On the pluriclosed flow on Oeljeklaus-Toma manifolds

ON THE PLURICLOSED FLOW ON OELJEKLAUS-TOMA MANIFOLDS

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Abstract. We investigate the pluriclosed flow on Oeljeklaus-Toma manifolds. We parametrize left-invariant pluriclosed metrics on Oeljeklaus-Toma manifolds and we classify the ones which lift to an algebraic soliton of the pluriclosed flow on the universal covering. We further show that the pluriclosed flow starting from a left-invariant pluriclosed metric has a long-time solution ω_t which once normalized collapses to a torus in the Gromov-Hausdorff sense. Moreover the lift of $\frac{1}{1+t}\omega_t$ to the universal covering of the manifold converges in the Cheeger-Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$ where $\tilde{\omega}_\infty$ is an algebraic soliton.

1. Introduction

Oeljeklaus-Toma manifolds are a very interesting class of complex manifolds introduced and firstly studied in [17]. These manifolds are defined as compact quotients of the type

$$M = \frac{\mathbb{H}^r \times \mathbb{C}^s}{U \ltimes \mathcal{O}_{\mathbb{K}}}$$

where $\mathbb{H} \subseteq \mathbb{C}$ is the upper half-plane, $\mathcal{O}_{\mathbb{K}}$ is the ring of algebraic integers of an algebraic extension $\mathbb{K}$ of $\mathbb{Q}$ satisfying $[\mathbb{K} : \mathbb{Q}] = r+2s$ and U is a free subgroup of rank r of $\mathcal{O}_{\mathbb{K}}^{*,+}$ satisfying some compatible conditions. The action of $U \ltimes \mathcal{O}_{\mathbb{K}}$ on $\mathbb{H}^r \times \mathbb{C}^s$ is defined via some embeddings of $\mathbb{K}$ in $\mathbb{R}$ and $\mathbb{C}$. Oeljeklaus-Toma manifolds have a rich geometric structure. For instance, they have a natural structure of $\mathbb{T}^{r+2s}$-torus bundle over a $\mathbb{T}^r$ and a structure of solvmanifold [13], i.e. they are always compact quotients of a solvable Lie group by a lattice. The Poincaré metric[1] $\omega_{\mathbb{H}^r} = \sqrt{-1}\sum_{a=1}^r \frac{dz_a \wedge d\bar{z}_a}{4(\Im\mathfrak{m} z_a)^2}$ induces a degenerate metric ω_∞ on M which has a central role in the study of geometric flows on these manifolds. The pair (r, s) is called the *type* of the manifold. The case of type $(r, s) = (1, 1)$ corresponds to the Inoue-Bombieri surfaces [11].

In [2, 7, 29, 32] the Chern-Ricci flow [10, 28] on Oeljeklaus-Toma manifolds M of type $(r, 1)$ is studied. Accordingly to the results in [2, 7, 29, 32], under some assumptions on the initial Hermitian metric, the flow has a long-time solution ω_t such that $(M, \frac{\omega_t}{1+t})$ converges in the Gromov-Hausdorff sense to an r-dimensional torus $\mathbb{T}^r$ as $t \to \infty$. The result can be adapted to Oeljeklaus-Toma manifolds of arbitrary type by assuming the initial metric to be left-invariant with respect to the structure of solvmanifold. Moreover, a result of Lauret in [14] allows us to give a characterization of left-invariant Hermitian metrics on an Oeljeklaus-Toma manifold which lift to an algebraic soliton of the Chern-Ricci flow on the universal covering of the manifold (see Proposition 4.1 in the present paper).

Following the same approach, we focus on the pluriclosed flow on Oeljeklaus-Toma manifolds when the initial pluriclosed Hermitian metric is left-invariant. The pluriclosed flow is a geometric flow of pluriclosed metrics, i.e. of Hermitian metrics having the fundamental form $\partial\bar{\partial}$-closed, introduced by Streets and Tian in [21]. The flow belongs to the family of the Hermitian curvature flows [20] and evolves an initial pluriclosed metric along the $(1, 1)$-component of the Bismut-Ricci form. Namely, on a Hermitian manifold (M, ω) there always exists a unique metric connection ∇^B, called the *Bismut connection*, preserving the complex structure and such that

$$\omega(T^B(\cdot,\cdot), J\cdot) \quad \text{is a 3-form}\,,$$

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[1] In the whole paper we identify a Hermitian metric with its fundamental form.

where T^B is the torsion of ∇^B. The *Bismut-Ricci form* of ω is then defined as

$$\rho_B(X,Y) := \sqrt{-1}\sum_{i=1}^n R_B(X,Y,X_i,\bar X_i)\,,$$

where R_B is the curvature tensor of ∇^B and $\{X_i\}$ is a unitary frame of ω. ρ_B is always a closed real form. Given a pluriclosed Hermitian metric ω on M, the *pluriclosed flow* is then defined as the geometric flow of pluriclosed metrics governed by the equation

$$\partial_t\omega_t = -\rho_B^{1,1}(\omega_t)\,,\quad \omega_{|t=0}=\omega\,.$$

The pluriclosed flow was deeply studied in literature, see for instance [3, 5, 6, 9, 12, 19, 22, 23, 24, 25, 26, 27] and the references therein.

Our main result is the following

Theorem 1.1. *Let ω be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold M. Then the pluriclosed flow starting from ω has a long-time solution ω_t such that $(M,\frac{\omega_t}{1+t})$ converges in the Gromov-Hausdorff sense to $(\mathbb{T}^s,d)$. Moreover, ω lifts to an expanding algebraic soliton on the universal covering of M if and only if it is diagonal and the first s diagonal components coincide. Finally, $(\mathbb{H}^s\times\mathbb{C}^s,\frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbb{H}^s\times\mathbb{C}^s,\tilde\omega_\infty)$ where $\tilde\omega_\infty$ is an algebraic soliton.*

Here we recall that a left-invariant Hermitian metric ω on a Lie group G with a left-invariant complex structure is an *algebraic soliton* for a geometric flow of left-invariant Hermitian metrics if $\omega_t = c_t\varphi_t^*(\omega)$ solves the flow, where $\{c_t\}$ is a positive scaling and $\{\varphi_t\}$ is a family of automorphims of G preserving the complex structure. Moreover the distance d in the statement is the distance induced by $3\omega_\infty$ on the torus base of M. Now we describe the condition *diagonal* appearing in the statement of Theorem 1.1. The existence of a pluriclosed metric on an Oeljeklaus-Toma manifold imposes some restrictions, see [1, Corollary 3]. In particular, the manifold has type (s,s) and admits a left-invariant $(1,0)$-coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ satisfying

$$\begin{cases} d\omega^k = \frac{\sqrt{-1}}{2}\omega^k\wedge\bar\omega^k & k=1,\dots,s\,,\\ d\gamma^i = \sum_{k=1}^s\lambda_{ki}\,\omega^k\wedge\gamma^i - \sum_{k=1}^s\lambda_{ki}\bar\omega^k\wedge\gamma^i & i=1,\dots,s\,,\end{cases}$$

with

$$\Im\mathfrak{m}\,\lambda_{ki} = -\frac14\,\delta_{ik}\,.$$

By ω *diagonal* we mean that it takes a diagonal form with respect to such a coframe. The first part of Theorem 1.1 in the case of the Inoue-Bombieri surfaces is proved in [5, Corollary 3.18].

Theorem 1.1 provides a description of the long-time behavior of the solution ω_t to the pluriclosed flow as $t\to\infty$. For the definition of the convergence in the Gromov-Hausdorff sense we refer to Section 3 in the present paper, while here we briefly recall the definition of convergence in the Cheeger-Gromov sense: a sequence of pointed riemannian manifolds (M_k,g_k,p_k) *converges in the Cheeger-Gromov sense to a pointed riemannian manifold* (M,g,p) if there exists a sequence of open subsets A_k of M so that every compact subset of M eventually lies in some A_k, and a sequence of smooth maps $\phi_k\colon A_k\to M_k$ which are diffeomorphisms onto some open set of M_k which satisfy $\phi_k(p_k)=p$, such that

$$\phi_k^*(g_k)\to g \quad \text{smoothly on every compact subset, as } k\to\infty.$$

See [15, Section 6] for a deep analysis of Cheeger-Gromov convergence both in the general case and in the homogeneous one and [14, Section 5.1] for the case of Hermitian Lie groups.

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2. Definition of Oeljeklaus-Toma manifolds

We briefly recall the construction of Oeljeklaus-Toma manifolds [17].

Let $\mathbb{Q} \subseteq \mathbb{K}$ be an algebraic number field with $[\mathbb{K} : \mathbb{Q}] = r + 2s$ and $r, s \geq 1$. Let $\sigma_1, \ldots, \sigma_r \colon \mathbb{K} \to \mathbb{R}$ be the real embeddings of $\mathbb{K}$ and $\sigma_{r+1}, \ldots, \sigma_{r+2s} \colon \mathbb{K} \to \mathbb{C}$ be the complex embeddings of $\mathbb{K}$ satisfying $\sigma_{r+s+i} = \bar{\sigma}_{r+i}$, for every $i = 1, \ldots, s$. We denote by $\mathcal{O}_{\mathbb{K}}$ the ring of algebraic integers of $\mathbb{K}$ and by $\mathcal{O}_{\mathbb{K}}^*$ the group of units of $\mathcal{O}_{\mathbb{K}}$. Let

$$\mathcal{O}_{\mathbb{K}}^{*,+} = \{u \in \mathcal{O}_{\mathbb{K}}^* \quad | \quad \sigma_i(u) > 0\,, \quad \text{for every } i = 1, \ldots, r\}$$

be the group of totally positive units of $\mathcal{O}_{\mathbb{K}}$. The groups $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{K}}^{*,+}$ act on $\mathbb{H}^r \times \mathbb{C}^s$ as

$$a \cdot (z_1, \ldots, z_r, w_1, \ldots, w_s) = (z_1 + \sigma_1(a), \ldots, z_r + \sigma_r(a), w_1 + \sigma_{r+1}(a), \ldots, w_s + \sigma_{r+s}(a))\,, \quad \text{for all } a \in \mathcal{O}_{\mathbb{K}}$$

and

$$u \cdot (z_1, \ldots, z_r, w_1, \ldots, w_s) = (\sigma_1(u) z_1, \ldots, \sigma_r(u) z_r, \sigma_{r+1}(u) w_1, \ldots, \sigma_{r+s}(u) w_s)\,, \quad \text{for every } u \in \mathcal{O}_{\mathbb{K}}^{*,+}\,.$$

There always exists a free subgroup U of rank r of $\mathcal{O}_{\mathbb{K}}^{*,+}$ such that $\mathrm{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in $\mathbb{R}^r$, where $l \colon \mathcal{O}_{\mathbb{K}}^{*,+} \to \mathbb{R}^{r+s}$ is the logarithmic representation of units

$$l(u) = (\log \sigma_1(u), \ldots, \log \sigma_r(u), 2\log|\sigma_{r+1}(u)|, \ldots, 2\log|\sigma_{r+s}(u)|)$$

and $\mathrm{pr}_{\mathbb{R}^r} \colon \mathbb{R}^{r+s} \to \mathbb{R}^r$ is the projection on the first r coordinates. The action of $U \ltimes \mathcal{O}_{\mathbb{K}}$ on $\mathbb{H}^r \times \mathbb{C}^s$ is free, properly discontinuous and co-compact. An *Oeljeklaus-Toma manifold* is then defined as the quotient

$$M := \frac{\mathbb{H}^r \times \mathbb{C}^s}{U \ltimes \mathcal{O}_{\mathbb{K}}}$$

and it is a compact complex manifold having complex dimension $r + s$.

