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Novel class of susceptible–infectious–recovered models involving power-law interactions

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ABSTRACT

It is shown that the ordinary SIR (Susceptible–Infectious–Recovered) epidemic model exhibits features that are common to a class of compartmental models with power-law interactions. Within this class of theoretical models, the standard SIR model emerges as a singular non-integrable model. Various integrable models, whose solutions are defined explicitly or implicitly in terms of elementary functions, are discovered within the same class. A Hamiltonian dynamics with position-depending forces underlies a sub-class of these models. The general class of models is very flexible and capable of describing epidemics characterized by a finite or indefinite lifespan. In the last case, the compartment population distributions evolve in time exhibiting exponential or power-law tails.

1. Introduction

The mathematical description of the spread of an infectious disease within a population is a long standing. The ordinary SIR (Susceptible–Infectious–Recovered) Kermack–McKendrick model [1] dates back to about a century ago and represents the paradigm of describing interactions in compartmental population dynamics. This model, due to its simplicity and effectiveness, has continued to attract attention without ever setting and has made a comeback in recent decades [2–8], not only because of the appearance and spread of various epidemics and/or pandemics [9–12], but also due to its extreme versatility in analyzing new phenomenologies in our globalized and highly interconnected socio-economic world. The same SIR model, originally proposed to describe infectious diseases, now appears to be a highly interdisciplinary tool that is particularly suitable for describing the concept of the spreading of political, social, religious or scientific ideas, of social consensus, of fake news, of computer viruses, and of commercial, economic or financial products [13–16]. This interest in the ordinary SIR model has also inexorably highlighted, together with some of its mathematical limitations, its enormous potentiality for possible generalizations and extensions [17–25], especially for the purpose of describing old or emerging phenomenologies more clearly [26–31].

Let us consider the population of N individuals and indicate the fractions of Susceptible, Infectious and Recovered individuals respectively, with $x(t) = S(t)/N$, $y(t) = I(t)/N$ and $z(t) = R(t)/N$, so that $x(t) + y(t) + z(t) = 1$. According to the ordinary SIR model, which does not account for demography (births and deaths), the $0 < x, y, z < 1$ distributions obey the following system of coupled ordinary differential equations

$$\frac{d\,x}{d\,t} = -\gamma_1\, x\, y \quad , \tag{1}$$

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$$\frac{d\,y}{d\,t} = \gamma_1\, x\, y - \gamma_2 y \quad , \tag{2}$$

$$\frac{d\,z}{d\,t} = \gamma_2 y \quad . \tag{3}$$

The evolution equations of the system take on a bilinear form, and this very simple structure of the model is the reason for its success. Despite the simplicity of the above equations, the nature of the interactions between the compartments is such as to render the model non-integrable. The computational complexity of such equations emerges more clearly if, for instance, we consider the epidemic evolution equation of the $y = y(t)$ distribution function, which after decoupling the two first equations of the above system, reads

$$y\,\frac{d^2y}{d\,t^2} - \left(\frac{d\,y}{d\,t}\right)^2 + \gamma_1\, y^2 \left(\frac{d\,y}{d\,t} + \gamma_2\, y\right) = 0 \quad . \tag{4}$$

Starting from the latter equation and applying the Painleve test in Ref. [8], it has been shown that the ordinary SIR model is not-integrable in the general case.

The three distributions can be expressed, in terms of the auxiliary function $\zeta(t)$, as

$$x = \exp(-\zeta) \quad , \tag{5}$$

$$y = c - \gamma\,\zeta - \exp(-\zeta) \, , \tag{6}$$

$$z = 1 - c + \gamma\,\zeta \, , \tag{7}$$

where $\gamma = \gamma_2/\gamma_1$ and $c = 1 - z_0 - \gamma \ln x_0$, x_0 and z_0 are the initial values of the x and z distributions, respectively. The ζ function is obtained implicitly in the form

$$\gamma_1\,(t - t_0) = \int_{\zeta_0}^{\zeta} \frac{d\,w}{c - \gamma\, w - \exp(-w)} \tag{8}$$

and can only be evaluated numerically.

As previously remarked, the ordinary SIR model is not integrable and therefore does not possess a solution in closed form. For its applications in various fields of science, this feature does not create any problem because its solution can be obtained numerically. However, the analytical solution of a given generalized SIR model, when it exists, becomes important from a theoretical point of view because it makes the interaction mechanism between the populations of the various compartments more transparent. For instance, various significant quantities of the model e.g. the maximum value of the infectious population can be correlated with the model parameters through analytical formulas. For this reason, the search for models that possess solutions in closed form is a very active and topical field of research. Then the non-integrability of the ordinary SIR model is the important motivating factor of the present work. Here we show that the ordinary SIR model is a singular model that belongs to a large class of new SIR models all of which share the main features of the ordinary model. Within this class of generalized SIR models exist various integrable models that have closed-form explicit or implicit solutions,

The main purposes of this work are first to introduce the new large class of generalized SIR models and second to obtain the solutions of the integrable models and study their main properties in view of their application in fields where the ordinary SIR model is regularly employed.

2. General nonlinear model

To better understand the origin of the non-integrability of the ordinary SIR model, we first focalize on its three fundamental properties: (i) The interaction of the I and S populations is described by the product of two distinct contributions related to the two populations (ii) The interaction between the I and R populations depends exclusively on the contribution of the I population (iii) The contribution of each population to the interactions is described by a characteristic function of the population, which is imposed to be linear.

Hereafter, we consider the class of SIR models for which only the first two properties of the ordinary SIR model continue to hold. The third property is released, so that a new class of SIR models is obtained, starting from the equations that define the ordinary model and after performing, on the right-hand side of the equations, the substitutions $x \to f(x)$, and $y \to g(y)$, thereby obtaining the following general class of dynamic models

$$\frac{d\,x}{d\,t} = -\gamma_1\, f(x)\, g(y) \quad , \tag{9}$$

$$\frac{d\,y}{d\,t} = \gamma_1\, f(x)\, g(y) - \gamma_2\, g(y) \quad , \tag{10}$$

$$\frac{d\,z}{d\,t} = \gamma_2\, g(y) \quad , \tag{11}$$

where $f(x)$ and $g(y)$ are two arbitrary positive functions with $f(0) = 0$ and $g(0) = 0$. It is remarkable that the present generalized model is different from the other models already considered in literature [17], and the difference consists in the character of the contribution of the x and y populations, in the interaction between the x and y as well as the y and z compartments. The generalized character of the contributions of the x and y populations in the interactions is described by the functions $f(x)$ and $g(y)$. The above

general class of compartmental systems contains as a special case the ordinary bilinear model which is obtained when $f(x) = x$ and $g(y) = y$.

After introducing the $\lambda = \lambda(x)$ function defined up to an additive constant, through

$$\frac{d\lambda(x)}{dx} = \frac{1}{f(x)} \quad , \tag{12}$$

Eq. (9) becomes $d\lambda(x)/dt = -\gamma_1 g(y)$, which, after comparison with Eq. (11), permits the first integral of motion $\frac{dc(x,z)}{dt} = 0$ to be obtained in the form

$$c(x, z) = 1 - z - \gamma\, \lambda(x) \quad , \tag{13}$$

with

$$\gamma = \frac{\gamma_2}{\gamma_1} \quad . \tag{14}$$

The value of the first integral of motion $c = c(x_0, z_0)$, as obtained from the initial conditions, is given by

$$c = 1 - z_0 - \gamma\, \lambda(x_0) \quad . \tag{15}$$

Exploiting the latter equation and the relation $x + y + z = 1$ the x and y functions can be expressed in terms of the z function to obtain

$$x = \epsilon\,(-\zeta) \quad , \tag{16}$$

$$y = c - \gamma\,\zeta - \epsilon\,(-\zeta) \quad , \tag{17}$$

$$z = 1 - c + \gamma\,\zeta \quad . \tag{18}$$

In the above expressions of the three distributions, the $\zeta = \zeta(t)$ auxiliary function, essentially is the z distribution, after properly amplified and translated. The $\epsilon = \epsilon(w)$ function is defined as the inverse function of $\lambda(x)$ i.e. $\epsilon(w) = \lambda^{-1}(w)$. Therefore, after taking into account Eq. (12), the $\epsilon = \epsilon(w)$ function results to be the solution to the differential equation

$$\frac{d\,\epsilon}{d\,w} = f(\epsilon) \quad . \tag{19}$$

The evolution equation of ζ follows from Eq. (11), which, after taking into account Eqs. (17) and (18), assumes the form

$$\frac{1}{\gamma_1}\frac{d\,\zeta}{d\,t} = g\,(c - \gamma\zeta - \epsilon\,(-\zeta)) \quad . \tag{20}$$

The solution of the latter first-order ordinary differential equation can be written in implicit form as

$$\gamma_1\,(t - t_0) = \int_{\zeta_0}^{\zeta} \frac{d\zeta}{g\,(c - \gamma\zeta - \epsilon\,(-\zeta))} \quad . \tag{21}$$

The integrability of the system depends on the form of the $f(x)$ and $g(y)$ functions. In particular, as regards the ordinary SIR model, the linearity of $f(x) = x$, according to Eq. (19), imposes the exponential form on the $\epsilon(w) = \exp(w)$ function and this is what makes the system non-integrable, while the $g(y) = y$ function plays no role.

