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Detecting Stubborn Behaviors in Influence Networks: A Model-Based Approach for Resilient Analysis

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***Abstract*—The wide spread of on-line social networks poses new challenges in information environment and cybersecurity. A key issue is detecting stubborn behaviors to identify leaders and influencers for marketing purposes, or extremists and automatic bots as potential threats. Existing literature typically relies on known network topology and extensive centrality measures computation. However, the size of social networks and their often unknown structure could make social influence computation impractical.**

We propose a new approach based on opinion dynamics to estimate stubborn agents from data. We consider a DeGroot model in which regular agents adjust their opinions as a linear combination of their neighbors' opinions, whereas stubborn agents keep their opinions constant over time. We formulate the stubborn nodes identification and their influence estimation problems as a low-rank approximation problem. We then propose an interpolative decomposition algorithm for their solution. We determine sufficient conditions on the model parameters to ensure the algorithm's resilience to noisy observations. Finally, we corroborate our theoretical analysis through numerical results.

***Index Terms*—Network analysis and control; Identification; Social networks; Detection.**

I. Introduction

ON-LINE social networks have reached a large pervasiveness and relevance in opinion formation and content dissemination. Understanding the dynamics of information propagation within these networks is crucial for predicting inclinations and preferences in order to design targeting actions or to prevent critical issues. However, individuals often exhibit stubborn behaviors [1] and tend to resist changes, leading to the emergence of phenomena such as opinion polarization and the diffusion of fake news [2]. For this reason, a main research problem in opinion dynamics is the detection of stubborn individuals [2], [3], extremists, and automatic bots [2], [4], [5], providing tools to intervene and take specific countermeasures.

Most state-of-the-art inference methods are based on the extensive collection and processing of data, often requiring preliminary network knowledge [2], [5], [6] and the computation of centrality measures [7]. This approach implies multiple limitations, as observing and analyzing the complete structure of social networks can be difficult or totally impossible. Examples can be deliberative groups or forum discussions, where interpersonal influences can be inferred only a posteriori after deliberation [8], or platforms like *Truth* or *4chan*, for which there is no clear definition of social structure due to user anonymity and without the concept of “friends” or “followers”. Classical system identification techniques used in control theory (see [9] and references therein) also find limited applicability in this context. Indeed, without additional assumptions about the network structure, they typically require a number of observations proportional to the number of links, leading to a too high computational cost [10].

In this paper, we address the issue of detecting stubborn agents without network knowledge and without the need to reconstruct it. Specifically, we consider the classical French-DeGroot opinion dynamics model where some of the agents are stubborn and persistently express a fixed opinion. This is equivalent to the Friedkin-Johnsen model and is a well-established framework in sociology, already adopted to model opinion formation in deliberative groups [8] and validated with real experimental data [11]. In the asymptotic equilibrium, each non stubborn agent reaches an opinion that is a convex combination of the stubborn agents' ones. Weights of such combinations depend on the topology of interactions and form the so called influence matrix. We assume our data to be a noisy observation of the complete set of such equilibrium opinions for a number of different discussions (e.g. initial conditions) and our goal is to estimate the set of stubborn nodes and the influence matrix.

Our contribution is two-fold. First, we formulate the estimation problem model-based approach as an optimization problem tailored to be computationally efficiently solved by Interpolative Decomposition techniques. Second, we derive

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sufficient conditions on the model parameters and on the noise's size guaranteeing that any solution of the optimization problem will correctly determine the set of stubborn nodes and well approximate the influence matrix. Finally, we corroborate our analysis through numerical results.

II. Stubborn nodes detection and influence estimation in social networks

Let us first introduce some notational conventions adopted throughout the paper. Matrices are bold capital, vectors are bold lowercase and scalars or entries are not bold. $\boldsymbol{X}^\top$ and $\boldsymbol{x}^\top$ are the transpose of matrices $\boldsymbol{X}$ and vector $\boldsymbol{x}$ respectively. We denote the identity matrix by $\boldsymbol{I}$, where the dimension will be clear from the context. For a rectangular matrix $\boldsymbol{X}$, $\sigma_k(\boldsymbol{X})$ denotes its k-th largest singular value and $\operatorname{rank}(\boldsymbol{X})$ its rank. The spectral norm of a matrix is denoted by $\|\boldsymbol{X}\|_2$ and it coincides with the maximum singular value $\sigma_1(\boldsymbol{X})$. The Frobenius or Hilbert-Schmidt norm is denoted by $\|\boldsymbol{X}\|_F$. The maximum entry of $\boldsymbol{X}$ (in absolute value) is denoted by $\|\boldsymbol{X}\|_{\max} = \max_{(i,j)} |X_{ij}|$. A matrix $\boldsymbol{X}$ is row stochastic when its entries are non-negative and $\boldsymbol{X}\mathbb{1} = \mathbb{1}$. A matrix $\boldsymbol{X}$ is said to be Schur stable if the absolute value of all its eigenvalues is strictly smaller than 1. Given a matrix $\boldsymbol{X}$ in $\mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ where $\mathcal{V}$ is a finite set and given $\mathcal{A},\mathcal{B} \subseteq \mathcal{V}$, we indicate with $\boldsymbol{X}_{\mathcal{A}\mathcal{B}}$ the submatrix of $\boldsymbol{X}$ having rows in $\mathcal{A}$ and columns in $\mathcal{B}$. We use the notation $\boldsymbol{X}_{\mathcal{B}}$ for $\boldsymbol{X}_{\mathcal{V}\mathcal{B}}$.

A. Opinion dynamics model

We model social influence networks as finite directed weighted graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{Q})$ where the set of nodes $\mathcal{V}$ represents the agents, the set of (directed) edges $\mathcal{E}$ represents pairwise interactions, and the weight matrix $\boldsymbol{Q}$ in $\mathbb{R}_+^{\mathcal{V}\times\mathcal{V}}$ is such that $(i,j) \in \mathcal{E}$ if and only if $Q_{ij} > 0$, in which case Q_{ij} measures the strength of the direct influence of j on i.

Every agent i in $\mathcal{V}$ is endowed with a state $x_i(t)$ in $\mathbb{R}$ representing its opinion at time $t = 0, 1 \ldots$. Opinions vary in time according to the French-DeGroot model with stubborn agents. Precisely, let the set of nodes be partitioned into two disjoint sets: $\mathcal{V} = \mathcal{R}^* \cup \mathcal{S}^*$. Nodes in $\mathcal{S}^*$ (called stubborn agents) maintain their opinion fixed at all times, while nodes in $\mathcal{R}^*$ (regular agents) update their opinion as follows:

$$x_i(t+1) = \begin{cases} \sum_{j\in\mathcal{V}} P_{ij} x_j(t), & \forall i \in \mathcal{R}^* \\ x_i(t), & \forall i \in \mathcal{S}^* \end{cases} \tag{1}$$

where $P_{ij} = Q_{ij}/(\sum_h Q_{ih})$ is the nomalized weight of link (i,j). We assemble opinions of regular and stubborn agents into row vectors, respectively, $\boldsymbol{x}_{\mathcal{R}^*}(t)$ and $\boldsymbol{x}_{\mathcal{S}^*}$ (note that the latter does not depend on t), and all normalized weights P_{ij} in a matrix $\boldsymbol{P}$.

