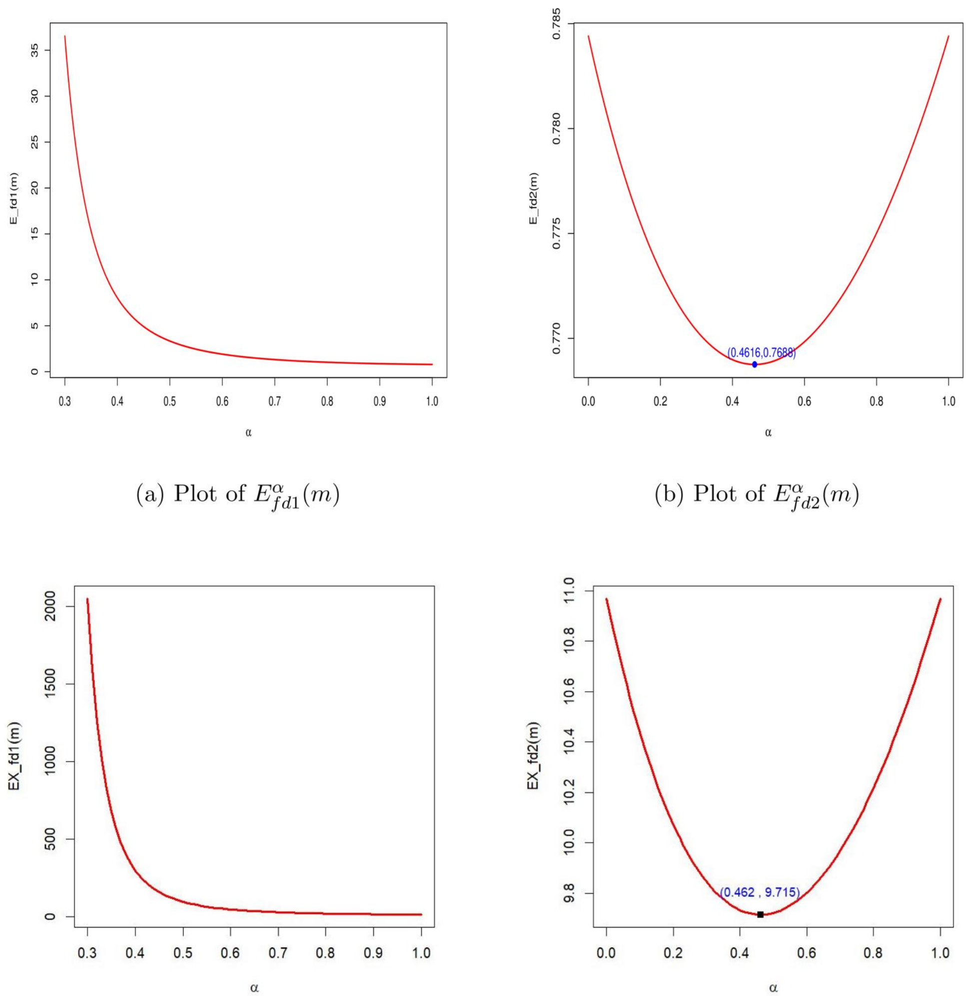
(d) Plot of $EX^{\alpha}_{fd2}(m)$

Figure 3. Plot of EFDEn and EFDEx in Example 3.

Table 6. The value of EFDEn and EFDEx when α changes in Example 4.

Measure	Expression (approx)	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
$E^{\alpha}_{fd1}(m)$	$(\alpha!)^{\frac{1}{\alpha}}\left(\frac{9}{10}\left(\log_2\left(\frac{10}{9}\right)\right)^{\frac{1}{\alpha}}+\frac{1}{100}(\log_2(300))^{\frac{1}{\alpha}}+\frac{9}{100}\left(\log_2\left(\frac{700}{9}\right)\right)^{\frac{1}{\alpha}}\right)$	36.540	1.905	0.870
$E^{\alpha}_{fd2}(m)$	$(\alpha!)\left(\frac{9}{10}\log_2\left(\frac{10}{9}\right)+100\log_2(300)+\frac{100}{9}\log_2\left(\frac{700}{9}\right)\right)$	801.271	797.740	858.675
$EX^{\alpha}_{fd1}(m)$	$0.1[\alpha!\log_2 30]^{\frac{1}{\alpha}}+0.99\left[\alpha!\log_2\frac{100}{99}\right]^{\frac{1}{\alpha}}$	13.999	1.175	0.569
$EX^{\alpha}_{fd2}(m)$	$\alpha!\left[0.1\log_2 30+0.99\log_2\frac{100}{99}\right]$	0.453	0.451	0.486

