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# **ALGEBRAIC COMBINATORICS**

Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi & Emanuele Ventura

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### Quasihomomorphisms from the integers into Hamming metrics

## Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi & Emanuele Ventura

ABSTRACT A function  $f : \mathbb{Z} \to \mathbb{Q}^n$  is a *c*-quasihomomorphism if the Hamming distance between f(x + y) and f(x) + f(y) is at most *c* for all  $x, y \in \mathbb{Z}$ . We show that any *c*-quasihomomorphism has distance at most some constant C(c) to an actual group homomorphism; here C(c) depends only on *c* and not on *n* or *f*. This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

### 1. INTRODUCTION

Let c be a nonnegative real number. A c-quasihomomorphism from a group G to a group H with a left-invariant metric d is a map  $f : G \to H$  such that  $d(f(xy), f(x)f(y)) \leq c$  for all x, y in G. A central question in geometric group theory, raised by Ulam in [17, Chapter 6], is whether there exists an actual homomorphism  $f' : G \to H$  such that d(f(x), f'(x)) is at most some constant C for all x. (Related questions were studied before Ulam, e.g. by Turing in his work on approximability of groups [16].) Different versions of Ulam's question are of interest: for example, C may be allowed to depend on c, G, (H, d) but not on f; G, (H, d) may be restricted to certain classes and C is only allowed to depend on c.

A well-known example where the answer to this question is negative is the case where  $G = H = \mathbb{Z}$  with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [15, 1], and the answer is yes only for those quasihomomorphisms that correspond to integers. In fact, this construction can be extended to construct completions of fields in general [11].

Much literature in this area focusses on *quasimorphisms*, which are quasihomomorphisms into the real numbers  $\mathbb{R}$  with the standard metric; we refer to [12] for a brief introduction. In particular, the concept of a quasimorphism features in bounded cohomology, see [13, 4, 6]. In another branch of the research on quasihomomorphisms H is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan's theorem on  $\varepsilon$ -representations of amenable groups [9]. For more

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recent results on quasihomomorphisms into nonabelian groups we refer to [7, 8, 5, 2] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [10].

QUESTION 1.1. Let  $c \in \mathbb{N}$ . Does there exist a constant C = C(c) such that the following holds: For all  $n \in \mathbb{N}$  and all functions  $f : \mathbb{Z} \longrightarrow \mathbb{C}^{n \times n}$  such that

for all  $x, y \in \mathbb{Z}$ ,  $\operatorname{rk}(f(x+y) - f(x) - f(y)) \leq c$ ,

there exists a matrix g such that

for all 
$$x \in \mathbb{Z}$$
,  $\operatorname{rk}(f(x) - xg) \leq C(c)$ ?

Here, G equals  $\mathbb{Z}$  and H equals  $\mathbb{C}^{n \times n}$ , both with addition, and the metric on H is defined by  $d(A, B) := \operatorname{rk}(A - B)$ . In [10, p1], the function  $R(\mathbb{Z}, c, \mathbb{C})$  denotes the minimal possible choice of C(c). Our main result is an affirmative answer to Question 1.1 in the special case where all matrices f(x) are assumed to be *diagonal*.

DEFINITION 1.2. Let (Q, +) be an abelian group. For an element  $v \in Q^n$ , the Hamming weight  $w_H(v)$  is the number of nonzero entries of v. For a pair of elements  $u, v \in Q^n$ , their Hamming distance is  $w_H(v-u)$ . This metric is clearly left-invariant, and indeed even bi-invariant.

DEFINITION 1.3. Let A be another abelian group. A function  $f : A \to Q^n$  is called a c-quasihomomorphism if

(1) for all 
$$x, y \in A$$
,  $w_H(f(x+y) - f(x) - f(y)) \leq c$ 

REMARK 1.4. The map diag :  $\mathbb{C}^n \to \mathbb{C}^{n \times n}$  is an isometric embedding from  $\mathbb{C}^n$  with the Hamming metric to  $\mathbb{C}^{n \times n}$  with the rank metric. This connects Definition 1.3 to Question 1.1.