3. Epidemic models with power-law interactions

To identify any integrable generalized SIR model within the large class of models introduced in the previous section, the simplest sub-class of models involving power-law interactions between the compartment populations is considered hereafter. It is worth mentioning that power-law interactions proportional to $x^\alpha\, y^\beta$, between the compartments of the x and y populations have already been considered in the literature to construct various models [17,18,32–37]. The power-law interactions was first introduced in theoretical biochemistry [38–41] and have been considered extensively for the construction of various complex model to describe reaction kinetics in chemistry but the search for integrable models or of models having Hamiltonian structures did not attract the attention of the researchers. The presence in the new model of the two further parameters α and β will permit to better describe, with respect the ordinary SIR model, the observed epidemic phenomenology.

In the present class of generalized SIR models the interactions between the population x and x are accounted for by posing $f(x) = x^\alpha$ and $g(y) = y^\beta$, and the class of power-law models is then described by

$$\frac{d\,x}{d\,t} = -\gamma_1\, x^\alpha\, y^\beta \quad , \tag{22}$$

$$\frac{d\,y}{d\,t} = \gamma_1\, x^\alpha\, y^\beta - \gamma_2\, y^\beta \quad , \tag{23}$$

$$\frac{d\,z}{d\,t} = \gamma_2\, y^\beta \quad . \tag{24}$$

The above three-compartment model is identified hereafter as the (α, β) model. The ordinary bilinear SIR model emerges as a particular case of the (α, β) general model and is indicated by $(1, 1)$.

The γ_1 and γ_2 constants behave as two pure scaling parameters. It is easy to verify that the scaled populations, $(x_s, y_s, z_s) = (\gamma^{-\frac{1}{\alpha}} x,\ \gamma^{-\frac{1}{\alpha}} y,\ \gamma^{-\frac{1}{\alpha}} z)$ with $\gamma = \gamma_2/\gamma_1$, evolve in the $t_s = \gamma^{\frac{\beta-1}{\alpha}}\ \gamma_2\, t$ scaled time, obeying the same Eqs. (22)–(24) with $\gamma_1 = \gamma_2 = 1$. On the other hand, the role of the two α and β parameters is more essential. The interactions between the three populations and therefore the dynamics of the system are univocally defined through the α and β shape parameters.

In the case of processes with infinite life the asymptotic behavior for $t \to +\infty$ of the functions $x(t)$, $y(t)$ and $z(t)$ follow from Eqs. (22)–(24)

$$x \approx x_\infty + \frac{x_\infty^\alpha}{\gamma - x_\infty^\alpha}\, h(t) \quad , \tag{25}$$

$$y \approx h(t) \quad , \tag{26}$$

$$z \approx 1 - x_\infty - \frac{\gamma}{\gamma - x_\infty^\alpha}\, h(t) \quad . \tag{27}$$

For $\beta = 1$ the model exhibits exponential tails

$$h(t) \propto \exp\left(-\frac{t}{\tau_1}\right) \quad , \quad \tau_1 = \frac{1}{\gamma_1(\gamma - x_\infty^\alpha)} \quad , \tag{28}$$

while for $\beta > 1$ the model exhibits Pareto power-law tails

$$h(t) = \left(\frac{t_\beta}{t}\right)^{\frac{1}{\beta-1}} \quad , \quad t_\beta = \frac{\tau_1}{\beta - 1} \quad . \tag{29}$$

From Eq. (24) it follows the normalization condition for the function $y(t)$

$$\int_0^\infty y(t)^\beta\, dt = \frac{z_\infty - z_0}{\gamma_2} \quad , \tag{30}$$

which does not introduce any constraints regarding the range of variability of the β parameter.

For $0 < \beta < 1$ the process exhibits a finite life i.e. $t_0 \le t \le t_e$ with $y(t_e) = 0$. The asymptotic behaviors for $t \to t_e$, of the functions $x(t)$, $y(t)$ and $z(t)$ are given by follow

$$x \approx x_e + \frac{x_e^\alpha}{x_e^\alpha - \gamma}\, A\, (t_e - t)^{\frac{1}{1-\beta}} \quad , \tag{31}$$

$$y \approx A\, (t_e - t)^{\frac{1}{1-\beta}} \quad , \tag{32}$$

$$z \approx 1 - x_e - \frac{\gamma}{x_e^\alpha - \gamma}\, A\, (t_e - t)^{\frac{1}{1-\beta}} \quad , \tag{33}$$

with $A = [\gamma_1(1-\beta)(x_e^\alpha - \gamma)]^{\frac{1}{1-\beta}}$ and $x_e = x(t_e)$. For $t > t_e$ it results $x = x_e$, $y = 0$ and $z = 1 - x_e$.

By comparison of Eqs. (22) and (24) it follows $\frac{dx}{dz} = -\frac{1}{\gamma} x^\alpha$ and after integration it obtains the first integral of motion in the form

$$z + \frac{\gamma}{1-\alpha}\, x^{1-\alpha} = z_0 + \frac{\gamma}{1-\alpha}\, x_0^{1-\alpha} \, . \tag{34}$$

The same first integral of motion in terms of the variables x and y follows from the latter its expression by taking into account that $x + y + z = 1$. Alternatively by comparing Eqs. (22) and (23) it obtains $\frac{dy}{dx} = \gamma\, x^{-\alpha} - 1$ and after integration it follows

$$y + x - \frac{\gamma}{1-\alpha}\, x^{1-\alpha} = y_0 + x_0 - \frac{\gamma}{1-\alpha}\, x_0^{1-\alpha} \, . \tag{35}$$

The evolution equation of the $y(t)$ epidemic population in the case of the (α, β) general class of models with power-law interactions can be easily obtained by uncoupling Eqs. (22) and (23), thus obtaining

$$y \frac{d^2 y}{d\, t^2} - \beta \left(\frac{d\, y}{d\, t}\right)^2 + \alpha \gamma_1^{\frac{1}{\alpha}}\, y^{1+\frac{\beta}{\alpha}} \left(\frac{d\, y}{d\, t} + \gamma_2 y^\beta\right)^{2-\frac{1}{\alpha}} = 0 \, , \tag{36}$$

$$y(t_0) = y_0 \quad , \quad y'(t_0) = (\gamma_1 x_0^\alpha - \gamma_2)\, y_0^\beta \quad . \tag{37}$$

A direct inspection of Eq. (36) reveals that the ordinary $(1, 1)$ model is not the simplest model contained in the (α, β) class of models.

From the expression of $f(x) = x^\alpha$ and after integration of Eqs. (12) and (19) with the $\lambda(1) = 0$ and $\epsilon(0) = 1$ conditions, the functions $\lambda(x)$ and $\epsilon(w)$ assume the expressions of the following Euler functions [42,43]

$$\lambda(x) = \nu(x^{1/\nu} - 1) \quad , \tag{38}$$

$$\epsilon(w) = \left(1 + \frac{1}{\nu}\, w\right)^\nu \quad , \tag{39}$$

where the ν parameter is given by

$$\nu = \frac{1}{1-\alpha} \quad . \tag{40}$$

Therefore, the populations of the three compartments given by Eqs. (16)–(18) become

$$x = \left(1 - \frac{1}{\nu}\,\zeta\right)^{\nu} \quad , \tag{41}$$

$$y = c - \gamma\,\zeta - \left(1 - \frac{1}{\nu}\,\zeta\right)^{\nu} \quad , \tag{42}$$

$$z = 1 - c + \gamma\,\zeta \ , \tag{43}$$

with

$$c = 1 - z_0 + \nu\,\gamma - \nu\,\gamma\,x_0^{\frac{1}{\nu}} \quad . \tag{44}$$

The evolution equation of ζ given by Eq. (20) reads

$$\frac{1}{\gamma_1}\,\frac{d\,\zeta}{d\,t} = \left(c - \gamma\,\zeta - \left(1 - \frac{1}{\nu}\,\zeta\right)^{\nu}\right)^{\beta} \quad , \tag{45}$$

whose formal solution, in implicit form, becomes

$$\gamma_1\,(t - t_0) = \Gamma_{\alpha,\beta}(\zeta) - \Gamma_{\alpha,\beta}(\zeta_0) \ , \tag{46}$$

with

$$\Gamma_{\alpha,\beta}(\zeta) = \int \frac{d\zeta}{\left(c - \gamma\,\zeta - \left(1 - \frac{1}{\nu}\,\zeta\right)^{\nu}\right)^{\beta}} \ . \tag{47}$$

The $\zeta(t)$ function is defined implicitly, and in the general case can be evaluated using numerical integration techniques. It clearly emerges, from Eq. (47), that the ordinary SIR model corresponding to $\alpha = 1$ and then to $\nu = \pm\infty$ is a limiting, singular and not-integrable model.

