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Some remarks on varieties whose twisted normal bundle is an instanton

Vincenzo Antonelli and Gianfranco Casnati

Abstract. Let $X \subseteq \mathbb{P}^N$ be a smooth variety with normal bundle $\mathcal{N}_X$. In this note we prove that if $\mathcal{N}_X \otimes \mathcal{O}_{\mathbb{P}^N}(-t)$ is an instanton with quantum number k in the sense of [**2**], then $\dim(X) = N-2$, $t = 1$ and, when $n \geq 4$, also $\lfloor \dim(X)/2 \rfloor \leq \deg(X)$. Moreover, we also discuss some methods for constructing smooth varieties such that $\mathcal{N}_X \otimes \mathcal{O}_{\mathbb{P}^N}(-1)$ is an instanton with $k \neq 0$, illustrating them with explicit examples and counterexamples.

1. Introduction

In this paper a *variety* X is a closed, integral subscheme of some projective space over a field $\mathbf{k}$.

The following definition mimics the one in [**16**] extending it to each variety besides $\mathbb{P}^N$: for a more detailed discussion motivating it, see [**2**] (and [**12**] for the particular case $k = 0$).

Definition 1.1. Let $X \subseteq \mathbb{P}^N$ be a variety of dimension $n \geq 2$ and let $\mathcal{O}_X(h) := \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1)$.

A rank r sheaf $\mathcal{E}$ on X is called h–instanton with quantum number k if

$$h^0(\mathcal{E}(-h)) = h^n(\mathcal{E}(-nh)) = 0, \qquad h^1(\mathcal{E}(-h)) = h^{n-1}(\mathcal{E}(-nh)) = k,$$
$$h^i(\mathcal{E}(-(i+1)h)) = h^j(\mathcal{E}(-jh)) = 0$$

for $1 \leq i \leq n-2$ and $2 \leq j \leq n-1$.

An h–instanton sheaf $\mathcal{E}$ with quantum number 0 is called h–Ulrich.

We only notice that the vanishings in the last displayed row of the above definition contribute to it only if $n \geq 3$. When $n = 2$ they are empty conditions. When $n \geq 3$ then $h^i(\mathcal{E}(-th)) = 0$ for $2 \leq t \leq n-1$ and $0 \leq i \leq n$.

On the one hand, rank two instanton bundles on $\mathbb{P}^3$ were first introduced due to their connection with the solutions of the Yang–Mills equations (see [**5**]) through the Atiyah–Penrose–Ward transformation and therefore with the physics of particles.

On the other hand, the existence of an instanton sheaf with fixed quantum number k on X is not obvious. E.g. the study of the algebraic counterpart of Ulrich

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sheaves, i.e. maximally generated maximal Cohen–Macaulay modules, started in the 80's of the past century and the problem of their existence on each variety is a wide open question which have been object of deep study since [**12**] appeared in 2003 (see [**6**] for more details about this case both from the algebraic viewpoint and the geometric one).

The existence of instanton and Ulrich sheaves is often also related to interesting geometric deep properties of the underlying variety X. E.g. if $X \subseteq \mathbb{P}^{n+1}$ is a hypersurface, then the existence of a locally Cohen–Macaulay h–instanton sheaf is equivalent to the existence of a representation of a power of the form defining X as the determinant of a suitable morphism of vector bundles of the same rank on $\mathbb{P}^{n+1}$ which are Steiner in the sense of [**11**] (see Definition 1.4).

Notice that every variety $X \subseteq \mathbb{P}^N$ comes equipped with certain distinctive vector bundles such as the *cotangent bundle*, i.e. the sheaf of differentials Ω^1_X, the *tangent bundle*, i.e. its dual $\mathcal{T}_X$, the *conormal bundle*, i.e. $\mathcal{C}_X := \mathcal{I}_X/\mathcal{I}_X^2$ where $\mathcal{I}_X$ is the ideal sheaf of X, and the *normal bundle*, i.e. its dual $\mathcal{N}_X$. Thus it is quite reasonable to ask if one of them is an instanton bundle with respect to $\mathcal{O}_X(h)$.

In [**8**] the second author, improving and generalizing a recent result in [**7**], shows that one can answer affirmatively to the above question only for $\mathcal{T}_X$ when either X is a twisted cubic curve in $\mathbb{P}^3$, or a Veronese surface in $\mathbb{P}^5$ and in these cases $\mathcal{T}_X$ is actually Ulrich.

Let us now focus on the normal bundle $\mathcal{N}_X$ of $X \subseteq \mathbb{P}^N$. The fact that $\mathcal{N}_X$ is never an h–instanton bundle trivially follows by combining the exact sequences

$$
\begin{gathered}
0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^N} \otimes \mathcal{O}_X \longrightarrow \mathcal{N}_X \longrightarrow 0, \\
0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{\oplus N+1} \longrightarrow \mathcal{T}_{\mathbb{P}^N}(-1) \longrightarrow 0.
\end{gathered} \tag{1.1}
$$

Indeed by composition we obtain a surjective morphism

$$
\mathcal{O}_X^{\oplus N+1} \twoheadrightarrow \mathcal{T}_{\mathbb{P}^N} \otimes \mathcal{O}_X(-h) \twoheadrightarrow \mathcal{N}_X(-h),
$$

hence $h^0(\mathcal{N}_X(-h)) \neq 0$. For this reason it is natural to raise the same question for the tensor products of $\mathcal{N}_X$ with suitable rank one coherent sheaves $\mathcal{L}$ on X.

The first result in this direction can be found in [**19**, Theorem 3.6] where the authors show that if $X \subseteq \mathbb{P}^N$ is the degeneracy locus of a matrix M of linear forms and the characteristic of $\mathbf{k}$ is zero, then there is a rank one sheaf $\mathcal{L}$ on X which is invertible outside the singular locus of X and such that $\mathcal{N}_X \otimes \mathcal{L}$ is Ulrich. In the particular case of codimension 2 varieties, then $\mathcal{L} \cong \mathcal{O}_X(-h)$.

If we restrict to smooth varieties and to sheaves $\mathcal{L} \cong \mathcal{O}_X(-th)$, there are even more precise results. Indeed, on the one hand, at least when $\mathbf{k} = \mathbb{C}$, in [**20**] the author proves that if $\mathcal{N}_X(-th)$ is Ulrich then either X is a linear subspace of $\mathbb{P}^N$, or $N = n+2$, $t = 1$ and $n = \dim(X) \leq 3$. On the other hand, for each non–degenerate complete intersection $X \subseteq \mathbb{P}^{n+2}$ the bundle $\mathcal{N}_X(-h)$ is never h–Ulrich.

