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## Small amplitude free vibration of soft materials and structures by high order finite elements

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### Summary

In the recent years, applications of soft material are widely spread in many mechanical, aeronautical, robotics engineering and biological applications. Many experimental and numerical studies have been conducted on hyperelastic media under various static and dynamics loads. Vibrations of hyperelastic materials have become more and more important due to their extreme elastic behavior: in this framework both geometrical and material non-linearities must be taken into account, leading to strongly nonlinear governing equations that result in a lack of closed-form solutions for vibrations problems. Studies in the last decade rely on FEM (Finite Element Method), that allow a wide range of investigations in terms of material properties and geometries, topology and weight optimization, frequency analysis and design phase of components.

In this work, finite element models for hyperelasticity and vibrations around nonlinear equilibrium states are based on Carrera Unified Formulation (CUF): the primary unknown variables are expressed by a polynomial expansion of kinematic models and arbitrary cross-section function/thickness functions (Node-Dependent Kinematics) [1]. Under the CUF formulation, the nonlinear governing equations in weak form are obtained adopting an index notation that allows the definitions of fundamental nuclei of physical quantities independently of the chosen polynomial expansion of the displacement field:

$$\text{Beam 1D models: } \mathbf{u}(x, y, z) = F_\tau(x, z)N_i(y)\mathbf{q}_{\tau i} \quad i = 1, 2, \dots, N_n \quad (1)$$

$$\text{Plate 2D models: } \mathbf{u}(x, y, z) = F_\tau(z)N_i(x, y)\mathbf{q}_{\tau i} \quad i = 1, 2, \dots, N_n \quad (2)$$

$$\text{Solid 3D models: } \mathbf{u}(x, y, z) = N_i(x, y, z)\mathbf{q}_i \quad i = 1, 2, \dots, N_n \quad (3)$$

Recently, CUF models for geometrical nonlinear analysis of linear elastic structures [2, 3] have been extended to including material non-linearities, in particular hyperelasticity, obtaining higher-order displacement-based models for nearly-incompressible and compressible soft materials [4].

To show the capabilities of the present implementation of hyperelastic models, we present here the nonlinear static analysis of a thick hollow sphere subjected to internal uniform pressure. In this case study, the analytic solution is known: radial displacement distribution obtained via parabolic hexahedral models is compared with the analytic reference, for various load conditions. The geometry, boundary conditions and material model and constants are taken from Jiang et al.[5]. Figure 1(a) shows the comparison of radial displacement distribution between analytical and numerical results, whereas Fig.1(b) shows the deformed configuration when  $p = 1$  Pa.

Afterwards, as done in Carrera et.al [6], we derive the governing equation of the free vibration problem (or undamped vibration problem) for hyperelastic structures by means of the principle of virtual displacements (PVD), written as:

$$\delta \mathcal{L}_{int} + \delta \mathcal{L}_{ine} = 0 \quad (4)$$

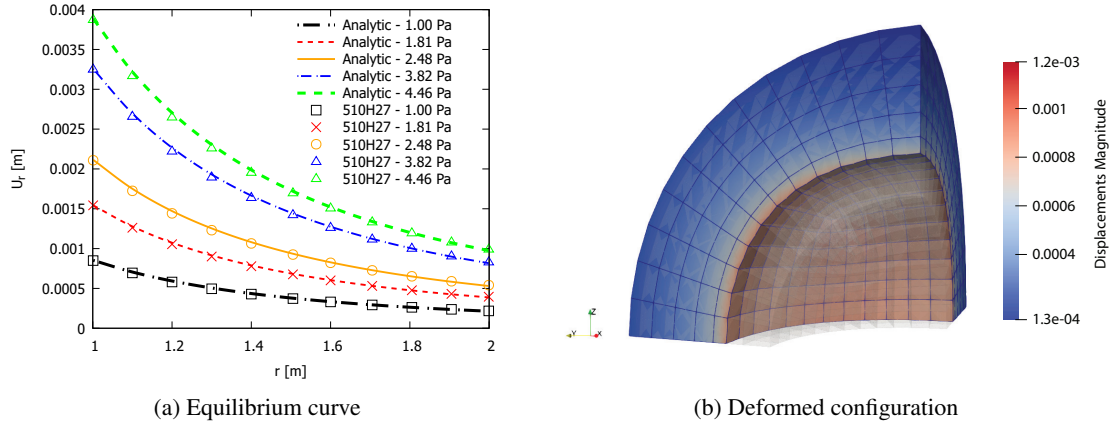


Figure 1: Thick hollow sphere: equilibrium curve and deformed configurations.