The structure of torus bundle of an Oeljeklaus-Toma manifold can be seen as follows: we have

$$\frac{\mathbb{H}^r \times \mathbb{C}^s}{\mathcal{O}_{\mathbb{K}}} = \mathbb{R}^r_+ \times \mathbb{T}^{r+2s}$$

and that the action of U on $\mathbb{H}^r \times \mathbb{C}^s$ induces an action on $\mathbb{R}^r_+ \times \mathbb{T}^{r+2s}$ such that, for every $x \in \mathbb{R}^r_+$ and $u \in U$, the induced map

$$u \colon (x, \mathbb{T}^{r+2s}) \mapsto (\sigma_1(u) x_1, \ldots, \sigma_r(u) x_r, \mathbb{T}^{r+2s})$$

is a diffeomorphism. Hence

$$M = \frac{\mathbb{R}^r_+ \times \mathbb{T}^{r+2s}}{U}$$

inherits the structure of a $\mathbb{T}^{r+2s}$-bundle over $\mathbb{T}^r$. We denote by π and F the projections

$$\pi \colon \mathbb{H}^r \times \mathbb{C}^s \to M\,, \quad F \colon M \to \mathbb{T}^r\,.$$

From the viewpoint of Lie groups, the universal covering of an Oeljeklaus-Toma manifold M has a natural structure of solvable Lie group G and the complex structure on M lifts to a left-invariant complex structure [13]. Therefore, Oeljeklaus-Toma manifolds can be seen as compact solvmanifolds with a left-invariant complex structure. The solvable structure on the universal covering of M can be described in terms of the existence of a left-invariant $(1,0)$-coframe $\{\omega^1, \ldots, \omega^r, \gamma^1, \ldots, \gamma^s\}$ such that

$$(1) \qquad \begin{cases} d\omega^k = \frac{\sqrt{-1}}{2} \omega^k \wedge \bar{\omega}^k & k = 1, \ldots, r\,, \\ d\gamma^i = \sum_{k=1}^r \lambda_{ki}\, \omega^k \wedge \gamma^i - \sum_{k=1}^r \lambda_{ki} \bar{\omega}^k \wedge \gamma^i & i = 1, \ldots, s\,, \end{cases}$$

where

$$\lambda_{ki} = \frac{\sqrt{-1}}{4} b_{ki} - \frac{1}{2} c_{ki}$$

and $b_{ki}, c_{ki} \in \mathbb{R}$ depend on the embeddings σ_j as

$$(2) \qquad \sigma_{r+i}(u) = \left(\prod_{k=1}^r (\sigma_k(u))^{\frac{b_{ki}}{2}} \right) e^{\sqrt{-1} \sum_{k=1}^r c_{ki} \log \sigma_k(u)}\,,$$

for any $u \in U$, $k = 1, \dots, r$ and $i = 1, \dots, s$. Since $U \subseteq \mathcal{O}_{\mathbb{K}}^*$, it is easy to see that

$$l(U) \subseteq \left\{ x \in \mathbb{R}^{r+s} \quad \middle| \quad \sum_{i=1}^{r+s} x_i = 0 \right\}.$$

This fact together with (2) implies that, for every $u \in U$,

$$\sum_{i=1}^{r} \log \sigma_i(u) \left(1 + \sum_{k=1}^{s} b_{ik} \right) = 0\,,$$

which, since $\mathrm{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in $\mathbb{R}^r$, is equivalent to

$$\sum_{k=1}^{s} b_{ik} = -1\,, \quad \text{for all } i = 1, \dots, r\,. \tag{3}$$

The dual frame $\{Z_1, \dots, Z_r, W_1, \dots, W_s\}$ to $\{\omega^1, \dots, \omega^r, \gamma^1, \dots, \gamma^s\}$ satisfies the following structure equations:

$$[Z_k, \bar{Z}_k] = -\frac{\sqrt{-1}}{2}(Z_k + \bar{Z}_k)\,, \quad [Z_k, W_i] = -\lambda_{ki} W_i\,, \quad [Z_k, \bar{W}_i] = \bar{\lambda}_{ki} \bar{W}_i\,,$$

for $k = 1, \dots, r$, $i = 1, \dots, s$. Consequently the Lie algebra $\mathfrak{g}$ of the universal covering of M splits as vector space as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{I}$$

where $\mathfrak{I}$ is an abelian ideal and $\mathfrak{h}$ is a subalgebra isomorphic to $\underbrace{\mathfrak{f} \oplus \dots \oplus \mathfrak{f}}_{r\text{-times}}$, where $\mathfrak{f}$ is the *filiform* Lie algebra $\mathfrak{f} = \langle e_1, e_2 \rangle$, $[e_1, e_2] = -\frac{1}{2} e_1$. The complex structure J induced on $\mathfrak{g}$ preserves both $\mathfrak{h}$ and $\mathfrak{I}$ and its restriction $J_{\mathfrak{h}}$ on $\mathfrak{h}$ satisfies

$$J_{\mathfrak{h}} = \underbrace{J_{\mathfrak{f}} \oplus \dots \oplus J_{\mathfrak{f}}}_{r\text{-times}}\,,$$

where $J_{\mathfrak{f}}$ is the complex structure on $\mathfrak{f}$ defined by $J_{\mathfrak{f}}(e_1) = e_2$. Moreover

$$[\mathfrak{h}^{1,0}, \mathfrak{I}^{0,1}] \subseteq \mathfrak{I}^{0,1}\,.$$

3. Convergence in the Gromov-Hausdorff sense

We briefly recall Gromov-Hausdorff convergence of metric spaces. The *Gromov-Hausdorff distance* between two metric spaces (X, d_X), (Y, d_Y) is the infimum of all positive ϵ for which there exist two functions $F \colon X \to Y$, $G \colon Y \to X$, not necessarily continuous, satisfying the following four properties

$$\begin{aligned} |d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| \le \epsilon\,, \quad d_X(x, G(F(x))) \le \epsilon\,, \\ |d_Y(y_1, y_2) - d_X(G(y_1), G(y_2))| \le \epsilon\,, \quad d_Y(y, F(G(y))) \le \epsilon\,, \end{aligned}$$

for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. If $\{d_t\}_{t \in [0,\infty)}$ is a 1-parameter family of distances on X, (X, d_t) *converges to* (Y, d_Y) *in the Gromov-Hausdorff sense* if the Gromov-Hausdorff distance between (X, d_t) and (Y, d) tends to 0 as $t \to \infty$.

Let $\{\omega_t\}_{t \in [0,\infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus-Toma manifold and let d_t be the induced distance on M. For a smooth curve γ on M, let $L_t(\gamma)$ be the length of γ with respect to ω_t. We further denote by $\mathcal{H}$ the foliation induced by $\mathfrak{h}$ on M.

Proposition 3.1. *Let $\{\omega_t\}_{t \in [0,\infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus-Toma manifold such that*

$$\lim_{t \to \infty} \omega_t = \omega_\infty$$

pointwise. Assume that there exist $T \in (0, \infty)$ and $C > 0$ such that

1. $L_t(\gamma) \le C L_0(\gamma)$, *for every smooth curve γ in M;*
2. $L_t(\gamma) \le (C/\sqrt{t}) L_0(\gamma)$, *for every smooth curve γ in M such that $\dot{\gamma} \in \ker \omega_\infty$.*

Assume further

3. *for every $\epsilon, \ell > 0$, there exists $T > 0$ such that $|L_t(\gamma) - L_\infty(\gamma)| < \epsilon$, for every $t > T$ and every curve γ in M tangent to $\mathcal{H}$ and such that $L_\infty(\gamma) < \ell$.*

Then (M, d_t) converges in the Gromov-Hausdorff sense to $(\mathbb{T}^r, d)$, where d is the distance induced by ω_∞ onto $\mathbb{T}^r$.

Proof. We follow the approach in [29, Section 5] and in [32, Proof of Theorem 1.1]. Let M be an Oeljeklaus-Toma manifold. Consider the structure of M as $\mathbb{T}^{r+2s}$-bundle over a $\mathbb{T}^r$. Let $F\colon M \to \mathbb{T}^r$ be the projection onto the base and let $G\colon \mathbb{T}^r \to M$ be an arbitrary map such that $F \circ G = \mathrm{Id}_{\mathbb{T}^r}$. We show that, for every $\epsilon > 0$, there exists $T > 0$ such that

$$|d_t(p,q) - d(F(p), F(q))| \le \epsilon\,, \tag{4}$$

$$|d(a,b) - d_t(G(a), G(b))| \le \epsilon\,, \tag{5}$$

$$d_t(p, G(F(p))) \le \epsilon\,, \tag{6}$$

$$d(a, F(G(a))) \le \epsilon\,, \tag{7}$$

for every $t \ge T$, $p, q \in M$, $a, b \in \mathbb{T}^r$ which implies the statement.

Note that (7) is trivial since

$$d(a, F(G(a))) = 0\,,$$

for every $a \in \mathbb{T}^r$.

Then, we show that (6) is satisfied. Let $p, q \in M$ be two points in the same fiber over $\mathbb{T}^r$. Assume $p = \pi(z, w)$. We denote with $\mathcal{L}_{(z,w)}$ the leaf of the foliation $\ker \omega_\infty$ on the universal covering of M passing through (z, w). We easily see that, for all $(z, w) \in \mathbb{H}^r \times \mathbb{C}^s$, $\mathcal{L}_{(z,w)} = \{z\} \times \mathbb{C}^s$. In view of [30, Section 2], for every $z \in \mathbb{H}^r$, $\pi(\{z\} \times \mathbb{C}^s)$ is the leaf of the foliation $\ker \omega_\infty$ on M passing through p and it is dense in the fiber $F^{-1}(F(p))$. Let B_R be the standard ball in $\mathbb{C}^s$ about the origin having radius R. We can choose R so that every point in $F^{-1}(F(p))$ has distance with respect to d_0 less than $\epsilon/2C$ to $\pi(\{z\} \times \bar{B}_R)$. On the other hand, given two points in $\pi(\{z\} \times \bar{B}_R)$, they can be joined with a curve γ in $F^{-1}(F(p))$ which is tangent to $\ker \omega_\infty$. Hence, for any such curve, condition 2. implies

$$L_t(\gamma) \le \frac{C'}{\sqrt{t}}\,,$$

for a uniform constant C' depending only on R. Let $p_0 = \pi(z, 0)$, γ_1 be a curve in $F^{-1}(F(p))$ connecting p with p_0 tangent to $\ker \omega_\infty$ and γ_2 be a curve connecting p_0 with q having minimal length with respect to d_0. Hence, by using 1., for t sufficiently large, we have

$$d_t(p,q) \le L_t(\gamma_1) + L_t(\gamma_2) \le \frac{C'}{\sqrt{t}} + C L_0(\gamma_2) \le \frac{C'}{\sqrt{t}} + \frac{\epsilon}{2} \le \epsilon\,,$$

i.e.

$$d_t(p,q) \le \epsilon$$

and (6) follows.

Next we show (4) and (5). First of all, we denote with g the riemannian metric on $\mathbb{T}^r$ induced by ω_∞, for an explicit expression of g see [32, Section 2], and we observe that

$$L_g(F(\gamma)) \le L_\infty(\gamma)\,,\ \text{for every curve } \gamma \text{ in } M, \tag{8}$$

and the equality holds if and only if

$$\dot\gamma \in \mathcal{Y} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{1}{2\sqrt{-1}} \left(Z_i - \bar{Z}_i\right) \;\middle|\; i = 1, \dots, r \right\}.$$

Let $p, q \in M$. We can find a curve γ in M connecting p with a point $\tilde{q}$ in the $\mathbb{T}^{r+2s}$-fiber containing q which is tangent to $\mathcal{Y}$ and such that $F(\gamma)$ is a minimal geodesic on $(\mathbb{T}^r, g)$, see for instance [29, Proof of Theorem 5.1] or [32, Proof of Theorem 1.1]. By applying 3. we have

$$d_t(p,q) \le d_t(p,\tilde{q}) + d_t(\tilde{q}, q) \le d_t(p,\tilde{q}) + \epsilon \le L_t(\gamma) + \epsilon \le L_\infty(\gamma) + 2\epsilon = L_g(F(\gamma)) + 2\epsilon = d(F(p), F(q)) + 2\epsilon\,,$$

for t big enough, i.e.

$$d_t(p,q) - d(F(p),F(q)) \leq 2\epsilon\,, \tag{9}$$

for t sufficiently large.