In the present short note we partially extend the quoted result from [**20**] to instanton bundles proving the following result in Section 2.

THEOREM 1.2. *Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension $n < N$ and let $\mathcal{O}_X(h) := \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1)$.*

Then $\mathcal{N}_X(-th)$ is an h–instanton bundle if and only if $t = 1$ and one of the following assertions holds.

(1) X is a linear subspace in $\mathbb{P}^N$: in this case $\mathcal{N}_X(-h) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus N-n}$.

(2) $N = n+2$ *and* $h^j(\mathcal{N}_X(-(j+2)h)) = 0$ *for* $0 \le j \le n-2$*: in this case the quantum number of* $\mathcal{N}_X(-h)$ *is* $h^1(\mathcal{N}_X(-2h))$*.*

Moreover, if the latter case occurs and $n \ge 4$*, then*

$$\left\lfloor \frac{n}{2} \right\rfloor \le \deg(X). \tag{1.2}$$

In view of the above result, in Section 3 we deal with the well known Serre correspondence between varieties of codimension 2 and vector bundles (see [**23, 4**]). More precisely, we recall that for each smooth variety $X \subseteq \mathbb{P}^{n+2}$ of dimension n there exists at least one rank r vector bundle $\mathcal{F}$ on $\mathbb{P}^{n+2}$ and an injective morphism $\varphi\colon \mathcal{O}_{\mathbb{P}^{n+2}}^{\oplus r-1} \to \mathcal{F}^\vee(1)$ such that the locus $D_{r-1}(\varphi)$ of points where $\mathrm{rk}(\varphi) \le r-1$ is exactly X (see Lemma 3.1).

In Section 4 we prove the following result extending [**20**, Proposition 2.2] to instanton bundles.

Theorem 1.3. *Let* $\mathcal{F}$ *be a rank* r *Steiner bundle with* $c_1(\mathcal{F}) = f$ *on* $\mathbb{P}^{n+2}$*,* $n \ge 2$*. Assume that* $\varphi\colon \mathcal{O}_{\mathbb{P}^{n+2}}^{\oplus r-1} \to \mathcal{F}^\vee(1)$ *is an injective morphism such that* $X = D_{r-1}(\varphi)$ *is smooth of dimension* n *and set* $\mathcal{O}_X(h) := \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^{n+2}}(1)$*.*

Then $\mathcal{N}_X(-h)$ *is* h*–instanton with quantum number* $(r-1)f$*.*

Trivially each linear determinantal smooth variety of codimension 2 gives examples of the above theorems. Nevertheless, we can even find non–trivial examples, i.e. varieties $X \subseteq \mathbb{P}^{n+2}$ such that $\mathcal{N}_X$ is an h–instanton with strictly positive quantum number. Several examples are exploited in Section 5.

1.1. Notation and first results. Throughout the whole paper we will work over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic. The projective space of dimension N over $\mathbf{k}$ will be denoted by $\mathbb{P}^N$: $\mathcal{O}_{\mathbb{P}^N}(1)$ will denote the hyperplane line bundle.

A projective scheme X is a closed subscheme of some projective space over $\mathbf{k}$: X is a variety if it is also integral. If X is a projective scheme its structure sheaf is denoted by $\mathcal{O}_X$, its canonical sheaf by ω_X, its sheaf of p–differentials by Ω_X^p, and we set $\mathcal{T}_X := (\Omega_X^1)^\vee$.

Let X be smooth. We denote by K_X each Cartier divisor such that $\omega_X \cong \mathcal{O}_X(K_X)$. As in [**14**], $A^r(X)$ denotes the group of cycles on X of codimension r modulo rational equivalence, hence $A^1(X)$ coincides with the usual Picard group of X. The Chern classes of a coherent sheaf $\mathcal{A}$ on X are elements in $A(X)$: in particular, when $\mathcal{A}$ is locally free $c_1(\mathcal{A})$ is identified with $\det(\mathcal{A})$. When $X = \mathbb{P}^N$, then $A^r(\mathbb{P}^N)$ is generated by the class of a linear variety of codimension r, hence we make the standard identification $A^r(\mathbb{P}^N) = \mathbb{Z}$.

We recall the following definition: see [**11**].

Definition 1.4. Let $N \ge 2$.

A rank r bundle $\mathcal{F}$ on $\mathbb{P}^N$ is called Steiner if it fits into an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus f} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{\oplus r+f} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1.3}$$

for some $f \ge 0$.

Steiner bundles can be characterized cohomologically as follows.

Lemma 1.5. *Let $N \ge 2$.*

A rank r bundle $\mathcal{F}$ on $\mathbb{P}^N$ is Steiner if and only if $h^i(\mathcal{F}(t)) = 0$ when either $0 \le i \le N-2$ and $t \le -i-1$ or $1 \le i \le N$ and $t \ge -i$.

Proof. In [**3**, Corollary 4.3] the authors show that the notion of Steiner bundle can be be characterized cohomologically using either [**3**, Definition 1.1] or [**3**, Proposition 4.1 (2)]. The latter is exactly the statement above. □

We will also need the following lemma.

Lemma 1.6. *Let X be a smooth projective variety of dimension $n \ge 2$ endowed with an ample and globally generated line bundle $\mathcal{O}_X(h)$.*

A rank two bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = (n+1)h + K_X$ is an h–instanton if and only if $h^i(\mathcal{E}(-(i+1)h)) = 0$ when $0 \le i \le n-2$.

Proof. Taking into account that in this paper we only consider as instanton what is called instanton with defect $\delta = 0$ in [**2**], the statement above is exactly [**2**, Proposition 6.7]. □

For further notation and all the other results used in the paper we tacitly refer to [**15**], unless otherwise stated.

2. The proof of Theorem 1.2

In this first section we prove Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. If $X \subseteq \mathbb{P}^N$ is a linear subspace, then $\mathcal{N}_X(-h) \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus N-n}$ which is trivially an h–instanton.

