Pseudo-Transient Continuation for Enhanced Quadratic Programming and Optimal Control

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I. Introduction

Quadratic programming (QP) solvers that join effectiveness with a simple implementation are becoming essential in the field of optimal control, specifically when dealing with real-time applications with strict timing constraints and limited computational resources. To address this need, we present a novel high-performance QP solution method based on pseudo-transient continuation (PTC). PTC is a numerical technique that transforms multivariate nonlinear equations into autonomous systems that converge to the solution sought. In our approach, we recast the general QP Karush-Kuhn-Tucker (KKT) conditions into a system of equations and employ PTC to solve the latter to attain the optimal solution. Importantly, we provide theoretical guarantees demonstrating the global convergence of our PTC-based solver to the optimal solution of any given QP. To showcase the effectiveness of PTC, we employ it within the domain of Model Predictive Control (MPC). Specifically, numerical simulations are carried out on the MPC control of a quadrotor – a demanding dynamical system – highlighting excellent results in accurately executing the control task and ensuring lower computational times compared to conventional QP solvers.

II. Pseudo-Transient Continuation for Quadratic Programming

Consider a multivariate nonlinear equation in the form

$$F(x)=0, \quad F:\mathbb{R}^n\to\mathbb{R}^n, \tag{1}$$

having a set of solutions $S=\{x\in\mathbb{R}^n : F(x)=0\}$. Pseudo-transient continuation (PTC) [1] seeks a functional $f(F(x)):\mathbb{R}^n\to\mathbb{R}^n$ such that the autonomous system

$$\dot{x}=f(F(x)) \tag{2}$$

has a set of equilibrium points $S^*\subseteq S$ and converges (at least locally) to one of such equilibria, i.e.,

$$f(F(x(t)))\to 0, \;\; x(t)\to x^*\in S^* \;\text{ for }\; t\to+\infty. \tag{3}$$

In this work, we leverage PTC to efficiently solve general convex quadratic programs (QPs) in the following form:

$$\min_y \frac{1}{2}y^\top Hy+c^\top y \quad \text{s.t.} \quad Cy=p, \quad Dy\le q, \tag{4}$$

where $y\in\mathbb{R}^n$ is the vector of decision variables; $H\in\mathbb{R}^{n\times n}$, $H=H^\top\succ 0$; $C\in\mathbb{R}^{N_E\times n}$ and $p\in\mathbb{R}^{N_E}$ represent the N_E equality constraints, C has full row rank; $D\in\mathbb{R}^{N_I\times n}$ and $q\in\mathbb{R}^{N_I}$ represent the N_I inequality constraints. Since the cost and the inequality constraints of (4) are convex functions, it admits a unique global minimum y^*. PTC allows to solve (4) for its global optimum with high computational performance. In this perspective, two fundamental steps have to be performed:

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1. by manipulating the Karush-Kuhn-Tucker (KKT) conditions associated with (4), recast them as a system of equations like (1), having as unique solution the global optimum of (4);
2. derive sufficient conditions characterizing F such that, for given functionals f, the global asymptotic convergence of (2) to its unique equilibrium is guaranteed.

These two steps are assessed in Sections II-A and II-B, respectively.

A. Conversion of QP KKT Conditions into a System of Equations

Let us consider the Lagrangian of (4), i.e.,

$$\mathcal{L}(y,\mu,\lambda)=\frac{1}{2}y^\top Hy+c^\top y-\mu^\top(Cy-p)-\lambda^\top(Dy-q), \tag{5}$$

where $\mu\in\mathbb{R}^{N_E}$ and $\lambda\in\mathbb{R}^{N_I}$ are the Lagrange multipliers. If a triple (y^*,μ^*,λ^*) satisfies the KKT conditions, i.e.,

$$\nabla_y\mathcal{L}(y,\mu,\lambda)=Hy+c-C^\top\mu-D^\top\lambda=0, \quad Cy=p, \tag{6a}$$
$$Dy\le q, \quad \lambda\le 0, \quad \lambda^\top(Dy-q)=0, \tag{6b}$$

then y^* is also the global minimum of (4). According to [2], conditions (6b) can be equivalently rewritten as

$$(D)_{i,\cdot}\,y=q_i \;\text{ if }\; \lambda_i\le 0, \quad (D)_{i,\cdot}\,y<q_i \;\text{ if }\; \lambda_i=0, \tag{7}$$

where $(D)_{i,\cdot}$ denotes the i-th row of D. (7) is also equivalent to the following system of piecewise affine equations [2]:

$$Dy=\phi_{[-\infty,q]}(Dy-\alpha\lambda), \tag{8}$$

where $\alpha\in\mathbb{R}_{>0}$ and $\phi_{[a,b]}(z)$ denotes the asymmetric saturation of the vector $z\in\mathbb{R}^{N_I}$ by $[a,b]$, whose expression is

$$\phi_{[a,b]}(z)\equiv\left[\phi_{[a_i,b_i]}(z_i)\right]_{i=1}^{N_I}, \; \phi_{[a_i,b_i]}(z_i)=\begin{cases} a_i & \text{if } z_i<a_i,\\ z_i & \text{if } a_i\le z_i\le b_i,\\ b_i & \text{if } z_i>b_i.\end{cases} \tag{9}$$