Starting from Eqs. (42) and (45), and after taking into account $d\,y/d\,t = (d\,y/d\,\zeta)(d\,\zeta/d\,t)$, we obtain $d\,y/d\,t = \gamma_1\,y^{\beta}\left((1 - \frac{1}{\nu}\,\zeta)^{\nu-1} - \gamma\right)$. The asymptotic $y_\infty = 0$, and the maximum y_M, values of y follows easily from the previous equation

$$y_M = 1 - z_0 - \frac{\gamma}{1-\alpha}\,x_0^{1-\alpha} + \frac{\alpha}{1-\alpha}\,\gamma^{1/\alpha} \ . \tag{48}$$

The last equation in the $\alpha \to 1$ reproduces the well-known result of the ordinary SIR model $y_M = 1 - z_0 - \gamma\ln x_0 - \gamma + \gamma\ln\gamma$.

The (α, β) general class of model is highly nonlinear, but its numerical integration does not introduce any further computational difficulties, compared to the ordinary bilinear $(1, 1)$ model. If we are satisfied with obtaining numerical solutions like that of the ordinary model, the new model represents an important advancement that can help to deal with power-law interactions between the compartments.

Fig. 1 reports the evolution of the fraction of infectious individuals $y(t)$ (left picture), of susceptible individuals $x(t)$ (right picture, decreasing curves), and of recovered individuals $z(t)$ (right picture, increasing curves) for five (α, β) models corresponding to different values of the α and β parameters. The ordinary SIR $(\alpha = 1, \beta = 1)$ model corresponds to the black curves. The values of the populations are obtained by numerical integration of Eqs. (22)–(24) after fixing the constant $\gamma_2 = 0.6$ and the initial conditions $x_0 = 0.9$, $y_0 = 0.1$ and $z_0 = 0$. The maximum value of the epidemic curve $y(t)$ decreases as the α parameter increases while the asymptotic behavior of $y(t)$ is not influenced by the parameter α. The shape of $y(t)$ and its tail, are highly influenced by the value of the β parameter. Fat tails correspond to high values of the β parameter.

4. A novel class of Hamiltonian SIR models

To better understand the dynamics described by Eq. (36) we first consider the case when $\beta \neq 1$ and introduce the new unknown function $Y = Y(t)$ through the transformation

$$Y = y^{1-\beta} \quad . \tag{49}$$

From Eq. (36) the evolution equation of Y follows in the form

$$\frac{d^2Y}{dt^2} + (1-\beta)^{\frac{1}{\alpha}-1}\,\alpha\,\gamma_1^{\frac{1}{\alpha}}\,Y^{\frac{\beta}{1-\beta}}\left(\frac{d\,Y}{d\,t} + (1-\beta)\gamma_2\right)^{2-\frac{1}{\alpha}} = 0 \ . \tag{50}$$

Interestingly, the $(\alpha = \frac{1}{2}, \beta)$ subclass of models is the one that has the simplest epidemic evolution equation. The models $(\frac{1}{2}, \beta)$ are governed by the Newton equation

$$\frac{d^2Y}{d\,t^2} = F(Y) \quad , \tag{51}$$

with

$$F(Y) = -k\,Y^{\frac{\beta}{1-\beta}} \quad , \tag{52}$$

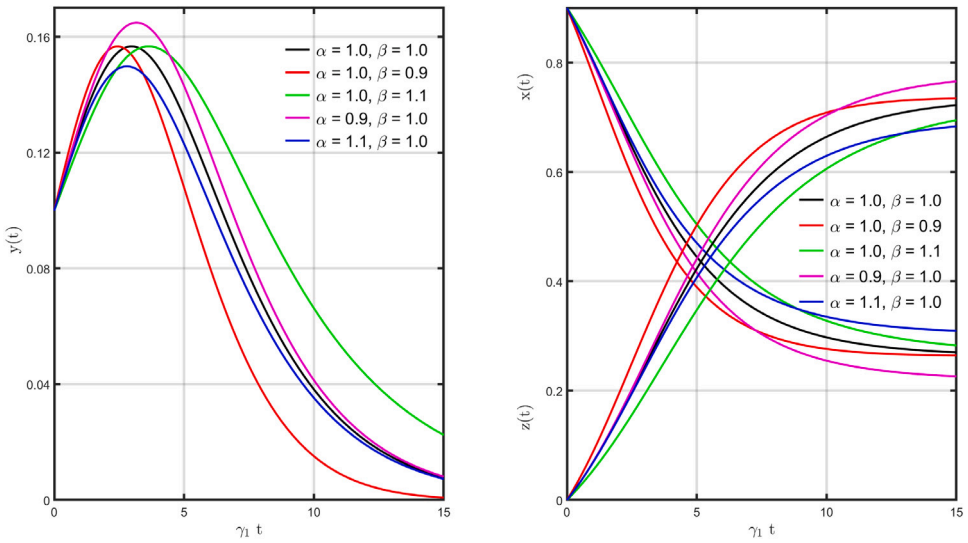

Fig. 1. Evolution of the fraction of infectious individuals $y(t)$ (left picture), of susceptible individuals $x(t)$ (right picture, decreasing curves), and of recovered individuals $z(t)$ (right picture, increasing curves) for five (α, β) models corresponding to different values of the α and β parameters. The ordinary SIR $(\alpha = 1, \beta = 1)$ model corresponds to the black curves. The values of the populations are obtained by numerical integration of Eqs. (22)–(24) after fixing the constant $\gamma_2 = 0.6$ and the initial conditions $x_0 = 0.9$, $y_0 = 0.1$ and $z_0 = 0$. The maximum value of the epidemic curve $y(t)$ decreases as the α parameter increases while the asymptotic behavior of $y(t)$ is not influenced by the parameter α. The shape of $y(t)$ and its tail, are highly influenced by the value of the β parameter. Fat tails correspond to high values of the β parameter.

and $k = \frac{1}{2}\gamma_1^2(1-\beta)$. Eqs. (51) and (52), describe the dynamics of a classical particle subjected to a one-dimension position-depending power-law force $F(Y) = -\frac{d\,V(Y)}{d\,Y}$, related to the potential

$$V(Y) = (1-\beta)k\,Y^{\frac{1}{1-\beta}} \quad . \tag{53}$$

The Hamiltonian of the system governed by the above Newton equation is a conserved quantity, given by the sum of the kinetic energy and the potential i.e.

$$H\left(Y, \frac{dY}{dt}\right) = \frac{1}{2}\left(\frac{d\,Y}{d\,t}\right)^2 + V(Y) \quad . \tag{54}$$

Upon integration of the equation $H\left(Y, \frac{dY}{dt}\right) = E_0$, the formal expression of the function $Y = Y(t)$ is obtained implicitly as

$$\frac{\sqrt{2}}{\gamma_1}(t - t_0) = \int_{Y_0}^{Y} \frac{d\,\mathrm{y}}{\sqrt{E_0 - V(\mathrm{y})}} \quad , \tag{55}$$

in terms of the power-law potential $V(Y)$, which is attractive for $0 < \beta < 1$ and repulsive for $\beta > 1$.

Let us introduce the new variables

$$X = (1-\beta)\,\gamma_1\left(\sqrt{x} - \gamma\right) \quad , \tag{56}$$

$$Z = (1-\beta)\,\gamma_1\left(\sqrt{x_0} - \gamma + \frac{z_0}{2\gamma} - \frac{z}{2\gamma}\right) \quad , \tag{57}$$

It is easy to verify after taking into account the expression of the first integral of motion given by Eq. (34) with $\alpha = \frac{1}{2}$ that results to be $X = Z$. The three evolution Eqs. (22)–(24) in terms of the new variables X, Y, and Z, transform as follows

$$\frac{d\,X}{d\,t} = -k\,Y^{\frac{\beta}{1-\beta}} \quad , \tag{58}$$

$$\frac{d\,Y}{d\,t} = X \quad , \tag{59}$$

$$\frac{d\,Z}{d\,t} = -k\,Y^{\frac{\beta}{1-\beta}} \quad . \tag{60}$$

The system described by the above evolution equation presents a double Hamiltonian structure. Indeed after introducing the Hamiltonian

$$H(P, Y) = \frac{1}{2}\,P^2 + (1-\beta)\,k\,Y^{\frac{\beta}{1-\beta}} \quad , \tag{61}$$

the Hamilton-equations

$$\frac{d\,Y}{d\,t} = \frac{\partial H}{\partial P} \quad , \tag{62}$$

$$\frac{d\,P}{d\,t} = -\frac{\partial H}{\partial Y} \quad . \tag{63}$$

assume the explicit form

$$\frac{d\,Y}{d\,t} = P \quad , \tag{64}$$

$$\frac{d\,P}{d\,t} = -k\,Y^{\frac{\beta}{1-\beta}} \quad , \tag{65}$$

and for $P = X$ or $P = Z$ reproduce the above obtained evolution Eqs. (58)–(60).