If $N = n+2$, then (1.1) yields

$$c_1(\mathcal{N}_X(-h)) = (n+1)h + K_X. \tag{2.1}$$

In particular, thanks to Lemma 1.6, $\mathcal{N}_X(-h)$ is an h–instanton bundle if and only if $h^j(\mathcal{N}_X(-(j+2)h)) = 0$ for $0 \le j \le n-2$. Moreover, if these vanishings occur, then the quantum number of $\mathcal{N}_X(-h)$ is $h^1(\mathcal{N}_X(-2h))$.

The proof of the converse implication can be obtained in the same way as in [**20**, Theorem 1].

Indeed one of the two ingredients of the proof therein is that equality

$$c_1(\mathcal{E})h^{n-1} = \frac{\mathrm{rk}(\mathcal{E})}{2}((n+1)h^n + K_X h^{n-1})$$

holds for each h–Ulrich bundle $\mathcal{E}$ on a smooth variety. In [**2**, Theorem 1.6] the authors prove that the same equality also holds for each h–instanton bundle regardless of the field $\mathbf{k}$ when $\mathcal{O}_X(h)$ is very ample.

The other ingredient is that the line bundle $\mathcal{O}_X((n+1)h + K_X)$ is globally generated, hence $(n+1)h^n + K_X h^{n-1} \ge 0$ by the Nakai–Moishezon criterion. This result holds regardless of the characteristic of $\mathbf{k}$ as well, thanks to [**22**].

Thus the converse implication can be proved repeating verbatim the aforementioned proof in [**20**].

In order to prove (1.2), assume that $\mathcal{N}_X(-h)$ is an h–instanton bundle and X is not a linear subspace in $\mathbb{P}^{n+2}$. Assume $\lfloor n/2 \rfloor > \deg(X)$: thanks to [**2**, Proposition 6.3] we deduce that $\mathcal{N}_X(-h)$ is aCM, whence it is actually Ulrich. It follows from [**20**, Theorem 1] that $n \le 3$, i.e. $\deg(X) \le 1$ thanks to the above inequality, a contradiction. Thus the proof is complete. □

REMARK 2.1. We are unable to show that $n \leq 3$ as in [**20**, Theorem 1], because the proof therein heavily depends on the fact that Ulrich bundles are aCM. We are only able prove the weaker bound (1.2). In particular, we do not know if there are non–degenerate varieties X with $n = \dim(X) \geq 4$ such that $\mathcal{N}_X(-h)$ is an h–instanton bundle.

Nevertheless, at least if $\mathbf{k} \cong \mathbb{C}$, we can look at the problem from a different perspective, suggesting the existence of a stronger bound $n \leq 3$.

Indeed if $n \geq 4$, then X would satisfy $\omega_X \cong \mathcal{O}_X(e)$ for some integer, thanks to [**14**, Theorem 2.2], hence there would be an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_X(c_1(\mathcal{E})) \longrightarrow 0$$

where $\mathcal{E}$ is a rank two vector bundle on $\mathbb{P}^{n+2}$ (for details see [**14**, Proposition 6.2] or the next Section 3). The above sequence is locally a Koszul complex, hence its restriction to X yields $\mathcal{O}_X \otimes \mathcal{E} \cong \mathcal{N}_X$.

Assume $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^{n+2}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^{n+2}}(a_2)$. If $\mathcal{N}_X(-h)$ is an h–instanton bundle, then

$$h^0(\mathcal{O}_X((a_i-2)h)) = 0, \qquad h^n(\mathcal{O}_X((a_i-n-1)h)) = h^0(\mathcal{O}_X((n+1-a_i+e)h)) = 0,$$

thanks to Definition 1.1, hence $n+2+e \leq a_i \leq 1$. Since

$$\mathcal{O}_X((n+1)h + K_X) \cong \mathcal{O}_X((n+1+e)h)$$

is globally generated by [**22**], we infer that $a_i = 1$ and $e = -n-1$, i.e. $X \cong \mathbb{P}^n$ embedded linearly in $\mathbb{P}^{n+2}$ by [**17**, Theorem 1], contradicting the non–degeneracy of X. We deduce that $\mathcal{E}$ is necessarily indecomposable if $\mathcal{N}_X(-h)$ is an h–instanton bundle, contradicting the Hartshorne conjecture when $n \geq 4$.

REMARK 2.2. If $\mathcal{N}_X(-h)$ is an instanton, then X has codimension 2, hence its quantum number can be computed using the Riemann–Roch theorem (see [**15**, Theorem A.4.1]), because

$$k = h^1(\mathcal{N}_X(-2h)) = -\chi(\mathcal{N}_X(-2h)),$$

taking into account that $c_2(\mathcal{N}_X) = dh^2$ (see [**15**, A.3.C7]) and computing $c_2(\mathcal{T}_X)$ from (1.1).

E.g. if $X \subseteq \mathbb{P}^5$ is a threefold of degree d, then

$$c_2(\mathcal{T}_X) = (15-d)h^2 + 6hK_X + K_X^2.$$

Thus the Riemann–Roch theorem yields

$$k = \frac{1}{6}(7d-71)d - 4h^2K_X - \frac{1}{3}hK_X^2. \tag{2.2}$$

A similar argument for a surface $X \subseteq \mathbb{P}^4$ of degree d returns the equality

$$k = \frac{1}{2}(2d-13)d - \frac{5}{2}hK_X - 2\chi(\mathcal{O}_X). \tag{2.3}$$

3. Codimension two subvarieties

Taking into account Theorem 1.2, it is then interesting to deal with the normal bundle of codimension two smooth subvarieties $X \subseteq \mathbb{P}^{n+2}$. Since every h–instanton bundle on a curve is automatically h–Ulrich we restrict to $n \geq 2$.

The ideal of such an X has a particular resolution as the following well known result shows (see [**23**]: see also [**4**]). In order to state it, we notice that

$$\mathcal{L}(X) := \{\ \ell \in \mathbb{Z} \mid \mathcal{O}_X((n+3-\ell)h + K_X) \text{ is globally generated }\}$$

is always non–empty: indeed, $\mathcal{O}_X((n+1)h+K_X)$ is globally generated thanks to [**22**], hence $\mathcal{L}(X)$ certainly contains at least all the integers not greater than 2.

LEMMA 3.1. *Let $X \subseteq \mathbb{P}^{n+2}$ be a locally complete intersection of dimension $n \ge 2$ and let $\mathcal{O}_X(h) := \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^{n+2}}(1)$.*

For each $\ell \in \mathcal{L}(X)$ there is an integer $r_\ell \ge 2$ and a rank r_ℓ bundle $\mathcal{F}_\ell$ on $\mathbb{P}^{n+2}$ with $c_1(\mathcal{F}_\ell) = r_\ell - \ell$ which fits into an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(-\ell)^{\oplus r_\ell - 1} \xrightarrow{\varphi} \mathcal{F}_\ell^\vee(1-\ell) \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Such a sequence is uniquely determined by the choice of ℓ.