DEFINITION 1.5. Let  $C \in \mathbb{N}$  and let  $f : A \to Q^n$  be a c-quasihomomorphism. A group homomorphism  $h : A \to Q^n$  is a C-approximation of f if the Hamming distance between f and h satisfies

for all 
$$x \in A$$
,  $w_H(f(x) - h(x)) \leq C$ .

We are ready to state our main result.

THEOREM 1.6 (Main Theorem). Let  $c \in \mathbb{N}$ . Then there exists a constant  $C = C(c) \in \mathbb{N}$ such that for all  $n \in \mathbb{N}$  and c-quasihomorphisms  $f : \mathbb{Z} \to \mathbb{Q}^n$ , we have:

for all 
$$x \in \mathbb{Z}$$
,  $w_H(f(x) - xf(1)) \leq C$ .

Moreover, we can take C = 28c.

REMARK 1.7. The coefficient 28 is probably not optimal. However, we certainly have that  $C(c) \ge c$ . Indeed, any map  $f : \mathbb{Z} \to \mathbb{Q}^n$  for which the only nonzero entries of f(x) are among the first c, is automatically a c-quasihomomorphism.

COROLLARY 1.8. Theorem 1.6 also holds with  $\mathbb{Q}$  replaced by any torsion-free abelian group Q, with the same value of C = C(c).

Proof. Suppose, for a contradiction, that we have a *c*-quasihomomorphism  $f : \mathbb{Z} \to Q^n$ but  $w_H(f(y) - yf(1)) > C$  for some  $y \in \mathbb{Z}$ . Since Q is torsion-free, the natural map  $\iota$  from Q into the  $\mathbb{Q}$ -vector space  $V := \mathbb{Q} \otimes_{\mathbb{Z}} Q$  is injective. Consequently,  $g := \iota^n \circ f$ is a *c*-quasihomomorphism  $\mathbb{Z} \to V^n$  with  $w_H(g(y) - yg(1)) > C$ . Now choose any  $\mathbb{Q}$ -linear function  $\xi : V \to \mathbb{Q}$  that is nonzero on the nonzero entries of g(y) - yg(1). Then  $h := \xi^n \circ g$  is a *c*-quasihomomorphism  $\mathbb{Z} \to \mathbb{Q}^n$  with  $w_H(h(y) - yh(1)) > C$ , a contradiction to Theorem 1.6. REMARK 1.9. As a referee kindly pointed out to us, our result fits in the broader context of  $\mathcal{G}$ -stability for a family  $\mathcal{G}$  of groups endowed with a bi-invariant metric; this was first introduced in [9] and further studied in [3] under the name of *Ulam stability*. Let  $\mathcal{G}$  be the family of groups  $\{\operatorname{GL}_n(\mathbb{C})\}_{n\geq 1}$  with the normalized rank metric, i.e.  $d(A, B) = \frac{1}{n} \operatorname{rk}(A - B)$ . Let  $\mathcal{G}_d$  be the subfamily of  $\mathcal{G}$  consisting of diagonal matrices. Theorem 1.6 shows that the abelian group  $\mathbb{Z}$  is uniformly  $\mathcal{G}_d$ -stable with a linear estimate.

Theorem 1.6 shows that for a *c*-quasihomomorphism  $f : \mathbb{Z} \to \mathbb{Q}^n$ , the group homomorphism  $\tilde{f} : \mathbb{Z} \to \mathbb{Q}^n$  defined by  $\tilde{f}(x) = xf(1)$  gives a *C*-approximation for some constant  $C \in \mathbb{N}$  independent on *n*. However,  $\tilde{f}$  need not be the homomorphism closest to *f*, as the next example shows.