Definition 14. (Entropy of a commonality function Q_Y) Shannon entropy is defined as the expected value of the information gained when learning about a single symbol of b such that $b \in 2^Y$ (Jiroušek and Shenoy 2018),



(a) Plot of $E^{\alpha}_{fd1}(m)$ (b) Plot of $E^{\alpha}_{fd2}(m)$

(c) Plot of $EX^{\alpha}_{fd1}(m)$ (d) Plot of $EX^{\alpha}_{fd2}(m)$

Figure 4. Plot of EFDEn and EFDEx in Example 4.

$$H(Q_Y) = \sum_{b \in 2^Y} (-1)^{|b|} Q_Y(b) \log_2(Q_Y(b)). \tag{15}$$

Definition 15. (Deng entropy of a commonality function Q_Y) The Deng entropy of Q_Y can be defined as follows (Xue and Deng 2022),

$$DE(Q_Y) = \sum_{b \in 2^Y} (-1)^{|b|} Q_Y(b) \log_2\left(\frac{Q_Y(b)}{2^{|b|} - 1}\right). \tag{16}$$

Definition 16. (fractional Entropy of a commonality function Q_Y) The fractional Entropy of Q_Y is given by

$$H_q(Q_Y) = \sum_{b \in 2^Y} Q_Y(b) \left[-\log_2(Q_Y(b))\right]^q, \quad 0 < q \le 1. \tag{17}$$

Note that the fractional Deng entropy of Q_Y can be defined as follows

$$DE_q(Q_Y) = \sum_{b \in 2^Y} Q_Y(b) \left[-\log_2 \left(\frac{Q_Y(b)}{2^{|b|} - 1} \right) \right]^q, \quad 0 < q \leq 1. \tag{18}$$

Example 5. Let us consider a FOD $Y = \{y_1, y_2, y_3\}$ and a BPA m such that

$$m_Y(\{y_1, y_2\}) = m_Y(\{y_1, y_3\}) = m_Y(\{y_2, y_3\}) = \frac{1}{3}.$$

Then, the Q_Y can be obtained as follows

$$\begin{aligned}
&Q_Y(\{y_1\}) = m_Y(\{y_1, y_2\}) + m_Y(\{y_1, y_3\}) + m_Y(Y) = \frac{2}{3},\\
&Q_Y(\{y_2\}) = m_Y(\{y_1, y_2\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{2}{3},\\
&Q_Y(\{y_3\}) = m_Y(\{y_1, y_3\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{2}{3},\\
&Q_Y(Y) = m_Y(Y) = 0,\\
&Q_Y(\{y_1, y_2\}) = Q_Y(\{y_1, y_3\}) = Q_Y(\{y_2, y_3\}) = \frac{1}{3}.
\end{aligned}$$

As a result, the fractional entropy and fractional Deng entropy of commonality function Q_Y are obtained as follows

$$\begin{aligned}
H_q(Q_Y) &= 2\left[-\log_2 \left(\frac{2}{3} \right) \right]^q + \left[-\log_2 \left(\frac{1}{3} \right) \right]^q,\\
DE_q(Q_Y) &= 2\left[-\log_2 \left(\frac{2}{3} \right) \right]^q + \left[-2\log_2 \left(\frac{1}{3} \right) \right]^q.
\end{aligned}$$

Example 6. Given frame of discernment $Y = \{y_1, y_2, y_3\}$, m is a mass function under Y as follows

$$m_Y(\{y_1\}) = m_Y(\{y_2\}) = m_Y(\{y_3\}) = \frac{1}{3}.$$

Then, the Q_Y can be obtained as follows

$$\begin{aligned}
&Q_Y(\{y_1\}) = m_Y(\{y_1\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_1, y_3\}) + m_Y(Y) = \frac{1}{3},\\
&Q_Y(\{y_2\}) = m_Y(\{y_2\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{1}{3},\\
&Q_Y(\{y_3\}) = m_Y(\{y_3\}) + m_Y(\{y_1, y_3\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{1}{3},\\
&Q_Y(\{y_1, y_2\}) = m_Y(\{y_1, y_2\}) + m_Y(Y) = 0,\\
&Q_Y(\{y_1, y_3\}) = m_Y(\{y_1, y_3\}) + m_Y(Y) = 0,\\
&Q_Y(\{y_2, y_3\}) = m_Y(\{y_2, y_3\}) + m_Y(Y) = 0,\\
&Q_Y(Y) = m_Y(Y) = 0.
\end{aligned}$$

The fractional entropy of commonality function Q_Y can be obtained as

$$H_q(Q_Y) = DE_q(Q_Y) = \left[-\log_2 \frac{1}{3} \right]^q,$$

As can be seen from this example, fractional entropy and fractional Deng entropy are equal.