The following result holds true [12].

Proposition 1: For a social influence network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{Q})$ and a partition $\mathcal{R}^* \cup \mathcal{S}^* = \mathcal{V}$ such that from every regular agent there exists a path in $\mathcal{G}$ to some stubborn agent, the matrix $\boldsymbol{I} - \boldsymbol{P}^T_{\mathcal{R}^*\mathcal{R}^*}$ is invertible and for every initial condition $\boldsymbol{x}(0) = (\boldsymbol{x}_{\mathcal{R}^*}(0), \boldsymbol{x}_{\mathcal{S}^*})$, the opinion dynamics (1) converges to a limit profile $\boldsymbol{x}^* = (\boldsymbol{x}_{\mathcal{R}^*}, \boldsymbol{x}_{\mathcal{S}^*})$ with

$$\boldsymbol{x}_{\mathcal{R}^*} = \boldsymbol{x}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*, \quad \boldsymbol{\Gamma}^* = \boldsymbol{P}^T_{\mathcal{R}^*\mathcal{S}^*}(\boldsymbol{I} - \boldsymbol{P}^T_{\mathcal{R}^*\mathcal{R}^*})^{-1} \in \mathbb{R}_+^{\mathcal{S}^*\times\mathcal{R}^*}. \tag{2}$$

The column-stochastic matrix $\boldsymbol{\Gamma}^*$ in (2) is referred to as the *influence matrix*: its entries Γ^*_{ij} measure the relative influence of stubborn agent i on the final opinion of regular agent j.

B. Stubborn nodes detection and influence estimation

For a social influence network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{Q})$, and a binary partition $\mathcal{V} = \mathcal{S}^* \cup \mathcal{R}^*$, consider the opinion dynamics as in Eq. (1) with distinct initial conditions representing different discussion topics l in $\mathcal{T}$. Stack the row vectors of the stubborn agents opinions in a matrix $\boldsymbol{X}_{\mathcal{S}^*}$ in $\mathbb{R}^{\mathcal{T}\times\mathcal{S}^*}$.

Assumption 1: The matrix $\boldsymbol{X}_{\mathcal{S}^*}$ has rank $k^* = |\mathcal{S}^*|$.

Assumption 2: There exists $\delta > 0$ such that for every v in $\mathcal{R}^*$, there exist $s_1 \neq s_2$ in $\mathcal{S}^*$ with $\min\{\Gamma^*_{s_1 v}, \Gamma^*_{s_2 v}\} \geq \delta$.

Assumption 1 requires that $|\mathcal{T}| \geq |\mathcal{S}^*|$, i.e., the number of topics is at least equal to the number of stubborn agents, and that the k^* column vectors that gather the initial conditions of each stubborn agent on the different topics are linearly independent. Assumption 2 requires instead that every regular agent is influenced by at least two stubborn agents and is thus stronger than the assumption in Proposition 1.

Both assumptions are necessary for the correct identification of the stubborn agents in a situation where the network is unknown and observations are only asymptotic. Indeed, if the profile of opinions of one stubborn agent s was a linear combination of the opinion profiles of other stubborn agents, s would be undistinguishable from a regular agent influenced by such stubborn agents through the same linear combination. Similarly, if a regular node v was influenced by just one stubborn node s, then the role of v and s could be interchanged with no possibility to distinguish their roles.

Notice that Assumption 1 holds true (with large probability) if stubborn nodes' opinions are considered exogenous inputs modeled as i.i.d. random variables, which is a common assumption in the opinion dynamics literature (see [13], [14]). Finally, observe that in many applicative scenarios for the detection of trolls and bots [2], [5], [6], [8] the number of stubborn nodes is a small percentage of the total number of nodes so that Assumption 1 is computationally feasible.

We assemble now the row vectors of the asymptotic opinions of all agents into a matrix $\boldsymbol{X}$ in $\mathcal{R}^{\mathcal{T}\times\mathcal{V}}$ so that $X_{\ell i}$ represents the asymptotic opinion of agent i in $\mathcal{V}$ under the discussion on topic ℓ in $\mathcal{T}$. It follows from Eq. (2) that

$$\boldsymbol{X}_{\mathcal{R}^*} = \boldsymbol{X}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*, \tag{3}$$

where $\boldsymbol{X}_{\mathcal{R}^*}$ and $\boldsymbol{X}_{\mathcal{S}^*}$ indicate the sub-matrices of $\boldsymbol{X}$ consisting of the columns in $\mathcal{R}^*$ and $\mathcal{S}^*$, respectively.

We now assume to have access to the observations

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{\Xi}, \tag{4}$$

where $\boldsymbol{\Xi}$ in $\mathbb{R}^{\mathcal{T}\times\mathcal{V}}$ is a noise matrix. Our goal is to identify the stubborn set $\mathcal{S}^*$ and estimate the influence matrix $\boldsymbol{\Gamma}^*$ starting from the observation of $\boldsymbol{Y}$, without any prior knowledge of the network. We formalize it as follows.

Problem 1 (Detection and estimation): For a tolerance $\varepsilon \geq 0$, consider the following low-rank approximation problem

$$\begin{array}{ll} \min_{\mathcal{S}\subseteq\mathcal{V}} |\mathcal{S}| & \\ \exists \boldsymbol{\Gamma} \in \mathbb{R}_{+}^{\mathcal{S}\times\mathcal{R}} & \text{s.t. } \|\boldsymbol{Y}_{\mathcal{R}} - \boldsymbol{Y}_{\mathcal{S}}\boldsymbol{\Gamma}\|_2 \leq \varepsilon,\ \|\boldsymbol{\Gamma}\|_{\max} \leq 1 \end{array} \tag{5}$$

We denote by $\widehat{\mathcal{S}}_\varepsilon$ a solution of the problem and with $\widehat{\boldsymbol{\Gamma}}_\varepsilon$ any corresponding matrix $\boldsymbol{\Gamma}$ of minimum 2-norm, satisfying the constraints.