EXAMPLE 1.10. Let c = 1 and  $n \ge 3$ . Define  $f : \mathbb{Z} \to \mathbb{Q}^n$  to be

(2) 
$$f(x) = \left( \left\lfloor \frac{2x}{5} \right\rfloor, \left\lfloor \frac{x}{5} \right\rfloor, \alpha_x, 0, \dots, 0 \right),$$

where  $\alpha_x \in \mathbb{Q}$  is arbitrary if  $5 \mid x$ , and  $\alpha_x = 0$  otherwise. Here  $\lfloor \rceil$  denotes rounding to the nearest integer. To check that f is a 1-quasihomomorphism (1) we work mod 5. For simplicity, restrict to the case n = 3. Then, for  $k \in \mathbb{Z}$ ,

$$f(5k) = (2k, k, \alpha_{5k}), \qquad f(5k+1) = (2k, k, 0),$$
  

$$f(5k+2) = (2k+1, k, 0), \qquad f(5k+3) = (2k+1, k+1, 0),$$
  

$$f(5k+4) = (2k+2, k+1, 0).$$

Let  $x = 5k + \ell_1$  and  $y = 5h + \ell_2$  with  $0 \leq \ell_1 \leq \ell_2 < 5$ . Then we can verify that

$$w_H(f(5(k+h) + (\ell_1 + \ell_2)) - f(5k + \ell_1) - f(5h + \ell_2)) \leq c = 1$$

in all cases. Roughly speaking, the check boils down to verifying that there are no cases where both  $\lfloor \frac{x+y}{5} \rfloor \neq \lfloor \frac{x}{5} \rfloor + \lfloor \frac{y}{5} \rfloor$  and  $\lfloor \frac{2(x+y)}{5} \rfloor \neq \lfloor \frac{2x}{5} \rfloor + \lfloor \frac{2y}{5} \rfloor$ , and moreover that if  $5 \mid x+y$ , then  $f(x+y) - f(x) - f(y) = (0, 0, \alpha_x)$  (because in this case,  $\frac{x}{5}$  is rounded down if and only if  $\frac{y}{5}$  is rounded up). Note that  $w_H(f(x) - xf(1)) \leq 3$  where equality is sometimes achieved (provided there is at least one  $x \neq 0$  for which we chose  $\alpha_x \neq 0$ ). However, there also exist 2-approximations of f. For instance, letting  $v = (\frac{2}{5}, \frac{1}{5}, 0, \dots, 0) \in \mathbb{Q}^n$ , one verifies that

$$w_H(f(x) - xv) \leq 2$$
 for all  $x \in \mathbb{Z}$ .

In [14], the authors show that for every 1-quasihomomorphism  $f : \mathbb{Z} \to \mathbb{Q}^n$ , and even for every 1-quasihomomorphism from  $\mathbb{Z}$  into the space of symmetric  $n \times n$ -matrices with the rank metric, there is a 2-approximation.

(This result is consistent with the second paragraph of [10], where a proof of the corresponding statement for general matrices is sketched. However, the above example shows that that proof is incomplete: viewing f as a map to the diagonal matrices, and assuming  $\alpha_0 = 0$  as is done in that paragraph, we obtain a counterexample to the statement in [10] that there exists either a subspace of codimension 1 living in the kernel of all matrices f(n+m) - f(n) - f(m) or else a subspace of dimension 1 containing all their images.)

On the other hand, the following shows that the best possible approximation of a given quasihomomorphism f is at most twice as close as the homomorphism  $x \mapsto xf(1)$ .

REMARK 1.11. Suppose that a map  $f : \mathbb{Z} \to \mathbb{Q}^n$  has a C'-approximation h. Then h(x) = xv for some  $v \in \mathbb{Q}^n$ , and

$$w_H(f(x) - xv) \leq C'$$
 for all  $x \in \mathbb{N}$ .

Substituting x = 1 yields  $w_H(f(1) - v) \leq C'$ . Thus

$$w_H(f(x) - xf(1)) \leq w_H(f(x) - xv) + w_H(xv - xf(1)) \leq 2C'.$$

REMARK 1.12. A result similar to Theorem 1.6 is easily proven in positive characteristic if we allow the constant C to depend on the characteristic. Let K be a field of characteristic p > 0, and let  $f : \mathbb{Z} \to K^n$  be a *c*-quasihomomorphism. Then there exists a constant C = C(p, c) such that  $w_H(f(x) - xf(1)) \leq C$ , for all  $x \in \mathbb{Z}$ .

