Quasihomomorphisms from the integers into Hamming metrics

ALGEBRAIC COMBINATORICS

Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi & Emanuele Ventura

Quasihomomorphisms from the integers into Hamming metrics

Volume 7, issue 3 (2024), p. 843-851.
https://doi.org/10.5802/alco.348

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 7, issue 3 (2024), p. 843–851
https://doi.org/10.5802/alco.348

Quasihomomorphisms from the integers into Hamming metrics

Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi
& Emanuele Ventura

Abstract A function $f : \mathbb{Z} \to \mathbb{Q}^n$ is a c-*quasihomomorphism* if the Hamming distance between $f(x+y)$ and $f(x)+f(y)$ is at most c for all $x, y \in \mathbb{Z}$. We show that any c-quasihomomorphism has distance at most some constant $C(c)$ to an actual group homomorphism; here $C(c)$ depends only on c and not on n or f. This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

1. Introduction

Let c be a nonnegative real number. A c-quasihomomorphism from a group G to a group H with a left-invariant metric d is a map $f : G \to H$ such that $d(f(xy), f(x)f(y)) \leqslant c$ for all x, y in G. A central question in geometric group theory, raised by Ulam in [17, Chapter 6], is whether there exists an actual homomorphism $f' : G \to H$ such that $d(f(x), f'(x))$ is at most some constant C for all x. (Related questions were studied before Ulam, e.g. by Turing in his work on approximability of groups [16].) Different versions of Ulam's question are of interest: for example, C may be allowed to depend on $c, G, (H, d)$ but not on f; $G, (H, d)$ may be restricted to certain classes and C is only allowed to depend on c.

A well-known example where the answer to this question is negative is the case where $G = H = \mathbb{Z}$ with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [15, 1], and the answer is yes only for those quasihomomorphisms that correspond to integers. In fact, this construction can be extended to construct completions of fields in general [11].

Much literature in this area focusses on *quasimorphisms*, which are quasihomomorphisms into the real numbers $\mathbb{R}$ with the standard metric; we refer to [12] for a brief introduction. In particular, the concept of a quasimorphism features in bounded cohomology, see [13, 4, 6]. In another branch of the research on quasihomomorphisms H is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan's theorem on ε-representations of amenable groups [9]. For more

Manuscript received 27th March 2023, revised 11th September 2023 and 21st November 2023, accepted 26th November 2023.

Keywords. Quasihomomorphisms, Hamming distance, Linear approximation.

Acknowledgements. JD,TS,NT,EV were partially or fully supported by Vici grant 639.033.514 from the Netherlands Organisation for Scientific Research (NWO) and Swiss National Science Foundation (SNSF) project grant 200021_191981. RE was supported by NWO Veni grant 016.Veni.192.113. EV is a member of GNSAGA of INdAM (Italy). We thank the reviewers for many useful suggestions.

ISSN: 2589-5486

recent results on quasihomomorphisms into nonabelian groups we refer to [7, 8, 5, 2] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [10].

Question 1.1. *Let $c \in \mathbb{N}$. Does there exist a constant $C = C(c)$ such that the following holds: For all $n \in \mathbb{N}$ and all functions $f : \mathbb{Z} \longrightarrow \mathbb{C}^{n\times n}$ such that*

$$\text{for all } x, y \in \mathbb{Z},\ \operatorname{rk}(f(x+y) - f(x) - f(y)) \leqslant c,$$

there exists a matrix g such that

$$\text{for all } x \in \mathbb{Z},\ \operatorname{rk}(f(x) - xg) \leqslant C(c)?$$

Here, G equals $\mathbb{Z}$ and H equals $\mathbb{C}^{n\times n}$, both with addition, and the metric on H is defined by $d(A, B) := \operatorname{rk}(A - B)$. In [10, p1], the function $R(\mathbb{Z}, c, \mathbb{C})$ denotes the minimal possible choice of $C(c)$. Our main result is an affirmative answer to Question 1.1 in the special case where all matrices $f(x)$ are assumed to be *diagonal*.

