# Stabilization of EFIE-IBC by Spatial Filtering

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***Abstract*—Impedance Boundary Condition (IBC) is a widely used approximation in the analysis of metasurfaces, and it greatly simplifies the design process. However, for some ranges of impedance values of practical interest in metasurface applications, the Integral Equation formulation has shown instabilities. This contribution proposes a way to improve that shortcoming; the method is based on the property of the involved operators and the nature of the IBC approximation.**

***Index Terms*—Metasurfaces, Integral Equations, Impedance Boundary Conditions**

## I. Introduction

The design and prototyping of electromagnetic metasurfaces often entails complex multiscale mathematical models to accurately represent unit cells (sub-wavelength details) and the associated macro-scale structure, with the related computational challenges. To simplify these complex models, Surface Impedance Boundary Conditions (IBC) are widely used at the macroscopic design level, and incorporated into the Maxwell boundary-value problem.

The addition of the IBC into the Integral Equation (IE) formulation has been shown to have problems for certain ranges of impedance values, though [1]; in particular, it suffers from poor conditioning under specific impedance conditions. This happens when the body under examination is considered impenetrable, and the IBC one-sided (a.k.a. opaque). Literature suggests using a stabilized CFIE-IBC [2] for finite-sized volumes, but leaving the EFIE-IBC formulation problematic for thin, flat structures. The problematic values are unfortunately those that support guided surface wave in metamaterials. This is somewhat obviated in [1], that proposes employing IBC only on thin, penetrable sheets and describing the rest as it is (e.g., a grounded dielectric substrate), of course increasing the computational complexity. This covers a large swat of metamaterial applications, but still leaves out IBC ranges that would enlarge the flexibility in design; also, it does not solve the problem for stand-alone thin screens without backing, and for cases (e.g., all-metal pillars) where a one-sided (impenetrable) description would be preferable,

This work focuses on addressing the above instabilities. Preliminary accounts of this endeavor were presented in the conference paper [3].

## II. Background

In the following, we will restrict our study to a thin open planar structure (denoted $\Sigma$) surrounded by vacuum; a layered and possibly grounded medium can be added at the only cost of employing the appropriate Green's function in the integral equation listed in the following. The IBC (1) is used to represent the effect of the metasurface patterning in the homogenization approximation,

$$\hat{n} \times \mathbf{E}(\mathbf{r}) = \hat{n} \times \left( Z_s \underline{\underline{\mathcal{I}}}(\hat{n} \times \mathbf{H}) \right)(\mathbf{r}) \qquad \mathbf{r} \in \Sigma \tag{1}$$

and for the sake of simplicity here we will restrict our analysis to scalar (isotropic) impedance; this results in the EFIE-IBC integral equation

$$\eta \hat{n} \times \mathcal{L}(\mathbf{J})(\mathbf{r}) + Z_s \hat{n} \times \mathbf{J}(\mathbf{r}) = \hat{n} \times \mathbf{E}^{inc}(\mathbf{r}) \qquad \mathbf{r} \in \Sigma \tag{2}$$

for the unknown equivalent current density $J$; in the above

$$\mathcal{L}(X)(\mathbf{r}) = jk_0 \int_S \left( \mathcal{I} + \frac{1}{k_0^2} \nabla\nabla\cdot \right) \mathcal{G}_0(\mathbf{r}, \mathbf{r}') \cdot X(\mathbf{r}') d\mathbf{r}' \tag{3}$$

$$\mathcal{G}_0(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk_0 R}}{4\pi R} \tag{4}$$

It is noted that the Magnetic Field Integral operator is not involved as the surface is planar.

The Method of Moments approach, with a discretization of the current $\mathbf{J}$ into a linear combination of Rao-Wilton-Glisson (RWG) functions [4], with the classical Galerkin test method yields to a matrix problem

$$\left[ \mathbf{L} + z_s \mathbf{G}^{\mathbf{\Lambda}} \right] \cdot \mathbf{J}^{\mathbf{\Lambda}} = \frac{1}{\eta} V^E \tag{5}$$

where, $\mathbf{L}$ is the tested EFIE operator, $G^{\Lambda}$ is the Gram matrix associated to the RWG basis, and $V^E$ denotes the right-hand vector of the linear system.

As seen in Fig. 1b, the inclusion of the IBC term substantially impacts the EFIE-IBC system stability, specifically when dealing with inductive IBC values.

## III. Spatial Filter Re-conditioning

To elaborate on the ill-conditioning, consider a Perfect Electric Conductor (PEC). The issue with PEC is represented by the EFIE (corresponding to Z = 0 in (2)), a first-kind equation. In this case, the singularity within the EFIE operator ensures that the problem is well-posed. Unfortunately, when introducing the EFIE-IBC, we add an identity operator weighted by the impedance value Zs. This alteration significantly affects the diagonal terms of the EFIE, leading to ill-conditioning for specific surface impedance ranges.