Example 7. Given a frame of discernment $Y = \{y_1, y_2, y_3\}$, m is a mass function under Y as follows

$$\begin{aligned} & m_Y(\{y_1\}) = m_Y(\{y_2\}) = m_Y(\{y_3\}) = m_Y(\{y_1, y_2\}) \\ & = m_Y(\{y_1, y_3\}) = m_Y(\{y_2, y_3\}) = m_Y(\{y_1, y_2, y_3\}) = \frac{1}{7}. \end{aligned}$$

Then, the Q_Y can be obtained as follows

$$\begin{aligned} & Q_Y(\{y_1\}) = m_Y(\{y_1\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_1, y_3\}) + m_Y(Y) = \frac{4}{7}, \\ & Q_Y(\{y_2\}) = m_Y(\{y_2\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{4}{7}, \\ & Q_Y(\{y_3\}) = m_Y(\{y_3\}) + m_Y(\{y_1, y_3\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{4}{7}, \\ & Q_Y(\{y_1, y_2\}) = m_Y(\{y_1, y_2\}) + m_Y(Y) = \frac{2}{7}, \\ & Q_Y(\{y_1, y_3\}) = m_Y(\{y_1, y_3\}) + m_Y(Y) = \frac{2}{7}, \\ & Q_Y(\{y_2, y_3\}) = m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{2}{7}, \\ & Q_Y(Y) = m_Y(Y) = \frac{1}{7}. \end{aligned}$$

The fractional entropy of commonality function Q_Y can be obtained as

$$H_q(Q_Y) = \frac{12}{7}\left[-\log_2 \frac{4}{7}\right]^q + \frac{6}{7}\left[-\log_2 \frac{2}{7}\right]^q + \frac{1}{7}\left[-\log_2 \frac{1}{7}\right]^q.$$

Also, the fractional Deng entropy of commonality function Q_Y is obtained as follows

$$DE_q(Q_Y) = \frac{12}{7}\left[-\log_2 \frac{4}{7}\right]^q + \frac{6}{7}\left[-\log_2 \frac{2}{21}\right]^q + \frac{1}{7}\left[-\log_2 \frac{1}{49}\right]^q.$$

Example 8. Given a frame of discernment $Y = \{y_1, y_2, y_3\}$, m is a mass function under Y as follows

$$\begin{aligned} & m_Y(\{y_1\}) = m_Y(\{y_2\}) = m_Y(\{y_3\}) = \frac{1}{19}, \\ & m_Y(\{y_1, y_2\}) = m_Y(\{y_1, y_3\}) = m_Y(\{y_2, y_3\}) = \frac{3}{19}, \\ & m_Y(\{y_1, y_2, y_3\}) = \frac{7}{19}. \end{aligned}$$

Then, the Q_Y can be obtained as follows

$$\begin{aligned} & Q_Y(\{y_1\}) = m_Y(\{y_1\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_1, y_3\}) + m_Y(Y) = \frac{14}{19}, \\ & Q_Y(\{y_2\}) = m_Y(\{y_2\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{14}{19}, \\ & Q_Y(\{y_3\}) = m_Y(\{y_3\}) + m_Y(\{y_1, y_3\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{14}{19}, \\ & Q_Y(\{y_1, y_2\}) = m_Y(\{y_1, y_2\}) + m_Y(Y) = \frac{10}{19}, \\ & Q_Y(\{y_1, y_3\}) = m_Y(\{y_1, y_3\}) + m_Y(Y) = \frac{10}{19}, \\ & Q_Y(\{y_2, y_3\}) = m_Y(\{y_2, y_3\}) + m_Y(Y) = \frac{10}{19}, \\ & Q_Y(Y) = m_Y(Y) = \frac{7}{19}. \end{aligned}$$

The fractional entropy of commonality function Q_Y can be obtained as

$$H_q(Q_Y) = \frac{42}{19}\left[-\log_2 \frac{14}{19}\right]^q + \frac{30}{19}\left[-\log_2 \frac{10}{19}\right]^q + \frac{7}{19}\left[-\log_2 \frac{7}{19}\right]^q.$$

Also, the fractional Deng entropy of commonality function Q_Y is obtained as follows

$$DE_q(Q_Y) = \frac{42}{19}\left[-\log_2 \frac{14}{19}\right]^q + \frac{30}{19}\left[-\log_2 \frac{10}{57}\right]^q + \frac{7}{19}\left[-\log_2 \frac{1}{19}\right]^q.$$