Next, using again (8), we obtain, for $p,q\in M$,

$$d(F(p),F(q)) \leq L_g(F(\gamma)) \leq L_\infty(\gamma) \leq L_t(\gamma)+\epsilon = d_t(p,q)+\epsilon\,,$$

for t big enough, where γ is curve which realizes the distance $d_t(p,q)$. Hence we obtain

$$d(F(p),F(q)) - d_t(p,q) \leq \epsilon\,. \tag{10}$$

By substituting $p=G(a)$ and $q=G(b)$ in (9) and (10) we infer

$$-\epsilon \leq d_t(G(a),G(b)) - d(a,b) \leq 2\epsilon$$

and (4) and (5) follow. □

4. The left-invariant Chern-Ricci flow on Oeljeklaus-Toma manifolds

Given a Hermitian manifold (M,ω), the Chern connection of ω is the unique connection ∇ on (M,ω) preserving both ω and the complex structure such that the $(1,1)$-component of its torsion tensor is vanishing. The *Chern-Ricci form* of ω is the real closed $(1,1)$-form

$$\rho_C(X,Y) := \sqrt{-1}\sum_{i=1}^n R_C(X,Y,X_i,\bar X_i)\,,$$

where R_C is the curvature tensor of ∇ and $\{X_i\}$ is a unitary frame of ω. The *Chern-Ricci flow* is then defined as the geometric flow

$$\partial_t\omega_t = -\rho_C(\omega_t)\,,\quad \omega_{|t=0}=\omega\,.$$

In this section we prove the following

Proposition 4.1. *Let ω be a left-invariant Hermitian metric on an Oeljeklaus-Toma manifold M. Then ω lifts to an expanding algebraic soliton for the Chern-Ricci flow on the universal covering of M if and only if it takes the following expression with respect to the coframe $\{\omega^1,\dots,\omega^r,\gamma^1,\dots,\gamma^s\}$ satisfying (1):*

$$\omega = \sqrt{-1}\left(A\sum_{i=1}^r \omega^i\wedge\bar\omega^i + \sum_{i,j=1}^s g_{r+i\overline{r+j}}\gamma^i\wedge\bar\gamma^j\right)\,. \tag{11}$$

Moreover, the Chern-Ricci flow starting from ω has a long-time solution $\{\omega_t\}$ such that $(M,\frac{\omega_t}{1+t})$ converges as $t\to\infty$ in the Gromov-Hausdorff sense to $(\mathbb{T}^r,d)$, where d is the distance induced by ω_∞ onto $\mathbb{T}^r$. Finally, $(\mathbb{H}^r\times\mathbb{C}^s,\frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbb{H}^r\times\mathbb{C}^s,\tilde\omega_\infty)$ where $\tilde\omega_\infty$ is an algebraic soliton.

The proof of Proposition 4.1 is based on the following Theorem of Lauret

Theorem 4.2 (Lauret [14])**.** *Let (G,J) be a Lie group with a left-invariant complex structure. Then the Chern-Ricci form of a left-invariant Hermitian metric ω on (G,J) does not depend on the Hermitian metric. Moreover, if $P\neq 0$ is the endomorphism associated to ρ_C with respect to ω, then the following are equivalent:*

1. *ω is an algebraic soliton of the Chern-Ricci flow,*
2. *$P=cI+D$, for some $D\in\mathrm{Der}(\mathfrak{g})$,*
3. *The eigenvalues of P are either 0 or c, for some $c\in\mathbb{R}$ with $c\neq 0$, $\ker P$ is an abelian ideal of the Lie algebra of G and $(\ker P)^\perp$ is a subalgebra.*

Proof of Proposition 4.1. Let M be an Oeljeklaus-Toma manifold. Since the Chern-Ricci form does not depend on the choice of the left-invariant Hermitian metric, it is enough to compute ρ_C for the "canonical metric"

$$\omega=\sqrt{-1}\left(\sum_{i=1}^{r}\omega^i\wedge\bar\omega^i+\sum_{j=1}^{s}\gamma^j\wedge\bar\gamma^j\right)\,. \tag{12}$$

We recall that the Chern-Ricci form of a left-invariant Hermitian metric $\omega=\sqrt{-1}\sum_{a=1}^n\alpha^a\wedge\bar\alpha^a$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$\rho_C(X,Y)=-\sum_{a=1}^{n}(\omega([[X,Y]^{0,1},X_a],\bar X_a)+\omega([[X,Y]^{1,0},\bar X_a],X_a))\,, \tag{13}$$

for every left-invariant vector fields X,Y on G, where $\{\alpha^i\}$ is a left-invariant unitary $(1,0)$-coframe with dual frame $\{X_a\}$ (see e.g. [31]). By applying (13) to the canonical metric (12) we have

$$\begin{aligned}\rho_C(X,Y)=-\sum_{a=1}^{r}\{\omega([[X,Y]^{0,1},Z_a],\bar Z_a)+\omega([[X,Y]^{1,0},\bar Z_a],Z_a)\}\\ -\sum_{b=1}^{s}\{\omega([[X,Y]^{0,1},W_b],\bar W_b)+\omega([[X,Y]^{1,0},\bar W_b],W_b)\}\,.\end{aligned}$$

Clearly,

$$\rho_C(Z_i,\bar Z_j)=0\,,\quad \text{for all } i\neq j\,,\quad \rho_C(W_i,\bar W_j)=0\,,\quad \text{for every } i,j=1,\dots,s\,.$$

Moreover, since $\mathfrak{I}$ is an abelian ideal and ω makes $\mathfrak{I}$ and $\mathfrak{h}$ orthogonal, we have:

$$\rho_C(Z_i,\bar W_j)=0\,,\quad \text{for all } i=1,\dots,r\,,\quad j=1,\dots,s\,.$$

Moreover we have

$$\omega([[Z_i,\bar Z_i]^{0,1},Z_a],\bar Z_a)=\frac{\sqrt{-1}}{4}\delta_{ia}\,,\quad \omega([[Z_i,\bar Z_i]^{1,0},\bar Z_a],Z_a)=\frac{\sqrt{-1}}{4}\delta_{ia}$$

and

$$\omega([[Z_i,\bar Z_i]^{0,1},W_b],\bar W_b)=\frac{1}{2}\lambda_{ib}\,,\quad \omega([[Z_i,\bar Z_i]^{1,0},\bar W_b],W_b)=-\frac{1}{2}\bar\lambda_{ib}$$

which imply

$$\rho_C(Z_i,\bar Z_i)=-\sqrt{-1}\left(\frac{1}{2}+\sum_{b=1}^{s}\mathfrak{Im}(\lambda_{ib})\right)=-\frac{\sqrt{-1}}{4}\,.$$

and, consequently,

$$\rho_C=-\omega_\infty\,,$$

where ω_∞ is the degenerate metric induced on M by the Poincaré metric on $\mathbb{H}^r$, namely,

$$\omega_\infty=\frac{\sqrt{-1}}{4}\sum_{i=1}^{r}\omega^i\wedge\bar\omega^i\,.$$

In general, we have that

$$P_i^j=(\rho_C)_{i\bar k}g^{\bar kj}=\begin{cases}-\frac{1}{4}g^{\bar ij} & \text{if } i\in\{1,\dots,r\}\,,\\ 0 & \text{otherwise}\ .\end{cases}$$

Then, part (3) of Theorem 4.2 readily implies that any left-invariant Hermitian metrics of the form (11) lifts to an expanding algebraic soliton on the universal covering of M with cosmological constant $c=\frac{1}{4A}$. Conversely, let ω be an algebraic soliton for the Chern-Ricci flow. Then, thanks to part (2) of Theorem 4.2, we have that

$$P-cI\in\mathrm{Der}(\mathfrak{g})\,.$$

On the other hand, we can easily see that, if $D \in \mathrm{Der}(\mathfrak{g})$, then $\mathfrak{h} \subseteq \ker D$, see proof of Corollary 5.4 in the present paper for the details. This readily implies that

$$-\frac{1}{4}g^{i\bar{i}} = -\frac{1}{4}g^{\bar{j}j} = c\,, \quad \text{for all } i,j = 1,\dots,r\,, \quad g^{\bar{i}j} = 0\,, \quad \text{for all } i \in \{1,\dots,r\}\,,\, j \neq i\,,$$

from which the claim follows.

Moreover, the Chern-Ricci flow evolves an arbitrary left-invariant Hermitian metric ω as $\omega_t = \omega + t\omega_\infty$ and $\frac{\omega_t}{1+t} \to \omega_\infty$ as $t \to \infty$. In order to obtain the claim regarding the Gromov-Hausdorff convergence, we show that $\frac{\omega_t}{1+t}$ satisfies conditions 1,2,3 in Proposition 3.1. Here we denote by $|\cdot|_t$ the norm induced by ω_t.

Condition 2 is trivially satisfied since $\omega_{t|\mathfrak{J}\oplus\mathfrak{J}} = \omega_0$, for every $t \geq 0$, and

$$L_t(\gamma) = \frac{1}{\sqrt{1+t}}L_0(\gamma)\,,$$

for every curve γ in M tangent to $\ker \omega_\infty$.

On the other hand, for a vector $v \in \mathfrak{h}$, we have

$$\frac{1}{\sqrt{1+t}}|v|_t \leq C|v|_0\,,$$

for a constant $C > 0$ independent on v. This, together with condition 2, guarantees condition 1.

In order to prove condition 3, let $\epsilon, \ell > 0$ and $T > 0$ be such that

$$\left|\frac{|v|_t}{\sqrt{1+t}} - |v|_\infty\right| \leq \frac{\epsilon}{\ell}\,,$$

for every $v \in \mathfrak{h}$ and $t \geq T$. Let γ be a curve in M tangent to $\mathcal{H}$ which is parametrized by arclength with respect to ω_∞ and such that $L_\infty(\gamma) < \ell$. Then

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left|\frac{1}{\sqrt{1+t}}|\dot\gamma|_t - |\dot\gamma|_\infty\right| da \leq \frac{\epsilon}{\ell} b \leq \epsilon\,,$$

since $b \leq \ell$.

For the last statement, we identify ω_t with its pull-back onto $\mathbb{H}^r \times \mathbb{C}^s$ and we fix as base point the identity element of $\mathbb{H}^r \times \mathbb{C}^s$. Firstly, we observe that the endomorphism D represented with respect to the frame $\{Z_1,\dots,Z_r,W_1,\dots,W_s\}$ by the following matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\mathfrak{J}} \end{pmatrix}$$

is a derivation of $\mathfrak{g}$. Moreover, we can construct

$$\exp(s(t)D) = \begin{pmatrix} \mathrm{I}_{\mathfrak{h}} & 0 \\ 0 & e^{s(t)}\mathrm{I}_{\mathfrak{J}} \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}, J)\,, \quad \text{for every } t \geq 0\,,$$

where $s(t) = \log(\sqrt{1+t})$ and define the 1-parameter family $\{\varphi_t\} \subseteq \mathrm{Aut}(\mathbb{H}^r \times \mathbb{C}^s, J)$ such that

$$d\varphi_t = \exp(s(t)D)\,, \quad \text{for every } t \geq 0\,.$$

Trivially, we see that

$$\varphi_t^*\frac{\omega_t}{1+t}(Z_i, \bar{Z}_j) = \sqrt{-1}\frac{1}{1+t}\left(g_{i\bar{j}} + \frac{t}{4}\delta_{ij}\right) \to \frac{\sqrt{-1}}{4}\delta_{ij} \quad \text{as } t \to \infty\,,$$

$$\varphi_t^*\frac{\omega_t}{1+t}(Z_i, \bar{W}_j) = \sqrt{-1}\frac{e^{s(t)}}{1+t}g_{i\overline{r+j}} \to 0 \quad \text{as } t \to \infty\,,$$

$$\varphi_t^*\frac{\omega_t}{1+t}(W_i, \bar{W}_j) = \sqrt{-1}\frac{e^{2s(t)}}{1+t}g_{r+i\overline{r+j}} \to \sqrt{-1}g_{r+i\overline{r+j}} \quad \text{as } t \to \infty\,.$$

These facts guarantee that

$$\varphi_t^*\frac{\omega_t}{1+t} \to \omega_\infty + \omega_{\mathfrak{J}\oplus\mathfrak{J}} \quad \text{as } t \to \infty\,,$$

hence, the assertion follows. □

5. Proof of the main result

In this section we prove Theorem 1.1.

The existence of pluriclosed metrics on Oeljeklaus-Toma manifolds was studied in [1], [8] and [18]. In particular from [1] it follows the following result.