PROOF. Notice that $c_1(\mathcal{N}_X) = (n+3)h + K_X$ thanks to (1.1). By definition the line bundle $\mathcal{O}_X((n+3-\ell)h+K_X)$ is globally generated. Assume there are $r_\ell - 1$ global sections generating $\mathcal{O}_X((n+3-\ell)h+K_X)$. Thus [**4**, Theorem 1.1] yields the existence of a rank r_ℓ vector bundle $\mathcal{F}$ on $\mathbb{P}^{n+2}$ such that $c_1(\mathcal{F}^\vee(1)) = \ell$, fitting into a unique sequence as in the statement because $h^i(\mathcal{O}_{\mathbb{P}^{n+2}}(-\ell)) = 0$ for $1 \le i \le 2$. In particular $c_1(\mathcal{F}) = r_\ell - \ell$. □

REMARK 3.2. Trivially one can take $r_\ell := h^0(\mathcal{O}_X((n+3-\ell)h+K_X)) + 1$.

The above result implies that every locally complete intersection $X \subseteq \mathbb{P}^{n+2}$ of codimension 2 satisfies the equality $X = D_{r-2}(\varphi)$ for a suitable morphism φ from a trivial bundle of rank $r-1$ to another rank r bundle $\mathcal{F}^\vee(1)$. In general X is singular along the locus $D_{r-3}(\varphi)$ whose expected dimension is $n-4$. Thus X is likely singular when $n \ge 4$.

Conversely, let $\mathcal{F}$ be a rank r bundle on $\mathbb{P}^{n+2}$, $n \ge 2$, with $c_1(\mathcal{F}) = f$ and assume that $\varphi\colon \mathcal{O}_{\mathbb{P}^{n+2}}^{\oplus r-1} \to \mathcal{F}^\vee(1)$ is injective and such that $X := D_{r-2}(\varphi) \subseteq \mathbb{P}^{n+2}$ has pure dimension n. If X is smooth, then it is certainly irreducible, because $n \ge 2$. In this case there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(f-r)^{\oplus r-1} \xrightarrow{\varphi} \mathcal{F}^\vee(f-r+1) \longrightarrow \mathcal{I}_X \longrightarrow 0. \tag{3.1}$$

The following result is well known and part of folklore.

PROPOSITION 3.3. *Let P be a smooth projective variety. Assume that the characteristic of $\mathbf{k}$ is 0.*

If $\mathcal{A}$ and $\mathcal{B}$ are bundles of respective ranks a and b on P such that $\mathrm{Hom}_{\mathcal{O}_P}(\mathcal{A},\mathcal{B})$ is globally generated and $\varphi\colon \mathcal{A} \to \mathcal{B}$ is general, then $D_r(\varphi)$ is either empty or has codimension $(a-r+1)(b-r+1)$ which is smooth outside $D_{r-1}(\varphi)$.

PROOF. E.g. see [**21**, Theorem 2.6]. □

In particular, on the one hand X is likely to be singular when $n \ge 4$. On the other hand, when $n \le 3$, the variety X is smooth of pure dimension n if the sheaf

$$\mathcal{F}^\vee(1)^{\oplus r-1} \cong \mathcal{H}om_{\mathbb{P}^{n+2}}(\mathcal{O}_{\mathbb{P}^{n+2}}(-1)^{\oplus r-1}, \mathcal{F}^\vee)$$

is globally generated, φ is general and the characteristic of $\mathbf{k}$ is 0. In this case $r - f = c_1(\mathcal{F}^\vee(1)) \ge 0$ necessarily.

PROPOSITION 3.4. *Let $\mathcal{F}$ be a rank r bundle on $\mathbb{P}^{n+2}$ with $n \ge 2$ and consider a morphism $\varphi\colon \mathcal{O}_{\mathbb{P}^{n+2}}^{\oplus r-1} \to \mathcal{F}^\vee(1)$ such that $X := D_{r-2}(\varphi) \subseteq \mathbb{P}^{n+2}$ is smooth of pure dimension n.*

If $c_1(\mathcal{F}) = f$, then

$$h^0(\omega_X) = (r-1)h^0(\mathcal{O}_{\mathbb{P}^{n+2}}(r-f-n-3)) \\ - h^0(\mathcal{F}(r-f-n-4)) + h^1(\mathcal{F}(r-f-n-4))$$

and the following assertions hold.

(1) *If $r \le f+n+2$, and $h^1(\mathcal{F}(r-f-n-4)) = 0$, then $h^0(\omega_X) = 0$.*
(2) *If $r \ge f+n+3$, then ω_X is globally generated.*
(3) *If $r \ge f+n+4$, then ω_X is very ample.*

Proof. By applying the functor $\mathcal{H}om_{\mathbb{P}^{n+2}}(-, \omega_{\mathbb{P}^{n+2}})$ to the exact sequence obtained by glueing together (3.1) and

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{3.2}$$

we obtain the exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(-n-3) \longrightarrow \mathcal{F}(r-f-n-4) \longrightarrow \mathcal{A} \longrightarrow 0, \\ 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(r-f-n-3)^{\oplus r-1} \longrightarrow \omega_X \longrightarrow 0. \end{gathered} \tag{3.3}$$

Thus ω_X is globally generated if $r \ge f+n+3$ and very ample when $r \ge f+n+4$.

Finally, we have $h^i(\mathcal{A}) = h^i(\mathcal{F}(r-f-n-4))$ for $0 \le i \le 2$ because $n \ge 2$, hence the assertion about $h^0(\omega_X)$. In particular, if $h^1(\mathcal{F}(r-f-n-4)) = 0$, then

$$0 \le h^0(\omega_X) \le (r-1)h^0(\mathcal{O}_{\mathbb{P}^{n+2}}(r-f-n-3)) - h^0(\mathcal{F}(r-f-n-4)) :$$

the first summand on the right vanishes if $r \le f+n+2$, hence $h^0(\omega_X) = 0$ necessarily. □

Remark 3.5. The cohomology of (3.1) yields

$$H^i_*(\mathcal{F}) \cong H^{n+2-i}_*(\mathcal{F}^\vee(-n-3)) \cong H^{n+2-i}_*(\mathcal{I}_X(r-f-n-4)),$$

by the Serre duality. Thus X is aCM if and only if the latter direct sum is zero in the range $2 \le i \le n+1$, i.e. if and only if $H^i_*(\mathcal{F}) = 0$ in the same range.

