On the Regularization of the High Order Time Domain EFIE with Quasi-Helmholtz Projectors

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On the Regularization of the High Order Time Domain EFIE with Quasi-Helmholtz Projectors

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***Abstract*—In this work, we present a quasi-Helmholtz projector-based preconditioning and stabilization of the time domain electric field integral equation when discretized with high-order basis functions and high-order convolution quadrature schemes. The proposed approach effectively cures the ill-conditioning at large time steps that the original scheme suffers from along with the late time instabilities caused by the static solenoidal nullspace. This stabilization is achieved through a generalization of the quasi-Helmholtz projectors adapted to the high-order space basis functions and the high-order implicit schemes used by the convolution quadrature method.**

I. Introduction

High-order basis functions coupled with integral formulations are increasingly used for modelling electromagnetic radiation and scattering from perfectly electrically conducting (PEC) objects in time domain. Among the different formulations, the time-domain electric field integral equation (TD-EFIE) is one of the most widespread due to its ability to accurately model time-domain PEC scattering problems in the absence of spurious resonances. When discretized in time via the convolution quadrature method [1], this formulation provides an efficient time-stepping scheme that is solved via the Marching-On-In-Time (MOT) scheme. Unfortunately, the discretization of the EFIE yields ill-conditioned matrix systems when the time steps are large (which is know as the low-frequency or large time step breakdown), in addition to being plagued by the solenoidal static nullspace responsible for the so-called direct current instability (DC instability). This work addresses all of the above-mentioned limitations by extending and adapting the quasi-Helmholtz projectors stabilization proposed in [2] for RWG basis functions, to higher-order basis functions using the newly introduced quasi-Helmholtz projectors defined in [3]. The resulting formulation is compatible with multi-stage implicit Runge-Kutta methods used in the convolution quadrature method and provides DC-stable solutions and well-conditioned systems.

II. Background and notations

This work addresses the problem of the electromagnetic scattering from a PEC object of surface boundary Γ illuminated by an electromagnetic field $(\boldsymbol{e}^{\mathrm{inc}}, \boldsymbol{h}^{\mathrm{inc}})(\boldsymbol{r}, t)$. The incident field induces a surface current $\boldsymbol{j}_\Gamma$ on Γ which can be computed by solving the TD-EFIE

$$\eta_0 \mathcal{T}(\boldsymbol{j}_\Gamma)(\boldsymbol{r}, t) = -\hat{\boldsymbol{n}}(\boldsymbol{r}) \times \boldsymbol{e}^{\mathrm{inc}}(\boldsymbol{r}, t), \quad \forall (\boldsymbol{r}, t) \in \Gamma \times \mathbb{R}, \quad (1)$$

where $\hat{\boldsymbol{n}}$ is the outpointing normal vector of Γ and η_0 is the characteristic impedance of the background. The TD-EFIE operator $\mathcal{T}$ reads

$$\mathcal{T}(f)(\boldsymbol{r}, t) = -\frac{1}{c_0}\frac{\partial}{\partial t}\mathcal{T}_{\mathrm{s}}(f)(\boldsymbol{r}, t) + c_0 \int_{-\infty}^{t} \mathcal{T}_{\mathrm{h}}(f)(\boldsymbol{r}, t')dt', \quad (2)$$

where c_0 is the speed of light in the background, and $\mathcal{T}_{\mathrm{s}}$ and $\mathcal{T}_{\mathrm{h}}$ are respectively the vector and scalar potentials [2].

In contrast with previous works, we consider a spatial discretization of the TD-EFIE operator using high-order div-conforming GWP basis functions [4]. We denote by $\left\{\boldsymbol{\psi}_n^{(q)}\right\}_{N_s}$ the set of the N_s GWP basis functions of order q. The convolution quadrature method consists in discretizing the time to solve (1) via the following MOT approach

$$\forall i \in \mathbb{N}, \quad \boldsymbol{T}_0 \boldsymbol{j}_i = \boldsymbol{e}_i^{\mathrm{inc}} - \sum_{j=1}^{i} \boldsymbol{T}_j \boldsymbol{j}_{i-j}\,, \quad (3)$$

where $\boldsymbol{j}_i$ is the array sequence expansion of the surface current on GWP at any time step, $\boldsymbol{e}_i^{\mathrm{inc}}$ is the array sequence of $-\eta_0^{-1}\hat{\boldsymbol{n}}\times \boldsymbol{e}^{\mathrm{inc}}$ tested with the rotated GWP basis functions and