Theorem 5.1 (Corollary 3, [1])**.** *An Oeljeklaus-Toma manifold of type (r,s) admits a pluriclosed metric if and only if $r = s$ and*

$$\sigma_j(u)|\sigma_{r+j}(u)|^2 = 1\,, \qquad \text{for every } j = 1,\dots,s \text{ and } u \in U\,. \tag{14}$$

Condition (14) in the previous Theorem can be rewritten in terms of the structure constants appearing in (1). Indeed, (1) together with (14) forces $b_{ki} \in \{0,-1\}$ and $b_{ki}b_{li} = 0$, for every $i,k,l = 1,\dots,s$ with $k \neq l$. In particular, using (3), for every fixed index $k \in \{1,\dots,s\}$, there exists a unique $i_k \in \{1,\dots,s\}$ such that

$$b_{ki_k} = -1\,, \quad b_{ki} = 0\,,$$

for all $i \neq i_k$ and, if $k \neq l$, then $i_k \neq i_l$. Hence, up to a reorder of the γ_j's, we may and do assume, without loss of generality, $i_k = k$, for every $k \in \{1,\dots,s\}$, i.e.

$$\lambda_{ki} = \begin{cases} -\frac{1}{2}c_{ki} & \text{if } i \neq k\,, \\ -\frac{1}{2}c_{kk} - \frac{\sqrt{-1}}{4} & \text{if } i = k\,. \end{cases} \tag{15}$$

Proposition 5.2 (Characterization of left-invariant pluriclosed metrics on Oeljeklaus-Toma manifolds)**.** *A left-invariant metric ω on an Oeljeklaus-Toma manifold admitting pluriclosed metrics is pluriclosed if and only if it takes the following expression with respect to a coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ satisfying* (1) *and* (15)*:*

$$\omega = \sqrt{-1}\sum_{i=1}^{s} A_i\omega^i\wedge\bar\omega^i + B_i\gamma^i\wedge\bar\gamma^i + \sqrt{-1}\sum_{r=1}^{k}\left(C_r\omega^{p_r}\wedge\bar\gamma^{p_r} + \bar C_r\gamma^{p_r}\wedge\bar\omega^{p_r}\right) \tag{16}$$

for some $A_1,\dots,A_s,B_1,\dots,B_s \in \mathbb{R}_+$, $C_1,\dots,C_k \in \mathbb{C}$, where $\{p_1,\dots,p_k\} \subseteq \{1,\dots,s\}$ are such that

$$\lambda_{jp_i} = 0\,,\ \text{for all } j \neq p_i\,,\ \text{for all } i = 1,\dots,k\,.$$

Proof. We assume $s > 1$ since the case $s = 1$ is trivial. Let

$$\omega = \sqrt{-1}\sum_{p,q=1}^{s} A_{p\bar q}\omega^p\wedge\bar\omega^q + B_{p\bar q}\gamma^p\wedge\bar\gamma^q + C_{p\bar q}\omega^p\wedge\bar\gamma^q + \bar C_{p\bar q}\gamma^q\wedge\bar\omega^p$$

be an arbitrary real left-invariant $(1,1)$-form on M, with $A_{p\bar p}, B_{p\bar p} \in \mathbb{R}$, for every $p = 1,\dots,s$, $A_{p\bar q}, B_{p\bar q} \in \mathbb{C}$, for all $p,q = 1,\dots,s$ with $p \neq q$, and $C_{p\bar q} \in \mathbb{C}$, for every $p,q = 1,\dots,s$.

From the structure equations (1), it easily follows

$$\begin{cases} \partial\bar\partial(\omega^p\wedge\bar\omega^q) \in \langle\omega^p\wedge\omega^q\wedge\bar\omega^p\wedge\bar\omega^q\rangle \\ \partial\bar\partial(\omega^p\wedge\bar\gamma^q) \in \langle\omega^i\wedge\omega^j\wedge\bar\omega^l\wedge\bar\gamma^m\rangle \\ \partial\bar\partial(\gamma^p\wedge\bar\gamma^q) \in \langle\omega^i\wedge\bar\omega^j\wedge\gamma^l\wedge\bar\gamma^m\rangle \end{cases} \tag{17}$$

and that ω is pluriclosed if and only if the following three conditions are satisfied

$$\sum_{p,q=1}^{s} A_{p\bar q}\partial\bar\partial(\omega^p\wedge\bar\omega^q) = 0\,; \tag{18}$$

$$\sum_{p,q=1}^{s} B_{p\bar q}\partial\bar\partial(\gamma^p\wedge\bar\gamma^q) = 0\,; \tag{19}$$

$$\sum_{p,q=1}^{s} C_{p\bar q}\partial\bar\partial(\omega^p\wedge\bar\gamma^q) = 0\,. \tag{20}$$

The first relation in (17) yields that (18) is satisfied if and only if

$$A_{p\bar q}=0\,,\ \text{for all } p\neq q\,.$$

Next we focus on (19). We have

$$\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=\partial\left(-\sum_{\delta=1}^s\lambda_{\delta p}\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q-\gamma^p\wedge\sum_{\delta=1}^s\bar\lambda_{\delta q}\bar\omega^\delta\wedge\bar\gamma^q\right)$$

and

$$\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=\sum_{\delta=1}^s(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\partial\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q-\bar\omega^\delta\wedge\partial\gamma^p\wedge\bar\gamma^q+\bar\omega^\delta\wedge\gamma^p\wedge\partial\bar\gamma^q\right)$$

which implies

$$\begin{aligned}\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=&\sum_{\delta=1}^s\frac{\sqrt{-1}}{2}(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^\delta\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q-\sum_{\delta=1}^s(\bar\lambda_{\delta q}-\lambda_{\delta p})\bar\omega^\delta\wedge\left(\sum_{a=1}^s\lambda_{ap}\omega^a\wedge\gamma^p\right)\wedge\bar\gamma^q\\&+\sum_{\delta=1}^s(\bar\lambda_{\delta q}-\lambda_{\delta p})\bar\omega^\delta\wedge\gamma^p\wedge\left(-\sum_{a=1}^s\bar\lambda_{aq}\omega^a\wedge\bar\gamma^q\right)\\=&\sum_{\delta=1}^s\frac{\sqrt{-1}}{2}(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^\delta\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q+\sum_{\delta,a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q\,.\end{aligned}$$

Finally, we get

$$\begin{aligned}\partial\bar\partial(\gamma^p\wedge\bar\gamma^q)=&\sum_{\delta=1}^s(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q\\&+\sum_{\delta\neq a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta\wedge\gamma^p\wedge\bar\gamma^q\end{aligned}$$

and that condition (19) is equivalent to

$$B_{p\bar q}\left(\sum_{\delta=1}^s(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\delta\neq a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta\right)=0\,,$$

for every $p,q=1,\dots,s$.

By using our conditions on the b_{ki}'s, it is easy to show that the quantity

$$\sum_{\delta=1}^s(\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\delta\neq a}(\lambda_{ap}-\bar\lambda_{aq})(\bar\lambda_{\delta q}-\lambda_{\delta p})\omega^a\wedge\bar\omega^\delta$$

is vanishing for $p=q$ and, consequently, there are no restrictions on the $B_{q\bar q}$'s. Now we observe that the real part of

$$(\bar\lambda_{pq}-\lambda_{pp})\left(\frac{\sqrt{-1}}{2}+\lambda_{pp}-\bar\lambda_{pq}\right)$$

is different from 0, for every p,q with $p\neq q$, which forces $B_{p\bar q}=0$, for $p\neq q$. Indeed, we have

$$\begin{aligned}\bar\lambda_{\delta q}-\lambda_{\delta p}&=\frac12(c_{\delta p}-c_{\delta q})-\frac{\sqrt{-1}}{4}(b_{\delta p}+b_{\delta q})\,,\\\frac{\sqrt{-1}}{2}+\lambda_{\delta p}-\bar\lambda_{\delta q}&=-\frac12(c_{\delta p}-c_{\delta q})+\frac{\sqrt{-1}}{2}\left(1+\frac{b_{\delta p}+b_{\delta q}}{2}\right)\end{aligned}$$

which implies

$$\Re\mathfrak{e}\left((\bar\lambda_{\delta q}-\lambda_{\delta p})\left(\frac{\sqrt{-1}}{2}+\lambda_{\delta q}-\bar\lambda_{\delta p}\right)\right)=-\frac{(c_{\delta p}-c_{\delta q})^2}{4}+\frac14\left(\frac{b_{\delta p}+b_{\delta q}}{2}\right)\left(1+\frac{b_{\delta p}+b_{\delta q}}{2}\right)\,. \tag{21}$$

Since $p \neq q$, we have
$$b_{pp} = -1\,, \quad b_{pq} = 0\,,$$
and so (21) computed for $\delta = q$ gives
$$\mathfrak{Re}\left(\left(\bar\lambda_{pq} - \lambda_{pp}\right)\left(\frac{\sqrt{-1}}{2} + \lambda_{pq} - \bar\lambda_{pp}\right)\right)\right) = \frac{1}{4}\left(-(c_{pp} - c_{pq})^2 - \frac{1}{4}\right) \neq 0\,,$$
as required. Therefore equation (19) is satisfied if and only if
$$B_{p\bar q} = 0\,, \text{ for all } p \neq q\,.$$

Next we focus on (20). We have
$$\partial\bar\partial(\omega^p \wedge \bar\gamma^q) = \partial\left(\frac{\sqrt{-1}}{2}\omega^p \wedge \bar\omega^p \wedge \bar\gamma^q - \omega^p \wedge \left(\sum_{\delta=1}^{s} \bar\lambda_{\delta q}\bar\omega^\delta \wedge \bar\gamma^q\right)\right)$$
and
$$\partial\bar\partial(\omega^p \wedge \bar\gamma^q) = \frac{\sqrt{-1}}{2}\left(-\frac{\sqrt{-1}}{2}\omega^p \wedge \omega^p \wedge \bar\omega^p \wedge \bar\gamma^q + \omega^p \wedge \bar\omega^p \wedge \left(-\sum_{\delta=1}^{s} \bar\lambda_{\delta q}\omega^\delta \wedge \bar\gamma^q\right)\right)$$
$$+\sum_{\delta=1}^{s} \frac{\sqrt{-1}}{2}\bar\lambda_{\delta q}\omega^p \wedge \omega^\delta \wedge \bar\omega^\delta \wedge \bar\gamma^q + \sum_{\delta=1}^{s} \bar\lambda_{\delta q}\omega^p \wedge \bar\omega^\delta \wedge \left(\sum_{a=1}^{s} \bar\lambda_{aq}\omega^a \wedge \bar\gamma^q\right)\,.$$
Hence we get
$$\partial\bar\partial(\omega^p \wedge \bar\gamma^q) = \sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2}\bar\lambda_{\delta q}\omega^p \wedge \bar\omega^p \wedge \omega^\delta \wedge \bar\gamma^q + \sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2}\bar\lambda_{\delta q}\omega^p \wedge \omega^\delta \wedge \bar\omega^\delta \wedge \bar\gamma^q$$
$$+\sum_{\substack{\delta,a\\ a\neq p}} \bar\lambda_{\delta q}\bar\lambda_{aq}\omega^p \wedge \bar\omega^\delta \wedge \omega^a \wedge \bar\gamma^q$$
and
$$\partial\bar\partial(\omega^p \wedge \bar\gamma^q) = \sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2}\bar\lambda_{\delta q}\omega^p \wedge \bar\omega^p \wedge \omega^\delta \wedge \bar\gamma^q + \sum_{\substack{a=1\\ a\neq p}}^{s} \bar\lambda_{pq}\bar\lambda_{aq}\omega^p \wedge \bar\omega^p \wedge \omega^a \wedge \bar\gamma^q$$
$$+\sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \frac{\sqrt{-1}}{2}\bar\lambda_{\delta q}\omega^p \wedge \omega^\delta \wedge \bar\omega^\delta \wedge \bar\gamma^q + \sum_{\substack{\delta,a\\ \delta\neq p\\ a\neq p}} \bar\lambda_{\delta q}\bar\lambda_{aq}\omega^p \wedge \bar\omega^\delta \wedge \omega^a \wedge \bar\gamma^q\,.$$