Hereafter, for notation clarity, we denote $\phi_{[-\infty,q]}$ as ϕ.

The KKT conditions (6) can be then rewritten as a system of equations as follows:

$$Hy+c-C^\top\mu-D^\top\lambda=0, \quad Cy-p=0, \tag{10a}$$
$$Dy-\phi(Dy-\alpha\lambda)=0. \tag{10b}$$

The system of equations (10) can be simplified as

$$y=G'D^\top\lambda+h', \tag{11a}$$
$$\mu=(CH^{-1}C^\top)^{-1}(p-CH^{-1}(D^\top\lambda-c)), \tag{11b}$$
$$G\lambda+h-\phi((G-\alpha I)\lambda+h)=0, \tag{11c}$$

where

$$\begin{aligned} G'&=H^{-1}-H^{-1}C^\top(CH^{-1}C^\top)^{-1}CH^{-1},\\ h'&=H^{-1}(C^\top(CH^{-1}C^\top)^{-1}(CH^{-1}c+p)-c),\\ G&=DG'D^\top, \quad h=Dh'. \end{aligned} \tag{12}$$

Only (11c) has to be solved for λ, since y and μ are function of λ only. Also, since (11) admits a unique solution (y^*,μ^*,λ^*), then λ^* is the unique solution of (11c).

B. Global Convergence of PTC for QPs Solution

To prove that any QP problem (4), when rewritten as a system of equations (11), can be effectively solved through PTC and exhibits

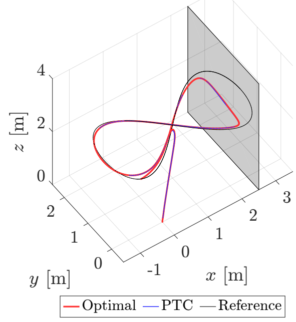

Figure 1. MPC quadrotor control: comparison of PTC closed-loop trajectory with the globally optimal one.

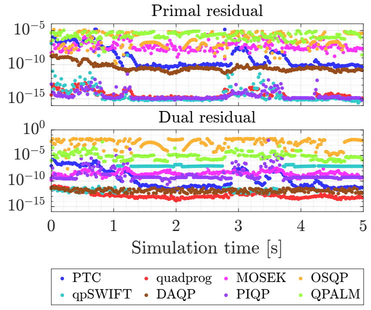

Figure 2. Primal and dual residuals on the MPC solution obtained by each solver.

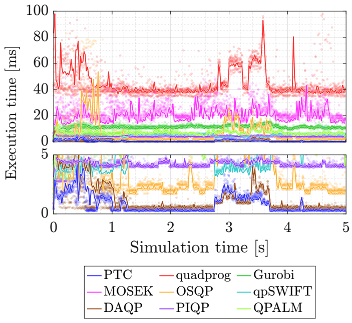

Figure 3. Execution time of each MPC control step, with detail of the range $[0, 5]$ ms.

Solver	Exec. time		Iterations	
	Max	Mean	Max	Mean
PTC	4.19	0.85	20	3
quadprog	98.04	45.02	90	7
Gurobi	14.23	10.96	24	16
MOSEK	43.20	20.17	49	24
OSQP	53.64	6.17	2875	329
qpSWIFT	6.33	4.73	18	14
DAQP	8.74	1.43	23	3
PIQP	4.98	4.28	13	11
QPALM	13.29	6.82	49	28

Table 1. Execution time (in ms) and iterations (single MPC control step).

global convergence to its optimum, we introduce Theorem 1, reporting a sufficient condition for the PTC autonomous system (2) to be globally asymptotically stable in the sense of Lyapunov.

Theorem 1. *Assume that: i) the multivariate nonlinear equation* (1)*, i.e., $F(x) = 0$, has a unique solution x^*; ii) there exists a symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$ such that*

$$(x - x^*)^\top M F(x) > 0, \quad \forall x \neq x^*. \tag{13}$$

Then, the system

$$\dot{x} = -\beta F(x), \quad \beta \in \mathbb{R}_{>0} \tag{14}$$

has a unique equilibrium point in x^* and such equilibrium is globally asymptotically stable (GAS).