Thinking of the identity operator in terms of an integral operator, it is apparent that its kernel is a Dirac delta; in turn, this means that its spectral representation is totally flat. Here and in the following, we will use the term "spectral" to mean the Fourier (Plane-Wave) spectrum. This is a 0-th order approximation of the homogenization implied in the IBC, and it appears to be justified only in a subset of cases.

Instructed by the above considerations, we undertake to control the spectrum of the solution, to avoid the spectral region associated with instabilities.

This is naturally done in the (Fourier, Plane-Wave) spectral domain; to keep the complexity at a minimum in this study, though, we resort to a set of basis functions that possess spectral resolution, i.e., mimicking a Fourier series. This is conveniently done on a rectangular domain, where basis functions with this spectral property, and the necessary div-conforming property are easily found as the vector eigenfunctions (modes) of the 2D Helmholtz eigenproblem (waveguide cross-section) [5] to build this second set. As already shown in [6] these entire-domain can be effectively expressed in terms of the underlying RWG discretization,

$$\mathbf{\Phi}_k(\mathbf{r}) = \sum_{n=1}^{N} \psi_{nk} \mathbf{\Lambda}_n(\mathbf{r}) \tag{6}$$

which induces the dual spatial-spectral representation of the current

$$\mathbf{J}^{\mathbf{\Lambda}}(\mathbf{r}) = \sum_{n=1}^{N} j_n^{\mathbf{\Lambda}} \mathbf{\Lambda}_n(\mathbf{r}) \qquad \mathbf{J}^{\mathbf{\Phi}}(\mathbf{r}) = \sum_{k=1}^{K} \mathrm{j}_k^{\mathbf{\Phi}} \mathbf{\Phi}_k(\mathbf{r}) \tag{7}$$

The latter is algebraically equivalent to basis change transformation (8),

$$\mathbf{\Phi}_k(\mathbf{r}) = \sum_{n=1}^{N} \psi_{nk} \mathbf{\Lambda}_n(\mathbf{r}) \quad \longrightarrow \quad j_k^{\mathbf{\Phi}} = \sum_{n=1}^{N} \psi_{nk} j_n^{\mathbf{\Lambda}} \tag{8}$$

explicit as in terms of the basis change matrix $\psi$ on (5) leads to the filtering EFIE-IBC matrix problem formulation

$$\psi^H \cdot \left[\mathbf{L} + z_s \mathbf{G}^{\mathbf{\Lambda}}\right] \cdot \psi \cdot \mathbf{J}^{\mathbf{\Phi}} = \frac{1}{\eta_0} \psi^H \cdot V^E, \tag{9}$$

with $\psi^H$ indicating the conjugate transpose of matrix $\psi$.

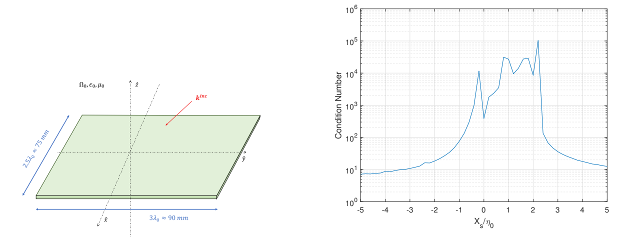
We notice that the same operations can be executed using Fast Fourier Transform (FFT), ensuring compatibility with fast methods. However, here we aim at assessing the effect of the spectral filtering, the detailed implementation of this aspect extending beyond the scope of our present work.

## IV. Numerical Results

To illustrate our filtering model, we will consider a XY plate of size $3\lambda_0 \times 2.5\lambda_0$ without thickness, centered at the origin and surrounded by vacuum. The plate is illuminated by a 10 GHz incident plane wave, with incident wave-vector $k^{inc} = [k_x^{inc}, k_y^{inc}, k_z^{inc}]$. This simplified model is used to approximate a thin planar structure once the isotropic IBC is applied.

A first approach, is to verify that our filter offers a "non-invasive" solution for non-problematic IBC values: as an example, we can select, from 1b, the capacitive value $Z_s^c = -j2.5\eta_0$. In this case, the solution is stable and accurate, so we can consider the difference between this reference solution and the obtained with the filtered equation. The condition number of the filtered problem remains very low, and the error with respect to the reference solution remains very low; it decreases for increasing spectral occupation of the basis, as expected.

Then, we consider a problematic case, viz. the inductive $Z_s = j2.5\eta_0$, with a clear bad conditioning, $\kappa^i = 37094$ (Fig. 1b).