Therefore
$$\partial\bar\partial(\omega^p \wedge \bar\gamma^q) = \sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2} + \bar\lambda_{pq}\right)\omega^p \wedge \bar\omega^p \wedge \omega^\delta \wedge \bar\gamma^q +$$
$$\sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2} - \bar\lambda_{\delta q}\right)\omega^p \wedge \omega^\delta \wedge \bar\omega^\delta \wedge \bar\gamma^q + \sum_{\substack{\delta\neq a\\ \delta\neq p\\ a\neq p}} \bar\lambda_{\delta q}\bar\lambda_{aq}\omega^p \wedge \bar\omega^\delta \wedge \omega^a \wedge \bar\gamma^q$$
and (20) is equivalent to
$$C_{p\bar q}\left(\sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2} + \bar\lambda_{pq}\right)\bar\omega^p \wedge \omega^\delta + \sum_{\substack{\delta=1\\ \delta\neq p}}^{s} \bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2} - \bar\lambda_{\delta q}\right)\omega^\delta \wedge \bar\omega^\delta + \sum_{\substack{\delta\neq a\\ \delta\neq p\\ a\neq p}} \bar\lambda_{\delta q}\bar\lambda_{aq}\bar\omega^\delta \wedge \omega^a\right) = 0\,,$$

for every $p,q=1,\dots,s$. Since

$$\lambda_{pq}\neq\pm\frac{\sqrt{-1}}{2}\,,\quad \text{for all}\ \ p,q=1,\dots,s\,,$$

the quantity

$$E_{p\bar q}:=\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}+\bar\lambda_{pq}\right)\bar\omega^p\wedge\omega^\delta+\sum_{\substack{\delta=1\\ \delta\neq p}}^{s}\bar\lambda_{\delta q}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{\delta q}\right)\omega^\delta\wedge\bar\omega^\delta+\sum_{\substack{\delta\neq a\\ \delta\neq p\\ a\neq p}}\bar\lambda_{\delta q}\bar\lambda_{aq}\bar\omega^\delta\wedge\omega^a$$

is vanishing if and only if

$$\lambda_{\delta q}=0\,,\qquad \text{for all}\ \delta\neq p\,.$$

Since $\lambda_{qq}\neq 0$, it follows

$$E_{p\bar q}\neq 0\,,\qquad \text{for every}\ p,q\ \text{with}\ p\neq q$$

and

$$E_{p\bar p}=0\ \text{if and only if}\ c_{\delta p}=0\,,\ \text{for all}\ \delta\neq p\,.$$

Hence the claim follows. □

Proposition 5.3. *Let*

$$\omega=\sqrt{-1}\sum_{i=1}^{s}A_i\omega^i\wedge\bar\omega^i+B_i\gamma^i\wedge\bar\gamma^i+\sqrt{-1}\sum_{r=1}^{k}\left(C_r\omega^{p_r}\wedge\bar\gamma^{p_r}+\bar C_r\gamma^{p_r}\wedge\bar\omega^{p_r}\right)\tag{22}$$

be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold, where the components are with respect to a coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ satisfying (1) and (15) and $\{p_1,\dots,p_k\}\subseteq\{1,\dots,s\}$ are such that

$$\lambda_{jp_i}=0\,,\ \textit{for all}\ j\neq p_i\,,\ \textit{for all}\ i=1,\dots,k\,.$$

Then, the $(1,1)$-part of the Bismut-Ricci form of ω takes the following expression:

$$\rho_B^{1,1}=-\sqrt{-1}\sum_{r=1}^{k}\frac{3}{4}\left(1+\frac{|C_r|^2}{A_{p_r}B_{p_r}-|C_r|^2}\right)\omega^{p_r}\wedge\bar\omega^{p_r}-\sqrt{-1}\sum_{i\notin\{p_1,\dots,p_k\}}\frac{3}{4}\omega^i\wedge\bar\omega^i$$

$$-\sqrt{-1}\sum_{r=1}^{k}\left(-\frac{3}{16}-\frac{c^2_{p_rp_r}}{4}-\frac{\sqrt{-1}c_{p_rp_r}}{4}\right)\frac{B_{p_r}C_r}{A_{p_r}B_{p_r}-|C_r|^2}\omega^{p_r}\wedge\bar\gamma^{p_r}+\quad\text{conjugates}\,.$$

Proof. We recall that the Bismut-Ricci form of a left-invariant Hermitian metric $\omega=\sqrt{-1}\sum_{a,b=1}^n g_{a\bar b}\alpha^a\wedge\bar\alpha^b$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$\rho_B(X,Y)=-\sum_{a,b=1}^{n}g^{a\bar b}\omega([[X,Y]^{1,0},X_a],\bar X_b)+g^{\bar ab}\omega([[X,Y]^{0,1},\bar X_a],X_b)+\sqrt{-1}\sum_{a,b=1}^{n}g^{a\bar b}\omega([X,Y],J[X_a,\bar X_b])\,,\tag{23}$$

for every left-invariant vector fields X,Y on G, where $\{\alpha^i\}$ is a left-invariant $(1,0)$-coframe with dual frame $\{X_a\}$ and $(g^{\bar ba})$ is the inverse matrix to $(g_{i\bar j})$ (see e.g. [31]). We apply (23) to a left-invariant Hermitian metric on an Oeljeklaus-Toma manifold of the form (22).

We have

$$g^{\bar is+i}=\begin{cases}0 & \text{if } i\notin\{p_1,\dots,p_k\}\,,\\ -\frac{C_i}{A_iB_i-|C_i|^2} & \text{otherwise}\,,\end{cases}\qquad g^{\bar ii}=\frac{B_i}{A_iB_i-|C_i|^2}\,,\qquad g^{\overline{s+i}s+i}=\frac{A_i}{A_iB_i-|C_i|^2}$$

and taking into account that the ideal $\mathfrak{I}$ is abelian, we have

$$\rho_B(X,Y)=-\sum_{i=1}^{4}\rho_i(X,Y)\,,$$

where

$$\rho_1(X,Y)=\sum_{a=1}^{s} g^{a\bar a}(\omega([[X,Y]^{1,0},Z_a],\bar Z_a)-\frac{\sqrt{-1}}{2}\omega([X,Y],Z_a-\bar Z_a)+\omega([[X,Y]^{0,1},\bar Z_a],Z_a))\,,$$

$$\rho_2(X,Y)=\sum_{a=1}^{s} g^{s+a\overline{s+a}}(\omega([[X,Y]^{1,0},W_a],\bar W_a)+\omega([[X,Y]^{0,1},\bar W_a],W_a))\,,$$

$$\rho_3(X,Y)=\sum_{r=1}^{k} g^{p_r\overline{s+p_r}}\left(\omega([[X,Y]^{1,0},Z_{p_r}],\bar W_{p_r})-\omega([X,Y],[Z_{p_r},\bar W_{p_r}])\right)+g^{\overline{p_r}s+p_r}\omega([[X,Y]^{0,1},\bar Z_{p_r}],W_{p_r})\,,$$

$$\rho_4(X,Y)=\sum_{r=1}^{k} g^{s+p_r\bar p_r}\left(\omega([[X,Y]^{1,0},W_{p_r}],\bar Z_{p_r})+\omega([X,Y],[W_{p_r},\bar Z_{p_r}]))\right)+g^{\overline{s+p_r}p_r}\omega([[X,Y]^{0,1},\bar W_{p_r}],Z_{p_r})\,.$$

Next we focus on the computation of $\rho_B(Z_i,\bar Z_j)$. Thanks to (1), we easily obtain that

$$\rho_B(Z_i,\bar Z_j)=0\,,\quad \text{for every } i,j=1,\dots,s\,,\ i\neq j\,.$$

On the other hand,

$$\rho_1(Z_i,\bar Z_i)=-\frac{\sqrt{-1}}{2}\sum_{a=1}^{s} g^{a\bar a}\left(-\frac{\sqrt{-1}}{2}\omega(Z_i+\bar Z_i,Z_a-\bar Z_a)\right)=\frac{\sqrt{-1}}{2}g^{i\bar i}A_i=\frac{\sqrt{-1}}{2}\left(\frac{A_iB_i}{A_iB_i-|C_i|^2}\right).$$

Moreover, we have

$$\begin{aligned}\rho_2(Z_i,\bar Z_i)&=-\frac{\sqrt{-1}}{2}\sum_{a=1}^{s} g^{s+a\overline{s+a}}(\omega([Z_i,W_a],\bar W_a)+\omega([\bar Z_i,\bar W_a],W_a)\\&=-\sqrt{-1}\sum_{a=1}^{s} g^{s+a\overline{s+a}}\mathfrak{Re}\,\omega([Z_i,W_a],\bar W_a)\,.\end{aligned}$$

Using (1), we have

$$\omega([Z_i,W_a],\bar W_a)=-\sqrt{-1}\lambda_{ia}B_a\,,$$

$$\mathfrak{Re}\,\omega([Z_i,W_a],\bar W_a)=\frac{B_ab_{ia}}{4}=-\frac{B_a}{4}\delta_{ia}\,.$$

Then,

$$\rho_2(Z_i,\bar Z_i)=\sqrt{-1}\frac{g^{s+i\overline{s+i}}B_i}{4}=\frac{\sqrt{-1}}{4}\frac{A_iB_i}{A_iB_i-|C_i|^2}\,.$$

Next we observe that

$$\rho_3(Z_i,\bar Z_i)+\rho_4(Z_i,\bar Z_i)=0$$

which implies

$$\rho_B(Z_i,\bar Z_i)=\begin{cases}-\sqrt{-1}\frac{3}{4}\left(1+\frac{|C_r|^2}{A_{p_r}B_{p_r}-|C_r|^2}\right) & \text{if there exists } r=1,\dots,k \text{ such that } i=p_r\,,\\ -\sqrt{-1}\frac{3}{4} & \text{if } i\notin\{p_1,\dots,p_k\}\,.\end{cases}\tag{24}$$

We have

$$\begin{aligned}\rho_3(Z_i,\bar Z_i)=&\sum_{j=1}^{k} g^{p_j\overline{s+p_j}}\omega([Z_i,\bar Z_i],[Z_{p_j},\bar W_{p_j}])=-\frac{\sqrt{-1}}{2}\sum_{j=1}^{k} g^{p_j\overline{s+p_j}}\bar\lambda_{p_jp_j}\omega(Z_i+\bar Z_i,\bar W_{p_j})\\=&\begin{cases}0 & \text{if } i\notin\{p_1,\dots,p_k\}\,,\\ \frac{1}{2}g^{i\overline{s+i}}\bar\lambda_{ii}C_i & \text{otherwise}\,.\end{cases}\end{aligned}$$

We compute the three addends in the expression of ρ_4 separately:

$$\omega([[Z_i,\bar Z_i]^{1,0},W_{p_j}],\bar Z_{p_j}) = -\frac{1}{2}\lambda_{ip_j}\bar C_{p_j} = \begin{cases}0 & \text{if } i\notin\{p_1,\dots,p_k\} \quad \text{or } i\neq p_j\,,\\ -\frac12\lambda_{ii}\bar C_i & \text{otherwise}\,;\end{cases}$$

$$\omega([Z_i,\bar Z_i],[W_{p_j},\bar Z_{p_j}]) = \frac{1}{2}\lambda_{p_jp_j}g_{\bar i s+p_j} = \begin{cases}0 & \text{if } i\notin\{p_1,\dots,p_k\} \quad \text{or} \quad i\neq p_j\,,\\ \frac12\lambda_{ii}\bar C_i & \text{otherwise}\,;\end{cases}$$

$$\omega([[Z_i,\bar Z_i]^{0,1},\bar W_{p_j}],Z_{p_j}) = \frac{1}{2}\bar\lambda_{ip_j}g_{\overline{s+p_j}p_j} = \begin{cases}0 & \text{if } i\neq p_j\,,\\ \frac12\bar\lambda_{ii}C_i & \text{otherwise}\,.\end{cases}$$

It follows

$$\rho_3(Z_i,\bar Z_i)=\rho_4(Z_i,\bar Z_i)=0 \quad \text{if } i\notin\{p_1,\dots,p_k\}\,,$$

and, for $i\in\{p_1,\dots,p_k\}$,

$$\rho_3(Z_i,\bar Z_i)+\rho_4(Z_i,\bar Z_i) = -\frac12 g^{i\overline{s+i}}\bar\lambda_{ii}C_i - g^{s+i\bar i}\frac12\lambda_{ii}\bar C_i + g^{s+i\bar i}\frac12\lambda_{ii}\bar C_i + g^{\overline{s+i}i}\frac12\bar\lambda_{ii}C_i = 0\,.$$