The underlying Hamiltonian dynamics for the $\beta = 1$ case, corresponding to the $(\frac{1}{2}, 1)$ model, is obtained starting from Eq. (36) and by introducing the $Y = -\ln(y)$ transformation. In this case the position depending force is given by $F(Y) = k_1\,\exp(-Y)$ with $k_1 = \frac{1}{2}\,\gamma_1^2$. The related potential assumes the form $V(Y) = -k_1\,\exp(-Y)$ while the Hamiltonian of the system is given by $H = \frac{1}{2}\left(\frac{dY}{dt}\right)^2 + k_1\,\exp(Y)$. The variables canonically conjugate to Y are given by $X = \gamma_1\left(\sqrt{x} - \gamma\right)$ and $Z = \gamma_1\left(\sqrt{x_0} - \gamma + \frac{z_0}{2\gamma} - \frac{z}{2\gamma}\right)$ and results $P = X = Z$ with $P = \frac{dY}{dt}$. The Hamiltonian of the system assumes the form $H = \frac{1}{2}P^2 + k_1\,\exp(Y)$ while the Hamilton equations become $\frac{dY}{dt} = P$, $\frac{dP}{dt} = k_1\exp(-Y)$.

Within the (α, β) general class of SIR models, the $(\frac{1}{2}, \beta)$ subclass with $\beta > 0$ is particularly interesting because it describes Hamiltonian systems. Furthermore all the other models within the (α, β) general class including the standard $(1, 1)$ SIR model are non-Hamiltonian models.

In the section below, our task will be to investigate the existence, within the $(\frac{1}{2}, \beta)$ class of models, of any integrable model admitting closed-form solution.

5. Solution of the Hamiltonian SIR models

5.1. General case

The main task of the present section is to investigate the existence of exact analytical closed-form explicit or implicit solutions within the $(\frac{1}{2}, \beta)$ class of Hamiltonian SIR models, described through Eqs. (22)–(24) after posing $\alpha = \frac{1}{2}$. After recalling that $z = 1-x-y$ we focus on the evolution equations of the variables x and y which can be written in the form

$$\frac{d\,\sqrt{x}}{d\,t} = -\frac{1}{2}\,\gamma_1\;y^{\beta} \quad , \tag{66}$$

$$\frac{d\,y}{d\,t} = \gamma_1\,(\sqrt{x} - \gamma)\,y^{\beta} \quad . \tag{67}$$

It will be more convenient for the purposes of the discussion that follows to introduce instead of the function $\zeta(t)$ the new auxiliary function $-1 < \xi(t) < 1$ function through

$$\xi = \frac{\gamma - 1}{\eta} + \frac{1}{2\eta}\,\zeta \quad , \tag{68}$$

with

$$\eta = \sqrt{y_0 + (\sqrt{x_0} - \gamma)^2} \quad . \tag{69}$$

The solutions of the system of Eqs. (22)–(24) given by Eqs. (41)–(43), can be expressed easy in terms of the ξ function assuming the forms

$$x = (\gamma - \eta\,\xi)^2 \quad , \tag{70}$$

$$y = \eta^2\,(1 - \xi^2\,) \quad , \tag{71}$$

$$z = 1 - \eta^2 - \gamma^2 + 2\gamma\eta\,\xi \quad . \tag{72}$$

Furthermore, the introduction of the characteristic time $\tau = \frac{2}{\eta^{2\beta-1}\gamma_1}$, permits the evolution equation of ξ to be written, after taking into account Eq. (45), in the following simple form

$$\tau\,\frac{d\,\xi}{d\,t} = (1 - \xi^2)^{\beta} \quad . \tag{73}$$

The initial value of ξ is given by $\xi_0 = (\gamma - \sqrt{x_0})/\eta$. So that the solution of the system of coupled Eqs. (22)–(24) with $\alpha = \frac{1}{2}$, is reduced to the solution of the first order Eq. (73).

Next task is to study the range of variability of the parameter γ so that the model can correctly describe an epidemic when the initial rate of infected is positive and according to Eq. (67) results to be $\sqrt{x_0} - \gamma > 0$. The first integral of motion as given by Eq. (35) for $x = x_\infty$, after some simple algebra can be written in the form

$$(\sqrt{x_0} - \sqrt{x_\infty}\,)(2\gamma - \sqrt{x_0} - \sqrt{x_\infty}) = y_0 \quad . \tag{74}$$

After noting that in the above equation the r.h.s. and the first factor in the l.h.s. are positive we can conclude that also the second factor in l.h.s of this equation must be positive i.e. $2\gamma - \sqrt{x_0} - \sqrt{x_\infty} > 0$. This latter inequality can be written as $\gamma - \sqrt{x_\infty} > \sqrt{x_0} - \gamma$ and after taking into account that $\sqrt{x_0} - \gamma > 0$ it follows $\gamma - \sqrt{x_\infty} > 0$ and then $\sqrt{x_\infty} < \gamma$.

The value $\xi_\infty = 1$ of ξ, follows from Eq. (73): After taking into account Eq. (70) and the above obtained condition $\sqrt{x_\infty} < \gamma$, the value of x_∞ can be obtained from $\sqrt{x_\infty} = \gamma - \eta$. From Eq. (66) it follows that $\sqrt{x}$ is a positive e monotonically decreasing function which, for $t \to \infty$, reaches its minimum value $\sqrt{x_\infty} = \gamma - \eta$. The inequality $\gamma - \eta > 0$, after taking into account the expression of η implies the condition

$$\gamma > \frac{x_0 + y_0}{2\sqrt{x_0}} \quad . \tag{75}$$

The latter condition regarding the variability range of the parameter γ guarantees that $\gamma > \eta$ so that x_∞ is a monotonically increasing function of the variable γ or monotonically decreasing function of the reproductive ratio $1/\gamma$. The monotonicity of x_∞ imply that $z_\infty = 1 - x_\infty$ is also a monotonic function of the parameter γ.

From Eq. (67) follows that when $\gamma < \sqrt{x_0}$, for $t < t_{_M}$ it $y(t)$ is an increasing function which reaches its maximum $y_{_M}$ at $t = t_{_M}$ and after for $t > t_{_M}$ decreases monotonically and approaches the zero value for $t \to \infty$. From Eqs. (70) and (71) after posing $\xi = 0$ follows that

$$\gamma = \sqrt{x_{_M}} \ , \qquad \eta = \sqrt{y_{_M}} \ , \tag{76}$$

$x_{_M}$ being the value of x at $t = t_{_M}$. The two latter relationships permits to explain the physical meaning of the parameters γ and η in terms of $x_{_M}$ and $y_{_M}$ respectively. From $\sqrt{x_\infty} = \gamma - \eta$ follows that

$$\sqrt{x_\infty} = \sqrt{x_{_M}} - \sqrt{y_{_M}} \quad . \tag{77}$$

Regarding the predictability of the present class of models, it might introduce the following

Theorem 1. *In the framework of $(\frac{1}{2}, \beta)$ class of Hamiltonian models, the asymptotic value of the fraction of susceptible population x_∞, depends only on the measurable quantities which are the initial value of the fraction of susceptible population x_0, the initial value of the fraction of infectious population y_0, and on the maximum value of the fraction of infected population $y_{_M}$ and is given by*

$$\sqrt{x_\infty} = \sqrt{x_0} - \sqrt{y_{_M}} - \sqrt{y_{_M} - y_0} \quad . \tag{78}$$

Proof. The above simple formula can be obtained after inserting in Eq. (69) the expressions of $\gamma = \sqrt{x_{_M}}$ and $\eta = \sqrt{y_{_M}}$, and after taking into account Eq. (77).

Theorem 2. *In the framework of $(\frac{1}{2}, \beta)$ class of Hamiltonian models, the maximum value of the fraction of the infectious population is upper-bounded i.e.*

$$y_{_M} < \left(\frac{x_0 + y_0}{2\sqrt{x_0}} \right)^2 \quad . \tag{79}$$

Proof. The above inequality follows Eq. (78) after taking into account that $x_\infty > 0$.

From Eq. (67) follows that for $\sqrt{x_0} - \gamma < 0$ for any $t > t_0$ it results $\frac{dy}{dt} < 0$ so that $y(t)$ decreases monotonically and approaches the zero value for $t \to \infty$. Then the threshold value of the parameter γ is given by $\sqrt{x_0}$.

