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A note on *p*-Kähler structures on compact quotients of Lie groups

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Abstract

A p-Kähler structure on a complex manifold of complex dimension n is given by a d-closed transverse real (p, p)-form. In the paper, we study the existence of p-Kähler structures on compact quotients of simply connected Lie groups by discrete subgroups endowed with an invariant complex structure. In particular, we discuss the existence of p-Kähler structures on nilmanifolds, with a focus on the case $p = 2$ and complex dimension $n = 4$. Moreover, we prove that a $(n-2)$-Kähler almost abelian solvmanifold of complex dimension $n \geq 3$ has to be Kähler.

Keywords p-Kähler structure · Nilmanifold · Almost abelian solvmanifold

1 Introduction

A p-Kähler structure on a complex manifold (X, J) of complex dimension n is given by a d-closed transverse real (p, p)-form Ω. The p-Kähler structures have been introduced and studied in [2–4]. Recently, their behavior under small deformations of the complex structure has been studied in [20].

Some obstructions to their existence were determined in [14], where the authors extended the definition to non-integrable almost complex manifolds, and in [22], on nilmanifolds with nilpotent complex structures. In [5], Alessandrini and Bassanelli conjectured that if X is p-Kähler then it is q-Kähler for all $p \leq q < n$.

For $p = 1, n-1$, transversality is equivalent to positive definiteness, so in the first case we find the Kähler condition, whereas in the latter this property is equivalent to the balanced one. In complex dimension 3, these are all the possible cases, and both have been thoroughly studied, so we will consider higher dimension, where more cases arise.

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Examples of 2-Kähler structures on compact non-Kähler complex manifolds were constructed in complex dimension 5, using a smooth proper modification of $\mathbb{P}^5$ [5, Section 4], and lately, it was proved that non-compact examples exist in any dimension greater than 2, as products of $\mathbb{C}^m$ and a balanced complex manifold of complex dimension 3 [1, Theorem 5.3]. On the other hand, the existence of a 2-Kähler structure on a compact (non-Kähler) complex manifold of complex dimension 4 is quite restrictive. As a far as we know, no examples in the literature are known.

In Sect. 2, after recalling a few definitions and some known results on p-Kähler structures, we use the symmetrization process described in [7] (see also [8, 23]), to prove that on compact quotients of simply connected Lie groups by lattices, endowed with invariant complex structures, the existence of p-Kähler structures implies the existence of invariant ones (Lemma 2.4). By invariant p-Kähler structure, we mean one induced by a left-invariant one on G or, equivalently, by a p-Kähler structure on the Lie algebra $\mathfrak{g}$ of G. Furthermore, we find some necessary conditions to the existence of p-Kähler structures on Lie algebras endowed with a complex structure J such that there exists a J-invariant ideal of codimension 2 (Proposition 2.5) that we will use in the nilpotent case.

Section 3 is devoted to the nilpotent case. We show some obstructions to the existence of p-Kähler structures when the nilmanifold is endowed with a quasi-nilpotent complex structure J, namely such that the center of the associated nilpotent Lie algebra has a non-trivial J-invariant subspace, with a focus on $p = 2$ (Proposition 3.3). Later on, we study nilmanifolds of complex dimension 4 and prove that they do not admit 2-Kähler structures, unless they are tori (Theorem 3.7). We use this result as the first step of induction to prove that a nilmanifold of complex dimension greater then 3 endowed with an invariant quasi-nilpotent complex structure cannot admit 2-Kähler structures unless it is a torus (Theorem 3.8).

Finally, we consider the almost abelian case. We recall that a Lie group is called almost abelian if its Lie algebra has a codimension one abelian ideal. In Sect. 4, we prove that for almost abelian solvmanifolds of complex dimension $n \geq 3$, the $(n-2)$-Kähler condition implies Kähler (Theorem 4.2). This gives yet another case in complex dimension 4 of compact complex manifolds for which the existence of a 2-Kähler structure forces the manifold to be Kähler.

2 Preliminaries on *p*-Kähler structures

Let V be a complex vector space of dimension n, and let us denote by $\Lambda^{p,q} := \Lambda^{p,q} V^*$ the space of (p,q)-forms over V. In the following lines, we will recall a few positivity notions for differential forms.

Definition 2.1 (1) A (n,n)-form ν on V is *positive* if $\nu = c\, i\alpha^1 \wedge \alpha^{\bar{1}} \wedge \cdots \wedge i\alpha^n \wedge \alpha^{\bar{n}}$, where $c \in \mathbb{R}_{\geq 0}$ and $\left\{\alpha^j\right\}_{j=1}^n$ is a basis of $\Lambda^{1,0}$. If $c > 0$, we will call ν a *volume form*.

(2) A $(q,0)$-form ψ on V is called *simple* if $\psi = \mu_1 \wedge \cdots \wedge \mu_q$, with $\mu_1, \ldots, \mu_q \in \Lambda^{1,0}$. A (k,k)-form Ω is called *transverse* if for every nonzero simple $(n-k,0)$-form ψ,

$$i^{(n-k)^2}\, \Omega \wedge \psi \wedge \bar{\psi} = \Omega \wedge i\mu_1 \wedge \bar{\mu}_1 \wedge \cdots \wedge i\mu_{n-k} \wedge \bar{\mu}_{n-k}$$

is a volume form.

(3) A (k,k)-form Ω on V is said to be *positive definite* if for all $0 \neq \eta \in \Lambda^{n-k,0}$, the (n,n)-form

$$i^{(n-k)^2}\, \Omega \wedge \eta \wedge \bar{\eta}$$

is a volume form.

(4) A (k, k)-form Ω on V is *strongly positive* if it can be written as

$$\Omega = i^{k^2} \sum_j \psi_j \wedge \bar{\psi}_j,$$

with $\psi_j \in \Lambda^{k,0}$ simple.

It follows from Definition 2.1 that every strongly positive form is positive definite and every positive definite form is transverse. Moreover, every transverse form is real. Notice that for $k = 1, n-1$, a $(k, 0)$-form is always simple (see for example [13]). In fact, for $k = 1, n-1$, (2), (3) and (4) in Definition 2.1 are all equivalent.

Using the previous notion of tranversality, one can introduce the following

Definition 2.2 Let (X, J) be a complex manifold of complex dimension n and let $1 \leq p < n$. A *p-Kähler* structure on X is given by a d-closed real (p, p)-form Ω such that, at every point $x \in X$, $\Omega_x \in \Lambda^{p,p}(T_x X)$ is transverse.

For $p = 1$, a transverse form is nothing but the fundamental form associated with a Hermitian metric on (X, J). This means that a 1-Kähler structure actually gives a Kähler metric on (X, J). On the other hand, when $p = n-1$, we know by [18] that every strongly positive $(n-1, n-1)$-form can be written as the $(n-1)$th power of a strongly positive $(1, 1)$-form and so determines a Hermitian metric. It follows that the datum of a $(n-1)$-Kähler structure is equivalent to that of a balanced Hermitian metric. Note that these are the only two cases where p-Kähler structures have metric meaning (cf. [4, Proposition 2.1]).