Now, we focus on the calculation of $\rho_B(Z_i,\bar W_j)$. We have

$$\begin{aligned}\rho_1(Z_i,\bar W_j) &= \sum_{a=1}^s g^{a\bar a}\bar\lambda_{ij}\left(-\frac{\sqrt{-1}}{2}\omega(\bar W_j,Z_a-\bar Z_a)+\omega([\bar W_j,\bar Z_a],Z_a)\right)\\ &= \begin{cases}0 & \text{otherwise}\,,\\ \sqrt{-1}g^{i\bar i}C_i\bar\lambda_{ii}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{ii}\right) & \text{if } i=j\in\{p_1,\dots,p_k\}\,,\end{cases}\end{aligned}$$

and since $\mathfrak{I}$ is abelian

$$\rho_2(Z_i,\bar W_j)=0\,.$$

Furthermore

$$\begin{aligned}\rho_3(Z_i,\bar W_j) &= \sum_{j=1}^k g^{\overline{p_j}s+p_j}\omega([[Z_i,\bar W_j]^{0,1},\bar Z_{p_j}],W_{p_j}) = -\sqrt{-1}\sum_{j=1}^k g^{\overline{p_j}s+p_j}\bar\lambda_{ij}\bar\lambda_{p_jp_j}g_{\overline{s+j}s+p_j}\\ &= \begin{cases}0 & \text{otherwise}\,,\\ -\sqrt{-1}\bar\lambda_{jj}^2g^{\bar j s+j}B_j & \text{if } i=j\in\{p_1,\dots,p_k\}\end{cases}\end{aligned}$$

and

$$\begin{aligned}\rho_4(Z_i,\bar W_j) &= \sum_{j=1}^k g^{s+p_j\bar p_j}\omega([Z_i,\bar W_j],[W_{p_j},\bar Z_{p_j}]) = \sqrt{-1}\sum_{j=1}^k g^{s+p_j\bar p_j}\bar\lambda_{ij}\lambda_{p_jp_j}g_{\overline{s+j}s+p_j}\\ &= \begin{cases}0 & \text{otherwise}\,,\\ \sqrt{-1}g^{s+j\bar j}\bar\lambda_{jj}\lambda_{jj}B_j & \text{if } i=j\in\{p_1,\dots,p_k\}\,.\end{cases}\end{aligned}$$

It follows that $\rho_B(Z_i,\bar W_j)\neq 0$ if and only if $i=j\in\{p_1,\dots,p_k\}$. In such a case, we have

$$\rho_B(Z_j,\bar W_j) = -\sqrt{-1}\left(g^{s+j\bar j}B_j\left(|\lambda_{jj}|^2-\bar\lambda_{jj}^2\right)+g^{j\bar j}C_j\bar\lambda_{jj}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{jj}\right)\right)\,.$$

Since

$$g^{s+j\bar j}B_j = -\frac{B_jC_j}{A_jB_j-|C_j|^2} \quad \text{and} \quad g^{j\bar j}C_j = \frac{B_jC_j}{A_jB_j-|C_j|^2}\,,$$

we infer

$$\rho_B(Z_j,\bar W_j) = -\sqrt{-1}\left(\bar\lambda_{jj}\left(\frac{\sqrt{-1}}{2}-\bar\lambda_{jj}\right)-\left(|\lambda_{jj}|^2-\bar\lambda_{jj}^2\right)\right)\frac{B_jC_j}{A_jB_j-|C_j|^2}\,.$$

Taking into account that $\lambda_{jj}=-\frac{\sqrt{-1}}{4}-\frac{c_{jj}}{2}$, we obtain

$$\rho_B(Z_j,\bar W_j) = -\sqrt{-1}\left(-\frac{3}{16}-\frac{c_{jj}^2}{4}-\frac{\sqrt{-1}c_{jj}}{4}\right)\frac{B_jC_j}{A_jB_j-|C_j|^2}$$

and the claim follows. □

Corollary 5.4. *Let ω be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold M. Then ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if it takes the following diagonal expression with respect to a coframe $\{\omega^1,\dots,\omega^s,\gamma^1,\dots,\gamma^s\}$ satisfying* (1) *and* (15)*:*

$$\omega=\sqrt{-1}\sum_{i=1}^{s}A\omega^i\wedge\bar\omega^i+B_i\gamma^i\wedge\bar\gamma^i\,. \tag{25}$$

Proof. Let ω be a pluriclosed left-invariant metric on an Oeljeklaus-Toma manifold M. In view of [14, Section 7], ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if

$$\rho_B^{1,1}(\cdot,\cdot)=c\omega(\cdot,\cdot)+\frac{1}{2}\left(\omega(D\cdot,\cdot)+\omega(\cdot,D\cdot)\right)\,,$$

for some $c\in\mathbb{R}_-$ and some derivation D of $\mathfrak{g}$ such that $DJ=JD$.

Assume that ω takes the expression in formula (25). Proposition 5.3 implies that ρ_B is represented with respect to the basis $\{Z_1,\dots,Z_s,W_1,\dots,W_s\}$ by the matrix

$$P=-\frac{3}{4A}\begin{pmatrix}\mathrm{I}_{\mathfrak{h}} & 0\\ 0 & 0\end{pmatrix}\,.$$

Since

$$\frac{3}{4A}\begin{pmatrix}0 & 0\\ 0 & \mathrm{I}_{\mathfrak{I}}\end{pmatrix}$$

induces a symmetric derivation on $\mathfrak{g}$, ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M and the first part of the claim follows.

In order to prove the second part of the statement, we need some preliminary observations on derivations D of $\mathfrak{g}$ that commute with J, i.e. such that

$$D(\mathfrak{g}^{1,0})\subseteq\mathfrak{g}^{1,0}\,,\quad D(\mathfrak{g}^{0,1})\subseteq\mathfrak{g}^{0,1}\,.$$

We can write

$$DZ_i=\sum_{j=1}^{s}k^i_jZ_j+m^i_jW_j\qquad\text{and}\qquad D\bar Z_i=\sum_{j=1}^{s}l^i_j\bar Z_j+r^i_j\bar W_j\,.$$

Since D is a derivation, we have, for all $i=1,\dots,s$,

$$D[Z_i,\bar Z_i]=[DZ_i,\bar Z_i]+[Z_i,D\bar Z_i]\,.$$

On the other hand

$$\begin{aligned}D[Z_i,\bar Z_i]&=-\frac{\sqrt{-1}}{2}\left(\sum_{j=1}^{s}k^i_jZ_j+l^i_j\bar Z_j+m^i_jW_j+r^i_j\bar W_j\right)\,,\\ [DZ_i,\bar Z_i]&=-\frac{\sqrt{-1}}{2}k^i_i(Z_i+\bar Z_i)-\sum_{j=1}^{s}m^i_j\lambda_{ij}W_j\,,\\ [Z_i,D\bar Z_i]&=-\frac{\sqrt{-1}}{2}l^i_j(Z_i+\bar Z_i)+\sum_{j=1}^{s}r^i_j\bar\lambda_{ij}\bar W_j\end{aligned}$$

and

$$\begin{aligned}0&=D[Z_i,\bar Z_i]-[DZ_i,\bar Z_i]-[Z_i,D\bar Z_i]\\ &=-\frac{\sqrt{-1}}{2}\sum_{j\neq i}k^i_jZ_j+l^i_j\bar Z_j+\frac{\sqrt{-1}}{2}l^i_iZ_i+\frac{\sqrt{-1}}{2}k^i_i\bar Z_i+\sum_{j=1}^{s}m^i_j\left(\lambda_{ij}-\frac{\sqrt{-1}}{2}\right)W_j-r^i_j\left(\frac{\sqrt{-1}}{2}+\bar\lambda_{ij}\right)\bar W_j\end{aligned}$$

which forces $DZ_i,D\bar Z_i=0$, for all $i=1,\dots,s$. It follows that $D_{|\mathfrak{h}}=0$.

Moreover, for all $I, I' \in \mathfrak{J}$, we have

$$0 = D[I, I'] = [DI, I'] + [I, DI'] ,$$

which implies

$$[DI, I'] = -[I, DI'] .$$

Assume

$$DW_i = \sum_{j=1}^{s} k_j^{s+i} Z_j + m_j^{s+i} W_j \quad \text{and} \quad D\bar{W}_i = \sum_{j=1}^{s} l_j^{s+i} \bar{Z}_j + r_j^{s+i} \bar{W}_j ,$$

then

$$[DW_i, \bar{W}_i] = \sum_{j=1}^{s} k_j^{s+i} [Z_j, \bar{W}_i] \in \mathfrak{J}^{0,1} \quad \text{and} \quad [W_i, D\bar{W}_i] = \sum_{j=1}^{s} l_j^{s+i} [W_i, \bar{Z}_j] \in \mathfrak{J}^{1,0} .$$

This implies

$$DW_i = \sum_{j=1}^{s} m_j^{s+i} W_j , \quad D\bar{W}_i = \sum_{j=1}^{s} r_j^{s+i} \bar{W}_j ,$$

i.e. $D(\mathfrak{J}) \subseteq \mathfrak{J}$. Moreover, for all $i = 1, \dots, s$, we have that

$$D[Z_i, W_i] = -\lambda_{ii} DW_i = -\sum_{j=1}^{s} \lambda_{ii} m_j^{s+i} W_j ,$$

while $[DZ_i, W_i] = 0$ and

$$[Z_i, DW_i] = -\sum_{j=1}^{s} m_j^{s+i} \lambda_{ij} W_j .$$

Using again the fact that D is a derivation, we have

$$DW_i = \sum_{j \in J_i} m_j W_j$$

where

$$J_i = \{ j \in \{1, \dots, s\} \quad | \quad \lambda_{ii} = \lambda_{ij} \} .$$

With analogous computations, we infer

$$D\bar{W}_i = \sum_{j \in J_i} r_j^{s+i} \bar{W}_j .$$

Clearly, $i \in J_i$. On the other hand, for all $i = 1, \dots, s$, we know that $\Im\mathfrak{m}(\lambda_{ii}) \neq 0$, while, for all $i \neq j$, $\lambda_{ij} \in \mathbb{R}$. This guarantees that, for all $i = 1, \dots, s$,

$$J_i = \{i\} .$$

This allows us to write

$$DW_i = m_i^{s+i} W_i , \quad D\bar{W}_i = r_i^{s+i} \bar{W}_i .$$

From the relations above, we obtain that

$$\mathrm{Der}(\mathfrak{g})^{1,0} = \{ E \in \mathrm{End}(\mathfrak{g})^{1,0} \quad | \quad \mathfrak{h} \subseteq \ker(E) , \ E(\langle W_i \rangle) \subseteq \langle W_i \rangle , \quad \text{for all } i = 1, \dots, s \} .$$

First of all, we suppose that ω is a pluriclosed Hermitian metric which takes the following diagonal expression with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15):

$$\omega = \sqrt{-1} \sum_{i=1}^{s} A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i .$$

such that there exist $i,j\in\{1,\dots,s\}$ such that $A_i\neq A_j$ and we suppose that ω is an algebraic soliton. Thanks to the facts regarding derivations proved before, we have that

$$-\sqrt{-1}\frac{3}{4}=\rho_B(Z_i,\bar Z_i)=c\omega(Z_i,\bar Z_i)+\frac12\left(\omega(DZ_i,\bar Z_i)+\omega(Z_i,D\bar Z_i)\right)=\sqrt{-1}cA_i\,,$$
$$-\sqrt{-1}\frac{3}{4}=\rho_B(Z_j,\bar Z_j)=c\omega(Z_j,\bar Z_j)+\frac12\left(\omega(DZ_j,\bar Z_j)+\omega(Z_j,D\bar Z_j)\right)=\sqrt{-1}cA_j\,,$$

which is impossible, since $A_i\neq A_j$.

