On a Novel Calderón Preconditioning Strategy Based on High-Order Quasi-Helmholtz Projectors

On a Novel Calderón Preconditioning Strategy Based on High-Order Quasi-Helmholtz Projectors

Johann Bourhis[(1)], Damiano Franzò[(1)], Adrien Merlini[(2)], and Francesco P. Andriulli[(1)]

[(1)] Politecnico di Torino, Turin, Italy
[(2)] IMT Atlantique, Brest, France

***Abstract*—The first of Calderón identities indicates that the operator of the electric field integral equation (EFIE) can potentially act as a preconditioner for itself. However, the discretization of this preconditioning operator via the boundary element method (BEM) requires constructing a dual space commonly defined on a barycentric refinement of the orginal mesh. This contribution introduces a novel Calderón strategy that leverages high-order quasi-Helmholtz projectors. Unlike the standard strategies, which are based on barycentric refinements, a dual space is obtained by combinating functions of higher order. This strategy allows the development of a discretization of the Calderón identity that is suitable for arbitrary order. The well-conditioning of the proposed formulation is investigated, and numerical results are presented to corroborate the theory.**

***Index Terms*—Electric field integral equation (EFIE), boundary element method (BEM), high-order, Calderón identity, quasi-Helmholtz projectors**

I. Introduction

The electric field integral equation (EFIE) allows the modeling of scattering and radiation problems from perfectly electrically conducting (PEC) objects [1]. Its discretization via the boundary element method (BEM) leads to a matrix system to be solved for finding the numerical solution's underlying expansion coefficients. In that context, a high-order discretization strategy [2] can be pursued to generate faster convergence rates to the physical solution with respect to the size of the system. Concurrently, ensuring a low condition number is desirable feature to ensure numerical accuracy and stability of the process and reduce the number of iterations when using iterative solvers [3]. Unfortunately, matrices arising from the discretization of the EFIE are known to suffer from h-refinement and low-frequency breakdowns. For these reasons, preconditioning the EFIE has been a widely investigated topic that has received considerable attention over the past decades (see [3] and references therein). Among the different possible strategies, this work focuses on techniques based on quasi-Helmholtz decompositions [3], [4] and on the Calderón identities [5]–[7]. Some relationships between these two approaches have been explored and most require the construction of a dual space [7], [8] for the discretization of the Calderón identity.

In this contribution, we propose a different strategy using a normalized version of the high-order quasi-Helmholtz projectors [4] to effectively construct the dual space. Similarly to what was done in [6], in the case of barycentric refinement, the solenoidal and non-solenoidal contributions of the GWP functions of order p are rotated and respectively mapped onto the non-solenoidal and solenoidal contributions of the GWP functions of order higher than p. Numerical experiments on canonical geometries showcase the stable condition numbers of the new formulation in the h-refinement and low-frequency regimes.

II. Background and notation

Consider a closed and simply connected triangular mesh of E edges, C cells, and V vertices modeling the boundary surface Γ of a PEC object. Given the incident electric field $\boldsymbol{E}^{\mathrm{inc}}$ that impinges on Γ, the EFIE allows finding the induced surface current density $\boldsymbol{J}$ and reads $\mathcal{T}\boldsymbol{J} = -\eta^{-1}\widehat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{inc}}$, with $\mathcal{T} = jk\,\mathcal{T}_s - (jk)^{-1}\,\mathcal{T}_h$, where

$$\left(\mathcal{T}_s\boldsymbol{J}\right)(\boldsymbol{r}) = \widehat{\boldsymbol{n}} \times \int_\Gamma G(\boldsymbol{r},\boldsymbol{r'})\boldsymbol{J}(\boldsymbol{r'})\,\mathrm{d}S(\boldsymbol{r'})\,, \tag{1}$$

$$\left(\mathcal{T}_h\boldsymbol{J}\right)(\boldsymbol{r}) = \widehat{\boldsymbol{n}} \times \mathbf{grad}_\Gamma \int_\Gamma G(\boldsymbol{r},\boldsymbol{r'})\mathrm{div}_\Gamma\boldsymbol{J}(\boldsymbol{r'})\,\mathrm{d}S(\boldsymbol{r'})\,, \tag{2}$$