The following result gives an obstruction to the existence of p-Kähler structures on compact complex manifolds.

Proposition 2.3 ([14]) *Let (X, J) be a compact complex manifold. Suppose there exists a $(2n-2p-1)$-form β on X such that*

$$0 \neq (d\beta)^{n-p,n-p} = \sum_j c_j \, \psi_j \wedge \overline{\psi_j},$$

where $c_j \in \mathbb{R}$ have the same sign and ψ_j are simple $(n-p, 0)$-forms. Then (X, J) does not admit any p-Kähler structure.

Let now consider as a complex manifold (X, J) the compact quotient $X = \Gamma\backslash G$ of a simply connected Lie group G by a discrete subgroup Γ endowed with an invariant complex structure J, i.e., a complex structure induced by a complex structure on $\mathfrak{g}$.

Next we prove that the existence of a p-Kähler structure on $(X = \Gamma\backslash G, J)$ implies the existence of an invariant one.

Lemma 2.4 *If $(X = \Gamma\backslash G, J)$ admits a p-Kähler structure Ω, then $(\mathfrak{g}, J)$ has a p-Kähler structure.*

Proof Let ν be a volume element on X induced by a bi-invariant one on the Lie group G (the existence of such a volume form was proved in [19]). After rescaling, we can suppose that X has volume equal to 1. Given the p-Kähler structure Ω, by symmetrization, we can define the (p, p)-form Ω_ν on $\mathfrak{g}$, given by

$$\Omega_\nu(Y_1, \ldots, Y_{2p}) = \int_{x \in X} \Omega_x(Y_1|_x, \ldots, Y_{2p}|_x)\nu,$$

for every $Y_j \in \mathfrak{g}$, where $Y_j|_x$ is the value at the point $x \in X$ of the projection on X of the left-invariant vector field Y_j on the Lie group G. By [7] (see also [8, 23]), the symmetrization commutes with the differential d, so $d\Omega_\nu = 0$. We only need to show that Ω_ν is still transverse, i.e., that $i^{(n-p)^2}\,\Omega_\nu \wedge \psi \wedge \bar{\psi}$ is a volume form on $\mathfrak{g}$, for every simple $(n-p,0)$-form ψ on $\mathfrak{g}$. This follows from the fact that for every differential forms α and β on X we have

$$(\alpha_\nu \wedge \beta)_\nu = \alpha_\nu \wedge \beta_\nu.$$

In fact, using $\psi = \psi_\nu$ we find that $i^{(n-p)^2}\,\Omega_\nu \wedge \psi \wedge \bar{\psi}$ is a volume form, as it is the integral on X of $i^{(n-p)^2}\,\Omega \wedge \psi \wedge \bar{\psi}$, positive by hypothesis. □

We now state some general restriction to the existence of a p-Kähler structure on a Lie algebra. From now on, $\mathfrak{g}^{1,0}$ (respectively, $\mathfrak{g}^{0,1}$) will denote the i-eigenspace (respectively, $-i$-eigenspace) of J as an endomorphism of $\mathfrak{g}^*$.

Proposition 2.5 *If a p-Kähler Lie algebra $(\mathfrak{g}, J, \Omega)$ of complex dimension $n \geq 3$, with $p < n-1$, admits a closed nonzero $(1,0)$-form α, then it has a p-Kähler J-invariant ideal of codimension 2.*

Proof Let $\{\alpha^1, \dots \alpha^n\}$ be a basis of $\mathfrak{g}^{1,0}$ such that $\alpha^1 := \alpha$. Consider its dual basis $\{Z_1, \dots, Z_n\}$. Consider a basis $\{e_1, \dots, e_{2n}\}$ of $\mathfrak{g}$ such that $Z_1 = e_1 - i\, e_2$. Clearly, $Je_1 = e_2$ and the dual elements of e_1, e_2 in $\mathfrak{g}^*$ are closed differential 1-forms on $\mathfrak{g}$, so that the subspace $\mathfrak{h} := \operatorname{span}\{e_3 \dots, e_{2n}\}$ is a J-invariant ideal of $\mathfrak{g}$ and $\mathfrak{h}^{1,0}$ is generated by $\alpha^2, \dots, \alpha^n$. Let us denote with $d_\mathfrak{h}$ the exterior derivative of $\mathfrak{h}$. Then, for every form $\beta \in \Lambda_\mathfrak{h}$ one has that $d\beta = d_\mathfrak{h}\beta + \tilde{d}\beta$, with $\tilde{d}\beta \in I\left(\alpha^1, \alpha^{\bar{1}}\right)$. Let Ω be a p-Kähler structure on $(\mathfrak{g}, J)$. We will prove that its restriction $\Omega_\mathfrak{h} \in \Lambda_\mathfrak{h}^{p,p}$ to $\mathfrak{h}$ is a p-Kähler structure on $(\mathfrak{h}, J|_\mathfrak{h})$. We can write

$$\Omega = \Omega_\mathfrak{h} + \alpha^1 \wedge \eta + \alpha^{\bar{1}} \wedge \bar{\eta} + i\,\alpha^{1\bar{1}} \wedge \vartheta,$$

with $\eta \in \Lambda_\mathfrak{h}^{p-1,p}$, $\vartheta \in \Lambda_\mathfrak{h}^{p-1,p-1}$, $\Omega_\mathfrak{h}$ and ϑ real. Therefore,

$$\begin{aligned} d\Omega &= d\Omega_\mathfrak{h} - \alpha^1 \wedge d\eta - \alpha^{\bar{1}} \wedge d\bar{\eta} + i\,\alpha^{1\bar{1}} \wedge d\vartheta \\ &= d_\mathfrak{h}\Omega_\mathfrak{h} + \tilde{d}\Omega_\mathfrak{h} - \alpha^1 \wedge d\eta - \alpha^{\bar{1}} \wedge d\bar{\eta} + i\,\alpha^{1\bar{1}} \wedge d\vartheta, \end{aligned}$$

with $d_\mathfrak{h}\Omega_\mathfrak{h} \in \Lambda_\mathfrak{h}^{2p+1}$ and $d\Omega - d_\mathfrak{h}\Omega_\mathfrak{h} \in I\left(\alpha^1, \alpha^{\bar{1}}\right)$, so that if Ω is d-closed, $\Omega_\mathfrak{h}$ must be $d_\mathfrak{h}$-closed. It remains to prove that $\Omega_\mathfrak{h}$ is transverse, namely that, for all

$$\phi = i\mu_1 \wedge \overline{\mu_1} \wedge \cdots \wedge i\mu_{n-1-p} \wedge \overline{\mu_{n-1-p}}, \quad \mu_j \in \mathfrak{h}^{1,0},$$

the $(n-1, n-1)$-form $\Omega_\mathfrak{h} \wedge \phi$ is a volume form on $\mathfrak{h}$. Since Ω is transverse, $\Omega \wedge i\,\alpha^{1\bar{1}} \wedge \phi$ is a volume form on $\mathfrak{g}$. One easily sees that

$$\Omega \wedge i\,\alpha^{1\bar{1}} \wedge \phi = \Omega_\mathfrak{h} \wedge i\,\alpha^{1\bar{1}} \wedge \phi,$$

yielding that $\Omega_\mathfrak{h} \wedge \phi$ is a volume form on $\mathfrak{h}$, as wanted. □