To see this, we observe that for all  $u, v \in \mathbb{Z}$  with  $u \ge 1$ , we have

$$w_H(f(uv) - uf(v)) \leq (u - 1)c.$$

This follows by repeatedly applying the inequality  $w_H(f(uv) - f((u-1)v) - f(v)) \leq c$ if u > 1; the case u = 1 is trivial.

For x = kp + r with  $k \in \mathbb{Z}$  and  $0 \leq r \leq p - 1$ , we have

$$w_H(f(x) - xf(1)) = w_H(f(kp + r) - rf(1));$$

here we have used that pf(1) = 0. We rewrite the latter as

$$w_H(f(kp+r) - f(kp) - f(r) + f(kp) + f(r) - rf(1)).$$

We have  $w_H(f(kp+r)-f(kp)-f(r)) \leq c$ ;  $w_H(f(kp)) \leq (p-1)c$  using our observation with u = p, v = k; and also  $w_H(f(r) - rf(1)) \leq (p-2)c$  (in the case r > 0). In total, this gives  $w_H(f(x) - xf(1)) \leq 2(p-1)c$ , so we can take C = 2(p-1)c.

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3, we apply this auxiliary result to the component functions of a *c*-quasihomomorphism  $\mathbb{Z} \to \mathbb{Q}^n$  to prove the Main Theorem.

### 2. Almost homomorphisms are almost zero

Let A be a finite abelian group and let H be a torsion-free abelian group. The only homomorphism  $A \to H$  is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

PROPOSITION 2.1. Let a be a positive integer, A an abelian group of order a, H a torsion-free abelian group,  $q \in [0, 1]$ , and  $f : A \to H$  a map. Suppose that the zero set

$$Z(f) := \{ b \in A \mid f(b) = 0 \}$$

has cardinality at most qa. Then the problem set

$$P(f) := \{ (b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \}$$

has cardinality at least  $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$ .

The contraposition of this statement says that if P(f) is a small fraction of  $a^2$ , so that f can be thought of as an (additive) "almost homomorphism"  $A \to H$ , then q must be close to 1 so that f is essentially zero.

*Proof.* Since H is torsion-free, it embeds into the  $\mathbb{Q}$ -vector space  $V := \mathbb{Q} \otimes_{\mathbb{Z}} H$ . By basic linear algebra, there exists a  $\mathbb{Q}$ -linear function  $\xi : V \to \mathbb{Q}$  such that  $\xi(f(b)) \neq 0$  for all  $b \notin Z(f)$ , so that  $Z(\xi \circ f) = Z(f)$ . Since  $P(\xi \circ f) \subseteq P(f)$ , it suffices to prove the proposition for  $\xi \circ f$  instead of f. In other words, we may assume from the beginning that  $H = \mathbb{Q}$ .

Set

$$B := \{ b \in A \mid f(b) > 0 \}.$$

Let  $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$  be the distinct values in f(B), and for each  $i = 1, \ldots, k$  set

$$B_i := \{b \in B \mid f(b) = \lambda_i\}$$
 and  $n_i := |B_i|$ 

as well as  $n := n_1 + \dots + n_k = |B|$ .

Now for each  $c \in B_1$  and each  $b \in B$  we have

$$f(b) + f(c) = f(b) + \lambda_1 > \lambda_1$$

so that the left-hand side is not in f(B) and in particular not equal to f(b+c). We have thus found  $n_1(n_1 + \cdots + n_k)$  pairs  $(b,c) \in P(f)$  with  $c \in B_1$ .

Next, suppose (b, c) is a pair with  $c \in B_2$ ,  $b \in B$ , and  $(b, c) \notin P(f)$ . Then

$$f(b+c) = f(b) + f(c) > f(c) = \lambda_2$$

and hence  $b+c \in B_1$ . But given c, there are at most  $n_1$  values of b with  $b+c \in B_1$ . (Note that here we have used that A is a group.) Hence we have at least  $n_2(n_2 + \cdots + n_k)$  pairs  $(b, c) \in P(f)$  with  $c \in B_2$ .