Example 3.6. Assume that the characteristic of $\mathbf{k}$ is 0 and $2 \le n \le 3$.

Let $\mathcal{G}$ be any Steiner bundle on $\mathbb{P}^{n+2}$ of rank r and $c_1(\mathcal{G}) = g$. In what follows we consider $\mathcal{F} := \mathcal{G}^\vee(1-t)$ where $t \ge 0$. The bundle $\mathcal{F}$ satisfies $H^i_*(\mathcal{F}) = 0$ for $2 \le i \le n+1$ and $\mathcal{F}^\vee(1) \cong \mathcal{G}(t)$ is globally generated because $\mathcal{G}$ is Steiner (see (1.3)), hence we can construct smooth varieties $X \subseteq \mathbb{P}^N$ whose sheaf of ideals fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(-rt-g)^{\oplus r-1} \longrightarrow \mathcal{G}(t-rt-g) \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Thanks to Remark 3.5 all such varieties are aCM.

Indeed we can actually recover a determinantal resolution of $\mathcal{I}_X$ simply by taking the mapping cone of the above sequence with (1.3). In this case we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(-rt-g)^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^{n+2}}(t-rt-g-1)^{\oplus g} \\ \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(t-rt-g)^{\oplus g+r} \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

There are many varieties of codimension 2 which are not aCM. E.g. in [**9**] the author classifies the arithmetically Buchsbaum ones in $\mathbb{P}^{n+2}$ which are not of general type, listing certain locally free resolutions of their sheaves of ideals. Some of them are not in the shape described in Lemma 3.1. In the following example we show how to recover a locally free resolution with that shape.

EXAMPLE 3.7. Assume that the characteristic of $\mathbf{k}$ is 0 and consider the arithmetically Buchsbaum $K3$ surfaces $X \subseteq \mathbb{P}^4$ of degree $d = 8$ whose sheaf of ideals fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \Omega^1_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{I}_X \longrightarrow 0. \tag{3.4}$$

In order to compute an integer ℓ and the corresponding sheaf $\mathcal{F}_\ell$ defined in Lemma 3.1 we apply the functor $\mathcal{H}om_{\mathbb{P}^4}(-, \omega_{\mathbb{P}^4})$ to the sequence above glued with (3.2). Thus we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{T}_{\mathbb{P}^4}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4} \longrightarrow \omega_X \longrightarrow 0.$$

In particular we can take $\ell = 4$ in Lemma 3.1 and, consequently, $r_\ell = 7$. Thus there is a vector bundle $\mathcal{F}_4$ of rank 8 fitting into an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 7} \longrightarrow \mathcal{F}_4^\vee(-3) \longrightarrow \mathcal{I}_X \longrightarrow 0. \tag{3.5}$$

The cohomologies of (3.4) and (3.5) yield that $h^q(\mathcal{F}_4^\vee(p+1))$ in the range $-4 \le p \le 0$ is as follows

0	0	0	0	0	$q = 4$
1	0	0	0	0	$q = 3$
0	0	0	0	0	$q = 2$
0	0	1	0	0	$q = 1$
0	0	0	0	15	$q = 0$
$p = -4$	$p = -3$	$p = -2$	$p = -1$	$p = 0$	

Table 1: Values of $h^q(\mathcal{F}_4^\vee(p+1))$ in the range $-4 \le p \le 0$

Thanks to [**1**, Beilinson's theorem (strong form)], we know that $\mathcal{F}_4^\vee(1)$ fits into an exact sequence of the following form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \Omega^2_{\mathbb{P}^4}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}^{\oplus 15} \longrightarrow \mathcal{F}_4^\vee(1) \longrightarrow 0.$$

4. The proof of Theorem 1.3.

In this section we prove Theorem 1.3 stated in the introduction. Let $X \subseteq \mathbb{P}^{n+2}$ be a smooth variety of dimension n and set $\mathcal{O}_X(h) := \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1)$. Thanks to (2.1) and Lemma 1.6, in order to show that $\mathcal{N}_X(-h)$ is an h–instanton bundle it suffices to check that $H^j(\mathcal{N}_X(-(j+2)h)) = 0$ for all $0 \le j \le n-2$.

In order to compute such vanishings we will use the isomorphism

$$H^j(\mathcal{N}_X(th)) \cong \mathrm{Ext}^{j+1}\big(\mathcal{I}_X, \mathcal{I}_X(t)\big) \tag{4.1}$$

for $0 \le j \le n$ and $t \in \mathbb{Z}$ (see [**18**, Remark 2.2.6]).

Proof of Theorem 1.3. By applying the contravariant functor $\mathrm{Hom}(-, \mathcal{I}_X(-j-2))$ to (3.1) we obtain

$$\begin{aligned} &H^j(\mathcal{I}_X(r-f-j-2))^{\oplus r-1} \longrightarrow H^j(\mathcal{N}(-(j+2)h)) \\ &\qquad \longrightarrow H^{j+1}(\mathcal{F} \otimes \mathcal{I}_X(r-f-j-3)) \longrightarrow H^{j+1}(\mathcal{I}_X(r-f-j-2))^{\oplus r-1}. \end{aligned} \tag{4.2}$$

Thus

$$\begin{aligned} h^j(\mathcal{N}(-(j+2)h)) \le (r-1)&h^j(\mathcal{I}_X(r-f-j-2)) \\ &+ h^{j+1}(\mathcal{F} \otimes \mathcal{I}_X(r-f-j-3)). \end{aligned}$$

First we compute $h^j(\mathcal{I}_X(r-f-j-2))$. Tensoring (3.1) by $\mathcal{O}_{\mathbb{P}^{n+2}}(r-f-j-2)$ we obtain

$$h^j(\mathcal{I}_X(r-f-j-2)) \le h^j(\mathcal{F}^\vee(-j-1)) + (r-1)h^{j+1}(\mathcal{O}_{\mathbb{P}^{n+2}}(-j-2)).$$

The right–hand side of the above inequality vanishes in the range $0 \le j \le n$, because trivially $h^{j+1}(\mathcal{O}_{\mathbb{P}^{n+2}}(-j-2)) = 0$ and the first summand vanishes by Lemma 1.5. In particular we also deduce that

$$h^j(\mathcal{I}_X(r-f-j-2-t)) = 0$$

for $0 \le j \le n$ and $t \ge 0$, thanks to [**2**, Proposition 2.1]. Thus, tensoring (1.3) by $\mathcal{I}_X(r-f-j-3)$, the above equality implies

$$\begin{aligned} h^{j+1}(\mathcal{F} \otimes \mathcal{I}_X(r-f-j-3)) &\le (r+f)h^{j+1}(\mathcal{I}_X(r-f-j-3)) \\ &\quad + fh^{j+2}(\mathcal{I}_X(r-f-j-4)) = 0 \end{aligned}$$

for $0 \le j \le n-2$. Thus the proof that $\mathcal{N}_X(-h)$ is an instanton bundle is complete.