and where $\widehat{\boldsymbol{n}}$ is the outgoing normal vector to Γ, η is the characteristic impedance of the medium, $\mathbf{grad}_\Gamma$, $\mathbf{curl}_\Gamma := \widehat{\boldsymbol{n}} \times \mathbf{grad}_\Gamma$, and div_Γ are the surface gradient, surface curl and surface divergence operators, respectively, and $G(\boldsymbol{r},\boldsymbol{r'})$ is the Green's function. We define $\langle \boldsymbol{f}, \boldsymbol{g}\rangle = \int_\Gamma \boldsymbol{f}(\boldsymbol{r})\cdot\boldsymbol{g}(\boldsymbol{r})\,\mathrm{d}S(\boldsymbol{r})$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are scalar or vector functions, as well as the associated Gram matrix between two sets of (scalar or vector) functions $\{\boldsymbol{f}_i\}_{i=1}^I$ and $\{\boldsymbol{g}_j\}_{j=1}^J$ denoted by $\mathbf{G}_{\boldsymbol{f},\boldsymbol{g}} \in \mathbb{R}^{I\times J}$ with components $[\mathbf{G}_{\boldsymbol{f},\boldsymbol{g}}]_{ij} = \langle \boldsymbol{f}_i, \boldsymbol{g}_j\rangle$.

The EFIE is then discretized by employing GWP div-conforming basis functions within a Petrov-Galerkin procedure. Denoting $\{\boldsymbol{\psi}_i^{(p)}\}_{i=1}^{N_p}$ the set of GWP basis functions of order p, with $N_p = p(p+1)C + (p+1)E$, the resulting matrix system of size N_p reads $\mathbf{T}_p\boldsymbol{j}_p = \boldsymbol{e}_p$, with $\mathbf{T}_p = jk\mathbf{T}_{p|s} + (jk)^{-1}\mathbf{T}_{p|h}$, $[\mathbf{T}_{p|s}]_{ij} = \langle \widehat{\boldsymbol{n}} \times \boldsymbol{\psi}_i^{(p)}, \mathcal{T}_s\boldsymbol{\psi}_j^{(p)}\rangle$, $[\mathbf{T}_{p|h}]_{ij} = -\langle \widehat{\boldsymbol{n}} \times \boldsymbol{\psi}_i^{(p)}, \mathcal{T}_h\boldsymbol{\psi}_j^{(p)}\rangle$, and $[\boldsymbol{e}]_i = -\eta^{-1}\langle \widehat{\boldsymbol{n}} \times \boldsymbol{\psi}_i, \widehat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{inc}}\rangle$. We finally consider a normalization of the EFIE that consists in orthonormalizing the basis, which become $\widetilde{\mathbf{T}}_p\widetilde{\boldsymbol{j}}_p = \widetilde{\boldsymbol{e}}_p$, with $\widetilde{\mathbf{T}}_p = jk\widetilde{\mathbf{T}}_{p|s} + (jk)^{-1}\widetilde{\mathbf{T}}_{p|h}$, and where $\widetilde{\mathbf{T}}_{p|s} = \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}\mathbf{T}_{p|s}\mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}$, $\widetilde{\mathbf{T}}_{p|h} = \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}\mathbf{T}_{p|h}\mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}$, $\widetilde{\boldsymbol{j}}_p = \mathbf{G}^{1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}\boldsymbol{j}_p$, and $\widetilde{\boldsymbol{e}}_p = \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}\boldsymbol{e}_p$.

III. Normalized High-Order Quasi-Helmholtz Projectors

Consider the two finite element spaces of order $p+1$ continuous and order p discontinuous scalar functions, to

DOI: 10.1109/ICEAA61917.2024.10701907

which we respectively associate the sets of basis functions $\{\lambda_i^{(p+1)}\}_{i=1}^{L_p}$ and $\{\sigma_i^{(p)}\}_{i=1}^{S_p}$, with $L_p = p(p-1)C/2+pE+V$ and $S_p = (p+1)(p+2)C/2$. These basis functions can be, for instance, based on Lagrange interpolatory functions [2]. This allows defining the normalized non-solenoidal (Star) transformation matrix [4]

$$\widetilde{\mathbf{\Sigma}}_p = \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}} \mathbf{G}_{\mathrm{div}_\Gamma \boldsymbol{\psi}^{(p)},\sigma^{(p)}} \mathbf{G}^{-1/2}_{\sigma^{(p)},\sigma^{(p)}}\,, \tag{3}$$

and the normalized solenoidal (Loop) transformation matrix

$$\widetilde{\mathbf{\Lambda}}_p = \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}} \mathbf{G}_{\boldsymbol{\psi}^{(p)},\mathbf{curl}_\Gamma \lambda^{(p+1)}} \mathbf{G}^{-1/2}_{\lambda^{(p+1)},\lambda^{(p+1)}}\,. \tag{4}$$