3 *p*-Kähler structures on nilmanifolds

We will now discuss the case where X is a nilmanifold, i.e., a compact quotient $\Gamma\backslash G$ of a simply connected nilpotent Lie group G by a lattice Γ endowed with an invariant complex structure. Lemma 2.4 allows us to restrict to the study of p-Kähler structures on the nilpotent

Lie algebra $\mathfrak{g}$ of G. From [21], we know that for every complex structure J on a nilpotent Lie algebra $\mathfrak{g}$, there exists a basis $\{\alpha^1, \ldots, \alpha^n\}$ of $\mathfrak{g}^{1,0}$ such that for all j,

$$d\alpha^{j+1} \in I\left(\alpha^1, \ldots, \alpha^j\right).$$

In particular, $d\alpha^1 = 0$, so Proposition 2.5 holds.

We recall that given a complex structure J on a real nilpotent Lie algebra $\mathfrak{g}$ of dimension $2n$, one can define the ascending series adapted to J as follows

$$\mathfrak{a}_0(J) = \{0\},$$
$$\mathfrak{a}_k(J) = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J), [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J)\}, \quad \text{for } k \geq 1.$$

Then J is said to be:

- *strongly non-nilpotent (SnN)* if $\mathfrak{a}_1(J) = \{0\}$;
- *quasi-nilpotent* if $\mathfrak{a}_1(J) \neq \{0\}$. In this case, the ascending series adapted to J stabilizes, namely there exists a positive integer t such that $\mathfrak{a}_t(J) = \mathfrak{a}_l(J)$ for all $l \geq t$ and we can distinguish between two subcases:
  - J is *weakly non-nilpotent* if $\mathfrak{a}_t(J) \neq \mathfrak{g}$;
  - J is *nilpotent* if $\mathfrak{a}_t(J) = \mathfrak{g}$, or equivalently, if there is a basis $\{\alpha^1, \ldots, \alpha^n\}$ of $\mathfrak{g}^{1,0}$ satisfying

$$\begin{cases} d\alpha^1 = 0, \\ d\alpha^j \in \Lambda^2\left\langle \alpha^1, \ldots, \alpha^{j-1}, \overline{\alpha^1}, \ldots, \overline{\alpha^{j-1}} \right\rangle, & j = 2, \ldots, n. \end{cases}$$

The following result is a consequence of Proposition 2.3 and it gives an obstruction to the existence of p-Kähler structures.

Proposition 3.1 ([22]) *Let $\mathfrak{g}$ be a nilpotent Lie algebra of complex dimension n endowed with a nilpotent complex structure J. Given a basis $\{\alpha^1, \ldots, \alpha^n\}$ of $\mathfrak{g}^{1,0}$, let t be the positive integer such that*

$$d\alpha^j = 0, \quad \text{for } j = 1, \ldots, t, \quad \text{and} \quad d\alpha^j \neq 0 \quad \text{for } j = t+1, \ldots, n.$$

Then, there are no $(n-t)$-Kähler structures on $(\mathfrak{g}, J)$.

3.1 Quasi-nilpotent complex structures

We will now consider quasi-nilpotent complex structures, namely the case where the center ζ of $\mathfrak{g}$ has a J-invariant non-trivial subspace. We recall the following

Definition 3.2 ([15]) Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with a quasi-nilpotent complex structure J and $\mathfrak{b}$ be a J-invariant subspace of ζ of real dimension 2. If $\mathfrak{k}$ is a nilpotent Lie algebra of real dimension $2(n-1)$ endowed with a complex structure K such that $(\mathfrak{k}, K)$ is isomorphic to $(\mathfrak{g}/\mathfrak{b}, J|_{\mathfrak{g}/\mathfrak{b}})$, the pair $(\mathfrak{g}, J)$ is called a $\mathfrak{b}$-extension of $(\mathfrak{k}, K)$.

We can prove the following.

Proposition 3.3 *If $\mathfrak{g}$ is a nilpotent Lie algebra of real dimension $2n \geq 6$ endowed with a quasi-nilpotent complex structure J and admitting a p-Kähler structure, then $(\mathfrak{g}, J)$ is the $\mathfrak{b}$-extension of a $(p-1)$-Kähler nilpotent Lie algebra.*

Proof In [15], it was proved that for every pair $(\mathfrak{g}, J)$, where $\mathfrak{g}$ is a nilpotent Lie algebra and J is quasi-nilpotent complex structure, there exists a two-dimensional J-invariant subspace $\mathfrak{b}$ of $\mathfrak{a}_1(J) = \zeta \cap J\zeta$, where ζ is the center of $\mathfrak{g}$, such that $(\mathfrak{g}, J)$ is the $\mathfrak{b}$-extension of some nilpotent Lie algebra $\mathfrak{k}$ of codimension 2 endowed with a complex structure K. We can choose a basis $\{\alpha^1, \dots, \alpha^n\}$ of $\mathfrak{g}^{1,0}$ such that $\mathfrak{b}^{1,0}$ is generated by α^n. In this way, $\{\alpha^1, \dots, \alpha^{n-1}\}$ is a basis of $\mathfrak{k}^{1,0}$ and $d\alpha^j \in \Lambda^2_{\mathfrak{k}}$, for all j. Let Ω be a p-Kähler structure on $\mathfrak{g}$. Then there exist $\Omega_{\mathfrak{k}} \in \Lambda^{p,p}_{\mathfrak{k}}$ and $\omega \in \Lambda^{p-1,p-1}_{\mathfrak{k}}$ real, $\eta \in \Lambda^{p-1,p}_{\mathfrak{k}}$ such that

$$\Omega = \Omega_{\mathfrak{k}} + \eta \wedge \alpha^n + \bar{\eta} \wedge \alpha^{\bar{n}} + \omega \wedge i\,\alpha^{n\bar{n}}\,. \tag{3.1}$$

We will prove that ω is a $(p-1)$-Kähler form for $(\mathfrak{k}, K)$. The closure of the $(p-1, p-1)$-form ω follows from the fact that Ω is closed and

$$\begin{aligned} 0 = d\Omega =& d\Omega_{\mathfrak{k}} - \eta \wedge d\alpha^n - \bar{\eta} \wedge d\alpha^{\bar{n}} \\ &+ \left(d\eta + i\omega \wedge d\alpha^{\bar{n}}\right) \wedge \alpha^n + \left(d\bar{\eta} + i\omega \wedge d\alpha^n\right) \wedge \alpha^{\bar{n}} \\ &+ d\omega \wedge i\,\alpha^{n\bar{n}}\,. \end{aligned} \tag{3.2}$$