Example 9. Given a frame of discernment $Y = \{y_1, y_2, y_3\}$, m is a mass function under Y as follows

$$m_Y(\{y_1\}) = 0.9, \quad m_Y(\{y_1, y_2\}) = 0.01, \quad m_Y(\{y_1, y_2, y_3\}) = 0.09.$$

Then, the Q_Y can be obtained as follows

$$\begin{aligned}
&Q_Y(\{y_1\}) = m_Y(\{y_1\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_1, y_3\}) + m_Y(Y) = 1,\\
&Q_Y(\{y_2\}) = m_Y(\{y_2\}) + m_Y(\{y_1, y_2\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = 0.1,\\
&Q_Y(\{y_3\}) = m_Y(\{y_3\}) + m_Y(\{y_1, y_3\}) + m_Y(\{y_2, y_3\}) + m_Y(Y) = 0.09,\\
&Q_Y(\{y_1, y_2\}) = m_Y(\{y_1, y_2\}) + m_Y(Y) = 0.1,\\
&Q_Y(\{y_1, y_3\}) = m_Y(\{y_1, y_3\}) + m_Y(Y) = 0.09,\\
&Q_Y(\{y_2, y_3\}) = m_Y(\{y_2, y_3\}) + m_Y(Y) = 0.09,\\
&Q_Y(Y) = m_Y(Y) = 0.09.
\end{aligned}$$

The fractional entropy of commonality function Q_Y can be obtained as

$$H_q(Q_Y) = 0.2[-\log_2 0.1]^q + 0.36[-\log_2 0.09]^q.$$

Also, the fractional Deng entropy of commonality function Q_Y is obtained as follows

$$\begin{aligned}
DE_q(Q_Y) = 0.1&\left([-\log_2 0.1]^q + \left[-\log_2 \frac{1}{30}\right]^q\right)\\
+0.09&\left([-\log_2 0.09]^q + 2[-\log_2 0.03]^q + \left[-\log_2 \frac{9}{700}\right]^q\right).
\end{aligned}$$

Definition 17. (fractional joint Deng entropy of a commonality function $Q_{Y,Z}$ with state space $Y \times Z$)

$$DE_q(Q_{Y,Z}) = \sum_{b \in 2^{Y \times Z}} Q_{Y,Z}(b)\left[-\log_2\left(\frac{Q_{Y,Z}(b)}{(2^{|b\downarrow Y|}-1)(2^{|b\downarrow Z|}-1)}\right)\right]^q, \quad (19)$$

in which $b \downarrow Y$ and $b \downarrow Z$ represent the subsets of b that consist of elements present in the sets Y and Z, respectively.

Remark 5. Let m be a mass function with state space $Y \times Z$. If the cardinalities of all focal elements of m are 1, then

$$DE_q(Q_{Y,Z}) = \sum_{b \in 2^{Y \times Z}} Q_{Y,Z}(b)\left[-\log_2 Q_{Y,Z}(b)\right]^q, \ 0 < q \leq 1. \quad (20)$$

Definition 18. (Conditional fractional Deng entropy of a commonality function $Q_{Z|Y}$)

$$DE_q(Q_{Z|Y}) = \sum_{b\in 2^{Y\times Z}} Q_{Y,Z}(b)\left[-\log_2\left(\frac{Q_{Z|Y}(b)}{(2^{|b\downarrow Z|}-1)}\right)\right]^q, \quad 0 < q \leq 1. \tag{21}$$

Remark 6. New versions of fractional entropy for a commonality function Q_Y are presented as

$$H_\alpha(Q_Y) = \sum_{b\in 2^Y}(-1)^{|b|}Q_Y(b)Ln_\alpha Q_Y(b) \approx \alpha! H(Q_Y), \quad 0 < \alpha < 1, \tag{22}$$

Note that new versions of the fractional Deng entropy of a commonality function Q_Y can be defined as follows

$$FDE_\alpha(Q_Y) = \sum_{b\in 2^Y} Q_Y(b)\left[-Ln_\alpha\left(\frac{Q_Y(b)}{2^{|b|}-1}\right)\right]^{\frac{1}{\alpha}} \tag{23}$$

$$\approx [\alpha!]^{\frac{1}{\alpha}}\sum_{b\in 2^Y} Q_Y(b)\left[-\log_2\left(\frac{Q_Y(b)}{2^{|b|}-1}\right)\right]^{\frac{1}{\alpha}}, \; 0 < \alpha < 1.$$