It remains to compute $k := h^1(\mathcal{N}_X(-2h)) = h^{n-1}(\mathcal{N}_X(-(n+1)h))$. Arguing as above, (1.3) tensored by $\mathcal{O}_{\mathbb{P}^{n+2}}(r-f-n-1)$ yields $h^n(\mathcal{I}_X(r-f-n-1)) = 0$, hence (4.2) for $j = n-1$ returns

$$h^{n-1}(\mathcal{N}_X(-(n+1)h)) = h^n(\mathcal{F} \otimes \mathcal{I}_X(r-f-n-2)).$$

By combining the cohomologies of (1.3) and (3.1) tensored by $\mathcal{I}_X(r-f-n-2)$ and $\mathcal{I}_X(r-f-n-3)$ respectively, the same argument used above yields

$$\begin{aligned} h^n(\mathcal{F} \otimes \mathcal{I}_X(r-f-n-2)) &= fh^{n+1}(\mathcal{I}_X(r-f-n-3)) \\ &= f(r-1)h^{n+2}(\mathcal{O}_{\mathbb{P}^{n+2}}(-n-3)) = f(r-1). \end{aligned}$$

The statement is then completely proved. □

In what follows we will construct examples of smooth varieties $X \subseteq \mathbb{P}^N$ of dimension n such that $\mathcal{N}_X(-h)$ is an instanton.

In view of Theorems 1.2, 1.3 and Proposition 3.3 the first possible approach is to construct them starting from a Steiner bundle $\mathcal{F}$ such that $\mathcal{F}^\vee(1)$ is globally generated. We do not know the classification of such interesting bundles on $\mathbb{P}^N$. Nevertheless, we are able to classify below Steiner bundles $\mathcal{F}$ such that $\mathcal{F}^\vee(1)$ is regular, i.e. such that $h^i(\mathcal{F}(-i)) = 0$ for $1 \le i \le N$. Recall that regular sheaves are globally generated.

Lemma 4.1. *Let $\mathcal{F}$ be a Steiner bundle on $\mathbb{P}^N$. Then $\mathcal{F}^\vee(1)$ is regular if and only if $\mathcal{F} \cong \Omega^{N-1}_{\mathbb{P}^N}(N)^{\oplus a} \oplus \mathcal{O}^b_{\mathbb{P}^N}$ for some non–negative integers a and b.*

Proof. Let $\mathcal{F}$ be a Steiner bundle such that $\mathcal{F}^\vee(1)$ is regular. We set

$$b = h^{N-1}(\mathcal{F}(-N-1)), \qquad c = h^{N-1}(\mathcal{F}(-N)), \qquad a = h^{N-1}(\mathcal{F}(-N)).$$

Thanks to Lemma 1.5 and [**1**, Beilinson's theorem (weak form)] applied to the vector bundle $\mathcal{F}(-1)$ we obtain a spectral sequence whose $E_1^{p,q}$ sheaves are as in Table 2.

$\mathcal{O}_{\mathbb{P}^N}^{\oplus b}(-1)$	0	0	$q=N$
$\mathcal{O}_{\mathbb{P}^N}^{\oplus c}(-1)$	$\Omega_{\mathbb{P}^N}^{N-1}(N-1)^{\oplus a}$	0	$q=N-1$
0	0	0	$0\le q\le N-2$
$p=-N$	$p=-N+1$	$-N+2\le p\le 0$	

Table 2: Values of $E_1^{p,q}$ in the range $-N \le p \le 0$

The only possibly non–zero differential in this first page is

$$d_1^{-N,N-1}\colon \mathcal{O}_{\mathbb{P}^N}^{\oplus c}(-1) \longrightarrow \Omega_{\mathbb{P}^N}^{N-1}(N-1)^{\oplus a}.$$

Let us continue by considering the second page E_2^{pq} of the spectral sequence in Table 3.

$\mathcal{O}_{\mathbb{P}^N}^{\oplus b}(-1)$	0	0	$q=N$
$\ker(d_1^{-N,N-1})$	$\operatorname{coker}(d_1^{-N,N-1})$	0	$q=N-1$
0	0	0	$0\le q\le N-2$
$p=-N$	$p=-N+1$	$-N+2\le p\le 0$	

Table 3: Values of $E_2^{pq}=E_\infty^{pq}$ in the range $-N \le p \le 0$

Notice that the spectral sequence degenerates after the second page, thus $E_2^{pq} = E_\infty^{pq}$ and in particular $\ker(d_1^{-N,N-1})=0$. Again[**1**, Beilinson's theorem (weak form)] implies that

$$\bigoplus_{p=0}^{N} E_\infty^{-p,p} \cong \operatorname{coker}(\varphi)\oplus \mathcal{O}_{\mathbb{P}^N}^{\oplus b}(-1)$$

is the graded sheaf associated to the filtration $0\subseteq \mathcal{G}(-1)\subseteq \mathcal{F}(-1)$, where $\mathcal{G}(-1):=\operatorname{coker}(d_1^{-N,N-1})$. Thus we obtain the short exact sequences

$$0\longrightarrow \mathcal{G}(-1)\longrightarrow \mathcal{F}(-1)\longrightarrow \mathcal{O}_{\mathbb{P}^N}^{\oplus b}(-1)\longrightarrow 0,$$
$$0\longrightarrow \mathcal{O}_{\mathbb{P}^N}^{\oplus c}(-1)\longrightarrow \Omega_{\mathbb{P}^N}^{N-1}(N-1)^{\oplus a}\longrightarrow \mathcal{G}(-1)\longrightarrow 0.$$