Now suppose that ω is a pluriclosed metric on M which is not diagonal. So, we suppose that there exists $\tilde j=1,\dots,s$ such that $C_{\tilde j}\neq 0$. Then, assume that there exist a constant $c\in\mathbb{R}$ and $D\in\mathrm{Der}(\mathfrak{g})$ such that

$$(\rho_B)^{1,1}(\cdot,\cdot)=c\omega(\cdot,\cdot)+\frac12\left(\omega(D\cdot,\cdot)+\omega(\cdot,D\cdot)\right)\,,\quad DJ=JD\,.$$

On the other hand

$$0=\rho_B(W_{\tilde j},\bar W_{\tilde j})=c\omega(W_{\tilde j},\bar W_{\tilde j})+\frac12\left(\omega(DW_{\tilde j},\bar W_{\tilde j})+\omega(W_{\tilde j},D\bar W_{\tilde j})\right)=\sqrt{-1}cB_{\tilde j}+\frac{\sqrt{-1}}{2}(r^{s+\tilde j}_{\tilde j}+m^{s+\tilde j}_{\tilde j})B_{\tilde j}\,,$$
$$\rho_B(Z_{\tilde j},\bar W_{\tilde j})=c\omega(Z_{\tilde j},\bar W_{\tilde j})+\frac12\left(\omega(DZ_{\tilde j},\bar W_{\tilde j})+\omega(Z_{\tilde j},D\bar W_{\tilde j})\right)=\sqrt{-1}cC_{\tilde j}+\frac{\sqrt{-1}}{2}r^{s+\tilde j}_{\tilde j}C_{\tilde j}\,,$$
$$\rho_B(\bar Z_{\tilde j},W_{\tilde j})=c\omega(\bar Z_{\tilde j},W_{\tilde j})+\frac12\left(\omega(D\bar Z_{\tilde j},W_{\tilde j})+\omega(\bar Z_{\tilde j},DW_{\tilde j})\right)=-\sqrt{-1}c\bar C_{\tilde j}-\frac{\sqrt{-1}}{2}m^{s+\tilde j}_{\tilde j}\bar C_{\tilde j}\,,$$

which implies that

$$c=-\frac12(r^{s+\tilde j}_{\tilde j}+m^{s+\tilde j}_{\tilde j})\,,$$

On the other hand,

$$\rho_B(Z_{\tilde j},\bar W_{\tilde j})=\sqrt{-1}KC_{\tilde j}\,,$$

where

$$K=\left(\frac{3}{16}+\frac{c^2_{\tilde j\tilde j}}{4}+\frac{\sqrt{-1}c_{\tilde j\tilde j}}{4}\right)\frac{B_{\tilde j}}{A_{\tilde j}B_{\tilde j}-|C_{\tilde j}|^2}\,.$$

Then,

$$K=c+\frac12 r^{s+\tilde j}_{\tilde j}=-\frac12 m^{s+\tilde j}_{\tilde j}$$

and

$$\bar K=c+\frac12 m^{s+\tilde j}_{\tilde j}=-\frac12 r^{s+\tilde j}_{\tilde j}\,.$$

From this we obtain that

$$c=K+\bar K=2\mathfrak{Re}(K)>0\,.$$

On the other hand, we have

$$-\sqrt{-1}\frac34\left(1+\frac{|C_{\tilde j}|^2}{A_{\tilde j}B_{\tilde j}-|C_{\tilde j}|^2}\right)=\rho_B(Z_{\tilde j},\bar Z_{\tilde j})=c\omega(Z_{\tilde j},\bar Z_{\tilde j})+\frac12\left(\omega(DZ_{\tilde j},\bar Z_{\tilde j})+\omega(Z_{\tilde j},D\bar Z_{\tilde j})\right)=\sqrt{-1}cA_{\tilde j}\,,$$

which implies that c must be negative. From this the claim follows. □

Corollary 5.5. *Let ω be a pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold which takes the form* (16)*. Then the pluriclosed flow starting from ω is equivalent to the following system of ODEs:*

$$\begin{cases} A_i'=\frac34 & \text{if } i\notin\{p_1,\dots,p_k\}\,,\\ A_{p_r}'=\frac34\left(1+\frac{|C_r|^2}{A_{p_r}B_{p_r}-|C_r|^2}\right) & \text{for all } r=1,\dots,k\,,\\ B_j'=0 & \text{for all } j=1,\dots,s\,,\\ C_r'=-\left(\frac{3}{16}+\frac{c^2_{p_rp_r}}{4}+\frac{\sqrt{-1}c_{p_rp_r}}{4}\right)\frac{B_{p_r}C_r}{A_{p_r}B_{p_r}-|C_r|^2} & \text{for all } r=1,\dots,k\,.\end{cases}\tag{26}$$

Moreover, $|C_r|$ is bounded, for all $r=1,\dots,k$, the solution exists for all $t\in[0,+\infty)$ and $A_i\sim\frac34 t$, as $t\to+\infty$, for all $i=1,\dots,s$.

In particular,

$$\frac{\omega_t}{1+t} \to 3\omega_\infty$$

as $t \to \infty$*.*

Proof. Observe that, for every $r \in \{1,\dots,k\}$,

$$(|C_r|^2)' = -\left(\frac{3}{8} + \frac{c^2_{p_r p_r}}{2}\right) \frac{B_{p_r}|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \leq 0\,,$$

which guarantees that $|C_r|^2$ is bounded. On the other hand, denote, for all $r = 1,\dots,k$,

$$u_r = A_{p_r}B_{p_r} - |C_r|^2.$$

We have that

$$u_r' = A'_{p_r}B_{p_r} - (|C_r|^2)' = \frac{3}{4}B_{p_r} + \left(\frac{9}{8} + \frac{c^2_{p_r p_r}}{2}\right) \frac{B_{p_r}|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \geq 0\,.$$

This guarantees

$$A'_{p_r} = \frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2}\right) \leq \frac{3}{4}\left(1 + \frac{K}{u_r(0)}\right)\,,$$

where $K > 0$ such that $|C_r|^2 \leq K$, for all $t \geq 0$. This implies the long-time existence. As regards the last part of the statement, it is sufficient to prove that

$$\lim_{t\to+\infty} \frac{|C_r|^2}{u_r} = 0\,.$$

But,

$$u_r' \geq \frac{3}{4}B_{p_r}\,.$$

So,

$$u_r \geq \frac{3}{4}B_{p_r}t + u_r(0) \to +\infty\,,\ \ t \to +\infty\,.$$

Then,

$$\lim_{t\to+\infty} u_r(t) = +\infty\,,$$

and, since $|C_r|^2$ is bounded, the assertion follows. □

Proof of Theorem 1.1. Let ω be a left-invariant pluriclosed metric on an Oeljeklaus-Toma manifold. Corollary 5.5 implies that pluriclosed flow starting from ω has a long-time solution ω_t such that

$$\frac{\omega_t}{1+t} \to 3\omega_\infty \quad \text{as} \quad t \to \infty\,.$$

We show that $\frac{\omega_t}{1+t}$ satisfies conditions 1,2,3 in Proposition 3.1. Here we denote by $|\cdot|_t$ the norm induced by ω_t.

Taking into account that

$$\omega_{t|\mathfrak{I}\oplus\mathfrak{I}} = \omega_{0|\mathfrak{I}\oplus\mathfrak{I}}\,,$$

condition 2 follows.

Thanks to the fact that condition 2 holds,

$$\omega_{t|\mathfrak{h}\oplus\mathfrak{h}} = \sum_{i=1}^s A_i(t)\omega^i \wedge \bar\omega^i$$

with $\frac{A_i(t)}{1+t} \to \frac{3}{4}$ as $t \to \infty$ and there exist $C, T > 0$ such that, for every vector $v \in \mathfrak{h}$,

$$\frac{1}{\sqrt{1+t}}|v|_t \leq C|v|_0\,,$$

for every $t \geq T$, condition 1 is satisfied.

In order to prove Condition 3, let $\epsilon, \ell > 0$ and let γ be a curve in M tangent to $\mathcal{H}$ which is parametrized by arclength with respect to $3\omega_\infty$ and such that $L_\infty(\gamma) < \ell$. Let $v = \dot\gamma$ and $T > 0$ such that

$$\left|\frac{A_i(t)}{1+t} - \frac{3}{4}\right| \le \frac{3\epsilon^2}{4\ell^2}\,,$$

for $t \ge T$. Then

$$\left|\frac{1}{1+t}|v|_t^2 - |v|_\infty^2\right| \le \sum_{i=1}^{s}\left|\frac{A_i(t)}{1+t} - \frac{3}{4}\right||v_i|^2 \le \frac{\epsilon^2}{\ell^2}$$

and

$$|L_t(\gamma) - L_\infty(\gamma)| \le \int_0^b \left|\frac{1}{\sqrt{1+t}}|\dot\gamma|_t - |\dot\gamma|_\infty\right| da \le \frac{\epsilon}{\ell} b \le \epsilon\,,$$

since $b \le \ell$.

Now we show the last part of the statement, using the same argument as in Proposition 4.1, and we prove that $(\mathbb{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde\omega_\infty)$ where $\tilde\omega_\infty$ is an algebraic soliton. Again, here we are identifying ω_t with its pull-back onto $\mathbb{H}^s \times \mathbb{C}^s$ and we are fixing as base point the identity element of $\mathbb{H}^s \times \mathbb{C}^s$. It is enough to construct a 1-parameter family of biholomorphisms $\{\varphi_t\}$ of $\mathbb{H}^s \times \mathbb{C}^s$ such that

$$\varphi_t^* \frac{\omega_t}{1+t} \to \tilde\omega_\infty\,.$$

As we already observed, since $\mathfrak{I}$ is abelian the endomorphism represented by the matrix

$$D = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathfrak{I}} \end{pmatrix}$$

is a derivation of $\mathfrak{g}$ that commutes with the complex structure J. Then, we can consider

$$d\varphi_t = \exp(s(t)D) = \begin{pmatrix} I_{\mathfrak{h}} & 0 \\ 0 & e^{s(t)} I_{\mathfrak{I}} \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}, J)$$

where $s(t) = \log(\sqrt{1+t})$. Using $d\varphi_t$, we can define

$$\varphi_t \in \mathrm{Aut}(\mathbb{H}^s \times \mathbb{C}^s, J)\,.$$

For $i = 1, \ldots, s$ we have

$$\begin{aligned}
\frac{1}{1+t}(\varphi_t^*\omega_t)(Z_i, \bar Z_i) &= \frac{1}{1+t}\omega_t(Z_i, \bar Z_i) \to \frac{3}{4}\sqrt{-1}\,, \quad \text{as } t \to \infty\,,\\
\frac{1}{1+t}(\varphi_t^*\omega_t)(Z_i, \bar W_i) &= \frac{1}{\sqrt{1+t}}\omega_t(Z_i, \bar W_i) \to 0\,, \quad \text{as } t \to \infty\,,\\
\frac{1}{1+t}(\varphi_t^*\omega_t)(W_i, \bar W_i) &= \omega_t(W_i, \bar W_i) = \sqrt{-1}B_i(0)\,.
\end{aligned}$$

Then,

$$\frac{1}{1+t}\varphi_t^*\omega_t \to \tilde\omega_\infty\,, \quad \text{as } t \to \infty\,,$$

where

$$\tilde\omega_\infty = 3\,\omega_\infty + \omega_{|\mathfrak{I}\oplus\mathfrak{I}}\,.$$

Notice that $\tilde\omega_\infty$ is an algebraic soliton diagonal since $\omega_{|\mathfrak{I}\oplus\mathfrak{I}}$ is diagonal in view of Proposition 5.2. □

6. A generalization to semidirect product of Lie algebras

From the viewpoint of Lie groups, the algebraic structure of Oeljeklaus-Toma manifolds is quite rigid and some of the results in the previous sections can be generalized to semidirect product of Lie algebras.

In this section we consider a Lie algebra $\mathfrak{g}$ which is a semidirect product of Lie algebras

$$\mathfrak{g} = \mathfrak{h} \ltimes_\lambda \mathfrak{I}\,,$$

where $\lambda\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{I})$ is a representation. We further assume that $\mathfrak{g}$ has a complex structure of the form

$$J = J_{\mathfrak{h}} \oplus J_{\mathfrak{I}}$$

where $J_{\mathfrak{h}}$ and $J_{\mathfrak{I}}$ are complex structures on $\mathfrak{h}$ and $\mathfrak{I}$, respectively.