where  $\mathcal{L}_{int}$  is the internal work,  $\mathcal{L}_{ine}$  is the work done by inertia forces and  $\delta$  denotes the virtual variation. Adopting the same index notation for the full Green-Lagrange strain tensor, we can derive the FNs (fundamental nuclei) of internal forces vector, mass matrix and tangent stiffness matrix by linearization of the virtual variation of internal work:

$$\delta \mathcal{L}_{int} = \delta \mathbf{q}_{sj}^T \mathbf{F}_{int}^{\tau sij} \quad (5)$$

$$\delta \mathcal{L}_{ine} = \delta \mathbf{q}_{sj}^T \mathbf{M}^{\tau sij} \ddot{\mathbf{q}}_{\tau i} \quad (6)$$

$$\delta(\delta \mathcal{L}_{int}) = \delta \mathbf{q}_{sj}^T \mathbf{K}_T^{\tau sij} \delta \mathbf{q}_{\tau i} \quad (7)$$

All these FNs have the property that they are independent of the polynomial expansion chosen: higher-order model are rapidly defined by considering the summation over the indices. The proposed method solves the undamped vibration problem around a nonlinear equilibrium states computing first the static equilibrium problem by means of a Newton-Raphson iterative procedure coupled with the arc-length procedure [7], then solving the simplified equations of motion (the classical linear eigenvalue problem) adopting the tangent stiffness matrix at the point of interest:

$$(\mathbf{K}_T^{\tau sij} - \omega^2 \mathbf{M}^{\tau sij}) \mathbf{q}_{\tau i} = 0 \quad (8)$$

In this way, normal mode frequencies and shapes can be obtained. Contrary to the case of linear elastic material, for which low amplitude vibrations were observed (thus, linearization of the problem around a non-trivial equilibrium condition is legitimate), hyperelastic materials undergo large amplitude vibrations: even so, the small-amplitude behaviour under prestressed conditions represents an interesting research problem thanks to its various practical applications.

In this last paragraph, we report the actual numerical results obtained for an undamped vibration problem regarding the thick hyperelastic hollow sphere considered before, around the trivial equilibrium condition. Regarding the mass matrix definition, density is fixed at the classical value of natural rubber  $\rho = 1340 \text{ kg/m}^3$ . Results from the present model are compared with reference results obtained by the commercial finite element code ABAQUS. In Table 1 we report the comparison between reference results and numerical values of the first ten natural frequencies obtained by the present model, whereas in Fig.2 undamped vibration configurations are shown.

Mode	3D ABQ	CUF	Mode	3D ABQ	CUF
1	0.0795	0.0798	6	0.1837	0.1848
2	0.0795	0.0798	7	0.2083	0.2091
3	0.1519	0.1519	8	0.2185	0.2190
4	0.1837	0.1848	9	0.2185	0.2190
5	0.1837	0.1848	10	0.2484	0.2489

Table 1: Thick hollow sphere: Natural frequencies [Hz]

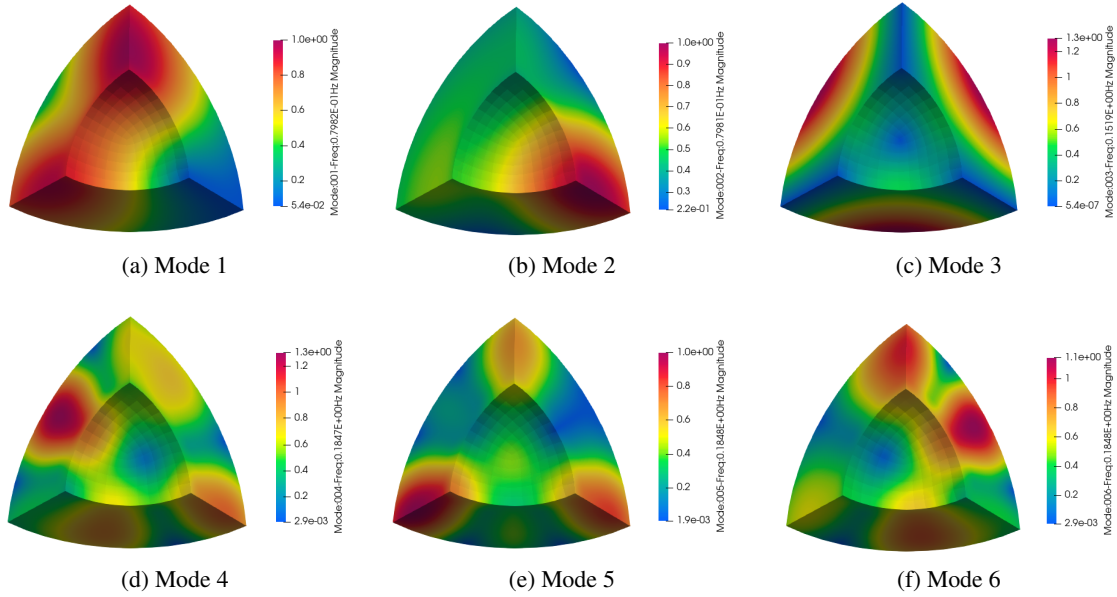


Figure 2: Thick hollow sphere: snapshots of normal modes of vibration.

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