Problem 1 is an optimization, with an external combinatorial optimization over a domain of exponential size $2^{|\mathcal{V}|}$. In spite of its complexity, we shall prove in the following sections how this problem can be efficiently attacked by Interpolative Decomposition (ID), a matrix decomposition method used to obtain approximate factorizations of low-rank matrices, specifically in the version proposed in [15]. ID allows optimizing over the domain specified in Eq. (5) (in particular to impose the non-negativity of the matrix $\boldsymbol{\Gamma}$) and, adopting the version in [15], to solve the optimization typically requiring only $O(|\mathcal{S}||\mathcal{R}||\mathcal{T}|)$ floating-point operations [16]. Numerical simulations based on ID will be presented in Section IV.

III. Theoretical analysis

We now present our main theoretical results regarding the behavior of solutions of Problem 1, particularly its consistency in the noise-free case and its robustness with respect to noise.

We first focus on the noise-free case. The following preliminary result ensures uniqueness of the partition $\mathcal{V} = \mathcal{R}^* \cup \mathcal{S}^*$ when a linear relation as Eq. (3) holds with $\boldsymbol{\Gamma}^*$ in $[0,1]^{\mathcal{S}\times\mathcal{R}}$.

Lemma 1: Let Assumptions 1 and 2 hold true. Let $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$ be a binary partition such that $k^* = |\mathcal{S}| = \operatorname{rank}(\boldsymbol{X}_{\mathcal{S}})$, $\mathcal{S} \neq \mathcal{S}^*$, and let $\boldsymbol{\Gamma}$ in $\mathbb{R}^{\mathcal{S}\times\mathcal{R}}$ be such that

$$\boldsymbol{X}_{\mathcal{R}} = \boldsymbol{X}_{\mathcal{S}}\boldsymbol{\Gamma}\,. \tag{6}$$

Then, there exists some (i,j) in $\mathcal{V}\times\mathcal{V}$ such that

$$\Gamma_{ij} \notin \left(-\frac{\delta}{2(1-\delta)}, 1 + \frac{\delta}{2(1-\delta)}\right)\,. \tag{7}$$

Proof: See Appendix A. ∎

A direct consequence of Lemma 1 is the following.

Proposition 2 (Noise free scenario): Let Assumptions 1 and 2 hold true. If $\boldsymbol{\Xi} = 0$ and $\varepsilon = 0$, then Problem 1 admits a unique optimal solution $(\widehat{\mathcal{S}}_0, \widehat{\boldsymbol{\Gamma}}_0) = (\mathcal{S}^*, \boldsymbol{\Gamma}^*)$.

Proof: First, note that, for $\varepsilon = 0$, the first constraint in Eq. (5) is equivalent to Eq. (6). Since Assumption 1 implies that Eq. (6) cannot be satisfied by any $\boldsymbol{\Gamma}$ if $|\mathcal{S}| < k^*$, there cannot be any feasible pair $(\mathcal{S}, \boldsymbol{\Gamma})$ for Problem 1 with $|\mathcal{S}| < k^*$ and the $\mathcal{S}$ columns of $\boldsymbol{X}$ must be independent, i.e. $\operatorname{rank}(\boldsymbol{X}_{\mathcal{S}}) = k^*$. On the other hand, it follows from Eq. (3) that $(\mathcal{S}^*, \boldsymbol{\Gamma}^*)$ is feasible for Problem 1 when $\boldsymbol{\Xi} = 0$ and $\varepsilon = 0$. Therefore, $(\mathcal{S}^*, \boldsymbol{\Gamma}^*)$ is an optimal solution for Problem 1.

Moreover, Lemma 1 implies that, if $|\mathcal{S}| = k^*$, $\mathcal{S} \neq \mathcal{S}^*$, then there is no $\boldsymbol{\Gamma}$ in $\mathbb{R}_{+}^{\mathcal{S}\times\mathcal{R}}$ s.t. $\|\boldsymbol{\Gamma}\|_{\max} \leq 1$ and Eq. (6) is satisfied. Hence, every optimal solution $(\widehat{\mathcal{S}}_0, \widehat{\boldsymbol{\Gamma}}_0)$ of Problem 1 is such that $\widehat{\mathcal{S}}_0 = \mathcal{S}^*$. Finally, observe that if $\widehat{\boldsymbol{\Gamma}}_0$ in $\mathbb{R}_{+}^{\mathcal{S}^*\times\mathcal{R}^*}$ is such that $\boldsymbol{X}_{\mathcal{R}^*} = \boldsymbol{X}_{\mathcal{S}^*}\widehat{\boldsymbol{\Gamma}}_0$, then $\boldsymbol{X}_{\mathcal{S}^*}(\boldsymbol{\Gamma}^* - \widehat{\boldsymbol{\Gamma}}_0) = 0$, so that Assumption 1 implies that $\widehat{\boldsymbol{\Gamma}}_0 = \boldsymbol{\Gamma}^*$. This proves that $(\mathcal{S}^*, \boldsymbol{\Gamma}^*)$ is the unique optimal solution for Problem 1. ∎

Let us denote by

$$\Delta := \Delta(\boldsymbol{X}, \mathcal{S}^*) = \frac{\|\boldsymbol{X}_{\mathcal{R}^*}\|_2}{\sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*})}\,,$$

the ratio between spectral norm of $\boldsymbol{X}_{\mathcal{R}^*}$ and the minimum positive singular value of $\boldsymbol{X}_{\mathcal{S}^*}$.

The following result provides sufficient conditions for every solution of Problem 1 to exactly detect the set of stubborn nodes $\mathcal{S}^*$ and accurately estimate the influence matrix $\boldsymbol{\Gamma}^*$.

Theorem 1: Let Assumption 1 and 2 hold true. If

$$\|\boldsymbol{\Xi}\|_2 \leq \gamma\sigma_{k^*}(\boldsymbol{X})/(2(1+\Delta))\,, \tag{8}$$

where $\gamma = \delta/((4-\delta)(1+\Delta) - 3\delta\Delta)$, and

$$\|\boldsymbol{\Xi}\|_2(1+\Delta) \leq \varepsilon \leq \gamma\sigma_{k^*}(\boldsymbol{X}) - \|\boldsymbol{\Xi}\|_2(1+\Delta)\,, \tag{9}$$

then every optimal solution $(\widehat{\mathcal{S}}_\varepsilon, \widehat{\boldsymbol{\Gamma}}_\varepsilon)$ of Problem 1 is such that

$$\widehat{\mathcal{S}}_\varepsilon = \mathcal{S}^*, \qquad \|\widehat{\boldsymbol{\Gamma}}_\varepsilon - \boldsymbol{\Gamma}^*\|_2 \leq \frac{\varepsilon + \|\boldsymbol{\Xi}\|_2(1+\Delta)}{\sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*})}\,. \tag{10}$$

Proof: The proof is obtained through intermediate steps (see Appendix B - D). ∎

Notice that Eq. (8) implies that the range of values of the admissible tolerance ε in Eq. (9) is nonempty. Thus, Theorem 1 guarantees that for sufficiently small noise, we can find tolerance values ε for which every optimal solution of Problem 1 correctly identifies the subset $\mathcal{S}^*$ and yields an influence matrix $\widehat{\boldsymbol{\Gamma}}_\varepsilon$ close to $\boldsymbol{\Gamma}^*$.