To prove that ω is transverse, fix a simple form $\psi \in \Lambda^{(n-1)-(p-1),0} = \Lambda^{n-p,0}$. For dimensional reasons, $\Omega_{\mathfrak{k}} \wedge \psi \wedge \bar{\psi} = \eta \wedge \psi \wedge \bar{\psi} = 0$, giving

$$i^{(n-p)^2}\Omega \wedge \psi \wedge \bar{\psi} = i^{(n-p)^2}\omega \wedge \psi \wedge \bar{\psi} \wedge i\,\alpha^{n\bar{n}}\,.$$

This is a volume form on $\mathfrak{g}$ because Ω is transverse, so $i^{(n-p)^2}\omega \wedge \psi \wedge \bar{\psi}$ is a volume form on $\mathfrak{k}$. □

As a consequence, we have the following

Corollary 3.4 *Let $\mathfrak{g}$ be a nilpotent Lie algebra of real dimension $2n$ endowed with a quasi-nilpotent complex structure J. If $(\mathfrak{g}, J)$ admits a 2-Kähler structure, then $(\mathfrak{g}, J)$ must be the $\mathfrak{b}$-extension of the $2(n-1)$-dimensional abelian Lie algebra. Moreover, the complex structure J has to be nilpotent.*

Proof The first part of the statement is given by Proposition 3.3 for $p = 2$. It follows that the complex structure equations of $(\mathfrak{g}, J)$ must be

$$\begin{cases} d\alpha^j = 0, \quad j = 1, \dots, n-1, \\ d\alpha^n \in \Lambda^2_{\mathfrak{k}}, \end{cases}$$

with $\Lambda^2_{\mathfrak{k}} = \Lambda^2\left\langle \alpha^1, \dots, \alpha^{n-1}, \alpha^{\bar{1}}, \dots, \alpha^{\overline{n-1}} \right\rangle$, meaning that J has to be nilpotent. □

3.2 2-Kähler structures on nilmanifolds of real dimension 8

In real dimension 8, we can actually prove that 2-Kähler nilmanifolds endowed with an invariant complex structures must be Kähler.

Proposition 3.5 *A (non-abelian) eight-dimensional nilpotent Lie algebra $\mathfrak{g}$ endowed with a quasi-nilpotent complex structure J does not admit 2-Kähler structures.*

Proof We only have to prove the statement for $\mathfrak{b}$-extensions $(\mathfrak{g}, J)$ of the six-dimensional abelian Lie algebra $\mathfrak{k}$. Namely, we can suppose to have a basis $\{\alpha^1, \dots, \alpha^4\}$ of $\mathfrak{g}^{1,0}$ such that

$d\alpha^j = 0$ for $j = 1, 2, 3$ and $d\alpha^4 \in \Lambda^2_{\mathfrak{k}}$. If $\partial\alpha^4 \neq 0$, it is in particular a $(2, 0)$-form on a space of complex dimension 3, hence simple, so that $\alpha^4 \wedge \overline{\partial\alpha^4}$ is a 3-form as in Proposition 2.3 and $(\mathfrak{g}, J)$ is not 2-Kähler. It remains to consider the case where $d\alpha^4 \in \Lambda^{1,1}_{\mathfrak{k}}$. Suppose there exists a 2-Kähler form Ω on $(\mathfrak{g}, J)$.

We can write Ω as in (3.1) and because $\mathfrak{k}$ is abelian, (3.2) reduces to

$$0 = d\Omega = -\eta \wedge d\alpha^4 - \bar{\eta} \wedge d\alpha^{\bar{4}} + i\,\omega \wedge d\alpha^{\bar{4}} \wedge \alpha^4 + i\,\omega \wedge d\alpha^4 \wedge \alpha^{\bar{4}}.$$

Since $-\eta \wedge d\alpha^4 - \bar{\eta} \wedge d\alpha^{\bar{4}} \in \Lambda^5_{\mathfrak{k}}$, we have $\omega \wedge d\alpha^4 = 0$. We already proved that since Ω is transverse, ω is a transverse $(1, 1)$-form, hence strongly positive, giving a contradiction. □

Strongly non-nilpotent complex structures on eight-dimensional nilpotent Lie algebras were classified in [15] and then refined in [16]. In particular, it turns out that, depending on the ascending type of $\mathfrak{g}$, the admissible complex structures are divided into two families as follows.

Proposition 3.6 [16, Thm. 3.3] *Let J be a strongly non-nilpotent complex structure on an eight-dimensional nilpotent Lie algebra $\mathfrak{g}$. Then, there exists a basis $\{\alpha^1, \ldots, \alpha^4\}$ of $\mathfrak{g}^{1,0}$ such that the complex structure equations are either given by*

$$\begin{cases} d\alpha^1 = 0, \\ d\alpha^2 = \varepsilon\,\alpha^{1\bar{1}}, \\ d\alpha^3 = \alpha^{14} + \alpha^{1\bar{4}} + a\,\alpha^{2\bar{1}} + i\,\delta\,\varepsilon\,b\,\alpha^{1\bar{2}}, \\ d\alpha^4 = i\,\nu\,\alpha^{1\bar{1}} + b\,\alpha^{2\bar{2}} + i\,\delta\left(\alpha^{1\bar{3}} - \alpha^{3\bar{1}}\right), \end{cases} \tag{3.3}$$

where $\delta = \pm 1$, $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $a \geq 0$ and the tuple (ε, ν, a, b) is one of the following:

$$(0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), \left(0, 1, 0, \frac{b}{|b|}\right), (0, 1, 1, b),$$
$$(1, 0, 0, 1), (1, 0, 1, |b|), (1, 1, a, b),$$

or given by

$$\begin{cases} d\alpha^1 = 0, \\ d\alpha^2 = \alpha^{14} + \alpha^{1\bar{4}}, \\ d\alpha^3 = a\,\alpha^{1\bar{1}} + \varepsilon\left(\alpha^{12} + \alpha^{1\bar{2}} - \alpha^{2\bar{1}}\right) + i\,\mu\left(\alpha^{24} + \alpha^{2\bar{4}}\right), \\ d\alpha^4 = i\,\nu\,\alpha^{1\bar{1}} - \mu\,\alpha^{2\bar{2}} + i\,b\left(\alpha^{1\bar{2}} - \alpha^{2\bar{1}}\right) + i\left(\alpha^{1\bar{3}} - \alpha^{3\bar{1}}\right), \end{cases} \tag{3.4}$$

where $a, b \in \mathbb{R}$, and the tuple $(\varepsilon, \mu, \nu, a, b)$ is one of the following:

$$(1, 1, 0, a, b), (1, 0, 1, a, b), (1, 0, 0, 0, b), (1, 0, 0, 1, b), (0, 1, 0, 0, 0), (0, 1, 0, 1, 0).$$

For both of the families (3.3) and (3.4), we can find a 3-form β as in Proposition 2.3 that gives us an obstruction to the existence of 2-Kähler forms on $(\mathfrak{g}, J)$. In the case of the family (3.3), we can take $\beta = b\,\alpha^{14\bar{1}} - a\,\alpha^{13\bar{2}}$. This gives

$$d\beta = (a^2 + b^2)\,\alpha^{12\bar{1}\bar{2}},$$

never zero because $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. If J belongs to the family (3.4), we choose $\beta = \alpha^{14\bar{1}} + (1 - \mu)\,\alpha^{12\bar{3}}$, giving

$$d\beta = (\varepsilon - \varepsilon\mu - \mu)\,\alpha^{12\bar{1}\bar{2}},$$

where $\varepsilon, \mu \in \{0, 1\}$ and $(\varepsilon, \mu) \neq (0, 0)$; hence, $d\beta$ is a nonzero $(2, 2)$-form as wanted.