The formal solution of Eq. (73) is given in implicit form by

$$\frac{t - t_0}{\tau} = I_{\frac{1}{2},\beta}(\xi) - I_{\frac{1}{2},\beta}(\xi_0) \ , \tag{80}$$

with

$$I_{\frac{1}{2},\beta}(\xi) = \int \frac{d\xi}{(1 - \xi^2)^\beta} \quad . \tag{81}$$

It should be noted that in general, the $\xi(t)$ function is defined implicitly and the possibility of obtaining $\xi = \xi(t)$ explicitly after inversion depends on the analytical form of the $I_{\frac{1}{2},\beta}(\xi)$ indefinite integral. The integrability of the systems is also strictly related to the possibility of computing the above indefinite integral and of obtaining it in terms of elementary functions. It is remarkable that the indefinite integral in Eq. (81) can be expressed formally, in terms of the ${}_2F_1$ hypergeometric function [44], as

$$I_{\frac{1}{2},\beta}(\xi) = \xi \ {}_2F_1\left(\frac{1}{2}, \beta; \frac{3}{2}; \xi^2\right) \quad . \tag{82}$$

Eqs. (70), (71), (72), (80) and (82) define the formal solutions of the $(\frac{1}{2}, \beta)$ sub-class of models.

The $(\frac{1}{2}, \beta)$ sub-class of models with $0 < \beta < 2$, in the general case, describes non-integrable dynamical systems, whose solutions are defined implicitly and can be obtained after numerical integration. Interestingly, when the β parameter assumes the three values $\beta = \frac{1}{2}, 1$, or $\frac{3}{2}$ the related models are integrable and their solutions can be obtained explicitly.

5.2. The $(\frac{1}{2}, \frac{1}{2})$ trigonometric model

The $I_{\frac{1}{2},\frac{1}{2}}(\xi)$ indefinite integral, corresponding to the model $(\frac{1}{2}, \frac{1}{2})$, can be computed easily [45] (page 93, 2.261), obtaining

$$I_{\frac{1}{2},\frac{1}{2}}(\xi) = \arcsin \xi \quad . \tag{83}$$

After introducing the dimensionless shifted time

$$T = \frac{\gamma_1}{2}\left(t - t_0 - \frac{2}{\gamma_1}\arcsin\frac{\sqrt{x_0}-\gamma}{\eta}\right) \quad , \tag{84}$$

with η defined in Eq. (69), the solution of Eq. (73) writes in the form

$$\xi = \sin T \quad . \tag{85}$$

The populations of the three compartments follow immediately from Eqs. (70)–(72) according to

$$x = (\gamma - \eta \sin T)^2 \quad , \tag{86}$$

$$y = \eta^2 \cos^2 T \quad , \tag{87}$$

$$z = 1 - \eta^2 - \gamma^2 + 2\gamma\eta \sin T \quad . \tag{88}$$

The above expressions hold for $\gamma > \eta$ or more explicitly for $\gamma > (x_0 + y_0)/2\sqrt{x_0}$.

The $(\frac{1}{2}, \frac{1}{2})$ model describes a process having a limited duration. Only for $t_i \le t \le t_f$ with

$$t_i = t_0 + \frac{2}{\gamma_1}\arcsin\frac{\sqrt{x_0}-\gamma}{\eta} - \frac{\pi}{\gamma_1} \quad , \tag{89}$$

$$t_f = t_0 + \frac{2}{\gamma_1}\arcsin\frac{\sqrt{x_0}-\gamma}{\eta} + \frac{\pi}{\gamma_1} \quad , \tag{90}$$

it results to be $y > 0$. It holds the condition that $t_i \le t_0 \le t_f$ and the duration of the life of the epidemic is given by $t_f - t_i = 2\pi/\gamma_1$. Square-root interactions in population dynamics have been previously considered in [35] and recently in [36,37]. The $(\frac{1}{2}, \frac{1}{2})$ model has been considered previously in [32].

It is remarkable that the underlying Newton dynamics of the $(\frac{1}{2}, \frac{1}{2})$ model is governed by the elastic force $F(Y) = -\frac{1}{2}\gamma_1^2 Y$.

5.3. The $(\frac{1}{2}, 1)$ logistic model

In the case where $\alpha = \frac{1}{2}$ and $\beta = 1$, the $I_{\frac{1}{2},1}(\xi)$ integral can be also computed easily [45] (page 73, 2.143, 2), obtaining

$$I_{\frac{1}{2},1}(\xi) = \operatorname{arctanh}(\xi) \quad . \tag{91}$$

After introducing the dimensionless shifted time

$$T = \frac{\gamma_1\eta}{2}\left(t - t_0 - \frac{2}{\gamma_1\eta}\operatorname{arctanh}\frac{\sqrt{x_0}-\gamma}{\eta}\right) \quad , \tag{92}$$

with η defined in Eq. (69), the solution of Eq. (73) writes in the form

$$\xi = \tanh T \quad . \tag{93}$$

The populations of the three compartments follow immediately from Eqs. (70)–(72) according to

$$x = (\gamma - \eta \tanh T)^2 \quad , \tag{94}$$

$$y = \frac{\eta^2}{\cosh^2 T} \quad , \tag{95}$$

$$z = 1 - \eta^2 - \gamma^2 + 2\gamma\eta \tanh T \quad . \tag{96}$$

The above expressions hold for $\gamma > \eta$ or more explicitly for $\gamma > (x_0 + y_0)/2\sqrt{x_0}$.

The $(\frac{1}{2}, 1)$ model describes an integrable system, with the solution defined explicitly, whose life has an infinite duration and presents exponential tails. Equation $\frac{dz}{dt} = \gamma_2 y$, suggests the y function should be interpreted as proportional to a probability density, while z should be interpreted as the cumulative distribution. In this case, z is the sigmoid function of the logistic distribution, and y is the corresponding logistic density function. The $(\frac{1}{2}, 1)$ has been considered recently in literature [33,34]. It is remarkable that the solutions described by Eqs. (94)–(96) are solutions also of the approximated version of the standard SIR model considered by Kermack and McKendrick [1].

5.4. A new integrable $(\frac{1}{2}, \frac{3}{2})$ model

In the case where $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$, the $I_{\frac{1}{2},\frac{3}{2}}(\xi)$ integral can be computed also easily [45] (page 94, 2.264, 5), obtaining

$$I_{\frac{1}{2},\frac{3}{2}}(\xi) = \frac{\xi}{\sqrt{1-\xi^2}} \quad . \tag{97}$$

After introducing the dimensionless shifted time

$$T = \frac{\gamma_1 y_0}{2}\left[1 + \left(\frac{\sqrt{x_0} - \gamma}{\sqrt{y_0}}\right)^2\right](t - t_0) + \frac{\sqrt{x_0} - \gamma}{\sqrt{y_0}} \quad , \tag{98}$$

with η defined in Eq. (69), the solution of Eq. (73) writes in the form

$$\xi = \frac{T}{\sqrt{1+T^2}} \quad . \tag{99}$$

The populations of the three compartments follow immediately from Eqs. (70)–(72) according to

$$x = \left(\gamma - \eta\frac{T}{\sqrt{1+T^2}}\right)^2 \quad , \tag{100}$$

$$y = \frac{\eta^2}{1+T^2} \quad , \tag{101}$$

$$z = 1 - \eta^2 - \gamma^2 + 2\gamma\eta\frac{T}{\sqrt{1+T^2}} \quad . \tag{102}$$

The above expressions hold for $\gamma > \eta$ or more explicitly for $\gamma > (x_0 + y_0)/2\sqrt{x_0}$.

The $(\frac{1}{2}, \frac{3}{2})$ model is new and describes a dynamic integrable system, admitting explicit closed-form solution. The process described by that model has a life with an infinite duration and presents power-law tails having the Pareto exponent $p = 2$. The epidemic curve describing the y infectious population is a Lorentzian function and is given by Eq. (100). The position-depending force, in the underlying Newton dynamics, is repulsive and has the expression $F(Y) = \frac{1}{2}\gamma_1^2 Y^{-3}$.

5.5. An infinite family of implicit integrable models

Besides, the above three integrable explicit models, the $(\frac{1}{2}, \beta)$ class of models contains an infinity of implicitly defined integrable models for which the β parameter is a positive integer or semi-integer.