Example 1: Let $\boldsymbol{\Xi}$ have i.i.d. Gaussian random entries with zero mean and variance ξ^2. Then, by [17, Corollary 5.35], $\|\boldsymbol{\Xi}\|_2 \leq 3\xi\sqrt{n}$ with high probability as n grows large. Let also $\boldsymbol{X}_{\mathcal{S}^*}$ have i.i.d. sub-Gaussian random entries with zero mean and unitary variance and assume that $|\mathcal{S}^*|/n \to \theta \in (0,1)$ and $|\mathcal{T}|/n \to \beta > \theta$. Then, [17, Theorem 5.31] implies that

$$\begin{aligned} (\sqrt{\beta}-\sqrt{\theta})\sqrt{n} + o(\sqrt{n}) &\leq \sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*}) \\ &\leq \|\boldsymbol{X}_{\mathcal{S}^*}\|_2 \leq (\sqrt{\beta}+\sqrt{\theta})\sqrt{n} + o(\sqrt{n})\,, \end{aligned}$$

as $n \to +\infty$, so that with high probability

$$\Delta = \frac{\|\boldsymbol{X}_{\mathcal{R}^*}\|_2}{\sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*})} \leq \frac{\|\boldsymbol{X}_{\mathcal{S}^*}\|_2\|\boldsymbol{\Gamma}^*\|_2}{\sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*})} \leq \frac{\sqrt{\beta}+\sqrt{\theta}}{\sqrt{\beta}-\sqrt{\theta}}\|\boldsymbol{\Gamma}^*\|_2(1+o(1))\,.$$

It follows that the right hand side of Eq. (8) satisfies

$$\frac{\gamma\sigma_{k^*}(\boldsymbol{X})}{2(1+\Delta)} \geq \frac{\delta\sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*})}{2(4-\delta)(1+\Delta)^2} \geq \frac{\delta(\sqrt{\beta}-\sqrt{\theta})\sqrt{n}}{2(4-\delta)(1+\Delta)^2}(1+o(1))\,.$$

Therefore, Eq. (8) is satisfied with high probability if

$$\xi \leq \frac{\delta(\sqrt{\beta}-\sqrt{\theta})^3}{6(4-\delta)(\sqrt{\beta}-\sqrt{\theta}+(\sqrt{\beta}+\sqrt{\theta})\|\boldsymbol{\Gamma}^*\|_2)^2}\,,$$

subjected to $\|\boldsymbol{\Gamma}^*\|_2$ being bounded. Due to $\boldsymbol{\Gamma}^*$ stochasticity,

$$\|\boldsymbol{\Gamma}^*\|_2^2 \leq \|\boldsymbol{\Gamma}^*\|_1\|\boldsymbol{\Gamma}^*\|_\infty = \max_{s\in\mathcal{S}^*}\sum_{j\in\mathcal{R}^*}\Gamma^*_{js}\,,$$

so that $\|\boldsymbol{\Gamma}^*\|_2$ is guaranteed to remain bounded whenever the total influence of every stubborn agent remains bounded.

Remark 1 (Observation at finite time): The case when the opinions are observed at finite time $t < +\infty$ can be cast into our framework by letting $\boldsymbol{\Xi} = \boldsymbol{Y} - \boldsymbol{X}$ be the difference between the opinion matrix at time t and the equilibrium

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opinion matrix. As it is known [18] that $\|\boldsymbol{\Xi}\|_2 \leq \lambda_{\max}^{t_{\rm obs}} \|\boldsymbol{X}(0) - \boldsymbol{X}\|_2$, where $\lambda_{\max} < 1$ denotes the dominant eigenvalue of the substochastic matrix $\boldsymbol{P}_{\mathcal{R}^*\mathcal{R}^*}$, one can readily derive sufficient conditions on t to ensure that Eq. (8) holds true, thus enabling the application of Theorem 1.

IV. Numerical results

Here, we illustrate some numerical simulations to corroborate our theoretical results. The method was implemented using the *interpolative* module from Python's *scipy.linalg* library. The metrics employed to assess the accuracy of the prediction include:

- relative error on $\boldsymbol{\Gamma}^*$ estimate, defined as

$$\text{err} := ||\boldsymbol{\Pi}[\boldsymbol{I}\ \widehat{\boldsymbol{\Gamma}}_\varepsilon] - [\boldsymbol{I}\ \boldsymbol{\Gamma}^*]||_2 / ||[\boldsymbol{I}\ \boldsymbol{\Gamma}^*||_2$$

  where $\boldsymbol{\Pi}$ is a permutation matrix which reorders the columns to put on the left the ones corresponding to $\mathcal{S}^*$.
- true positive rate (or sensitivity) and false positive rates

$$\text{TPR} = |\mathcal{S}^* \cap \widehat{\mathcal{S}}_\varepsilon| / |\mathcal{S}^*| \quad \text{FPR} = |\mathcal{R}^* \cap \widehat{\mathcal{S}}_\varepsilon| / |\mathcal{R}^*| \,.$$

We consider two scenarios for our simulations. In the first one we assume the observations to be taken at steady state, while in the second observations are taken at finite time. For the sake of simplicity, in all the simulations the initial opinions are randomly generated uniformly in the range $[0, 1]$. We indicate with n the number of nodes, k^* the number of stubborn agents, and m the number of observations.

A. Watts-Strogatz influence network with noisy observations

We consider a French-DeGroot influence system over a Watts-Strogatz random graph, where observations are taken at steady state and are corrupted by Gaussian noise with zero mean and variance ξ^2.

In the first set of simulations, we have chosen the size equal to $n = 200$ nodes with $k^* = 12$ stubborn nodes. Figure 1 displays the indices err, TPR, and FPR, as function both of variance ξ^2 and number of observations m, for a fixed tolerance ε satisfying Eq. (9). As expected from Theorem 1, we notice the existence of a threshold $\bar{\xi} > 0$ such that if $\xi < \bar{\xi}$ the algorithm succeeds both in the estimation of $\boldsymbol{\Gamma}^*$ and in the correct detection of stubborn nodes $\mathcal{S}^*$. Conversely, the stubborn nodes are overestimated and the relative error on $\boldsymbol{\Gamma}^*$ prediction is higher. However, simulations point out the robustness of the method that produces a good estimation of $\boldsymbol{\Gamma}^*$ even for variance values greater than $\bar{\xi}$. Finally, we observe that in case of high noise on data the stubborn nodes are overestimated (see the two indices TPR and FPR in Figure 1): all stubborn nodes are indeed correctly detected as stubborn (no false negative) together with some nodes that were instead regular (false positive).