Notice that $\mathcal{G}$ is Steiner thanks to [**3**, Example 4.5], hence

$$\operatorname{Ext}^1\big(\mathcal{O}_{\mathbb{P}^N}^{\oplus b}(-1),\mathcal{G}(-1)\big)\cong H^1(\mathcal{G})^{\oplus b}=0$$

by Lemma 1.5. It follows that the first sequence above splits, whence $\mathcal{F}\cong \mathcal{O}_{\mathbb{P}^N}^{\oplus b}\oplus\mathcal{G}$. If $\mathcal{F}^\vee(1)$ is regular then

$$c=h^{N-1}(\mathcal{G}(-N-1))=h^{N-1}(\mathcal{F}(-N-1))=h^1(\mathcal{F}^\vee)=0,$$

hence

$$\mathcal{F}\cong \Omega_{\mathbb{P}^N}^{N-1}(N)^{\oplus a}\oplus \mathcal{O}_{\mathbb{P}^N}^{\oplus b}.$$

Conversely, if $\mathcal{F}\cong \Omega_{\mathbb{P}^N}^{N-1}(N)^{\oplus a}\oplus \mathcal{O}_{\mathbb{P}^N}^{b}$, then it is easy to check that $\mathcal{F}$ is Steiner and $\mathcal{F}^\vee(1)$ is regular. $\square$

The above lemma motivates the following example.

EXAMPLE 4.2. Let $2 \le n \le 3$ and assume that the characteristic of $\mathbf{k}$ is 0. If we set
$$\mathcal{F}_{n;a,b} := \Omega^{n+1}_{\mathbb{P}^N}(n+2)^{\oplus a} \oplus \mathcal{O}^{\oplus b}_{\mathbb{P}^{n+2}}$$
thanks to Lemma 4.1 and Proposition 3.3 we can construct a family of smooth varieties $X_{n;a,b} \subseteq \mathbb{P}^{n+2}$ whose sheaf of ideals fits into the exact sequence
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+2}}(-a-an-b)^{\oplus an+2a+b-1} \longrightarrow \mathcal{F}^\vee_{n;a,b}(1-a-an-b) \longrightarrow \mathcal{I}_{X_{n;a,b}} \longrightarrow 0.$$

For each $a \ge 1$ the bundle $\mathcal{N}_{X_{n;a,b}}(-h)$ is h–instanton with quantum number
$$h^1(\mathcal{N}_{X_{n;a,b}}(-2h)) = (an+2a+b-1)a,$$
thanks to Theorem 1.3.

Thanks to Proposition 3.4 we now that $\omega_{X_{n;a,b}}$ is ample and globally generated when $a \ge 2$. It follows that $X_{n;a,b}$ is of general type for $a \ge 2$.

The surface $X_{2;1,0}$ is the isomorphic projection in $\mathbb{P}^4$ of the Veronese surface of $\mathbb{P}^5$. It is well known that $X_{2;1,0}$ supports h–Ulrich bundles of minimal rank two (see [**12**, Proposition 5.9]): notice that $\mathcal{T}_{X_{2;1,0}}$ is one of them (see [**8**]).

The threefold $X_{3;1,0}$ is the Palatini scroll (see [**9**]), which supports h–Ulrich bundles of ranks both one and two (see [**13**, Theorem 0.1 (2)]). Its general hyperplane section is $X_{2;1,1}$: trivially it still supports Ulrich bundles of ranks one and two.

The threefold $X_{3;1,1}$ is such that the linear system $|h+K_{X_{3;1,1}}|$ gives a birational morphism φ onto a cubic threefold $\overline{X} \subseteq \mathbb{P}^4$. Moreover, φ fails to be injective along a curve $C \subseteq \overline{X}$ which has degree 14 and genus 15. We do not know if such threefold $X_{3;1,1}$ actually supports h–Ulrich bundles.

5. Further examples

Theorem 1.3 gives a sufficient condition for $\mathcal{N}_X(-h)$ to be an instanton. We show in the following examples that such a condition is actually not necessary.

EXAMPLE 5.1. Let us consider the smooth threefold whose ideal sheaf fits into
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 9} \longrightarrow \Omega^3_{\mathbb{P}^5} \longrightarrow \mathcal{I}_X \longrightarrow 0. \tag{5.1}$$
In particular X is a $K3$ scroll of degree 9 over the surface $B = \{\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^5) \cap \mathbb{P}^8\}$. Thanks to Lemma 1.6, in order to show that $\mathcal{N}_X(-h)$ is an instanton bundle it is enough to check that $H^j(\mathcal{N}_X(-(j+2)h)) \cong 0$ for $j = 0, 1$. Arguing in the same way as in the proof of Theorem 1.3 one obtains the exact sequence
$$\begin{aligned} &H^j(\mathcal{I}_X(2-j))^{\oplus 9} \longrightarrow H^j(\mathcal{N}_X(-(j+2)h)) \\ &\qquad \longrightarrow H^{j+1}(\Omega^{3\vee}_{\mathbb{P}^5} \otimes \mathcal{I}_X(-j-2)) \longrightarrow H^{j+1}(\mathcal{I}_X(2-j))^{\oplus 9}. \end{aligned}$$
Let us start by computing the cohomology of $\mathcal{I}_X(2-j)$. Twisting (5.1) by $\mathcal{O}_{\mathbb{P}^5}(2-j)$ we get $H^i(\mathcal{I}_X(2-j)) \cong 0$ for all i and for $j = 0, 1$ since both $\mathcal{O}_{\mathbb{P}^5}(-j-2)^{\oplus 9}$ and $\Omega^3_{\mathbb{P}^5}(2-j)$ are acyclic in that range. Now we compute $H^{j+1}(\Omega^{3\vee}_{\mathbb{P}^5} \otimes \mathcal{I}_X(-j-2))$. In order to do so, recall that $\Omega^{3\vee}_{\mathbb{P}^5} \cong \Omega^2_{\mathbb{P}^5}(6)$ and let us consider the Euler sequences
$$0 \longrightarrow \Omega^p_{\mathbb{P}^5}(p) \longrightarrow \mathcal{O}_{\mathbb{P}^5}^{\oplus \binom{6}{p}} \longrightarrow \Omega^{p-1}_{\mathbb{P}^5}(p) \longrightarrow 0 \tag{5.2}$$
for $p = 1, 2$. If we tensor (5.2) by $\mathcal{I}_X(4-p-j)$ we obtain
$$H^{j+1}(\Omega^p_{\mathbb{P}^5} \otimes \mathcal{I}_X(4-j)) \cong H^j(\Omega^{p-1}_{\mathbb{P}^5} \otimes \mathcal{I}_X(4-j)) \cong 0$$

for $1 \le p \le 2$ and $0 \le j \le 1$ since $\mathcal{I}_X(4-p-j)$ is acyclic in this range, thus $\mathcal{N}_X(-h)$ is an instanton bundle. Its quantum number is $k = 6$ and can be computed by (2.2).