When $\beta = n$, with $n = 2, 3, 4, \ldots$ the $I_{\frac{1}{2},n}(\xi)$ integral, defined by means of Eq. (81), can be computed [45] (page 76, 2.149, 3), to obtain

$$I_{\frac{1}{2},n}(\xi) = \sum_{k=1}^{n-1} \frac{c_{nk}\,\xi}{(1-\xi^2)^{n-k}} + \frac{(2n-3)!!}{2^n(n-1)!}\ln\frac{1+\xi}{1-\xi} \quad , \tag{103}$$

with

$$c_{nk} = \frac{2^{-k}}{2n-1}\,\frac{(2n-1)(2n-3)...(2n-2k+1)}{(n-1)(n-2)...(n-k)} \quad , \tag{104}$$

When $\beta = n + \frac{1}{2}$, with $n = 2, 3, 4, \ldots$ the $I_{\frac{1}{2},n+\frac{1}{2}}(\xi)$ integral, can be computed [45] (page 98, 2.271, 6), to obtain

$$I_{\frac{1}{2},n+\frac{1}{2}}(\xi) = \sum_{k=0}^{n-1} \frac{1}{2k+1}\binom{n-1}{k}\frac{\xi^{2k+1}}{(1-\xi^2)^{\frac{2k+1}{2}}} \quad . \tag{105}$$

It is remarkable that the integrable $(\frac{1}{2}, 2)$ model, is implicitly defined having

$$I_{\frac{1}{2},2}(\xi) = \frac{1}{2}\,\frac{\xi}{1-\xi^2} + \frac{1}{4}\ln\frac{1+\xi}{1-\xi} \quad . \tag{106}$$

It is remarkable that the underlying hamiltonian process describing this model is governed by the Newton equation in presence of the repulsive Coulomb force $F(Y) = \gamma_1^2 Y^{-2}$.

The integrable $(\frac{1}{2}, \frac{5}{2})$ model corresponds to

$$I_{\frac{1}{2},\frac{5}{2}}(\xi) = \frac{1}{3}\,\frac{\xi(3-2\xi^2)}{(1-\xi^2)^{3/2}} \quad . \tag{107}$$

The inversion of the latter function requires the solution of a third-degree polynomial equation. Then the explicit solutions of the $(\frac{1}{2}, \frac{5}{2})$ model are somewhat complicated and are not reported here.

The solutions of all the other $(\frac{1}{2}, \beta)$ models with β being a positive integer or semi-integer, are defined implicitly and present power-law tails.

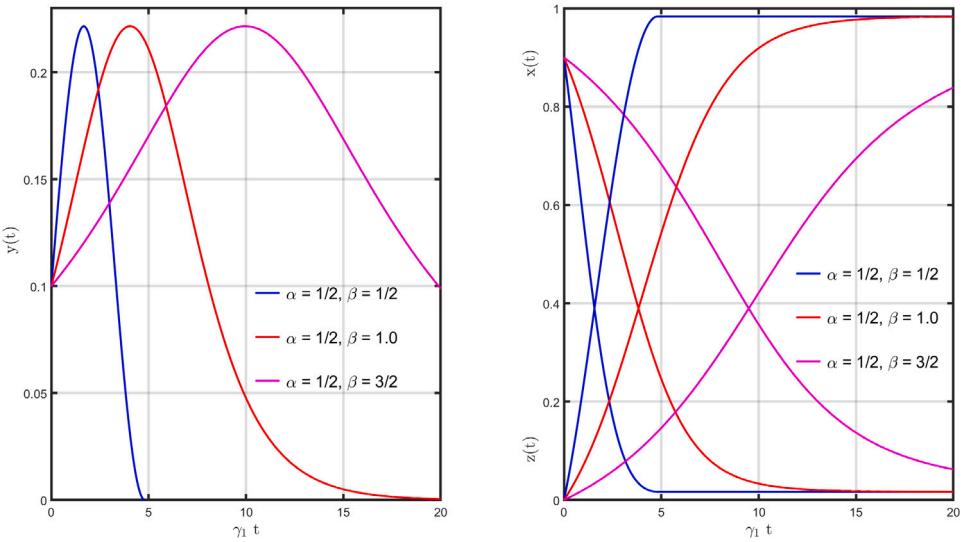

Fig. 2. Evolution of the fraction of infectious individuals $y(t)$ (left picture), of susceptible individuals $x(t)$ (right picture, decreasing curves), and of recovered individuals $z(t)$ (right picture, increasing curves), for the three $(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$, $(\alpha = \frac{1}{2}, \beta = 1)$, $(\alpha = \frac{1}{2}, \beta = \frac{3}{2})$ integrable Hamiltonian models. For all the models the initial conditions are fixed to $x_0 = 0.9$, $y_0 = 0.1$ and $z_0 = 0$, while the constant γ_2 is fixed to the value $\gamma_2 = 0.6$. The shape of the populations $x(t)$, $y(t)$, $z(t)$ and in particular of their tails, are highly influenced by the value of the β parameter. Exponential tails correspond to $\beta = 1$, while Pareto power-law tails with Pareto exponent $p = 2$ correspond to $\beta = \frac{3}{2}$. For $\beta = \frac{1}{2}$ the epidemic has a finite life and the tails are absent.

Fig. 2 shows the evolution of the fraction of infectious individuals $y(t)$ (left picture), of susceptible individuals $x(t)$ (right picture, decreasing curves), and of recovered individuals $z(t)$ (right picture, increasing curves), for the three $(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$, $(\alpha = \frac{1}{2}, \beta = 1)$, $(\alpha = \frac{1}{2}, \beta = \frac{3}{2})$ integrable Hamiltonian models. For all the models the initial conditions are fixed to $x_0 = 0.9$, $y_0 = 0.1$ and $z_0 = 0$, while the constant γ_2 is fixed to the value $\gamma_2 = 0.6$.

6. A further non-Hamiltonian integrable SIR model admitting implicit solutions

Hereafter is considered the $(2, \beta)$ class of non-Hamiltonian models. From Eq. (40) follows $\nu = -1$. After introducing the constants $b = 1 - z_o + \frac{\gamma}{x_0}$ and $s = \frac{1}{\sqrt{1 - \frac{4\gamma}{b^2}}}$, and the new function $\chi = \chi(t)$, through the relationship $\chi = s - \frac{bs}{2\gamma} - \frac{s}{\nu}\zeta$, the populations of the three compartments given by Eqs. (41)–(43) assumes the forms

$$x = \frac{2\gamma\, s}{b\, s + 2\gamma\, \chi} \ , \tag{108}$$

$$y = 1 - z_0 + \frac{\gamma}{x_0} - \frac{b}{2} - \frac{\gamma}{s}\chi - \frac{2\gamma\, s}{b\, s + 2\gamma\, \chi} \ , \tag{109}$$

$$z = z_0 - \frac{\gamma}{x_0} + \frac{b}{2} + \frac{\gamma}{s}\chi \ , \tag{110}$$

while the evolution equation of χ follows from Eq. (45)

$$\tau \frac{d\chi}{dt} = \left(\frac{1 - \chi^2}{s + \chi}\right)^{\beta} \ , \tag{111}$$

with $\tau = \frac{1}{\gamma_2} 2^{\beta - 1} (b^2 - 4\gamma)^{\frac{\beta - 1}{2}}$.

The formal solution of Eq. (111) is given by

$$\frac{t - t_0}{\tau} = J_{2,\beta}(\chi) - J_{2,\beta}(\chi_0) \ , \tag{112}$$

where the indefinite integral

$$J_{2,\beta}(\chi) = \int \left(\frac{s + \chi}{1 - \chi^2}\right)^{\beta} d\chi \ , \tag{113}$$

can be expressed in terms of a two-variable hypergeometric function. The integrability of the system is strictly related to the value of the β parameter. When β is a positive integer i.e. $\beta = n$, the integral defined in Eq. (113), is expressed in terms of rational and logarithmic functions so that an infinite class of integrable models $(2, n)$ with relevant solutions defined implicitly emerges. The

expressions of $J_{2,n}(\chi)$ for the first two models are given below

$$J_{2,1}(\chi) = \frac{1}{2}(s-1)\ln(1+\chi) - \frac{1}{2}(s+1)\ln(1-\chi), \tag{114}$$

$$J_{2,2}(\chi) = \frac{1+\frac{1}{2}(s^2+1)\chi}{1-\chi^2} + \frac{1}{4}(s^2-1)\ln\frac{1+\chi}{1-\chi}. \tag{115}$$

7. Approaching the ordinary SIR model without quadratures

Let us now consider the sequence of $(1-\frac{1}{n},1)$ models, defined by means of Eqs. (41)–(47), where $\alpha = 1-\frac{1}{n}$, n being a positive integer $n \geq 2$ and $\beta = 1$. The integrability of this infinite and discrete class of models is guaranteed by the fact that the $\Gamma_{1-\frac{1}{n},1}(\zeta)$ integral defined in Eq. (47), can be performed, in terms of logarithmic functions after obtaining the n roots of the $c-\gamma\zeta-\left(1-\frac{1}{n}\zeta\right)^n = 0$ polynomial equation. It is worth noting that the first element of the above sequence of models is just the already discussed $(\frac{1}{2},1)$ logistic model. The other models with $n>2$ are new and can be used for the construction of an algorithm that permits the solution of the ordinary bilinear model to be evaluated without performing any numerical integration. All the $(1-\frac{1}{n},1)$ models underestimate the value of the α parameter, i.e. $\alpha \to 1^-$, and converge to the $(1,1)$ ordinary model, when $n \to \infty$. The degree of accuracy in obtaining the approximate solution of the ordinary model, through the above sequence of models, is related to the value of the n integer.