Finally, the plot in Figure 2 shows the performance of the method with respect to network size. For a fixed fraction of stubborn nodes equal to 20% of the nodes, the plot shows the behavior of the relative error as a function of the number of nodes for different percentages of observations. The simulation corroborates our approach highlighting its scalability with respect to network size.

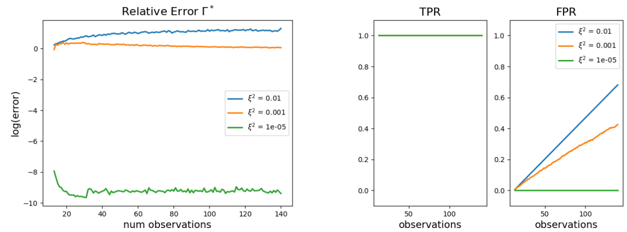


Fig. 1: Watts-Strogatz graph with $n = 200$, $k^* = 12$. Additional noise $\boldsymbol{\Xi} \sim N(0, \xi^2)$. Relative error on $\boldsymbol{\Gamma}^*$, TPR and FPR for stubborn nodes detection, for $\xi^2 \in \{10^{-2}, 10^{-3}, 10^{-5}\}$.

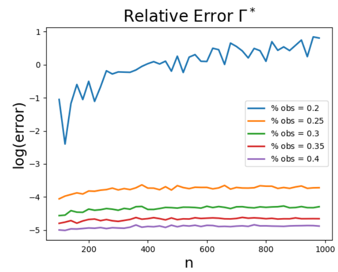
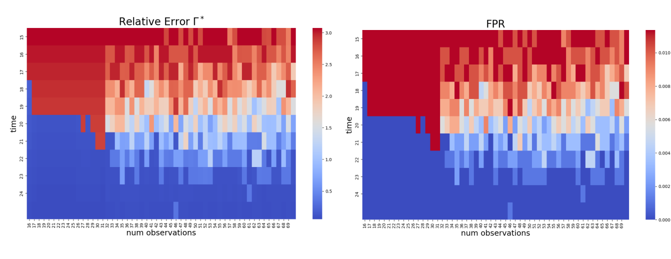


Fig. 2: Relative error as a function of size of the network n with $k^*/n = 0.2$ and $m/n \in \{0.2, 0.25, 0.3, 0.35, 0.4\}$.

B. A case with finite-time observations

Here, we collect opinions over a certain time interval for a Watts-Strogatz random graph with $n = 100$ nodes and $k^* = 12$ stubborn nodes. For the sake of simplicity, we assume no observation noise. Figure 3 illustrates the proposed method's performance, i.e., the average relative error on the estimation $\boldsymbol{\Gamma}^*$ and the average false positive rate as function of different parameters. The color intensity of each cell represents the magnitude of the error, where blue color indicates success and red color highlights error. We limit the time window shown to the interval $15 \leq t \leq 25$ observing that for $t > 24$ the behavior is comparable with the equilibrium one. Consistently with Remark 1, as the observation time increases, the method's performance improves. The TPR is not displayed in figure because all stubborn nodes are accurately detected. The FPR error shown in Fig. 3-(b) can instead be interpreted



(a) (b)

Fig. 3: Watts-Strogatz graph with $n = 100$, $k^* = 12$. (a) Relative error on $\boldsymbol{\Gamma}^*$ estimation; (b) FPR for stubborn nodes detection as function of number of observations m and $t_{\rm obs}$.

in terms of residual influence among regular agents far from the equilibrium.

V. Concluding remarks

In this paper, we have studied stubborn nodes' detection and influence matrix estimation in social networks though a model-based approach, casting the problem as a low-rank factorization. Moreover, we have provided a detailed theoretical analysis of method performance and robustness.

The applicability of the proposed method goes beyond the French-DeGroot model, to opinion dynamics models where few degrees of freedom can describe steady state opinions and observed opinions are sufficiently close to them. The linear relation between initial and final opinions can be seen as an approximation through linearization, with the nonlinear contributions incorporated into the error term. Moreover, for simplicity of exposition, we have restricted the scenario to agents communicating in synchronous way, assuming a static communication network. However, the model can be extended also to asynchronous communications [4], and to influence networks characterized by a continuous transition process that evolve toward oligarchic forms where the social power increasingly accumulates within a small group [3].

Future works will explore how bounds depend on network topology and apply the proposed framework to real datasets, comparing our approach with state-of-the-art methods.

Appendix

A. Proof of Lemma 1

Define $\mathcal{A} = \mathcal{R}^*\cap\mathcal{R}$, $\mathcal{B} = \mathcal{R}^*\cap\mathcal{S}$, $\mathcal{C} = \mathcal{S}^*\cap\mathcal{S}$, $\mathcal{D} = \mathcal{S}^*\cap\mathcal{R}$. Since $\mathcal{S} \neq \mathcal{S}^*$, $\mathcal{B}$ and $\mathcal{D}$ are nonempty. Since $|\mathcal{S}| = |\mathcal{S}^*|$, we have $|\mathcal{B}| = |\mathcal{D}|$. From Eq. (3) and Eq. (6), respectively, we get $\boldsymbol{X}_\mathcal{B} = \boldsymbol{X}_\mathcal{C}\boldsymbol{\Gamma}^*_{\mathcal{CB}}+\boldsymbol{X}_\mathcal{D}\boldsymbol{\Gamma}^*_{\mathcal{DB}}$ and $\boldsymbol{X}_\mathcal{D} = \boldsymbol{X}_\mathcal{B}\boldsymbol{\Gamma}_{\mathcal{BD}}+\boldsymbol{X}_\mathcal{C}\boldsymbol{\Gamma}_{\mathcal{CD}}$. So, $\boldsymbol{X}_\mathcal{B} - \boldsymbol{X}_\mathcal{C}\boldsymbol{\Gamma}^*_{\mathcal{CB}} = \boldsymbol{X}_\mathcal{D}\boldsymbol{\Gamma}^*_{\mathcal{DB}} = \boldsymbol{X}_\mathcal{B}\boldsymbol{\Gamma}_{\mathcal{BD}}\boldsymbol{\Gamma}^*_{\mathcal{DB}} + \boldsymbol{X}_\mathcal{C}\boldsymbol{\Gamma}_{\mathcal{CD}}\boldsymbol{\Gamma}^*_{\mathcal{DB}}$, that can be rearranged as

$$\boldsymbol{X}_\mathcal{B}(\boldsymbol{I} - \boldsymbol{\Gamma}_{\mathcal{BD}}\boldsymbol{\Gamma}^*_{\mathcal{DB}}) = \boldsymbol{X}_\mathcal{C}(\boldsymbol{\Gamma}^*_{\mathcal{CB}} + \boldsymbol{\Gamma}_{\mathcal{CD}}\boldsymbol{\Gamma}^*_{\mathcal{DB}}) .$$