Proof. We refer the reader to the full paper [3]. □

We then show, in Proposition 1, that the KKT system of equations (11) satisfies Theorem 1 and, thus, can be solved via PTC.

Proposition 1. *Let us consider* (11c) *and denote it as $F(\lambda) = 0$. Then, the system*

$$\dot{\lambda} = -\beta F(\lambda), \quad \beta \in \mathbb{R}_{>0} \tag{15}$$

has a unique equilibrium point λ^*, coinciding with the solution of (11c)*, and such equilibrium is GAS.*

Proof. We refer the reader to the full paper [3]. □

III. Application of PTC to Model Predictive Control

To assess the performance of PTC, we consider the following Model Predictive Control (MPC) problem:

$$\begin{aligned} \min_{\hat{u}_{\cdot|k}, \hat{x}_{\cdot|k}} \quad & J_k(\hat{u}_{\cdot|k}, \hat{x}_{\cdot|k}) \\ \text{s.t.} \quad & i = 0, 1, \dots, N_p - 1, \\ & \hat{x}_{0|k} = x_k, \quad \hat{x}_{i+1|k} = A_k \hat{x}_{i|k} + B_k \hat{u}_{i|k} + b_k, \quad (16a) \\ & \underline{u} \le \hat{u}_{i|k} \le \overline{u}, \quad \underline{x} \le \hat{x}_{i|k} \le \overline{x}, \quad (16b) \end{aligned}$$

$$J_k(\hat{u}_{\cdot|k}, \hat{x}_{\cdot|k}) = \sum_{i=0}^{N_p-1} \left(\|\hat{x}_{i|k} - x_{r,k+i}\|_Q^2 + \|\hat{u}_{i|k}\|_R^2 \right) + \sum_{i=1}^{N_p-1} \left(\|\hat{u}_{i|k} - \hat{u}_{i-1|k}\|_{R_\Delta}^2 \right) + \|\hat{x}_{N_p|k} - x_{r,k+N_p}\|_P^2, \tag{16c}$$

where $\hat{u}_{i|k} \in \mathbb{R}^{n_u}$, $\hat{x}_{i|k} \in \mathbb{R}^{n_x}$ are the inputs and states predicted i steps ahead at time k, respectively; x_r is the state reference trajectory; (16c) is the MPC cost function ($\|x\|_M^2 \equiv \frac{1}{2} x^\top M x$); (16a) are the prediction model constraints; (16b) are inputs and states constraints. The MPC optimal control problem (16) can be rewritten to match the QP formulation (4); thus, it can be fast solved for its global optimum by PTC with global convergence guarantees, in view of the results presented in Section II.

As nonlinear plant to control, we select the Euler-Lagrange quadrotor model in [4]. The control task is to track a lemniscate reference trajectory, partially crossing an infeasible region of space (see Figure 1). To deploy the linear MPC (16) to control the nonlinear continuous-time plant, we adopt the sequential quadratic programming (SQP) approach with real-time iteration (RTI) to discretize and linearize the plant.

IV. Simulations and Results

PTC is compared with the following conventional QP solvers: the active-set solvers quadprog and DAQP; the interior-point solvers Gurobi, MOSEK, qpSWIFT, and PIQP; the operator splitting solver OSQP; the augmented Lagrangian solver QPALM.

Simulations are performed with MATLAB® 2023b on a 13^{th} Gen Intel® Core™ i7 CPU at 1.7 GHz. The full source code is available online[1]. The PTC autonomous system (15) is numerically integrated using the explicit Runge-Kutta 2(3) method.

Figure 1 reports the quadrotor closed-loop trajectory, obtained by solving the MPC problem (16) with PTC. This trajectory is rather coincident with the globally optimal one (estimated with Gurobi by setting very low optimality tolerances).

To further assess the goodness of the obtained MPC solutions, Figure 2 reports the primal and dual residuals at each time instant and for every solver. We see that PTC consistently achieves low residuals that stay below the thresholds defined by the optimality criteria. On the contrary, some of the other solvers fail to deliver an acceptable dual residual.

Finally, Figure 3 compares, for each solver, the execution time of each MPC control step, reporting both the actual time values (solid lines) and Monte Carlo envelopes (scatter plots), obtained by randomly selecting 50 initial states x_0 from the feasible set $[\underline{x}, \overline{x}]$. Results are summarized in Table 1, including the number of iterations required by each solver. We observe that PTC outperforms all other solvers in terms of both maximum and average computational time.

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