We have proved the following.

Theorem 3.7 *A nilpotent Lie algebra $\mathfrak{g}$ of real dimension 8 endowed with a complex structure admits a 2-Kähler structure if and only if it is abelian.*

This result can actually be generalized to any dimension, when $\mathfrak{g}$ is endowed with a quasi-nilpotent complex structure J.

Theorem 3.8 *A non-abelian nilpotent Lie algebra of real dimension $2n \geq 8$ endowed with a quasi-nilpotent complex structure cannot be 2-Kähler.*

Proof We will prove the statement by induction on $n \geq 4$. The base of the induction is true by Theorem 3.7. Suppose the theorem is proved for every nilpotent Lie algebra of dimension $2(n-1)$ and consider $(\mathfrak{g}, J)$, with $\mathfrak{g}$ nilpotent and J quasi-nilpotent, admitting a 2-Kähler structure. We want to prove that $\mathfrak{g}$ is abelian. From Corollary 3.4, we know that the complex structure equations must be

$$\begin{cases} d\alpha^j = 0, & j = 1, \dots, n-1, \\ d\alpha^n \in \Lambda^2_{\mathfrak{k}} = \Lambda^2 \left\langle \alpha^1, \dots, \alpha^{n-1}, \alpha^{\bar{1}}, \dots, \alpha^{\overline{n-1}} \right\rangle . \end{cases} \tag{3.5}$$

for some basis $\left\{\alpha^1 \dots \alpha^n\right\}$ of $\mathfrak{g}^{1,0}$. Fix the dual basis $\left\{Z_j, \overline{Z_j}\right\}_{j=1}^n$ of $\mathfrak{g}_{\mathbb{C}}$ dual to $\left\{\alpha^j, \alpha^{\bar{j}}\right\}_{j=1}^n$ and consider the ideal $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}_{\mathbb{C}} = \operatorname{span}\left\{Z_2, \dots, Z_n, \overline{Z_2}, \dots, \overline{Z_n}\right\}$. Proposition 2.5 also holds so $\mathfrak{h}$, endowed with the complex structure $\tilde{J} := J|_{\mathfrak{h}}$, must be 2-Kähler as well. If $\tilde{J}$ is quasi-nilpotent, we can use the inductive hypothesis on $\mathfrak{h}$ to conclude that it is abelian. From (3.5), we know that the complex structure equations of $(\mathfrak{h}, \tilde{J})$ are

$$d_{\mathfrak{h}}\alpha^j = 0, \quad j = 2, \dots, n-1, \qquad d_{\mathfrak{h}}\alpha^n \in \Lambda^2_{\mathfrak{h}\cap\mathfrak{k}} = \Lambda^2 \left\langle \alpha^2, \dots, \alpha^{n-1}, \alpha^{\bar{2}}, \dots, \alpha^{\overline{n-1}} \right\rangle,$$

so $\tilde{J}$ is actually nilpotent and $\mathfrak{h}$ must be abelian. This, together with (3.5), gives

$$d\alpha^n = \alpha^1 \wedge \gamma_1 + \alpha^{\bar{1}} \wedge \gamma_2 + c\,\alpha^{1\bar{1}},$$

for some $\gamma_1 \in \Lambda^1_{\mathfrak{h}\cap\mathfrak{k}}$, $\gamma_2 \in \Lambda^{1,0}_{\mathfrak{h}\cap\mathfrak{k}}$ and $c \in \mathbb{C}$. We already saw that a 2-Kähler structure Ω on $(\mathfrak{g}, J)$ can be written as

$$\Omega = \Omega_{\mathfrak{h}} + \alpha^1 \wedge \eta + \alpha^{\bar{1}} \wedge \overline{\eta} + i\,\alpha^{1\bar{1}} \wedge \vartheta,$$

with $\Omega_{\mathfrak{h}} \in \Lambda^{2,2}_{\mathfrak{h}}$ real and transverse, $\eta \in \Lambda^{1,2}_{\mathfrak{h}}$, $\vartheta \in \Lambda^{1,1}_{\mathfrak{h}}$. Moreover, $d\Omega = 0$ if and only if

$$d\Omega_{\mathfrak{h}} = \alpha^1 \wedge d\eta + \alpha^{\bar{1}} \wedge d\overline{\eta} \in I\left(\alpha^{1\bar{1}}\right).$$

Similarly to Proposition 3.3, if $\Omega_{\mathfrak{h}\cap\mathfrak{k}}$ is the restriction of $\Omega_{\mathfrak{h}}$ to $\mathfrak{k}$, we have

$$\Omega_{\mathfrak{h}} = \Omega_{\mathfrak{h}\cap\mathfrak{k}} + \beta \wedge \alpha^n + \overline{\beta} \wedge \alpha^{\bar{n}} + \omega \wedge i\,\alpha^{n\bar{n}},$$

with $\Omega_{\mathfrak{h}\cap\mathfrak{k}} \in \Lambda^{2,2}_{\mathfrak{h}\cap\mathfrak{k}}$ and $\omega \in \Lambda^{1,1}_{\mathfrak{h}\cap\mathfrak{k}}$ real and transverse and $\beta \in \Lambda^{1,2}_{\mathfrak{h}\cap\mathfrak{k}}$. We get

$$\begin{aligned} d\Omega_{\mathfrak{h}} &= -\beta \wedge d\alpha^n - \overline{\beta} \wedge d\alpha^{\bar{n}} + i\omega \wedge d\alpha^n \wedge \alpha^{\bar{n}} - i\omega \wedge d\alpha^{\bar{n}} \wedge \alpha^n \\ &= \alpha^1 \wedge \left(\beta \wedge \gamma_1 + \overline{\beta} \wedge \overline{\gamma_2} + i\omega \wedge \gamma_1 \wedge \alpha^{\bar{n}} + i\omega \wedge \overline{\gamma_2} \wedge \alpha^n\right) \\ &\quad + \alpha^{\bar{1}} \wedge \left(\beta \wedge \gamma_2 + \overline{\beta} \wedge \overline{\gamma_1} + i\omega \wedge \gamma_2 \wedge \alpha^{\bar{n}} + i\omega \wedge \overline{\gamma_1} \wedge \alpha^n\right) \\ &\quad + c\,\alpha^{1\bar{1}} \wedge \left(-\beta + \overline{\beta} \wedge \overline{\gamma_1} + i\omega \wedge \alpha^{\bar{n}} - i\omega \wedge \alpha^n\right). \end{aligned} \tag{3.6}$$