Similarly, we find at least  $n_i(n_i + \cdots + n_k)$  pairs  $(b, c) \in P(f)$  with  $c \in B_i$ . In total, we have therefore found at least

(3) 
$$\sum_{i=1}^{k} n_i (n_i + \dots + n_k) \ge \frac{n(n+1)}{2}$$

pairs in P(f); see Figure 1.

Let  $B' := \{b' \in A \mid f(b') < 0\}$  and n' := |B'|. Repeating the same argument above with B' and n', we find at least n'(n'+1)/2 further pairs in P(f), disjoint from those found above. Since  $|Z(f)| \leq qa$ , we have  $n + n' \geq a(1-q)$ . Therefore

$$|P(f)| \ge \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2 + n'^2}{2} + \frac{n+n'}{2} \ge \left(\frac{n+n'}{2}\right)^2 + \frac{n+n'}{2},$$

where the second inequality is the Cauchy-Schwarz inequality

$$(n^2 + n'^2)\left(\frac{1}{2^2} + \frac{1}{2^2}\right) \ge \left(\frac{n}{2} + \frac{n'}{2}\right)^2$$

Since  $n + n' \ge a(1 - q)$ , we conclude that

$$|P(f)| \ge \left(\frac{a-qa}{2}\right) \left(\frac{a-qa}{2}+1\right).$$

REMARK 2.2. The lower bound in Proposition 2.1 is sharp. Let  $a = 2k + 1 \in \mathbb{Z}$ , consider  $A := \mathbb{Z}/a\mathbb{Z}$  and define  $f : A \to \mathbb{Z}$  as f(x) := the representative of  $x + a\mathbb{Z}$  in  $\{-k, \ldots, 0, \ldots, k\}$ . We take  $q = \frac{\mathbb{Z}(f)}{a} = \frac{1}{2k+1}$ . Then f(x+y) = f(x) + f(y) if and only if the right-hand side is still inside the interval  $\{-k, \ldots, k\}$ , and a straightforward count shows that this is the case for  $3k^2 + 3k + 1$  pairs  $(x, y) \in A^2$ . Hence P(f) has size k(k+1), which equals  $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$ .

A similar construction for a = 2k yields a problem set of size  $\frac{a^2}{4} = k^2$ , which equals the ceiling of the lower bound  $\frac{a^2}{4} - \frac{1}{4}$ .

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Below, we will use the following strengthening of Proposition 2.1:

PROPOSITION 2.3. Let a, A, H, q and f be as in Proposition 2.1. Furthermore, let  $p \in [0, \frac{1-q}{2})$  and let  $S \subseteq A$  be a subset of cardinality at most pa. Then the set

$$P_S(f) := \{ (b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \text{ and } b + c \notin S \}.$$

has cardinality at least  $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$ .

*Proof.* Keep the notation from the proof of Proposition 2.1. Recall n = |B| and n' = |B'|. Note that for a fixed b, there can be at most pa choices of c with  $b + c \in S$ . We then find at least  $n_i(n_i + \cdots + n_k - pa)$  pairs  $(b, c) \in P_S(f)$  with  $b \in B_i$ . Letting  $k' \leq k$  be the largest index for which the second factor  $(n_{k'} + \cdots + n_k - pa)$  is nonnegative, as in the proof of Proposition 2.1, we find that B contributes at least

(4) 
$$\sum_{i=1}^{k'} n_i (n_i + \dots + n_k - pa) = \sum_{i=1}^{k'} n_i (n - n_1 - \dots - n_{i-1} - pa) \\ \geqslant (n - pa)(n - pa + 1)/2$$

to  $P_S(f)$ ; see Figure 1. Similarly, B' contributes at least (n' - pa)(n' - pa + 1)/2, and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with n, n' replaced by n - pa, n' - pa.  $\square$ 

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write [a] for the set  $\{1, 2, \ldots, a\}$ .

COROLLARY 2.4. Let  $p, q \in [0,1]$  such that  $p < \frac{1-q}{2}$ . Let  $f : [2a] \to \mathbb{Q}$  such that:

- (1)  $|Z_a(f)| \leq qa$ , where  $Z_a(f) \coloneqq \{x \in [a] \mid f(x) = 0\}$  is the zero set of  $f|_{[a]}$ . (2)  $|NP(f)| \leq pa$ , where  $NP(f) \coloneqq \{x \in [a] \mid f(x+a) \neq f(x)\}$  is the nonperiodicity set.