$$\begin{aligned}
[\boldsymbol{T}_{\mathcal{L}}(s)]_{m,n} &= \int_\Gamma \hat{\boldsymbol{n}} \times \boldsymbol{\psi}_m^{(q)} \cdot \mathcal{L}\left(\mathcal{T}\left(\boldsymbol{\psi}_n^{(q)}\delta\right)\right)(s) d\Gamma\,, \\
\boldsymbol{T}_i &= \mathcal{Z}^{-1}\left(z \mapsto \boldsymbol{T}_{\mathcal{L}}\left(\mathbf{s}(z)\right)\right)_i\,,
\end{aligned} \quad (4)$$

where $\mathcal{L}$ is the Laplace transform, δ is the time Dirac delta, $\mathcal{Z}^{-1}$ is the inverse $\mathcal{Z}$-transform, and the matrix $\mathbf{s}(z)$ is determined by the Runge-Kutta method (in this paper, the Radeau IIA of stage $p = 2$) and the time step size Δt [2].

III. Time domain strategy

Although it accurately models the described electromagnetic problem and offers possibilities of very high-quality discretizations, this formulation is however limited by at least the following three drawbacks: (i) the EFIE operator $\mathcal{T}$ involves an unbounded number of non-vanishing matrices $\boldsymbol{T}_i$, rendering the overall computational complexity to quadratic with the number of time steps, (ii) the solution is only determined up to a constant solenoidal current because of the continuous nullspace of the EFIE operator generating DC-instability, and (iii) the matrix $\boldsymbol{T}_0$ becomes increasingly ill-conditioned when the time step size Δt is large, deteriorating the solution accuracy. All these limitations can be cured by time differentiation and time integration of the operator solenoidal and non-solenoidal components in a judicious recombination procedure

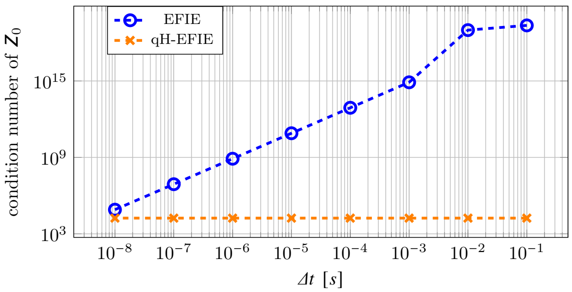

Fig. 1. Condition number as a function of the time step ($Ns = 1260$ and $q = 2$).

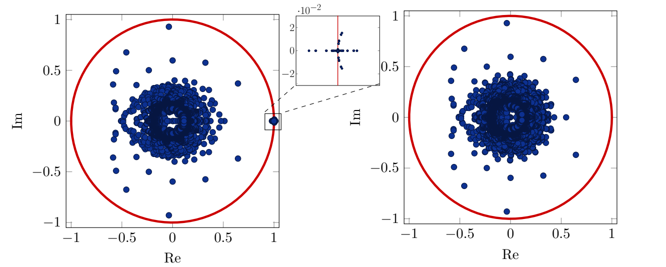

Fig. 2. Polynomial eigenvalue analyses of the TD EFIE operator (left) and the regularized one (right) with $\Delta t = 2\,\text{ns}$, $q = 2$ and $N_s = 840$. In this case, 722 eigenvalues are clustered around 1 (here, p times the number of loops).

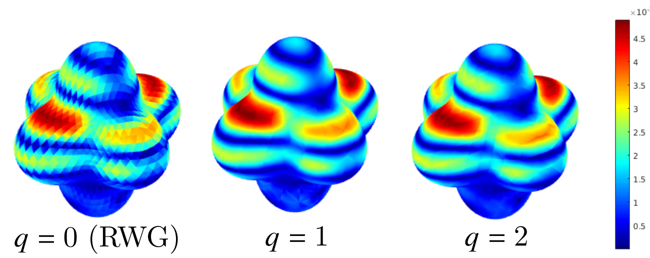

Fig. 3. Current intensity in Am^{-1} at $t = 0$ on a metallic scatterer surface. All simulations have a fixed number of space basis functions $N_s = 4200$, but different cell sizes and basis function orders.