Remark 7. Suppose Q_Y is a CF for Y. It holds that

$$FDE_\alpha(Q_Y) \geq [H_\alpha(Q_Y)]^{\frac{1}{\alpha}}.$$

Definition 19. (New version of fractional joint Deng entropy for a commonality function $Q_{Y,Z}$ with state space $Y \times Z$)

$$FDE_\alpha(Q_{Y,Z}) = \sum_{b\in 2^{Y\times Z}} Q_{Y,Z}(b)\left[-Ln_\alpha\left(\frac{Q_{Y,Z}(b)}{(2^{|b\downarrow Y|}-1)(2^{|b\downarrow Z|}-1)}\right)\right]^{\frac{1}{\alpha}}, \; 0 < \alpha < 1. \tag{24}$$

Remark 8. Let m be a mass function with state space $Y \times Z$. If the cardinalities of all focal elements of m are 1, then

$$FDE_\alpha(Q_{Y,Z}) \approx [\alpha!]^{\frac{1}{\alpha}} \sum_{b\in 2^{Y\times Z}} Q_{Y,Z}(b)\left[-\log_2 Q_{Y,Z}(b)\right]^{\frac{1}{\alpha}}, \; 0 < \alpha < 1. \tag{25}$$

Definition 20. (New version of Conditional fractional Deng entropy of $Q_{Z|Y}$)

$$FDE_\alpha(Q_{Z|Y}) = \sum_{b\in 2^{Y\times Z}} Q_{Z|Y}(b)Q_Y(b\downarrow Y)\left[-Ln_\alpha\left(\frac{Q_{Z|Y}(b)}{(2^{|b\downarrow Z|}-1)}\right)\right]^{\frac{1}{\alpha}} \tag{26}$$

$$= \sum_{b\in 2^{Y\times Z}} Q_{Y,Z}(b)\left[-Ln_\alpha\left(\frac{Q_{Z|Y}(b)}{(2^{|b\downarrow Z|}-1)}\right)\right]^{\frac{1}{\alpha}}, \; 0 < \alpha < 1.$$

Remark 9. Given a state space $Y \times Z$, then the decomposable property of fractional Deng entropy is $FDE_\alpha(Q_{Y,Z}) = FDE_\alpha(Q_{Z|Y}) + FDE_\alpha(Q_Y)$.

Remark 10. Given a state space $Y \times Z$, then the decomposable property of $H_\alpha(Q_Y)$ is $H_\alpha(Q_{Y,Z}) = H_\alpha(Q_{Z|Y}) + H_\alpha(Q_Y)$.

Example 10. Given a mass function $m_{Y,Z}$ under a given state space $Y \times Z$ with $Y = \{y, \bar{y}\}$ and $Z = \{z, \bar{z}\}$, m_Y is a mass function under Y as follows

$$m_Y(y) = 0.6,\ \ m_Y(\bar{y}) = 0.3,\ \ m_Y(y, \bar{y}) = 0.1.$$

Now let the conditional mass functions $m_{Z|y}$ and $m_{Z|\bar{y}}$ be as follows

$$m_{Z|y}(\{z\}) = 0.8,\ \ m_{Z|y}(\{\bar{z}\}) = 0.1,\ \ m_{Z|y}(\{z, \bar{z}\}) = 0.1,$$

$$m_{Z|\bar{y}}(\{z\}) = 0.3,\ \ m_{Z|\bar{y}}(\{\bar{z}\}) = 0.6,\ \ m_{Z|\bar{y}}(\{z, \bar{z}\}) = 0.1.$$

In Table 7, the values of the $H_\alpha(Q_{Y,Z})$ and $FDE_\alpha(Q_{Y,Z})$ are obtained for $\alpha = 0.5$. In addition, we remark that $H_\alpha(Q_{Y,Z}) = H_\alpha(Q_{Z|Y}) + H_\alpha(Q_Y)$. The other results in Table 7 are deduced directly from (Xue and Deng 2022).

Example 11. Given a mass function $m_{Y,Z}$ under a given state space $Y \times Z$ with $Y = \{y, \bar{y}\}$ and $Z = \{z, \bar{z}\}$, m_Y is a mass function under Y as follows

$$m_Y(y) = 0.6,\ \ m_Y(\bar{y}) = 0.4.$$

Now let the conditional mass functions $m_{Z|y}$ and $m_{Z|\bar{y}}$ be as follows

$$m_{Z|y}(\{z\}) = 0.8,\ \ m_{Z|y}(\{\bar{z}\}) = 0.2,$$

$$m_{Z|\bar{y}}(\{z\}) = 0.3,\ \ m_{Z|\bar{y}}(\{\bar{z}\}) = 0.7.$$

According to Table 8, we have obtained the values of $H_\alpha(Q_{Y,Z})$ and $FDE_\alpha(Q_{Y,Z})$ for $\alpha = 0.5$. Moreover, we conclude that $H_\alpha(Q_{Y,Z}) = H_\alpha(Q_{Z|Y}) + H_\alpha(Q_Y)$. The other results in Table 8 are deduced directly from (Xue and Deng 2022).