Then

$$|P(f)| \ge \frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a,$$

where

$$P(f) = \{(x, y) \in [a] \times [a] \mid f(x + y) \neq f(x) + f(y)\}.$$



FIGURE 1. On the left, a graphical proof of the inequality (3): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (4): the two expressions on top represent the area of the shaded region, while the bottom expression represents the area enclosed by the dashed line.

*Proof.* Let  $\tilde{f}$  be the restriction of f to the interval [a], and identify  $\mathbb{Z}/a\mathbb{Z}$  with [a] with the group operation  $\star$  defined by  $x \star y := x + y \pmod{a}$ .

Let S = NP(f), and apply Proposition 2.3 to  $\tilde{f}$ . We find that

$$P_S(\tilde{f}) = \{(b,c) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \mid \tilde{f}(b \star c) \neq \tilde{f}(b) + \tilde{f}(c) \text{ and } b \star c \notin S\}$$

has cardinality at least  $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$ . Since  $b \star c \notin S$  implies that  $\tilde{f}(b \star c) = f(b+c)$ , this set is contained in the problem set P(f).

#### 3. Proof of the main theorem

The main goal of this section is to prove Theorem 1.6. We start with some definitions. DEFINITION 3.1. Let 1 < a, and  $f : [2a] \to \mathbb{Q}$ . We define the following problem sets of f:

$$P(f) := \{ (x, y) \in [a] \times [a] \mid f(x + y) \neq f(x) + f(y) \},\$$

and

$$P_1(f) := \{ x \in [a] \mid f(x+1) \neq f(x) + f(1) \},\$$

and

$$P_a(f) := \{ x \in [a] \mid f(x+a) \neq f(x) + f(a) \}.$$

Furthermore, we recall that  $Z_a(f)$  denotes the zero set of  $f|_{[a]}$ :

$$Z_a(f) := \{ x \in [a] \mid f(x) = 0 \}.$$

The following proposition says that  $P_1(f), P_a(f), P(f)$  cannot be simultaneously small.

PROPOSITION 3.2. Let  $p, q \in (0, 1)$  such that 1 - q - 2p > 0,  $a \in \mathbb{N}$  with 1 < a, and let  $f : [2a] \to \mathbb{Q}$  such that  $f(a) \neq af(1)$ . Then at least one of the following holds:

- (i)  $|P_1(f)| > qa$ ,
- (ii)  $|P_a(f)| > pa$ ,
- (iii)  $|P(f)| \ge F(p,q)a^2$ ,

where

$$F(p,q) \coloneqq \frac{(1-q-2p)^2}{4}.$$

Proof. Without loss of generality we can assume f(a) = 0 and hence  $f(1) \neq 0$ . Indeed, suppose we have shown the statement for every  $\tilde{f}$  with  $\tilde{f}(a) = 0$ . Then for any  $f: [2a] \to \mathbb{Q}$  with  $f(a) \neq af(1)$ , we take  $\tilde{f}: [2a] \to \mathbb{Q}$  to be  $\tilde{f}(x) = af(x) - xf(a)$ . Now we observe that  $\tilde{f}(a) = 0 \neq a\tilde{f}(1)$ , and that  $P(f) = P(\tilde{f}), P_1(f) = P_1(\tilde{f}),$  $P_a(f) = P_a(\tilde{f})$ .

To prove the proposition we will assume that ((i)) and ((ii)) are false, and prove that then ((iii)) must hold. Write  $Z_a(f) = \{x_1, \ldots, x_m\}$ , where  $x_1 < \cdots < x_m$ . Note that for  $1 \leq i < m$ , one of the elements  $x_i, x_i + 1, \ldots, x_{i+1} - 1$  needs to be in  $P_1(f)$  since  $f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i)f(1)$ . Likewise, at least one of the elements  $1, 2, \ldots, x_1 - 1$  needs to be in  $P_1(f)$ . Thus we have

$$|Z_a(f)| \leqslant |P_1(f)| \leqslant qa,$$

and by assumption we have  $|NP(f)| = |P_a(f)| \leq pa$ . Now we can apply Corollary 2.4 to conclude.