Since $\mathrm{rank}(\boldsymbol{X}_\mathcal{S}) = |\mathcal{S}|$ and $\mathcal{B}\cup\mathcal{C} = \mathcal{S}$, the columns of $\boldsymbol{X}_\mathcal{B}$ and $\boldsymbol{X}_\mathcal{C}$ are linearly independent, so that the above implies

$$\boldsymbol{\Gamma}_{\mathcal{BD}}\boldsymbol{\Gamma}^*_{\mathcal{DB}} = \boldsymbol{I}\,, \qquad \boldsymbol{\Gamma}^*_{\mathcal{CB}} = -\boldsymbol{\Gamma}_{\mathcal{CD}}\boldsymbol{\Gamma}^*_{\mathcal{DB}}\,. \tag{11}$$

Now, for b in $\mathcal{B}$ and d in $\mathcal{D}$, let $\gamma^+_b = \max_{i\in\mathcal{D}}\Gamma_{bi}$, $\gamma^+_{bd} = \max_{i\in\mathcal{D}\setminus\{d\}}\Gamma_{bi}$, $\gamma^-_{bd} = \min_{i\in\mathcal{D}\setminus\{d\}}\Gamma_{bi}$. Now, we distinguish two cases. On the one hand, if $\Gamma^*_{cb} \geq \delta$, for some c in $\mathcal{C}$ and b in $\mathcal{B}$, then $\sum_{d\in\mathcal{D}}\Gamma^*_{db} \leq 1-\delta$, as $\boldsymbol{\Gamma}^*$ is column stochastic. Then, by the first equation in (11) we get

$$1 = \sum\nolimits_{d\in\mathcal{D}}\Gamma_{bd}\Gamma^*_{db} \leq \gamma^+_b\sum\nolimits_{d\in\mathcal{D}}\Gamma^*_{db} \leq \gamma^+_b(1-\delta)\,,$$

which implies that $\gamma^+_b \geq 1/(1-\delta) = 1+\delta/(1-\delta)$. On the other hand, if $\Gamma^*_{cb} < \delta$, for every c in $\mathcal{C}$ and b in $\mathcal{B}$, then, by Assumption 2, every column of $\boldsymbol{\Gamma}^*_{\mathcal{DB}}$ contains two entries not smaller than δ, and, since $|\mathcal{B}| = |\mathcal{D}|$, so does at least one of its rows, i.e., there exists d in $\mathcal{D}$, and $b \neq i$ in $\mathcal{B}$ such that $\Gamma^*_{db} \geq \delta$ and $\Gamma^*_{di} \geq \delta$. Since $\boldsymbol{\Gamma}^*$ is column stochastic, $\Gamma^*_{-db} = \sum_{j\neq d}\Gamma^*_{jb}$ and $\Gamma^*_{-di} = \sum_{j\neq d}\Gamma^*_{ji}$ satisfy $\Gamma^*_{-db} \leq 1-\delta$ $\Gamma^*_{-di} \leq 1-\delta$. Moreover, the first equation in (11) yields

$$\Gamma_{bd}\Gamma^*_{db} + \gamma^+_{bd}\Gamma^*_{-db} \geq 1\,, \qquad \Gamma_{bd}\Gamma^*_{di} + \gamma^-_{bd}\Gamma^*_{-di} \leq 0\,. \tag{12}$$

Now, if $\Gamma_{bd} \leq 0$, then $1 \leq \Gamma_{bd}\Gamma^*_{db} + \gamma^+_{bd}\Gamma^*_{-db} \leq \gamma^+_{bd}\Gamma^*_{-db} \leq \gamma^+_{bd}(1-\delta)$, so that $\gamma^+_{bd} \geq 1/(1-\delta) = 1+\delta/(1+\delta)$. Similarly, if $\gamma^+_{bd} \leq 0$, then $\Gamma_{bd} \geq 1+\delta/(1+\delta)$. Conversely, if $\Gamma_{bd} > 0$ and $\gamma^+_{bd} > 0$, then the second inequality in Eq. (12) implies that $\gamma^-_{bd} < 0$. Substituting now in Eq. (12) $\Gamma^*_{-db} = 1-\Gamma^*_{db}$ and $\Gamma^*_{-di} = 1-\Gamma^*_{di}$, we get $\frac{1-\gamma^+_{bd}(1-\Gamma^*_{db})}{\Gamma^*_{db}} \leq -\gamma^-_{bd}\frac{1-\Gamma^*_{di}}{\Gamma^*_{di}}$. Here we can distinguish two cases. First, if $\gamma^+_{bd} \leq 1$ we retrieve $-\gamma^-_{bd}\frac{1-\Gamma^*_{di}}{\Gamma^*_{di}} \geq 1$ which implies $-\gamma^-_{bd} \geq \frac{\delta}{1-\delta}$. Second, if $\gamma^+_{bd} > 1$ then necessarily, given that $\gamma^-_{bd} < 0$, the following inequality must hold $-\gamma^-_{bd} \geq \frac{\Gamma^*_{di}}{1-\Gamma^*_{di}}\frac{(1-\gamma^+_{bd}(1-\Gamma^*_{db}))_+}{\Gamma^*_{db}}$. Since $1 < \gamma^+_{bd} \leq \frac{1}{1-\Gamma^*_{db}}$, the above implies that $-\gamma^-_{bd} \geq \frac{\delta}{1-\delta}$.