(a) Scheme of the model

(b) Condition number of the EFIE-IBC on the model according to $Z_s = jX_s$

Fig. 1: XY plate of size $3\lambda_0 \times 2.5\lambda_0$ meshed with $N_e = 1334$ triangular cells ($N^{\Lambda} = 1953$ RWG basis functions)

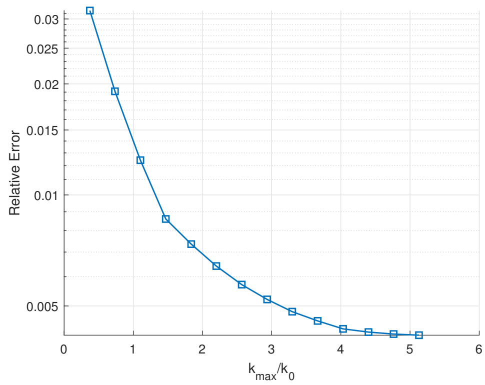



Fig. 2: Evolution of the relative error depending on the size of the filter for the capacitive example $Z_s = -j2.5\eta_0$

The behavior of the filtering on the EFIE-IBC for the new case is summarized in Figure 3. Since the standard formulation (5) is ill-conditioned, we cannot measure the distance between the original and filtered solution; hence we measure the distance between the original model and the filtered one via the residue of (5)

$$r = ||\mathbf{L} + z_s \mathbf{G}^{\mathbf{\Lambda}} \cdot \mathbf{J}^{\mathbf{\Lambda}} - \frac{1}{\eta} V^E||_2 \tag{10}$$

Hence, Fig. 3 reports on the same graph the condition number and and the (normalized) residue $r$.

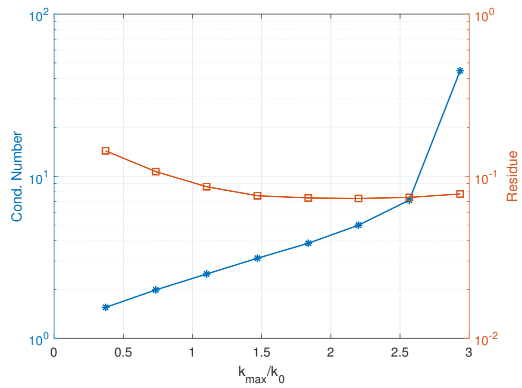
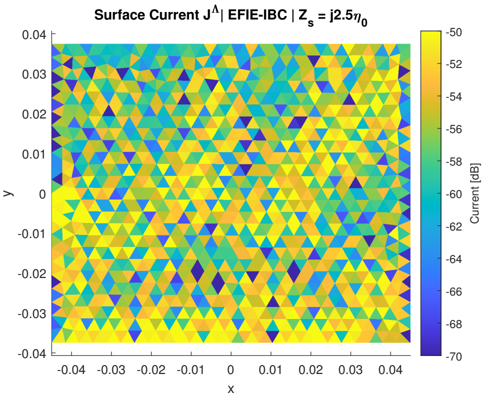


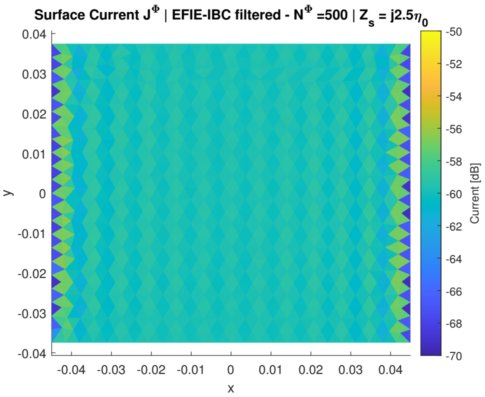

Fig. 3: Evolution of the residue and condition number depending on the size of the filter for the degenerative inductive example $Z_s = j2.5\eta_0$

We notice here encouraging results since the condition number, even if getting higher with the size of the filter, stays considerably lower than the one of the standard EFIE-IBC ($\approx 10^2$ order instead of the original $10^4$ conditioning). In the same time, the residue keeps relatively flat along the evolution of the filter size, showing the stability of the solution obtained.

Moreover, a comparison between the two electric currents can be done, with a suitable choice of spatial range; we choose here to fix it to $N^{\Phi} = 500$ based on the knowledge from figure 3. The figure 4 illustrates the advantage of the filtering method, by removing the "noise" of the solution $\mathbf{J}^{\Lambda}$ in figure 4a, to get the new solution $\mathbf{J}^{\Phi}$ (in figure 4b).




(a) Solution obtained using the standard EFIE-IBC ($N^{\Lambda} = 1953$ unknowns).




(b) Solution obtained with the EFIE-IBC filtering model for $k_{max}/k_0 = 2.39$

Fig. 4: Surface Current of the XY plate for an inductive $Z_s = j2.5\eta_0$

## ACKNOWLEDGMENT

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