We now prove Theorem 1.6.

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Proof of the Main Theorem. Consider a c-quasihomomorphism  $f = (f_1, \ldots, f_n)$ :  $\mathbb{Z} \to \mathbb{Q}^n$ . Our goal is to show that for every  $a \in \mathbb{Z}$  we have  $w_H(f(a) - af(1)) \leq C$  for some constant C depending only on c. We start with the case a > 0.

Write  $I_a := \{i \in [a] \mid f_i(a) \neq af_i(1)\}$ , and note that  $|I_a| = w_H(f(a) - af(1))$ . We will show that  $|I_a| \leq C'$  for some constant C' depending on c only. To this end, fix small parameters  $p, q \in (0, 1)$  (to be optimized over later) and write  $f_i^a := f_i|_{[2a]}$  for the restriction of  $f_i$  to [2a]. By Proposition 3.2, for every  $i \in I_a$ , we have

- (i)  $|P_1(f_i^a)| > qa$ , or
- (ii)  $|P_a(f_i^a)| > pa$ , or
- (iii)  $|P(f_i^a)| \ge F(p,q)a^2$ .

Let  $m_0$  be the number of coordinates  $i \in I_a$  such that (iii) holds. We define  $m_1$  and  $m_2$  analogously, for (i) and (ii) respectively.

By counting the number of triples  $(i, x, y) \in [n] \times [a] \times [a]$  such that  $f_i(x + y) - f_i(x) - f_i(y) \neq 0$  in two ways, we see that

$$\sum_{x=1}^{a} \sum_{y=1}^{a} w_H \left( f(x+y) - f(x) - f(y) \right) = \sum_{i=1}^{n} |P(f_i^a)| \ge \sum_{i \in I_a} |P(f_i^a)|$$

Because f is a c-quasihomomorphism, the very left-hand side is at most  $a^2c$ . On the other hand, the very right-hand side is at least  $m_0F(p,q)a^2$ , so

$$a^{2}c \geqslant \sum_{x=1}^{a} \sum_{y=1}^{a} w_{H} \left( f(x+y) - f(x) - f(y) \right) \geqslant \sum_{i \in I_{a}} |P(f_{i}^{a})| \geqslant m_{0}F(p,q)a^{2}.$$

So we obtain  $m_0 \leq \frac{c}{F(p,q)}$ . Similarly we find

$$ac \ge \sum_{x=1}^{a} w_H \left( f(x+1) - f(x) - f(1) \right) = \sum_{i=1}^{n} |P_1(f_i^a)| \ge \sum_{i \in I_a} |P_1(f_i^a)| > m_1 q a,$$

so that  $m_1 < \frac{c}{q}$ . Finally,

$$ac \ge \sum_{x=1}^{a} w_H \left( f(x+a) - f(x) - f(a) \right) = \sum_{i=1}^{n} |P_a(f_i^a)| \ge \sum_{i \in I_a} |P_a(f_i^a)| > m_2 p a.$$

So  $m_2 < \frac{c}{p}$ . But now  $|I_a| \leq m_0 + m_1 + m_2 < c(\frac{1}{F(p,q)} + \frac{1}{q} + \frac{1}{p}) =: C'$ . The case a = 0 is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \le c.$$

Finally, let us consider the case a < 0. Then

$$w_H(f(a) - af(1)) \leq w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) + w_H(f(-a) - (-a)f(1)) \leq 2c + C' =: C.$$

This completes the proof of the qualitative part of the Main Theorem. To obtain the explicit bound 28c, we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p,q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1 - q - 2p)^2}$$

This function is strictly convex for  $(p,q) \in \mathbb{R}^2_{>0}$ , so it has at most one minimum in the positive orthant. We find this by setting the partial derivatives to zero and solving for p, q. The minimum is  $\approx 27.6817$  and attained at  $(p,q) \approx (0.1167, 0.16500)$ .

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