We know that $d\Omega_{\mathfrak{h}}$ must be in the ideal generated by $\alpha^{1\bar{1}}$, so the second and third lines in (3.6) should vanish, giving

$$\beta \wedge \gamma_1 + \overline{\beta} \wedge \overline{\gamma_2} + i\omega \wedge \gamma_1 \wedge \alpha^{\overline{n}} + i\omega \wedge \overline{\gamma_2} \wedge \alpha^n = 0.$$

Notice that $\beta \wedge \gamma_1 + \overline{\beta} \wedge \overline{\gamma_2}$ lies in $\Lambda^4_{\mathfrak{h}\cap\mathfrak{k}}$, so this is equivalent to

$$\begin{cases} \beta \wedge \gamma_1 + \overline{\beta} \wedge \overline{\gamma_2} = 0, \\ \omega \wedge \gamma_1 = 0, \\ \omega \wedge \overline{\gamma_2} = 0. \end{cases} \tag{3.7}$$

Recall that ω is a transverse (1, 1)-form, hence positive definite. A direct consequence is that the last condition in (3.7) is equivalent to $\gamma_2 = 0$. This holds because if the (1, 0)-form γ_2 is nonzero, for any fixed $\psi \in \Lambda^{n-2,0}$ the positive definiteness of ω implies

$$0 = \omega \wedge \gamma_2 \wedge \psi \wedge \overline{\gamma_2} \wedge \overline{\psi} = \tilde{c}\,\mathrm{Vol}_{\mathfrak{g}},$$

for some positive constant $\tilde{c} \in \mathbb{R}$, giving a contradiction. We can also prove that the second condition in (3.7) is equivalent to γ_1 being zero. Indeed, the 1-form γ_1 can be written as $\gamma_1^{1,0} + \gamma_1^{0,1}$, and $\omega \wedge \gamma_1 = 0$ is equivalent to

$$\omega \wedge \gamma_1^{1,0} = 0, \quad \omega \wedge \gamma_1^{0,1} = 0.$$

The same argument used for γ_2 can then be used for $\gamma_1^{1,0}$ and $\gamma_1^{0,1}$, to conclude that (3.7) implies $\gamma_1^{1,0} = \gamma_1^{0,1} = 0$, namely $\gamma_1 = 0$. It follows that $d\alpha^n = c\,\alpha^{1\bar{1}}$, so in particular $d\eta \in I\left(\alpha^{1\bar{1}}\right)$ and

$$d\Omega_{\mathfrak{h}} = \alpha^1 \wedge d\eta + \alpha^{\bar{1}} \wedge d\overline{\eta} = 0.$$

From the first line of (3.6), we get $\omega \wedge d\alpha^n = 0$. Using again the positive definiteness of ω, we obtain that $d\alpha^n = 0$ and $\mathfrak{g}$ is abelian. □

4 ($n-2$)-Kähler almost abelian solvmanifolds

In this section, we will discuss the case where (X, J) is an almost abelian solvmanifold, i.e., a compact quotient $\Gamma\backslash G$ of a simply connected almost abelian Lie group G by a lattice Γ endowed with an invariant complex structure J. Lemma 2.4 allows us to restrict to the study of p-Kähler structures on unimodular almost abelian Lie algebras.

Let $\mathfrak{g}$ be an almost abelian Lie algebra of real dimension $2n$ and denote with $\mathfrak{a}$ its codimension one abelian ideal. Let (J, g) be a Hermitian structure on $\mathfrak{g}$ and denote by $\mathfrak{a}_1$ the J-invariant space $\mathfrak{a} \cap J\mathfrak{a}$. Then there exists a unitary basis $\{e_1, \ldots, e_{2n}\}$ such that $\mathfrak{a} = \mathrm{span}\,\{e_1, \ldots, e_{2n-1}\}$, $\mathfrak{a}_1 = \mathrm{span}\,\{e_2, \ldots, e_{2n-1}\}$ and $Je_j = e_{2n+1-j}$, for $j = 1, \ldots, n$. The matrix associated with $\mathrm{ad}_{e_{2n}}\big|_{\mathfrak{a}}$ in this basis can be written as

$$\begin{pmatrix} \lambda & 0 \\ v & A \end{pmatrix},$$

with $\lambda \in \mathbb{R}$, $v \in \mathfrak{a}_1$, $A = (a_{j,k})_{j,k=2}^{2n-1} \in \mathfrak{gl}\,(\mathfrak{a}_1)$. We will refer to such a basis $\{e_1 \ldots e_{2n}\}$ as *adapted* to the Hermitian structure (J, g). By [6, 17], the integrability of J is equivalent to

$AJ_1 = J_1 A$, with $J_1 := J|_{\mathfrak{a}_1}$, so that A must satisfy

$$a_{2n+1-j,k} = \begin{cases} -a_{j,2n+1-k} & k = 2, \dots, n \\ a_{j,2n+1-k} & k = n+1, \dots, 2n-1 \end{cases} \qquad j = 2, \dots, n.$$

If $\{e^1 \dots e^{2n}\}$ is the dual basis to $\{e_1 \dots e_{2n}\}$, we have that $\alpha^j = e^j + i e^{2n+1-j}$, for $j = 1 \dots n$, is a basis of $(1,0)$-forms on $\mathfrak{g}$ and the complex structure equations of $(\mathfrak{g}, J)$ are

$$\begin{cases} d\alpha^1 = \frac{i}{2} \lambda \, \alpha^{1\bar{1}}, \\ d\alpha^j = \frac{i}{2} w_j \, \alpha^{1\bar{1}} + \frac{\alpha^1 - \alpha^{\bar{1}}}{2} \wedge \sum_{k=2}^{n} b_{jk} \alpha^k, \quad j = 2, \dots, n, \end{cases} \tag{4.1}$$

with $w_j = v_j + i\, v_{2n+1-j}$ and $b_{jk} := i\, a_{j,k} - a_{2n+1-j,k} = i\, a_{j,k} + a_{j,2n+1-k}$.

Remark 4.1 Notice that $\mathfrak{g}$ is unimodular if and only if $\lambda = -\mathrm{tr}(A)$. Moreover, (J, g) is Kähler if and only if $v = 0$ and $A = -A^t$ (see [17] and [10, Lemma 3.6]), while it is balanced if and only if $v = 0$ and $\mathrm{tr}(A) = 0$ [9, Lemma 3.1].

We will now prove the following.