We then define the normalized quasi-Helmholtz projectors $\mathbf{P}^{\widetilde{\Sigma}}_p = \widetilde{\mathbf{\Sigma}}_p\big(\widetilde{\mathbf{\Sigma}}_p^\mathsf{T}\widetilde{\mathbf{\Sigma}}_p\big)^+\widetilde{\mathbf{\Sigma}}_p^\mathsf{T}$, $\mathbf{P}^{\widetilde{\Lambda}}_p = \widetilde{\mathbf{\Lambda}}_p\big(\widetilde{\mathbf{\Lambda}}_p^\mathsf{T}\widetilde{\mathbf{\Lambda}}_p\big)^+\widetilde{\mathbf{\Lambda}}_p^\mathsf{T}$, where "+" denotes the Moore-Penrose pseudo-inverse. By construction, these projectors are orthogonal to each other, yielding $\mathbf{P}^{\widetilde{\Lambda}}_p\mathbf{P}^{\widetilde{\Sigma}}_p = \mathbf{0}$, and, in conjunction, the following property holds

$$\mathbf{P}^{\widetilde{\Lambda}}_p\widetilde{\mathbf{T}}_{p|h} = \widetilde{\mathbf{T}}_{p|h}\mathbf{P}^{\widetilde{\Lambda}}_p = \mathbf{0}\,. \tag{5}$$

IV. Preconditioning the EFIE with the Novel Calderón Strategy

In this work, we exploit the Calderón identity $\mathcal{T}^2 = -\frac{1}{4}\mathcal{I} + \mathcal{K}^2$, where $\mathcal{I}$ is the identity operator and $\mathcal{K}$ is the compact operator defined in [5]. This is an integral operator of the second kind that is known to have a bounded spectrum. Nevertheless, the discretization of the Calderón identity requires the definition of a space that is dual to the GWP space of order p.

Leveraging the normalized high-order quasi-Helmholtz projectors, we first build a transformation matrix that provides the desired mapping of GWP functions of order $r > p$ to a space that is dual to the GWP functions of order p. On the one hand, the non-solenoidal functions of order p are rotated and then mapped onto the solenoidal functions of order r via the transformation matrix

$$\widetilde{\mathbf{\Theta}}^{\widetilde{\Lambda}}_{r,p} = \mathbf{P}^{\widetilde{\Lambda}}_r\widetilde{\mathbf{G}}^{\mathrm{mix}}_{r,p}\mathbf{P}^{\widetilde{\Sigma}}_p\,. \tag{6}$$

with the normalized mixed-Gram matrix

$$\widetilde{\mathbf{G}}^{\mathrm{mix}}_{r,p} = \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(r)},\boldsymbol{\psi}^{(r)}} \mathbf{G}_{\boldsymbol{\psi}^{(r)},\hat{\boldsymbol{n}}\times\boldsymbol{\psi}^{(p)}} \mathbf{G}^{-1/2}_{\boldsymbol{\psi}^{(p)},\boldsymbol{\psi}^{(p)}}\,. \tag{7}$$

On the other hand, the solenoidal functions of order p are rotated and then mapped onto the GWP functions of order r via the transformation matrix

$$\widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p} = \widetilde{\mathbf{G}}^{\mathrm{mix}}_{r,p}\mathbf{P}^{\widetilde{\Lambda}}_p\,. \tag{8}$$

Moreover, the following property holds

$$\mathbf{P}^{\widetilde{\Lambda}}_r\widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p} = \mathbf{0}\,, \tag{9}$$

which implies that $\widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p}$ actually maps the rotated solenoidal functions of order p onto the non-solenoidal functions of order r. This also implies that

$$\left(\widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p}\right)^\mathsf{T}\widetilde{\mathbf{\Theta}}^{\widetilde{\Lambda}}_{r,p} = \mathbf{0}\,. \tag{10}$$