A second $(1+\frac{1}{n-1},1)$ sequence of models, can be introduced through Eqs. (41)–(47), when is posed $\alpha = 1+\frac{1}{n-1}$ with $n \geq 2$ a positive integer and $\beta = 1$. The evaluation of ζ needs the calculation of the indefinite integral $\Gamma_{1+\frac{1}{n-1},1}(\zeta)$, defined in Eq. (47). This integral can be expressed in terms of logarithmic functions, after solving the n degree polynomial equation, $(c-\gamma\zeta)\left(1+\frac{1}{n-1}\zeta\right)^{n-1} - 1 = 0$. All the $(1+\frac{1}{n-1},1)$ models overestimate the value of the α parameter i.e. $\alpha \to 1^+$, and converge to the $(1,1)$ ordinary model, when $n \to \infty$. The first element of the second sequence of models corresponding to $n=1$ defines the $(2,1)$ model which is integrable and admits an implicitly defined solution that can easily be obtained. This second sequence of models can also be used to approach the ordinary SIR model.

The numerical solutions of the two sequences of models, $(1-\frac{1}{n},1)$ and $(1+\frac{1}{n-1},1)$, can be combined to construct of a more sophisticate algorithm to numerically determine the solution of the ordinary SIR model, without involving numerical integration techniques. The possibility of obtaining the approximate solution of the ordinary SIR model by using the two above-defined sequences of models, in which the α parameter is underestimated and overestimated, respectively, permits the error of the obtained solution to be quantified.

Let us consider the first two models of the above-defined sequences of models $(1-\frac{1}{n},1)$ and $(1+\frac{1}{n-1},1)$, corresponding to $n=2$ i.e. the already studied $(\alpha=\frac{1}{2},\beta=1)$ and $(\alpha=2,\beta=1)$ integrable models. Fig. 3 reports the evolution of the fraction of infectious individuals $y(t)$ (left picture), of susceptible individuals $x(t)$ (right picture, decreasing curves), and of recovered individuals $z(t)$ (right picture, increasing curves) for four different models. The black curves corresponding to the ordinary SIR model $(\alpha=1,\beta=1)$ are obtained numerically. The blue curves corresponding to the model $(\alpha=\frac{1}{2},\beta=1)$ and the purple curves corresponding to the model $(\alpha=2,\beta=1)$ are obtained analytically. For these three models, the constant γ_2 is fixed to the value $\gamma_2 = 0.6$ while the initial conditions are fixed to $x_0 = 0.9$, $y_0 = 0.1$, and $z_0 = 0$. The red curves correspond to the mixed model and are obtained as a semi-sum of the corresponding curves of the models $(\alpha=\frac{1}{2},\beta=1)$ and $(\alpha=2,\beta=1)$. The agreement between the mixed model and the ordinary SIR model can be improved by choosing $n>2$ and by involving higher order models within the sequences of models, $(1-\frac{1}{n},1)$ and $(1+\frac{1}{n-1},1)$.

8. Epimythion

Let us summarize the main results obtained in the present work. The (α,β) general class of SIR models introduced in Section 3 and described by Eqs. (22)–(24) represents a two-parameter generalization of the $(1,1)$ ordinary SIR model. This class of models has the important feature to account for power-law interactions between the three SIR compartments. The three-compartment populations of the (α,β) models are expressed through Eqs. (41)–(43), in terms of a unique function obeying the evolution Eq. (45). The evolution equation of the epidemic i.e. of the infected population is obtained in the general case and is given by Eq. (36). In the general case, the solutions of (α,β) models as in the case of the ordinary $(1,1)$ SIR model, can be obtained after numerical integration of the first-order differential Eq. (45).

In Section 4 it is shown that the sub-class $(\frac{1}{2},\beta)$ of models describes Hamiltonian systems whose infectious and recovered populations are canonically conjugated. This is a new and remarkable result especially if it is taken into account that the ordinary SIR model describes non-Hamiltonian dynamics.

In Section 5 it is proven that within the sub-class of Hamiltonian $(\frac{1}{2},\beta)$ models exist three integrable models corresponding to $\beta = 1/2, 1, 3/2$ admitting explicit solutions and an infinity of other integrable models admitting implicit solutions. These integrable models are very versatile and can describe epidemics with a finite $(\beta=\frac{1}{2})$ or an infinite duration of life and, in this latter case, the compartment populations evolve showing exponential tails $(\beta=1)$ or power-law Pareto tails with Pareto exponent $p = 1/(\beta-1)$ The above integrable models can be used directly in the analysis of empirical data while preliminary numerical integrations are required for the case of the ordinary $(1,1)$ model.

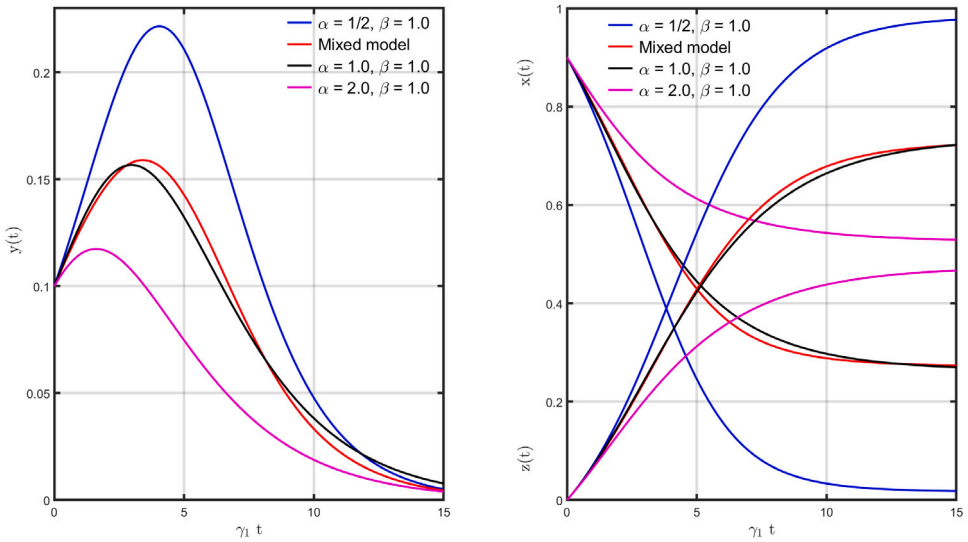

Fig. 3. Evolution of the fraction of infectious individuals $y(t)$ (left picture), of susceptible individuals $x(t)$ (right picture, decreasing curves), and of recovered individuals $z(t)$ (right picture, increasing curves) for four different models. The black curves corresponding to the ordinary SIR model $(\alpha = 1, \beta = 1)$ are obtained numerically. The blue curves corresponding to the model $(\alpha = \frac{1}{2}, \beta = 1)$ and the purple curves corresponding to the model $(\alpha = 2, \beta = 1)$ are obtained analytically. For these three models, the constant γ_2 is fixed to the value $\gamma_2 = 0.6$ while the initial conditions are fixed to $x_0 = 0.9$, $y_0 = 0.1$, and $z_0 = 0$. The red curves corresponding to the mixed model are obtained as a semi-sum of the corresponding curves of the models $(\alpha = \frac{1}{2}, \beta = 1)$ and $(\alpha = 2, \beta = 1)$.

The (α, β) general class of models contains the further non-Hamiltonian, integrable $(2, 1)$ model whose solution can be expressed implicitly in closed form as shown in Section 6.

The class of (α, β) models emerges as a two-parameter generalization of the original Kermack–McKendrick $(1, 1)$ model proposed about 100 ago and this fact represents the strength and at the same time the limitation of the present class of generalized SIR models. The original SIR model is still used and represents the paradigm par excellence for describing the interaction between populations. Its generalization like the one proposed here by introducing two new parameters α and β represents an important step forward for the description of empirical data. At the same time, over the last 100 years, new phenomenologies have been observed in population dynamics that have required modifications of the original SIR model by introducing new interaction terms or new population compartments beyond the three original Kermack–McKendrick ones or by modifying the mathematical typology of the description of compartmental dynamics. All these improvements to the Kermack–McKendrick model were made without changing its general connotations and the original way of describing the interaction between populations. In this sense, all the improvements introduced and already tested in the case of the original SIR model to better describe the new phenomenologies observed in the last decades can also be adopted for the class of (α, β) models. For instance, the present deterministic class of (α, β) models, can easily be generalized to obtain its stochastic or fractal counterparts. Further generalizations can be considered to incorporate spatial diffusion and time delay in the population dynamics. New highly topical phenomenologies emerging in recent years regarding multi-pathogen or multi-strain pandemics require the extension of (α, β) models to be able to treat multi-compartment populations in the presence of power-law interactions.