B. Conditions for exact recovery of number of stubborn nodes

Lemma 2: If Assumption 1 holds true, then

$$\|\boldsymbol{Y}_{\mathcal{R}^*} - \boldsymbol{Y}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 \leq \|\boldsymbol{\Xi}\|_2(1+\|\boldsymbol{\Gamma}^*\|_2)\,, \qquad \|\boldsymbol{\Gamma}^*\|_2 \leq \Delta\,. \tag{13}$$

Proof: It follows from Eq. (3) and Eq. (4) that

$$\begin{aligned}\|\boldsymbol{Y}_{\mathcal{R}^*} - \boldsymbol{Y}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 &\leq \|\boldsymbol{Y}_{\mathcal{R}^*} - \boldsymbol{X}_{\mathcal{R}^*}\|_2 + \|\boldsymbol{X}_{\mathcal{R}^*} - \boldsymbol{X}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 \\ &\quad + \|\boldsymbol{X}_{\mathcal{S}^*}\boldsymbol{\Gamma}^* - \boldsymbol{Y}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 \\ &\leq \|\boldsymbol{\Xi}\|_2(1+\|\boldsymbol{\Gamma}^*\|_2)\,.\end{aligned}$$

Moreover, $\|\boldsymbol{X}_{\mathcal{R}^*}\|_2 = \|\boldsymbol{X}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 \geq \sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*})\|\boldsymbol{\Gamma}^*\|_2$,

where the last inequality follows from the fact that $\boldsymbol{X}_{\mathcal{S}^*}$ is left invertible and $\sigma_{k^*}(\boldsymbol{X}_{\mathcal{S}^*}) = \|\boldsymbol{X}^{-1}_{\mathcal{S}^*}\|_2^{-1}$. Thus, $\|\boldsymbol{\Gamma}^*\|_2 \leq \Delta$. ■

The following result determines conditions under which, for any solution of Problem 1, the number of stubborn nodes is not over or underestimated.

Proposition 3: Suppose Assumption 1 holds true and let $(\widehat{\mathcal{S}}_\varepsilon, \widehat{\boldsymbol{\Gamma}}_\varepsilon)$ be a solution of Problem 1. Then:

(i) if $\varepsilon > \|\boldsymbol{\Xi}\|_2(1+\Delta)$, then $|\widehat{\mathcal{S}}_\varepsilon| \leq k^*$;
(ii) if $\varepsilon < \sigma_{k^*}(\boldsymbol{X}) - \|\boldsymbol{\Xi}\|_2$, then $|\widehat{\mathcal{S}}_\varepsilon| \geq k^*$.

Proof: (i): It follows from Eq. (13) and the assumption on ε in Problem 1 that $\|\boldsymbol{Y}_{\mathcal{R}^*} - \boldsymbol{Y}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 \leq \varepsilon$. Consequently, optimality of $\widehat{\mathcal{S}}_\varepsilon$ yields $|\widehat{\mathcal{S}}_\varepsilon| \leq |\mathcal{S}^*| = k^*$.

(ii): By contradiction, assume that $|\widehat{\mathcal{S}}_\varepsilon| < k^*$. From characterization of singular values (see Section 7.4.2 in [19]) and Weyl's Inequality (see (3) in [20]), we have that

$$\begin{aligned}\|\boldsymbol{Y}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{Y}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 &\geq \sigma_{|\widehat{\mathcal{S}}_\varepsilon|+1}(\boldsymbol{Y}) \geq \sigma_{k^*}(\boldsymbol{Y}) \\ &\geq \sigma_{k^*}(\boldsymbol{X}) - \|\boldsymbol{\Xi}\|_2 > \varepsilon\,.\end{aligned}$$

This contradicts the assumption made on ε in Problem 1 . ■

Proposition 4: Let Assumption 1 and 2 be satisfied. If

$$\|\boldsymbol{\Xi}\|_2 \leq \sigma_{k^*}(\boldsymbol{X})/(2(1+\Delta)) \tag{14}$$

$$\varepsilon \in [\|\boldsymbol{\Xi}\|_2(1+\Delta), \sigma_{k^*}(\boldsymbol{X}) - \|\boldsymbol{\Xi}\|_2(1+\Delta)] \tag{15}$$

then every solution $(\widehat{\mathcal{S}}_\varepsilon, \widehat{\boldsymbol{\Gamma}}_\varepsilon)$ of Problem 1 is such that

(i) $|\widehat{\mathcal{S}}_\varepsilon| = k^*$.
(ii) $\|\boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 \leq \varepsilon + \|\boldsymbol{\Xi}\|_2(1+\Delta)$
(iii) The columns of $\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}$ are linearly independent.

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Proof: (i) follows directly from Proposition 3.

(ii) Applying Eq. (4) and triangular inequality, we estimate:

$$\begin{aligned}\|\boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 &\leq \|\boldsymbol{Y}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{Y}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 + \|\boldsymbol{\Xi}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{\Xi}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2\\ &\leq \varepsilon + \|\boldsymbol{\Xi}_{\widehat{\mathcal{R}}_\varepsilon}\|_2 + \|\boldsymbol{\Xi}_{\widehat{\mathcal{S}}_\varepsilon}\|_2\|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 \leq \varepsilon + \|\boldsymbol{\Xi}\|_2(1 + \|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2)\\ &\leq \varepsilon + \|\boldsymbol{\Xi}\|_2(1+\Delta)\,,\end{aligned}$$

where the last inequality follows from the fact that since $|\widehat{\mathcal{S}}_\varepsilon| = |\mathcal{S}^*|$ and given that from Lemma 2, given ε as in Eq. (15), $(\mathcal{S}^*, \boldsymbol{\Gamma}^*)$ is a feasible solution of Problem 1, then by optimality it must hold $\|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 \leq \|\boldsymbol{\Gamma}^*\|_2 \leq \Delta$.

(iii) If the columns of $\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}$ were linearly dependent, then, $\|\boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 \geq \sigma_{k^*}(\boldsymbol{X})$. This coupled with the inequality in (ii) contradicts the assumptions on ε. Finally, condition in Eq. (14) guarantees that interval Eq. (15) is non-empty. ■

C. The estimation of $\boldsymbol{\Gamma}^*$

Lemma 3: Suppose the assumptions in Proposition 4 to be true, and let $(\widehat{\mathcal{S}}_\varepsilon, \widehat{\boldsymbol{\Gamma}}_\varepsilon)$ be a solution of Problem 1, then

$$\sigma_{k^*}(\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}) \geq \frac{\sigma_{k^*}(\boldsymbol{X}) - \|\boldsymbol{\Xi}\|_2(1+\Delta) - \varepsilon}{1+\Delta} \tag{16}$$

Proof: Let $\boldsymbol{Z}$ be the best rank $k^* - 1$ approximation of $\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}$. Characterization of singular values [19] yields $\|\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon} - \boldsymbol{Z}\|_2 = \sigma_{k^*}(\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon})$. This equality and Proposition 4 - (ii) imply

$$\begin{aligned}\sigma_{k^*}(\boldsymbol{X}) &\leq \|[\boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon}\ \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}] - \boldsymbol{Z}[\widehat{\boldsymbol{\Gamma}}_\varepsilon\ \boldsymbol{I}]\|_2\\ &\leq \|\boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 + \|(\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon} - \boldsymbol{Z})\|_2\|[\widehat{\boldsymbol{\Gamma}}_\varepsilon\ \boldsymbol{I}]\|_2\\ &\leq \varepsilon + \|\boldsymbol{\Xi}\|_2(1+\Delta) + \sigma_{k^*}(\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon})(1+\Delta).\end{aligned}$$