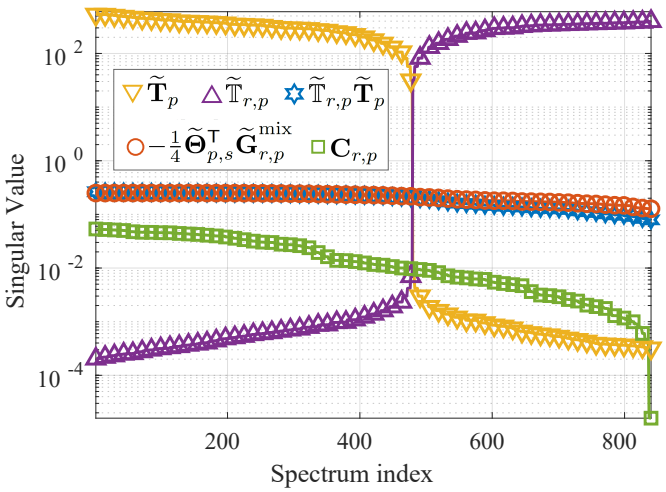

Fig. 1. Spectra of the matrices in (12). Results are obtained for the simulation of the unit sphere at 10^6 Hz with $p = 2$ and $h = 0.30$ m.

After defining the mapping matrix

$$\widetilde{\mathbf{\Theta}}_{r,p} = \widetilde{\mathbf{\Theta}}^{\widetilde{\Lambda}}_{r,p} + \widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p}\,, \tag{11}$$

we propose the following preconditioned formulation

$$\widetilde{\mathbb{T}}_{r,p}\widetilde{\mathbf{T}}_p\widetilde{\boldsymbol{j}}_p = \widetilde{\mathbb{T}}_{r,p}\widetilde{\boldsymbol{e}}_p\,, \tag{12}$$

with the preconditioner

$$\widetilde{\mathbb{T}}_{r,p} = \widetilde{\mathbf{\Theta}}^{\mathsf{T}}_{r,p}\widetilde{\mathbf{T}}_r\widetilde{\mathbf{\Theta}}_{r,p}\,. \tag{13}$$

Furthermore, we establish from (5) that

$$\widetilde{\mathbf{T}}_{r|h}\widetilde{\mathbf{\Theta}}^{\widetilde{\Lambda}}_{r,p} = \mathbf{0}\,, \tag{14}$$

$$\widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p}\widetilde{\mathbf{T}}_{p|h} = \mathbf{0}\,. \tag{15}$$

Consequently, by developing (12) and setting (14) and (15) we get

$$\begin{aligned}\widetilde{\mathbb{T}}_{r,p}\widetilde{\mathbf{T}}_p = \widetilde{\mathbf{\Theta}}^{\mathsf{T}}_{r,p}\widetilde{\mathbf{T}}_{r|s}\widetilde{\mathbf{\Theta}}^{\widetilde{\Lambda}}_{r,p}\widetilde{\mathbf{T}}_{p|h} + \widetilde{\mathbf{\Theta}}^{\mathsf{T}}_{r,p}\widetilde{\mathbf{T}}_{r|h}\widetilde{\mathbf{\Theta}}^{\widetilde{\Sigma}}_{r,p}\widetilde{\mathbf{T}}_{p|s} \\ - k^2\widetilde{\mathbf{\Theta}}^{\mathsf{T}}_{r,p}\widetilde{\mathbf{T}}_{r|s}\widetilde{\mathbf{\Theta}}_{r,p}\widetilde{\mathbf{T}}_{p|s}\end{aligned} \tag{16}$$

which indicates that the proposed formulation does not suffer from low-frequency breakdown on simply connected geometries. Furthermore, numerical experiments (a subset of which are presented in Section V) show that choosing $r = p + 2$ leads to a stable discretization.