It is possible to repeat the same argument to deal with other varieties. E.g. taking into account [**10**, Theorem] and Lemma 4.1, it is quite natural to inspect the case of codimension two arithmetically Buchsbaum threefolds and surfaces. Such varieties have been classified in [**9**], when not of general type.

We summarise what we obtain in the tables 4 and 5, the quantum numbers being computed by (2.2) and (2.3). Notice that cases (a), (e), (2) and (3) can be directly obtained using Theorem 1.3.

	Resolution of $\mathcal{I}_X$	$\mathcal{N}_X(-h)$ **instanton**	**Charge**
(a)	$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 4} \to \Omega^1_{\mathbb{P}^5}(-2) \to \mathcal{I}_X \to 0$	yes	4
(b)	$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 9} \to \Omega^3_{\mathbb{P}^5} \to \mathcal{I}_X \to 0$	yes	6
(c)	$0 \to \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \to \Omega^3_{\mathbb{P}^5}(-1) \to \mathcal{I}_X \to 0$	yes	7
(d)	$0 \to \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 10} \to \Omega^3_{\mathbb{P}^5}(-1) \oplus \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{I}_X \to 0$	yes	5
(e)	$0 \to \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 5} \to \Omega^1_{\mathbb{P}^5}(-3) \oplus \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{I}_X \to 0$	yes	5
(f)	$0 \to \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \to \Omega^1_{\mathbb{P}^5}(-3) \to \mathcal{I}_X \to 0$	??	??

Table 4: Arithmetically Buchsbaum threefolds of non-general type

	Resolution of $\mathcal{I}_X$	$\mathcal{N}_X(-h)$ **instanton**	**Charge**
(1)	$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 5} \to \Omega^2_{\mathbb{P}^4} \to \mathcal{I}_X \to 0$	yes	5
(2)	$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 3} \to \Omega^1_{\mathbb{P}^4}(-1) \to \mathcal{I}_X \to 0$	yes	3
(3)	$0 \to \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 4} \to \Omega^1_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \to \mathcal{I}_X \to 0$	yes	4
(4)	$0 \to \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 6} \to \Omega^2_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \to \mathcal{I}_X \to 0$	??	??
(5)	$0 \to \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-5) \to \Omega^2_{\mathbb{P}^4}(-1) \to \mathcal{I}_X \to 0$	yes	6
(6)	$0 \to \mathcal{O}_{\mathbb{P}^4}(-4)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-5) \to \Omega^1_{\mathbb{P}^4}(-2) \to \mathcal{I}_X \to 0$	yes	3

Table 5: Arithmetically Buchsbaum surfaces of non-general type

In view of the cases (f) and (4) in the table 4 and 5 above, it might be perhaps natural to ask the following question

QUESTION 5.2. *Let X be a codimension two smooth, arithmetically Buchsbaum variety not of general type. Is $\mathcal{N}_X(-h)$ an instanton bundle?*

In the following examples we show that the Question 5.2 has no general answer for varieties of general type.

EXAMPLE 5.3. Consider any non–degenerate complete intersection $X \subseteq \mathbb{P}^{n+2}$ of two hypersurfaces the bundle $\mathcal{N}_X(-h)$. It is easy to check that $h^0(\mathcal{N}_X(-2h)) \neq 0$, hence $\mathcal{N}_X(-h)$ cannot be an h–instanton.

EXAMPLE 5.4. Let the characteristic of $\mathbf{k}$ be 0. Thanks to Proposition 3.3 we can consider the smooth surface in $\mathbb{P}^4$ whose ideal sheaf admits the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-7) \longrightarrow \Omega^1_{\mathbb{P}^4}(-3) \longrightarrow \mathcal{I}_X \longrightarrow 0. \tag{5.3}$$

If we apply the functor $\mathcal{H}om_{\mathbb{P}^4}(-,\omega_{\mathbb{P}^4})$ to the sequence above glued with (3.2) as in Example 3.7, we deduce that X is of general type. Let us verify that $\mathcal{N}_X(-h)$ is not an instanton, by showing that $h^0(\mathcal{N}_X(-2h)) > 0$. Let us apply the contravariant functor $\mathrm{Hom}(-,\mathcal{I}_X(-2))$ to (5.3) obtaining the exact sequence

$$\begin{aligned} H^0(\Omega^3_{\mathbb{P}^4}(6)\otimes\mathcal{I}_X) \longrightarrow & H^0(\mathcal{I}_X(3))^{\oplus 2} \oplus H^0(\mathcal{I}_X(5)) \longrightarrow H^0(\mathcal{N}_X(-2h)) \\ & \longrightarrow H^1(\Omega^3_{\mathbb{P}^4}(6)\otimes\mathcal{I}_X) \longrightarrow H^1(\mathcal{I}_X(3))^{\oplus 2} \oplus H^1(\mathcal{I}_X(5)). \end{aligned}$$

We start by computing $h^0(\Omega^3_{\mathbb{P}^4}(6)\otimes\mathcal{I}_X)$. If we tensor (5.2) by $\mathcal{I}_X(3)$ we get

$$h^0(\Omega^3_{\mathbb{P}^4}(6)\otimes\mathcal{I}_X) \le 10h^0(\mathcal{I}_X(3)).$$

Thus, taking the cohomology of (5.3) twisted by $\mathcal{O}_{\mathbb{P}^4}(3)$ we obtain $h^0(\mathcal{I}_X(3)) = 0$, hence $h^0(\Omega^3_{\mathbb{P}^4}(6)\otimes\mathcal{I}_X) = 0$. Tensoring sequence (5.3) by $\mathcal{O}_{\mathbb{P}^4}(5)$ one can directly obtain $h^0(\mathcal{I}_X(5)) = 8$, thus $h^0(\mathcal{N}_X(-2h)) > 8$ and $\mathcal{N}_X(-h)$ is not an instanton bundle.

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