Theorem 4.2 *Let $(\mathfrak{g}, J)$ be a unimodular almost abelian Lie algebra of real dimension $2n \geq 6$ endowed with a complex structure J. If $(\mathfrak{g}, J)$ admits a $(n-2)$-Kähler structure, then $(\mathfrak{g}, J)$ is Kähler.*

Proof We know that there exists a basis of $(1,0)$-forms $\alpha^j = e^j + i e^{2n+1-j}$, for $j = 1 \dots n$, such that the complex structure equations of $(\mathfrak{g}, J)$ are given by (4.1). Suppose that $(\mathfrak{g}, J)$ admits a $(n-2)$-Kähler form Ω. Then, we can write

$$\Omega = \Theta + \alpha^1 \wedge \eta + \alpha^{\bar{1}} \wedge \overline{\eta} + i\, \alpha^{1\bar{1}} \wedge \vartheta, \tag{4.2}$$

with $\Theta \in \Lambda^{n-2,n-2}_{\mathfrak{a}_1}$, $\vartheta \in \Lambda^{n-3,n-3}_{\mathfrak{a}_1}$ both real and transverse and $\eta \in \Lambda^{n-3,n-2}_{\mathfrak{a}_1}$. The restriction Θ of Ω to $\mathfrak{a}_1$ is a $(n-2, n-2)$-transverse form on a space of complex dimension $n-1$, so it is strictly positive and so there exists a new basis $\{\beta^2, \dots, \beta^n\}$ of $\mathfrak{a}_1^{1,0}$ such that

$$\Theta = \left(i(\beta^{2\bar{2}} + \dots + \beta^{n\bar{n}}) \right)^{n-2}$$

[18]. We can then consider as basis $\{f_j\}$ of $\mathfrak{a}_1$ the dual basis of the basis $\{f^j\}$ of $\mathfrak{a}_1^*$, given by

$$f^j := \frac{\beta^j + \overline{\beta} j}{2}, \quad f^{2n+1-j} := -i \frac{\beta^j - \overline{\beta}^j}{2}, \quad j = 2 \dots n.$$

Note that we still have $Jf_j = f_{2n+1-j}$ for $j = 2 \dots n$. We can complete $\{f_2, \dots, f_{2n-1}\}$ to a basis $\{f_1, \dots, f_{2n}\}$ of $\mathfrak{g}$ just taking $f_1 = e_1$ and $f_{2n} = e_{2n}$. Then we have $\mathfrak{a} = \mathrm{span}\{f_1, \dots, f_{2n-1}\}$ and $Jf_1 = f_{2n}$. Therefore, the complex structure equations of $(\mathfrak{g}, J)$ are still of the form

$$\begin{cases} d\beta^1 = \frac{i}{2} \tilde{\lambda} \beta^{1\bar{1}}, \\ d\beta^j = \frac{i}{2} \tilde{w}_j \beta^{1\bar{1}} + \frac{\beta^1 - \beta^{\bar{1}}}{2} \wedge \sum_{k=2}^{n} \tilde{b}_{jk} \beta^k, \quad j = 2, \dots, n, \end{cases}$$

with $\tilde{\lambda} = \lambda$ and $\beta^1 = \alpha^1$. The pair $(\mathfrak{g}, J)$ is then determined by the matrix associated with $\mathrm{ad}_{e_{2n}}\big|_{\mathfrak{a}}$ in this new basis

$$\begin{pmatrix} \lambda & 0 \\ \tilde{v} & \tilde{A} \end{pmatrix},$$

with $\tilde{v} \in \mathfrak{a}_1$, $\tilde{A} = (\tilde{a}_{j,k})_{j,k=2}^{2n-1} \in \mathfrak{gl}(\mathfrak{a}_1)$, and with $\tilde{w}_j = \tilde{v}_j + i\,\tilde{v}_{2n+1-j}$ and $\tilde{b}_{jk} := i\,\tilde{a}_{j,k} - \tilde{a}_{2n+1-j,k} = i\,\tilde{a}_{j,k} + \tilde{a}_{j,2n+1-k}$. Recall that the integrability of J is equivalent to the condition $\tilde{A} J_1 = J_1 \tilde{A}$. Therefore, $\tilde{A}$ must satisfy

$$\tilde{a}_{2n+1-j,k} = \begin{cases} -\tilde{a}_{j,2n+1-k} & k = 2, \dots, n \\ \tilde{a}_{j,2n+1-k} & k = n+1, \dots, 2n-1 \end{cases} \qquad j = 2, \dots, n.$$

Moreover,

$$d\Theta = i\left(\beta^1 - \beta^{\bar{1}}\right) \wedge \rho + i\beta^{1\bar{1}} \wedge \gamma,$$
$$d\eta = i\left(\beta^1 - \beta^{\bar{1}}\right) \wedge \phi + \beta^{1\bar{1}} \wedge \psi$$

with $\rho \in \Lambda_{\mathfrak{a}_1}^{n-2,n-2}$ and $\gamma \in \Lambda_{\mathfrak{a}_1}^{2n-5}$ real forms. Notice that ρ and ϕ depend on $\tilde{A}$, while γ and ψ depend on $\tilde{v}$. We have

$$0 = d\Omega = d\Theta + \lambda\frac{i}{2}\beta^{1\bar{1}} \wedge (\eta + \bar{\eta}) - \beta^1 \wedge d\eta - \beta^{\bar{1}} \wedge \overline{d\eta},$$
$$= i\beta^{1\bar{1}} \wedge \left(\gamma + \frac{\lambda}{2}(\eta + \bar{\eta}) + \phi + \bar{\phi}\right) + \left(\beta^1 - \beta^{\bar{1}}\right) \wedge \rho,$$

or equivalently

$$\rho = 0, \qquad \gamma + \frac{\lambda}{2}(\eta + \bar{\eta}) + \phi + \bar{\phi} = 0. \tag{4.3}$$

As a first consequence, we have that the theorem is true when $\tilde{A} = 0$. In this case, $\rho = 0$, $\phi = 0$ and $\mathfrak{g}$ is unimodular if and only if $\lambda = 0$, so that (4.3) reduces to $\gamma = 0$, namely $d\Theta = 0$. Because $d\beta^{1\bar{1}} = 0$ and $d\varphi \in I(\beta^{1\bar{1}})$ for all $\varphi \in \Lambda_{\mathfrak{g}}$, one has

$$0 = d\Theta = d\left(i(\beta^{1\bar{1}} + \cdots + \beta^{n\bar{n}})\right)^{n-2};$$

hence, $d(\beta^{1\bar{1}} + \cdots + \beta^{n\bar{n}}) = 0$ (see [12]) and $(\mathfrak{g}, J)$ is Kähler. We can now prove that when $\tilde{A} \neq 0$ the vanishing of ρ implies that $\tilde{A} = -\tilde{A}^t$. Indeed, we have $d\Theta = i^{n-2}(n-2)\left(\beta^{2\bar{2}} + \cdots + \beta^{n\bar{n}}\right)^{n-3} \wedge d(\beta^{2\bar{2}} + \cdots + \beta^{n\bar{n}})$. For $j = 2 \dots n$,

$$d\beta^{j\bar{j}} = \frac{i}{2}\beta^{1\bar{1}} \wedge \left(\tilde{w}_j \overline{\beta} j - \overline{\tilde{w}_j}\beta^j\right) + \frac{\beta^1 - \beta^{\bar{1}}}{2} \wedge \sum_{k=2}^{n}\left(\tilde{b}_{jk}\beta^{k\bar{j}} - \overline{\tilde{b}_{jk}}\beta^{j\bar{k}}\right).$$