V. Numerical Results

To illustrate the effects of the preconditioning on the spectrum of the EFIE, we display in Fig. 1 the singular values of the matrices involved in (12). The experiment is done for simulation of the unit sphere at 1 MHz with $p = 2$ and average cell diameter $h = 0.30$ m. We firstly observe that the EFIE matrices $\mathbf{T}_p$ and $\mathbb{T}_{r,p}$ exhibit analogous spectra characterized by an interchange between solenoidal (small singular values) and non-solenoidal (high singular values) contributions. Secondly, the preconditioned matrix $\mathbb{T}_{r,p}\mathbf{T}_p$ has a spectrum similar to that of $-\frac{1}{4}\widetilde{\mathbf{\Theta}}^{\mathsf{T}}_{r,p}\widetilde{\mathbf{G}}^{\mathrm{mix}}_{r,p}$, while the singular

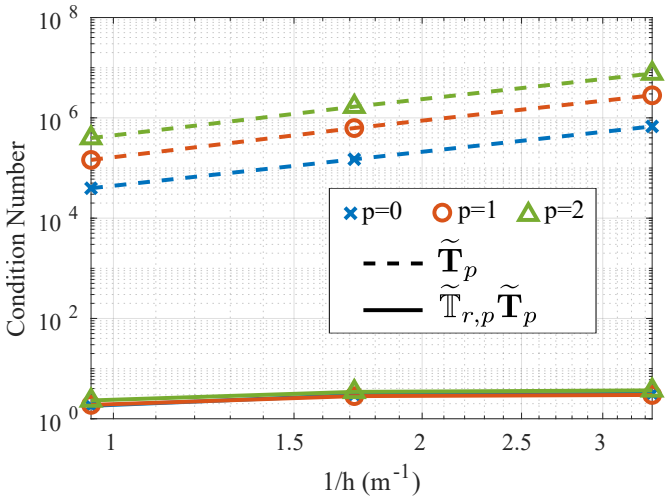

Fig. 2. Condition number as a function of the average cell diameter inverse. Results are obtained for the simulation of the unit sphere at 10^6 Hz.

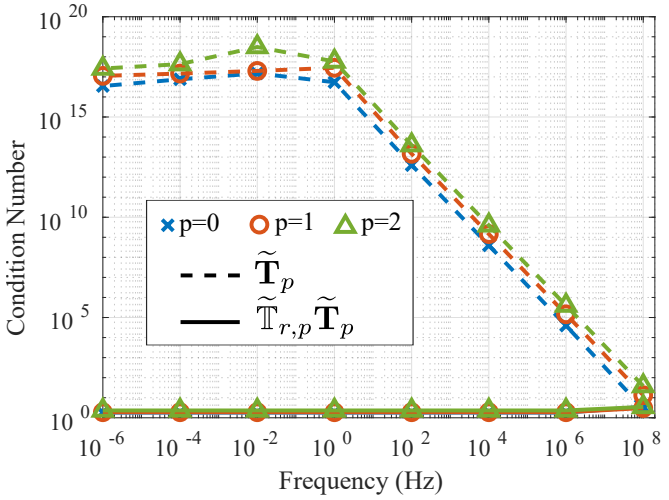

Fig. 3. Condition number as a function of the frequency. Results are obtained for the simulation of the unit sphere with $h = 0.30$ m.

values of the remainder $\mathbf{C}_{r,p} := \mathbb{T}_{r,p}\mathbf{T}_p + \frac{1}{4}\widetilde{\mathbf{\Theta}}^{\mathsf{T}}_{r,p}\widetilde{\mathbf{G}}^{\text{mix}}_{r,p}$ are sensibly smaller.

In addition, we computed the condition number as a function of h in Fig. 2 (with frequency $f = 1$ MHz), and as a function of the frequency f in Fig. 3 (with $h = 0.30$ m). In both experiments, we observe that the preconditioned matrix has a condition number that is almost constant, while the original matrix has a condition number that grows quadratically as h and as f tend towards zero.

Finally, we show the performance of the proposed strategy in terms of the number of iterations carried out by iterative solvers for finding the solution of the matrix system. For this study, we employ the transpose free quasi-minimal residual (TFQMR) solver [9]. Fig. 4 displays the residual norm as a function of the number of iterations when simulating at 3 GHz a 0.18 m section of coil of 55 mm diameter that is meshed with triangles of average diameter $h = 20$ mm.