relying on the quasi-Helmholtz projectors [2]. Using the high-order quasi-Helmholtz projectors on the non-solenoidal and solenoidal subspaces, respectively denoted by $\boldsymbol{P}_q^{\Sigma}$ and $\boldsymbol{P}_q^{\Lambda H}$ [3], and the regularization strategy proposed in [2], the space discretized Laplace regularized operator is in the form

$$\boldsymbol{T}_{\mathcal{L}}^{\text{reg}} = \left(\frac{c_0}{sa}\boldsymbol{P}_q^{\Lambda H} + \boldsymbol{P}_q^{\Sigma}\right)\boldsymbol{T}_{\mathcal{L}}\left(\boldsymbol{P}_q^{\Lambda H} + \frac{sa}{c_0}\boldsymbol{P}_q^{\Sigma}\right), \tag{5}$$

where a is the maximum length of the scatterer. In addition to providing a well-conditioned system, the associated regularized continuous operator is not affected by static nullspace, which cures the DC instability. The MOT generated by this regularization reads

$$\begin{aligned} \boldsymbol{T}_0^{\text{reg}}\boldsymbol{y}_i &= \frac{c_0}{a}\boldsymbol{P}_q^{\Lambda H}\boldsymbol{e}_i^{\text{Prim}} + \boldsymbol{P}_q^{\Sigma}\boldsymbol{e}_i^{\text{inc}} - \sum_{j=1}^{N_{\text{conv}}}\boldsymbol{T}_j^{\text{reg}}\boldsymbol{y}_{i-j}\,, \\ \boldsymbol{j}_i &= \boldsymbol{P}_q^{\Lambda H}\boldsymbol{y}_i + \boldsymbol{P}_q^{\Sigma}\left(\boldsymbol{Z}_1^s\boldsymbol{y}_i + \boldsymbol{Z}_2^s\boldsymbol{y}_{i-1}\right), \end{aligned} \tag{6}$$

where $\boldsymbol{e}^{\text{Prim}}$ is the array of $-\eta_0^{-1}\hat{\mathbf{n}}\times\int_{-\infty}^{t}\boldsymbol{e}^{\text{inc}}$ tested with rotated GWP basis functions at different time steps, $\boldsymbol{Z}_1^s$ and $\boldsymbol{Z}_2^s$ can be analytically determined and derived from a discretized time differentiated operator [2], and

$$\boldsymbol{T}_i^{\text{reg}} = \mathcal{Z}^{-1}\left(z \mapsto \boldsymbol{T}_{\mathcal{L}}^{\text{reg}}\left(\mathrm{s}\left(z\right)\right)\right)_i. \tag{7}$$

Finally, the integration matrix sequence $\left\{\boldsymbol{T}_i^{\text{reg}}\right\}_i$ converges exponentially to $\mathbf{0}$ as soon as $i > \frac{a}{\Delta t c_0}$. Therefore, the right-hand side summation in (6) can be truncated to N_{conv} elements reducing the MOT computational cost linearly with the number of time steps N_t.

IV. Numerical Results

The proposed formulation (6) is tested and validated on a discretized sphere. First, the condition number of $\boldsymbol{T}_0$ and $\boldsymbol{T}_0^{\text{reg}}$—the matrices responsible for the conditioning of (3) and (6)—are evaluated as a function of the time step and displayed in Fig 1. As expected, the regularized operator yields well-conditioned matrices, while the standard formulation yields a condition number that grows as the time step increases. Next, the polynomial eigenvalue analyses of both discretized operators are shown in Fig 2. The regularized operator has all its eigenvalues inside the unit circle in the complex plane revealing a stable operator, whereas the EFIE operator suffers from DC instability and has some eigenvalues that cluster around 1. Finally, the MOT solutions at $t = 0$ of 3 different space basis orders are shown in Fig 3, for a metallic scatterer illuminated by a modulated Gaussian plane wave

$$\boldsymbol{e}^{\text{inc}}(\boldsymbol{r}, t) = \cos(2\pi f_0 \tau(\boldsymbol{r}, t))\exp\left(-\frac{\tau(\boldsymbol{r}, t)^2}{2\sigma_0^2}\right)\hat{\boldsymbol{x}}\,, \tag{8}$$

where $\tau(\boldsymbol{r}, t) = t + \frac{\hat{\boldsymbol{z}}\cdot\boldsymbol{r}}{c_0}$, $\sigma_0 = 6/(2\pi f_0)$ and $f_0 = 200\,\text{MHz}$. In this experiment, the number of space basis functions is maintained constant ($N_s = 4200$), imposing larger cell sizes for the simulations with higher-order basis functions.

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