6. Application to pattern recognition

In this section, we assess the performance of all EFDEns on a pattern recognition problem, using the Iris data set given in Dua and Graff (2017). The data set comprises 150 samples. Each

Table 7. Mass functions and commonality functions.

b	$m_Y^{\uparrow(Y,Z)}$	$m_{y,Z}$	$m_{\bar{y},Z}$	$Q_{m_Y^{\uparrow(Y,Z)}}$	$Q_{m_{Z\|Y}}$	$Q_{m_{Y,Z}}$
$\{(y,z)\}$				0.7	0.9	0.63
$\{(y,\bar{z})\}$				0.7	0.2	0.14
$\{(y,z),(y,\bar{z})\}$	0.6			0.7	0.1	0.07
$\{(\bar{y},z)\}$				0.4	0.4	0.16
$\{(\bar{y},\bar{z})\}$				0.4	0.7	0.28
$\{(\bar{y},z),(\bar{y},\bar{z})\}$	0.3			0.4	0.1	0.04
$\{(y,z),(\bar{y},z)\}$				0.1	0.36	0.036
$\{(y,z),(\bar{y},\bar{z})\}$				0.1	0.63	0.063
$\{(y,z),(\bar{y},z)\}$				0.1	0.08	0.008
$\{(y,\bar{z}),(\bar{y},\bar{z})\}$				0.1	0.14	0.014
$\{(y,z),(y,\bar{z}),(\bar{y},z)\}$			0.3	0.1	0.04	0.004
$\{(y,z),(y,\bar{z}),(\bar{y},\bar{z})\}$			0.6	0.1	0.07	0.007
$\{(y,z),(\bar{y},z),(\bar{y},\bar{z})\}$		0.8		0.1	0.09	0.009
$\{(y,\bar{z}),(\bar{y},z),(\bar{y},\bar{z})\}$		0.1		0.1	0.02	0.002
$Y \times Z$	0.1	0.1	0.1	0.1	0.01	0.001
$H_{0.5}$				**0.49**	**0.29**	**0.78**
$FDE_{0.5}$				**0.43**	**1.92**	**2.35**

Table 8. Mass functions and commonality functions.

b	$m_Y^{\uparrow(Y,Z)}$	$m_{y,Z}$	$m_{\bar{y},Z}$	$Q_{m_Y^{\uparrow(Y,Z)}}$	$Q_{m_{Z\|Y}}$	$Q_{m_{Y,Z}}$
$\{(y,z)\}$				0.6	0.8	0.48
$\{(y,\bar{z})\}$				0.6	0.2	0.12
$\{(y,z),(y,\bar{z})\}$	0.6			0.6		
$\{(\bar{y},z)\}$				0.4	0.3	0.12
$\{(\bar{y},\bar{z})\}$				0.4	0.7	0.28
$\{(\bar{y},z),(\bar{y},\bar{z})\}$	0.4			0.4		
$\{(y,z),(\bar{y},z)\}$					0.24	
$\{(y,z),(\bar{y},\bar{z})\}$					0.56	
$\{(y,z),(\bar{y},z)\}$					0.06	
$\{(y,\bar{z}),(\bar{y},\bar{z})\}$					0.14	
$\{(y,z),(y,\bar{z}),(\bar{y},z)\}$			0.3			
$\{(y,z),(y,\bar{z}),(\bar{y},\bar{z})\}$			0.7			
$\{(y,z),(\bar{y},z),(\bar{y},\bar{z})\}$		0.8				
$\{(y,\bar{z}),(\bar{y},z),(\bar{y},\bar{z})\}$		0.2				
$Y \times Z$						
$H_{0.5}$				**0.860**	**0.69**	**1.55**
$FDE_{0.5}$				**2.4**	**0.8**	**3.2**

Table 9. The interval numbers of the statistical model.

	Attributes			
Item	SL	SW	PL	PW
Se	[4.3, 5.8]	[2.3, 4.4]	[1.0, 1.9]	[0.1, 0.6]
Ve	[4.9, 7.0]	[2.0, 3.4]	[3.0, 5.1]	[1.0, 1.8]
Vi	[4.9, 7.9]	[2.2, 3.8]	[4.5, 6.9]	[1.4, 2.5]
Se, Ve	[4.9, 5.8]	[2.3, 3.4]	NA	NA
Se, Vi	[4.9, 5.8]	[2.3, 3.8]	NA	NA
Ve, Vi	[4.9, 7.0]	[2.2, 3.4]	[4.5, 5.1]	[1.4, 1.8]
Se, Ve, Vi	[4.9, 5.8]	[2.3, 3.4]	NA	NA