CRediT authorship contribution statement

G. Kaniadakis: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Giorgio Kaniadakis reports was provided by Polytechnic of Turin. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proc. R. Soc. Lond. Ser. A 115 (772) (1927) 700–721.
[2] A. Ianni, N. Rossia, Describing the COVID-19 outbreak during the lockdown: fitting modified SIR models to data eur, Phys. J. Plus 135 (2020) 885.
[3] Kazumi Omata, Nonequilibrium statistical mechanics of a susceptible-infected-recovered epidemic model, Phys. Rev. E 96 (2017) 022404.
[4] F. Radicchi, G. Bianconi, Epidemic plateau in critical susceptible-infected-removed dynamics with nontrivial initial conditions, Phys. Rev. E 102 (2020) 052309.
[5] F.G. Schmitt, An algorithm for the direct estimation of the parameters of the SIR epidemic model from the I(t) dynamicals, Eur. Phys. J. Plus 137 (2022) 57.
[6] V. Ram, L.P. Schaposnik, A modifed age-structured SIR model for COVID-19 type viruses, Sci. Rep. 11 (2021) 15194.
[7] F. Zarei, S. Moghimi-Araghi, F. Ghanbarnejad, Act solution of generalized cooperative susceptible-infected-removed (SIR) dynamics, Phys. Rev. E 100 (2019) 012307.
[8] N.A. Kudryashova, M.A. Chmykhova, M. Vigdorowitsch, Analytical features of the SIR model and their applications to COVID-19, Appl. Math. Model. 90 (2021) 466–473.
[9] S. Bhattacharjee, S.T. Liao, D. Paul, S. Chaudhuri, Inference on the dynamics of COVID-19 in the United States, Sci. Rep. 12 (2022) 2253.
[10] N. Ukaj, S. Scheiner, C. Hellmich, Toward hereditary epidemiology: A temporal Boltzmann approach to COVID-19 fatality trends, Appl. Phys. Rev. 8 (2021) 041417.
[11] G. Kaniadakis, M.M. Baldi, T.S. Deisboeck, G. Grisolia, D. Hristopulos, A.M. Scarfone, A. Sparavigna, T. Wada, U. Lucia, The κ-statistics approach to epidemiology, Sci. Rep. 10 (2020) 19949.
[12] L. Stone, R. Olinky, A. Huppert, Seasonal dynamics of recurrent epidemics, Nature 446 (2007) 7135533.
[13] L.M.A. Bettencourta, A. Cintron-Arias, D.I. Kaiserd, C. Castillo-Chavezb, The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models, Physica A 364 (2006) 513–536.
[14] S.M. Reia, J.E. Fontanari, A SIR epidemic model for citation dynamics, Eur. Phys. J. Plus 136 (2021).
[15] G. Fibich, Bass-SIR model for diffusion of new products in social networks, Phys. Rev. E 94 (2016) 032305.
[16] H.S. Rodrigues, Application of SIR epidemiological model: new trends, Int. J. Appl. Math. Inform. 10 (2016) 92.
[17] A. Korobeinikov, P.K. Maini, N-linear incidence and stability of infectious disease models, Math. Med. Biol. 22 (2005) 113–128.
[18] A. Taghvaei, T.T. Georgiou, L. Norton, A. Tannenbaum, Fractional SIR epidemiological models, Sci. Rep. 10 (2020) 20882.
[19] J. Satsuma, R. Willox, A. Ramani, B. Grammaticos, A.S. Carstea, Extending the SIR epidemic model, Physica A 336 (2004) 369–375.
[20] N. Sene, SIR epidemic model with Mittag-Leffler fractional derivative, Chaos Solitons Fractals 137 (2020) 109833.
[21] J. Hindes, M. Assaf, I.B. Schwartz, Outbreak size distribution in stochastic epidemic models, Phys. Rev. Lett. 128 (2022) 078301.
[22] M.R. SidiAmmi, M. Tahiri, M. Tilioua, A. Zeb, I. Khan, M. Andualem, Global analysis of a time fractional order spatio-temporal SIR model, Sci. Rep. 12 (2022) 5751.
[23] Z. Cao, X. Liu, X. Wen, L. Liu, L. Zu, A regime-switching SIR epidemic model with a ratio-dependent incidence rate and degenerate difusion, Sci. Rep. 9 (2019) 10696.
[24] M.L. Bertotti, M. Delitala, On the qualitative analysis of the solutions of a mathematical model of social dynamics, Appl. Math. Lett. 19 (10) (2006) 1107–1112.
[25] G. Toscani, A multi-agent approach to the impact of epidemic spreading on commercial activities, Math. Models Methods Appl. Sci. 32 (10) (2022) 1931–1948.
[26] T. Lazebnik, Computational applications of extended SIR models: A review focused on airborne pandemics, Ecol. Model. 483 (2023) 110422, http://dx.doi.org/10.1016/j.ecolmodel.2023.110422.
[27] B.V. Ram, L.P. Schaposnik, A modified age-structured sir model forcovid-19 type viruses, Sci. Rep. 11 (2021) 15194.
[28] C.F. Majid, M. Gray, A.M. Deshpande, S. Ramakrishnan, M. Kumar, S. Ehrlich, Non-pharmaceutical interventions as a control to mitigate the spread of epidemics: Analysis using a spatiotemporal pde model -19data, ISA Trans. 124 (2022) 215–224.
[29] T. Odagaki, New compartment model for COVID-19, Sci. Rep. 13 (2023) 5409, http://dx.doi.org/10.1038/s41598-023-32159-6.
[30] A. Kartono, S.V. Karimah, S.T. Wahyudi, A.A. Setiawan, I. Sofian, Forecasting the long-term trends of coronavirus disease 2019 (COVID-19) epidemic using the susceptible-infectious-recovered (SIR) model, Infect. Dis. Rep. 13 (2021) 668–684, http://dx.doi.org/10.3390/idr13030063.
[31] T.R. Liu, J. Huang, Z.L. He, Y. Zhang, N. Yan, C.J.P. Zhang, W.K. Ming, A real-world data validation of the value of early-stage SIR modeling to public health, Sci. Rep. 13 (2023) 9164, http://dx.doi.org/10.1038/s41598-023-36386-9.
[32] R.E. Mickens, An exactly solvable model for the spread of disease, College Math. J. 43 (2) (2012) 114–121.
[33] W.G. Faris, The SIR model of an epidemic, 2021, arXiv:2104.12029.
[34] M. Turkyilmazoglu, A restricted epidemic SIR model with elementary solutions, Physica A 600 (2022) 127570.
[35] W.M. Liu, et al., Influence of nonlinear incidence rates apon the behavior of SIRS epidemiology models, J. Math. Biol. 23 (1986) 187–204, http://dx.doi.org/10.1007/BF00276956.
[36] G. Kaniadakis, Novel predator–prey model admitting exact analytical solution, Phys. Rev. E 106 (2022) 044401.
[37] S.L. da Silva, A. Carbone, G. Kaniadakis, Quantifying population dynamics via a geometric mean predator–prey model, Chaos 33 (2023) 083102.
[38] M.A. Savageau, E.O. Voit, Recasting nonlinear differential equations as S-systems: a canonical nonlinear form, Math. Biosci. 87 (1987) 83.
[39] M.A. Savageau, Introduction to S-systems and the underlying power-law formalism, Math. Comput. Modelling 11 (1988) 546551.
[40] E.O. Voit, Canonical Nonlinear Modelling: S-System Approach To Understanding Complexity, Van Nostrand Reinhold, New York, 1991.
[41] E.O. Voit, M.A. Savageau, Equivalence between S-systems and Volterra systems, Math. Biosci. 78 (1986) 47.
[42] L. Euler, De serie lambertina plurimiscue eius insignibus proprietatibus, in: Acta Academiae Scientiarum Petropolitanae (1779: II, 1783), 29–51, Sankt Peterburg, Leonardi Euleri Opera Omnia, in: Series Prima Opera Mathematica, IV, 1921, pp. 350–369.
[43] G. Kaniadakis, M.L. Lissia, Editorial of news and expectations in thermostatistics - proceedings of the 2nd sardinian international conference on news and expectations in thermostatistics (NEXT2003), Villasimius (Cagliari) Italy, 21-28 2003, Physica A 340 (1-3) (2004) XV–XIX.
[44] https://www.wolframalpha.com/calculators/integral-calculator/.
[45] L.S. Gradshteyn, L.M. Ryzbik, Table of Integrals, Sixth, in: Series and Products, Academic Press, USA, 2000.