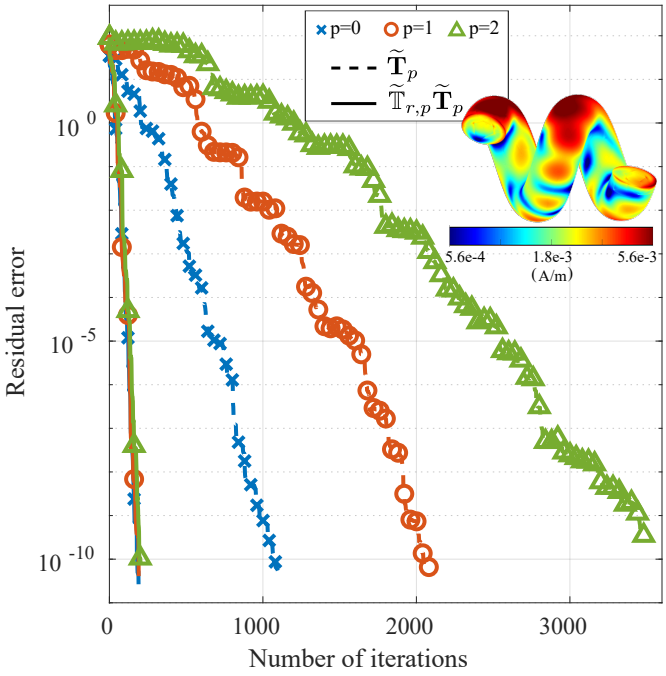

Fig. 4. Residual error of the iterates as a function of the number of iterations of TFQRM and surface current density (in A/m) obtained for $p = 2$.

When leveraging the new strategy, the number of iterations of TFQMR is substantially lower than when solving the original formulation and varies little with regards to the order and size of the system.

Acknowledgment

The work of this paper has received funding from the EU H2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement n° 955476 (project COMPETE), from the European Innovation Council (EIC) through the European Union's Horizon Europe research Programme under Grant 101046748 (Project CEREBRO), and from the Italian Ministry of University and Research within the Program FARE, CELER, under Grant R187PMFXA4.

References

[1] W. C. Gibson, *The method of moments in Electromagnetics*. Champan & Hall/CRC Press, 2021.

[2] R. Graglia, D. Wilton, and A. Peterson, "Higher order interpolatory vector bases for computational electromagnetics," *IEEE Transactions on Antennas and Propagation*, vol. 45, no. 3, pp. 329–342, Mar. 1997.

[3] S. B. Adrian, A. Dély, D. Consoli, A. Merlini, and F. P. Andriulli, "Electromagnetic Integral Equations: Insights in Conditioning and Preconditioning," *IEEE Open Journal of Antennas and Propagation*, vol. 2, pp. 1143–1174, 2021.

[4] J. Bourhis, A. Merlini, and F. P. Andriulli, "High-Order Quasi-Helmholtz Projectors: Definition, Analyses, Algorithms," *IEEE Transactions on Antennas and Propagation*, vol. 72, no. 4, pp. 3572–3579, 2024.

[5] F. P. Andriulli, K. Cools, H. Bagci, F. Olyslager, A. Buffa, S. Christiansen, and E. Michielssen, "A Multiplicative Calderón Preconditioner for the Electric Field Integral Equation," *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 8, pp. 2398–2412, 2008.

[6] F. Valdés, F. P. Andriulli, K. Cools, and E. Michielssen, "High-order Div- and Quasi Curl-Conforming Basis Functions for Calderón Multiplicative Preconditioning of the EFIE," *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 4, pp. 1321–1337, Apr. 2011.

[7] S. H. Christiansen and J.-C. Nédélec, "A Preconditioner for the Electric Field Integral Equation Based on Calderon Formulas," *SIAM Journal on Numerical Analysis*, vol. 40, no. 3, pp. 1100–1135, 2003.
[8] A. Buffa and S. H. Christiansen, "A dual finite element complex on the barycentric refinement," *Comptes Rendus Mathematique*, vol. 340, no. 6, pp. 461–464, 2005.
[9] R. W. Freund, "A Transpose-Free Quasi-Minimal Residual Algorithm for Non-Hermitian Linear Systems," *SIAM Journal on Scientific Computing*, vol. 14, no. 2, pp. 470–482, 1993.