categorized into one of three classes, Iris Setosa (*Se*), Iris Versicolor (*Ve*) and Iris Virginica (*Vi*). Additionally, each sample is characterized by four attributes: the sepal length in cm (SL), the sepal width in cm (SW), the petal length in cm (PL) and the petal width in cm (PW). We select 40 samples for each kind of Iris and then we use the method of max-min values to generate a model of interval numbers. In particular, for a fixed attribute, we study the interval of variability in a single class and then we intersect the intervals of more classes. The model of interval numbers is shown in Table 9. In the third and fourth columns, NA is used for (Se, Ve) instead of a specific interval, indicating that there is no intersection between the intervals of *Se* and *Ve*, and similarly for the cases of (Se, Vi) and (Se, Ve, Vi).

Suppose that the selected instance is $[6.3, 2.7, 4.9, 1.8, \text{Iris Virginica}]$. From the dataset, we know that the selected instance belongs to the class Iris Virginica and our purpose is to classify it in the right way. We generate four BPAs, one for each attribute, by using a method based on the similarity of interval numbers which was proposed by Kang et al. (2012). Given two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ their similarity is denoted by $S(A, B)$ and is defined as

$$S(A,B) = \frac{1}{1 + \beta\ D(A,B)}, \tag{27}$$

where $\beta > 0$ is the coefficient of support, we set $\beta = 5$, and $D(A, B)$ is the distance of intervals A and B that is defined in Tran and Duckstein (2002) as

$$D^2(A,B) = \left[\left(\frac{a_1 + a_2}{2}\right) - \left(\frac{b_1 + b_2}{2}\right)\right]^2 + \frac{1}{3}\left[\left(\frac{a_2 - a_1}{2}\right)^2 + \left(\frac{b_2 - b_1}{2}\right)^2\right].$$

For each attribute, we can get seven values of similarity by choosing as A the intervals given in Table 9 and as B the corresponding singleton of the selected instance. Then, by normalizing the obtained values, we get four BPAs which are reported in Table 10. Without any additional information, we can evaluate a final BPA by averaging of four values related to each attribute. In this way, we get the final BPA in Table 11.

Now, based on the BPA in Table 11, we can evaluate the PPT whose definition is recalled in (5). The results are

$$PPT(Se) = 0.1826, \quad PPT(Ve) = 0.4131, \quad PPT(Vi) = 0.4043$$

Hence, the focal element with the highest PPT is the class Ve, which is not the same as our sample, $[6.1, 3.0, 4.9, 1.8, \text{Iris Virginica}]$. However, we guess by computing PPTs, relating to each EFDEn, the correct estimation would be held. Let us fix the value $\alpha = 0.5$. We evaluate the EFDEn of BPAs given in Table 10 and we obtain the results shown in Table 12. Since a higher value of EFDEn means higher uncertainty, we normalize $E^{\alpha}_{fdi}(m)$ by using softmax function. All normalized $E^{\alpha}_{fdi}(m)$ are presented in Table 13. Based on the weights in Table 13, we get the weighted version of the final BPAs. The results are shown in Table 14 in which BPA$_i$ is related to $E^{\alpha}_{fdi}(m), \quad i = 1, 2$.

Finally, based on the BPAs in Table 14, we evaluate the PPT of the classes and we get our desired result shown in Table 15.

Table 10. BPAs based on kang's method.

Item	Attributes SL	SW	PL	PW
$m(Se)$	0.1035	0.0863	0.0624	0.0966
$m(Ve)$	0.1751	0.1555	0.1839	0.2418
$m(Vi)$	0.1471	0.1252	0.1817	0.2903
$m(Se, Ve)$	0.1330	0.1705	NA	NA
$m(Se, Vi)$	0.1330	0.1242	NA	NA
$m(Ve, Vi)$	0.1751	0.1677	0.5719	0.3713
$m(Se, Ve, Vi)$	0.1330	0.1705	NA	NA

Table 11. Final BPA.

Class	Final BPA
$m(Se)$	0.0872
$m(Ve)$	0.1891
$m(Vi)$	0.1861
$m(Se, Ve)$	0.0759
$m(Se, Vi)$	0.0643
$m(Ve, Vi)$	0.3215
$m(Se, Ve, Vi)$	0.0759

Table 12. Extended fractional Deng entropies of BPAs in Table 10.

Methods	Attributes SL	SW	PL	PW
$E^{\alpha}_{fd1}(m)$	12.571576	13.089082	5.079195	5.034744
$E^{\alpha}_{fd2}(m)$	3.420982	3.532220	2.227733	2.178458

Table 13. The normalized EFDEn.