The following assumptions are all satisfied in the case of an Oeljeklaus-Toma manifold:

i. $\mathfrak{h}$ has $(1,0)$-frame such that $\{Z_1,\dots,Z_r\}$ such that $[Z_k,\bar Z_k] = -\frac{\sqrt{-1}}{2}(Z_k+\bar Z_k)$, for all $k=1,\dots,r$ and the other brackets vanish;

ii. $\mathfrak{I}$ is a $2s$-dimensional abelian Lie algebra and $J_{\mathfrak{I}}$ is a complex structure on $\mathfrak{I}$;

iii. $\lambda(\mathfrak{h}^{1,0}) \subseteq \mathrm{End}(\mathfrak{I})^{1,0}$;

iv. $\mathfrak{I}$ has a $(1,0)$-frame $\{W_1,\dots W_s\}$ such that $\lambda(Z)\cdot \bar W_r = \lambda_r(Z)\bar W_r$, for every $r=1,\dots,s$, where $\lambda_r \in \Lambda^{1,0}(\mathfrak{h})$;

v. $\sum_{a=1}^s \Im\mathfrak{m}(\lambda_a(Z_i))$ is constant on i.

vi. $\mathfrak{I}$ has a $(1,0)$-frame $\{W_1,\dots W_s\}$ such that $\lambda(Z)\cdot W_r = \lambda'_r(Z)W_r$, for every $r=1,\dots,s$, where $\lambda'_r \in \Lambda^{1,0}(\mathfrak{h})$ and $\sum_{a=1}^s \Im\mathfrak{m}(\lambda'_a(Z_i))$ is constant on i.

Note that condition i. is equivalent to require that $\mathfrak{h} = \underbrace{\mathfrak{f}\oplus\cdots\oplus\mathfrak{f}}_{r\text{-times}}$ equipped with the complex structure $J_{\mathfrak{h}} = \underbrace{J_{\mathfrak{f}}\oplus\cdots\oplus J_{\mathfrak{f}}}_{r\text{-times}}$, while in condition iv. the existence of $\{W_r\}$ and λ_r is equivalent to require that

$$\lambda(Z)\circ\lambda(Z') = \lambda(Z')\circ\lambda(Z)\,,$$

for every $Z, Z' \in \mathfrak{h}^{1,0}$.

The computations in Section 5 can be used to study solutions to the flow

$$\partial_t \omega_t = -\rho_B^{1,1}(\omega_t) \tag{27}$$

in semidirect products of Lie algebras (this flow coincides to the pluriclosed flow only when the initial metric is pluriclosed). We have the following

Proposition 6.1. *Let $\mathfrak{g} = \mathfrak{h}\ltimes_\lambda \mathfrak{I}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J = J_{\mathfrak{h}}\oplus J_{\mathfrak{I}}$ and let ω be a Hermitian metric on $\mathfrak{g}$ making $\mathfrak{h}$ and $\mathfrak{I}$ orthogonal. Then the Bismut Ricci-form of ω satisfies $\rho^{1,1}_{B|\mathfrak{h}\oplus\mathfrak{I}} = \rho^{1,1}_{B|\mathfrak{I}\oplus\mathfrak{I}} = 0$.*

If i-iv hold and $\omega_{|\mathfrak{h}\oplus\mathfrak{h}}$ is diagonal with respect to the frame $\{Z_i\}$ then the $(1,1)$-component of the Bismut-Ricci form of ω does not depend on ω and the solution to the flow (27) starting from ω takes the following expression

$$\omega_t = \omega - t\rho_B^{1,1}(\omega)\,.$$

If i $-$ iv *and* vi *hold and $\omega_{|\mathfrak{h}\oplus\mathfrak{h}}$ is a multiple of the canonical metric with respect to the frame $\{Z_i\}$, then ω is a soliton for flow (27) with cosmological constant $c = \frac{1}{2} + \sum_{a=1}^s \Im\mathfrak{m}(\lambda'_a(Z_i))$.*

The previous Proposition does not cover the case when properties i-iv are satisfied and the restriction to $\mathfrak{h}\oplus\mathfrak{h}$ of the initial Hermitian inner product

$$\omega = \sqrt{-1}\sum_{a,b=1}^r g_{a\bar b}\,\omega^a\wedge\bar\omega^b + \sqrt{-1}\sum_{a,b=1}^s g_{r+a\overline{r+b}}\,\gamma^a\wedge\bar\gamma^b$$

is not diagonal with respect to $\{Z_i\}$. In this case flow (27) evolves only the components $g_{i\bar{i}}$ of ω along $\omega^i \wedge \bar{\omega}^i$ via the ODE

$$\partial_t g_{i\bar{i}} = \frac{1}{4}\sum_{a=1}^{r} g^{\bar{a}a}\mathfrak{Re}\, g_{i\bar{a}} - \frac{1}{2}\sum_{c,d=1}^{s} g^{\overline{r+d}r+c}\left\{\omega([Z_i,W_c],\bar{W}_d)+\omega([\bar{Z}_i,\bar{W}_c],W_d)\right\}$$

where $g_{i\bar{i}}$ depends on t. Note that the quantities $-\frac{1}{2}\sum_{c,d=1}^{s} g^{\overline{r+d}r+c}\left\{\omega([Z_i,W_c],\bar{W}_d)+\omega([\bar{Z}_i,\bar{W}_c],W_d)\right\}$ appearing in the evolution of $g_{i\bar{i}}$ are independent on t.

The same computations as in Section 4 imply the following

Proposition 6.2. *Let $\mathfrak{g}=\mathfrak{h}\ltimes_\lambda \mathfrak{I}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J=J_{\mathfrak{h}}\oplus J_{\mathfrak{I}}$. Assume that properties* i, ii, iii *are satisfied and let ω be a left-invariant Hermitian metric on $\mathfrak{g}$. Then*

$$\rho_{C|\mathfrak{I}\oplus\mathfrak{I}}=\rho_{C|\mathfrak{h}\oplus\mathfrak{I}}=0\,,$$

while $\rho_{C|\mathfrak{h}\oplus\mathfrak{h}}$ is diagonal with respect to $\{Z_1,\dots,Z_r\}$.

If further also iv. *holds, then*

$$\rho_C(Z_i,\bar{Z}_i)=-\sqrt{-1}\left(\frac{1}{2}-\sum_{a=1}^{s}\mathfrak{Im}(\lambda_a(Z_i))\right),\quad \textit{for all } i=1,\dots,r\,.$$

If, in addition, v. holds, then ω is a soliton for the Chern-Ricci flow with cosmological constant $c=\frac{1}{2}-\sum_{a=1}^{s}\mathfrak{Im}(\lambda_a(Z_i))$ if and only if $\omega_{\mathfrak{h}\oplus\mathfrak{h}}$ is a multiple of the canonical metric on $\mathfrak{h}$ with respect to the frame $\{Z_i\}$ and $\omega_{\mathfrak{h}\oplus\mathfrak{I}}=0$.

References

[1] D. Angella, A. Dubickas, A. Otiman, J. Stelzig: On metric and cohomological properties of Oeljeklaus-Toma manifolds. arXiv:2201.06377.

[2] D. Angella, V. Tosatti, Leafwise flat forms on Inoue-Bombieri surfaces. arXiv:2106.16141.

[3] R.M. Arroyo, R.A. Lafuente, The long-time behavior of the homogeneous pluriclosed flow. *Proc. Lond. Math. Soc.* (3), **119** (2019)(1): 266–289.

[4] J.-M. Bismut, A local index theorem for non-Kähler manifolds. *Math. Ann.* **284** (1989), no. 4, 681–699.

[5] J. Boling, Homogeneous Solutions of Pluriclosed Flow on Closed Complex Surfaces. *J. Geom. Anal.* **26** (2016), no. 3, 2130–2154.

[6] N. Enrietti, A. Fino, L. Vezzoni, The pluriclosed flow on nilmanifolds and Tamed symplectic forms. *J. Geom. Anal.* **25** (2015), no. 2, 883–909.

[7] S. Fang, V. Tosatti, B. Weinkove, T. Zheng, Inoue surfaces and the Chern-Ricci flow. *J. Funct. Anal.* **271** (2016), no. 11, 3162–3185.

[8] A. Fino, H. Kasuya, L. Vezzoni, SKT and tamed symplectic structures on solvmanifolds. *Tohoku Math. J. (2)* **67** (2015), no. 1, 19–37.

[9] M. Garcia-Fernandez, J. Jordan, J. Streets, Non-Kähler Calabi-Yau geometry and pluriclosed flow. arXiv:2106.13716.

[10] M. Gill, Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds. *Comm. Anal. Geom.* **19** (2011), 277–303

[11] M. Inoue, On surfaces of Class VII_0. *Invent. Math.* **24** (1974), no.4, 269–320.

[12] J. Jordan, J. Streets, On a Calabi-type estimate for pluriclosed flow. *Adv. Math.*, **366** (2020), Article ID: 107097, p.18.

[13] H. Kasuya, Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds. *Bull. Lond. Math. Soc.* **45** (2013), no. 1, 15–26.

[14] J. Lauret, Curvature flows for almost-hermitian Lie groups. *Trans. Amer. Math. Soc.* **367** (2015), no. 10, 7453–7480.

[15] J. Lauret, Convergence of homogeneous manifolds, *J. Lond. Math. Soc.*, II. Ser. **86** (2012), No. 3, 701–727.

[16] J. Lauret, E.A. Rodríguez Valencia, On the Chern-Ricci flow and its solitons for Lie group. *Math. Nachr.* **288** (2015), no. 13, 1512–1526.

[17] K. Oeljeklaus, M. Toma, Non-Kähler compact complex manifolds associated to number fields. *Ann. Inst. Fourier (Grenoble).* **55** (2005), no. 1, 161–171

[18] A. Otiman, Special Hermitian metrics on Oeljeklaus-Toma manifolds. arXiv:2009.02599.

[19] M. Pujia, L. Vezzoni, A remark on the Bismut-Ricci form on 2-step nilmanifolds. *C. R. Math. Acad. Sci. Paris* **356** (2018), no. 2, 222–226.

[20] J. Streets, G. Tian, Hermitian curvature flow. *J. Eur. Math. Soc. (JEMS)* **13** (2011), no. 3, 601–634.

[21] J. Streets, G. Tian, A parabolic flow of pluriclosed metrics. *Int. Math. Res. Notices* (2010), 3101–3133.

[22] J. Streets, G. Tian, Regularity results for pluriclosed flow. *Geom. Topol.* **17** (2013), no. 4, 2389–2429.

[23] J. Streets, Classification of solitons for pluriclosed flow on complex surfaces. *Math. Ann.* , **375** (2019), no. 3–4, 1555–1595.
[24] J. Streets, Pluriclosed flow, Born-Infeld geometry, and rigidity results for generalized Kähler manifolds. *Comm. Partial Differential Equations* **41** (2016), no. 2, 318–374.
[25] J. Streets, Pluriclosed flow and the geometrization of complex surfaces. *Prog. Math.*, **333** (2020), 471–510.
[26] J. Streets, Pluriclosed flow on generalized Kähler manifolds with split tangent bundle. *J. Reine Angew. Math.*, **739**(2018), 241–276.
[27] J. Streets, Pluriclosed flow on manifolds with globally generated bundles. *Complex Manifolds* **3** (2016), 222–230.
[28] V. Tosatti, B. Weinkove, On the evolution of a Hermitian metric by its Chern-Ricci form. *J. Differential Geom.* **99** (2015), no.1, 125–163.
[29] V. Tosatti, B. Weinkove, The Chern-Ricci flow on complex surfaces. *Compos. Math.* **149** (2013), no. 12, 2101–2138.
[30] S. Verbitsky, Surfaces on Oeljeklaus-Toma Manifolds. `arXiv:1306.2456`.
[31] L. Vezzoni, A note on canonical Ricci forms on 2-step nilmanifolds. *Proc. Amer. Math. Soc.* **141** (2013), no. 1, 325–333.
[32] T. Zheng, The Chern-Ricci flow on Oeljeklaus-Toma manifolds, *Canad. J. Math.* **69** (2017), no. 1, 220–240.

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