■

Proposition 5: Suppose the assumptions in Proposition 4 to be true, and let $(\widehat{\mathcal{S}}_\varepsilon, \widehat{\boldsymbol{\Gamma}}_\varepsilon)$ be a solution of Problem 1. Then there exists a matrix $\bar{\boldsymbol{\Gamma}}$ such that $\boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon} = \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}\bar{\boldsymbol{\Gamma}}$, which satisfies

$$\|\widehat{\boldsymbol{\Gamma}}_\varepsilon - \bar{\boldsymbol{\Gamma}}\|_2 \leq \frac{(1+\Delta)(\varepsilon + \|\boldsymbol{\Xi}\|_2(1+\Delta))}{\sigma_{k^*}(\boldsymbol{X}) - \|\boldsymbol{\Xi}\|_2(1+\Delta) - \varepsilon}\,. \tag{17}$$

Proof:

Existence of $\bar{\boldsymbol{\Gamma}}$ follows from the fact that, thanks to Proposition 4, the rank of $\boldsymbol{X}$ is $|\widehat{\mathcal{S}}_\varepsilon| = k^*$ and the columns of $\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}$ are linearly independent.

Moreover, the following series of equations holds true

$$\begin{aligned}\boldsymbol{Y}_{\widehat{\mathcal{R}}_\varepsilon} - \boldsymbol{Y}_{\widehat{\mathcal{S}}_\varepsilon}\widehat{\boldsymbol{\Gamma}}_\varepsilon &= \boldsymbol{X}_{\widehat{\mathcal{R}}_\varepsilon} + \boldsymbol{\Xi}_{\widehat{\mathcal{R}}_\varepsilon} - (\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon} + \boldsymbol{\Xi}_{\widehat{\mathcal{S}}_\varepsilon})\widehat{\boldsymbol{\Gamma}}_\varepsilon\\ &= \boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}\bar{\boldsymbol{\Gamma}} + \boldsymbol{\Xi}_{\widehat{\mathcal{R}}_\varepsilon} - (\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon} + \boldsymbol{\Xi}_{\widehat{\mathcal{S}}_\varepsilon})\widehat{\boldsymbol{\Gamma}}_\varepsilon\,,\end{aligned}$$

from which, using triangular inequality and recalling that $\|\boldsymbol{Y}_{\mathcal{R}^*} - \boldsymbol{Y}_{\mathcal{S}^*}\boldsymbol{\Gamma}^*\|_2 \leq \varepsilon$, we obtain $\|\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}(\bar{\boldsymbol{\Gamma}} - \widehat{\boldsymbol{\Gamma}}_\varepsilon)\|_2 \leq \varepsilon + \|\boldsymbol{\Xi}_{\widehat{\mathcal{R}}_\varepsilon}\|_2 + \|\boldsymbol{\Xi}_{\widehat{\mathcal{S}}_\varepsilon}\|_2\|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2$. Since $\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}$ is left invertible and recalling that $\|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 \leq \Delta$, it follows $\|\widehat{\boldsymbol{\Gamma}}_\varepsilon - \bar{\boldsymbol{\Gamma}}\|_2 \leq \frac{\varepsilon + \|\boldsymbol{\Xi}\|_2(1+\Delta)}{\sigma_{k^*}(\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon})}$. Finally, using Eq. (16), we conclude the proof. ■

D. Proof of Theorem 1

Notice that all assumptions of Proposition 4 hold true due to Eq. (8), Eq. (9) and the fact that $\gamma \leq 1$. In particular, there exists a matrix $\bar{\boldsymbol{\Gamma}}$ as in Proposition 5 satisfying Eq. (17). Using the upper bound on ε in Eq. (9) inside Eq. (17), it holds

$$\|\widehat{\boldsymbol{\Gamma}}_\varepsilon - \bar{\boldsymbol{\Gamma}}\|_2 \leq \frac{\gamma(1+\Delta)\sigma_{k^*}(\boldsymbol{X})}{(1-\gamma)\sigma_{k^*}(\boldsymbol{X})} < \frac{\delta}{2(1-\delta)} \tag{18}$$

where last inequality follows from the way γ has been defined.

If, by contradiction, $\widehat{\mathcal{S}}_\varepsilon \neq \mathcal{S}^*$, given that from Proposition 4 $\boldsymbol{X}_{\widehat{\mathcal{S}}_\varepsilon}$ has linearly independent columns, then from Lemma 1 and the fact that $\widehat{\boldsymbol{\Gamma}}_\varepsilon$ has all entries in $[0,1]$, we would have $\|\widehat{\boldsymbol{\Gamma}}_\varepsilon - \bar{\boldsymbol{\Gamma}}\|_2 \geq \|\widehat{\boldsymbol{\Gamma}}_\varepsilon - \bar{\boldsymbol{\Gamma}}\|_{\max} \geq \frac{\delta}{2(1-\delta)}$ contradicting Eq. (18).

Then, $\widehat{\mathcal{S}}_\varepsilon = \mathcal{S}^*$. It follows that

$$\begin{aligned}\boldsymbol{Y}_{\mathcal{R}^*} - \boldsymbol{Y}_{\mathcal{S}^*}\widehat{\boldsymbol{\Gamma}}_\varepsilon &= \boldsymbol{X}_{\mathcal{R}^*} + \boldsymbol{\Xi}_{\mathcal{R}^*} - (\boldsymbol{X}_{\mathcal{S}^*} + \boldsymbol{\Xi}_{\mathcal{S}^*})\widehat{\boldsymbol{\Gamma}}_\varepsilon\\ &= \boldsymbol{X}_{\mathcal{S}^*}\boldsymbol{\Gamma}^* + \boldsymbol{\Xi}_{\mathcal{R}^*} - (\boldsymbol{X}_{\mathcal{S}^*} + \boldsymbol{\Xi}_{\mathcal{S}^*})\widehat{\boldsymbol{\Gamma}}_\varepsilon\,,\end{aligned}$$

so that $\|\boldsymbol{X}_{\mathcal{S}^*}(\boldsymbol{\Gamma}^* - \widehat{\boldsymbol{\Gamma}}_\varepsilon)\|_2 \leq \varepsilon + \|\boldsymbol{\Xi}_{\mathcal{R}^*}\|_2 + \|\boldsymbol{\Xi}_{\mathcal{S}^*}\|_2\|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2$. The fact that $\boldsymbol{X}_{\mathcal{S}^*}$ is left-invertible and $\|\widehat{\boldsymbol{\Gamma}}_\varepsilon\|_2 \leq \|\boldsymbol{\Gamma}^*\|_2 \leq \Delta$ now yield Eq. (10).

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