Methods	Attributes SL	SW	PL	PW
Normalized $E^{\alpha}_{fd1}(m)$	0.00027	0.00016	0.48868	0.51089
Normalized $E^{\alpha}_{fd2}(m)$	0.11552	0.10335	0.38094	0.40019

Table 14. Final weighted BPA.

Class	Final weighted BPA_1	Final weighted BPA_2
$m(Se)$	7.989529e-02	0.08331220
$m(Ve)$	2.134721e-01	0.20312736
$m(Vi)$	2.372085e-01	0.21535537
$m(Se, Ve)$	6.320910e-05	0.03299475
$m(Se, Vi)$	5.579557e-05	0.02820607
$m(Ve, Vi)$	4.692418e-01	0.40400949
$m(Se, Ve, Vi)$	6.320910e-05	0.03299475

Table 15. PPT of the classes.

Methods	PPT(Se)	PPT(Ve)	PPT(Vi)	Final decision
$E^{\alpha}_{fd1}(m)$	0.07997586	0.4481457	**0.4718784**	*Vi*
$E^{\alpha}_{fd2}(m)$	0.1249109	0.4326277	**0.4424614**	*Vi*

Table 16. The recognition rate.

β	Method	Recognition Rate
$\beta = 0.001$	Kang	46.6%
	$E^{\alpha}_{fd1}(m)$	**61.5%**
	$E^{\alpha}_{fd2}(m)$	60.7%
	$EX^{\alpha}_{fd1}(m)$	**61.5%**
	$EX^{\alpha}_{fd2}(m)$	33.5%
$\beta = 1$	Kang	95.1%
	$E^{\alpha}_{fd1}(m)$	**95.7%**
	$E^{\alpha}_{fd2}(m)$	**96.6%**
	$EX^{\alpha}_{fd1}(m)$	62.2%
	$EX^{\alpha}_{fd2}(m)$	33.5%
$\beta = 5$	Kang	94.4%
	$E^{\alpha}_{fd1}(m)$	**96.9%**
	$E^{\alpha}_{fd2}(m)$	**96.9%**
	$EX^{\alpha}_{fd1}(m)$	**96.9%**
	$EX^{\alpha}_{fd2}(m)$	33.5%
$\beta = 10$	Kang	95.1%
	$E^{\alpha}_{fd1}(m)$	**96.9%**
	$E^{\alpha}_{fd2}(m)$	**96.9%**
	$EX^{\alpha}_{fd1}(m)$	**96.9%**
	$EX^{\alpha}_{fd2}(m)$	33.5%
$\beta = 1000$	Kang	94.4%
	$E^{\alpha}_{fd1}(m)$	**96.9%**
	$E^{\alpha}_{fd2}(m)$	**96.9%**
	$EX^{\alpha}_{fd1}(m)$	**96.6%**
	$EX^{\alpha}_{fd2}(m)$	33.5%
$\beta = 10000$	Kang	91.9%
	$E^{\alpha}_{fd1}(m)$	**96.3%**
	$E^{\alpha}_{fd2}(m)$	**96.3%**
	$EX^{\alpha}_{fd1}(m)$	**96.3%**
	$EX^{\alpha}_{fd2}(m)$	33.5%

In Table 16, we present the recognition rates of Kang and EFDEn methods, with α set to 0.5. Changing α does not have any effect on our results. However, altering the value of β in Equation (27) would correspondingly modify the results. Each recognition rate is an average of 1000 simulations. Based on Table 16 it is clear that for $\beta > 1$, $E^{\alpha}_{fd1}(m)$, $E^{\alpha}_{fd2}(m)$, and $EX^{\alpha}_{fd1}(m)$ outperform the Kang method and all of them classify the Iris dataset with the same accuracy.

7. Conclusions

In this paper the measures of EFDEn and EFDEx have been introduced as the extended versions of Deng entropy and extropy. Several numerical examples are presented to demonstrate the effectiveness and applicability of these newly proposed measures. Finally, we have discussed a classification problem using a data set to underscore the significance of these measures in pattern recognition. This study has provided the framework for further research aimed at evaluating the performance attributes of fractional entropy.

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Authors' contributions

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ORCID

Saeid Tahmasebi http://orcid.org/0000-0003-4025-6791
Francesco Buono http://orcid.org/0000-0002-3569-4052

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Abbreviations

The following abbreviations are used in this manuscript

BPA	Basic probability assignment
DST	Dempster-Shafer theory of evidence
EFDEn	Extended Fractional Deng Entropy
EFDEx	Extended Fractional Deng Extropy
FOD	Frame of discernment
PPT	Pignistic probability transformation