For some $c_n \in \mathbb{R}$, we can write

$$\left(\beta^{2\bar{2}} + \cdots + \beta^{n\bar{n}}\right)^{n-3} = c_n \sum_{2 \le l < m \le n} \beta^{2\bar{2}\cdots\widehat{l\bar{l}}\cdots\widehat{m\bar{m}}\cdots n\bar{n}}$$

so that

$$d\Theta = i\beta^{1\bar{1}} \wedge \gamma + \tilde{c_n}\left(\beta^1 - \beta^{\bar{1}}\right) \wedge \sum_{2\le l<m\le n} \beta^{2\bar{2}\cdots\widehat{l\bar{l}}\cdots\widehat{m\bar{m}}\cdots n\bar{n}} \wedge \sum_{j,k=2}^{n}\left(\tilde{b}_{jk}\beta^{k\bar{j}} - \overline{\tilde{b}_{jk}}\beta^{j\bar{k}}\right)$$

with $\tilde{c_n} \in \mathbb{C}$. We can rewrite the last sum as

$$\begin{aligned}
&\sum_{j=2}^{n}\beta^{i\bar{j}}\left(\tilde{b}_{jj} - \overline{\tilde{b}_{jj}}\right) + \sum_{2\le j<k\le n}\left(\left(\tilde{b}_{jk}\,\alpha^{k\bar{j}} - \overline{\tilde{b}_{jk}}\beta^{j\bar{k}}\right) + \left(b_{kj}\beta^{j\bar{k}} - \overline{b_{kj}}\beta^{k\bar{j}}\right)\right)\\
&\quad = \sum_{j=2}^{n}\beta^{j\bar{j}}\left(\tilde{b}_{jj} - \overline{\tilde{b}_{jj}}\right) + \sum_{2\le j<k\le n}\left(\beta^{j\bar{k}}\left(\tilde{b}_{kj} - \overline{\tilde{b}_{jk}}\right) + \beta^{k\bar{j}}\left(\tilde{b}_{jk} - \overline{\tilde{b}_{kj}}\right)\right)\\
&\quad = \sum_{j=2}^{n} 2i\,\tilde{a}_{jj}\beta^{j\bar{j}} + \sum_{2\le j<k\le n}\left(\beta^{j\bar{k}}\left(\tilde{b}_{kj} - \overline{\tilde{b}_{jk}}\right) + \beta^{k\bar{j}}\left(\tilde{b}_{jk} - \overline{\tilde{b}_{kj}}\right)\right).
\end{aligned}$$

It follows that up to a complex constant, ρ equals

$$\sum_{l\ne m=2}^{n} 2i\tilde{a}_{ll}\beta^{2\bar{2}\cdots\widehat{m\bar{m}}\cdots n\bar{n}} + \sum_{2\le l<m\le n}\beta^{2\bar{2}\cdots\widehat{l\bar{l}}\cdots\widehat{m\bar{m}}\cdots n\bar{n}} \wedge \left(\beta^{l\bar{m}}\left(\tilde{b}_{ml} - \overline{\tilde{b}_{lm}}\right) + \beta^{m\bar{l}}\left(\tilde{b}_{lm} - \overline{\tilde{b}_{ml}}\right)\right).$$

If $\rho = 0$, one gets

$$\begin{cases}\displaystyle\sum_{l\ne m}\tilde{a}_{ll} = 0, & m = 2\ldots n,\\ \tilde{b}_{lm} = \overline{\tilde{b}_{ml}}, & 2\le l<m\le n,\end{cases}$$

namely

$$\tilde{a}_{ll} = 0,\quad \tilde{a}_{lm} = -\tilde{a}_{ml},\quad \tilde{a}_{2n+1-l,m} = \tilde{a}_{2n+1-m,l} = -\tilde{a}_{m,2n+1-l}.$$

This, together with the conditions on $\tilde{A}$ given by the integrability of the complex structure, is enough to conclude that $\tilde{A} = -\tilde{A}^t$. Therefore, the matrix associated with $\mathrm{ad}_{e_{2n}}\big|_{\mathfrak{a}}$ in the basis $\{f_1, \ldots, f_{2n-1}\}$ is

$$C = \begin{pmatrix}\lambda & 0\\ \tilde{v} & \tilde{A}\end{pmatrix},$$

with $\tilde{A}$ antisymmetric. Because $\tilde{A} \ne 0$, the Jordan form of C is

$$\mathrm{Jord}(C) = \begin{pmatrix}\lambda & \delta\; 0\cdots 0\\ 0 & \mathrm{Jord}(\tilde{A})\end{pmatrix},$$

where $\mathrm{Jord}(\tilde{A})$ is the Jordan form of $\tilde{A}$, $\delta = 0$ if λ is an eigenvalue of $\tilde{A}$ and $\delta = 1$ otherwise. It follows that C is similar to a matrix

$$D = \begin{pmatrix}\lambda & 0\\ 0 & \tilde{D}\end{pmatrix},$$

with $\tilde{D}$ antisymmetric and $\mathrm{Jord}(\tilde{D}) = \mathrm{Jord}(\tilde{A})$. Consider the almost abelian Lie algebra $\tilde{\mathfrak{g}}$ with abelian ideal $\tilde{\mathfrak{a}} = \mathrm{span}\{\tilde{e}_1 \ldots \tilde{e}_{2n-1}\}$ and such that the matrix of $\mathrm{ad}_{\tilde{e}_{2n}}\big|_{\tilde{\mathfrak{a}}}$ is D. By [11, Proposition 1], $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$. As mentioned above (Remark 4.1), this gives the existence of a Kähler metric on $\tilde{\mathfrak{g}}$. □

Remark 4.3 In complex dimension 4, the theorem states that there are no 2-Kähler almost abelian solvmanifolds that are non-Kähler.

Remark 4.4 In the last part of the proof, we found a sufficient condition for an almost abelian unimodular Lie algebra $\mathfrak{g}$ to be Kähler. Let (J, g) be a Hermitian structure on $\mathfrak{g}$ and $\{e_1 \dots e_{2n}\}$ be an adapted basis to (J, g). If $\mathrm{ad}_{e_{2n}}\big|_{\mathfrak{a}}$ is conjugated to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ v & A \end{pmatrix},$$

with $\lambda \in \mathbb{R}$, $v \in \mathfrak{a}_1$, $A \in \mathfrak{so}(\mathfrak{a}_1)$, $\left[A,\ J|_{\mathfrak{a}_1}\right] = 0$, and A has same rank of $(v\ A)$, then $(\mathfrak{g}, J)$ is Kähler.

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