Definition 1.2. *Let $(Q, +)$ be an abelian group. For an element $v \in Q^n$, the* Hamming weight *$w_H(v)$ is the number of nonzero entries of v. For a pair of elements $u, v \in Q^n$, their* Hamming distance *is $w_H(v-u)$. This metric is clearly left-invariant, and indeed even bi-invariant.*

Definition 1.3. *Let A be another abelian group. A function $f : A \to Q^n$ is called a c-quasihomomorphism if*

$$\text{for all } x, y \in A,\ w_H(f(x+y) - f(x) - f(y)) \leqslant c. \tag{1}$$

Remark 1.4. The map $\operatorname{diag} : \mathbb{C}^n \to \mathbb{C}^{n\times n}$ is an isometric embedding from $\mathbb{C}^n$ with the Hamming metric to $\mathbb{C}^{n\times n}$ with the rank metric. This connects Definition 1.3 to Question 1.1.

Definition 1.5. *Let $C \in \mathbb{N}$ and let $f : A \to Q^n$ be a c-quasihomomorphism. A group homomorphism $h : A \to Q^n$ is a C-approximation of f if the Hamming distance between f and h satisfies*

$$\text{for all } x \in A,\ w_H(f(x) - h(x)) \leqslant C.$$

We are ready to state our main result.

Theorem 1.6 (Main Theorem). *Let $c \in \mathbb{N}$. Then there exists a constant $C = C(c) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and c-quasihomorphisms $f : \mathbb{Z} \to \mathbb{Q}^n$, we have:*

$$\text{for all } x \in \mathbb{Z},\ w_H(f(x) - xf(1)) \leqslant C.$$

Moreover, we can take $C = 28c$.

Remark 1.7. The coefficient 28 is probably not optimal. However, we certainly have that $C(c) \geqslant c$. Indeed, any map $f : \mathbb{Z} \to \mathbb{Q}^n$ for which the only nonzero entries of $f(x)$ are among the first c, is automatically a c-quasihomomorphism.

Corollary 1.8. *Theorem 1.6 also holds with $\mathbb{Q}$ replaced by any torsion-free abelian group Q, with the same value of $C = C(c)$.*

Proof. Suppose, for a contradiction, that we have a c-quasihomomorphism $f : \mathbb{Z} \to Q^n$ but $w_H(f(y) - yf(1)) > C$ for some $y \in \mathbb{Z}$. Since Q is torsion-free, the natural map ι from Q into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} Q$ is injective. Consequently, $g := \iota^n \circ f$ is a c-quasihomomorphism $\mathbb{Z} \to V^n$ with $w_H(g(y) - yg(1)) > C$. Now choose any $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ that is nonzero on the nonzero entries of $g(y) - yg(1)$. Then $h := \xi^n \circ g$ is a c-quasihomomorphism $\mathbb{Z} \to \mathbb{Q}^n$ with $w_H(h(y) - yh(1)) > C$, a contradiction to Theorem 1.6. ☐

Remark 1.9. As a referee kindly pointed out to us, our result fits in the broader context of $\mathcal{G}$-*stability* for a family $\mathcal{G}$ of groups endowed with a bi-invariant metric; this was first introduced in [9] and further studied in [3] under the name of *Ulam stability*. Let $\mathcal{G}$ be the family of groups $\{\mathrm{GL}_n(\mathbb{C})\}_{n\geqslant 1}$ with the normalized rank metric, i.e. $d(A,B) = \frac{1}{n}\,\mathrm{rk}(A-B)$. Let $\mathcal{G}_d$ be the subfamily of $\mathcal{G}$ consisting of diagonal matrices. Theorem 1.6 shows that the abelian group $\mathbb{Z}$ is uniformly $\mathcal{G}_d$-stable with a linear estimate.

Theorem 1.6 shows that for a c-quasihomomorphism $f:\mathbb{Z}\to\mathbb{Q}^n$, the group homomorphism $\tilde{f}:\mathbb{Z}\to\mathbb{Q}^n$ defined by $\tilde{f}(x) = xf(1)$ gives a C-approximation for some constant $C\in\mathbb{N}$ independent on n. However, $\tilde{f}$ need not be the homomorphism closest to f, as the next example shows.

Example 1.10. Let $c=1$ and $n\geqslant 3$. Define $f:\mathbb{Z}\to\mathbb{Q}^n$ to be

$$f(x) = \left(\left\lfloor \frac{2x}{5}\right\rceil, \left\lfloor \frac{x}{5}\right\rceil, \alpha_x, 0, \dots, 0\right), \tag{2}$$

where $\alpha_x\in\mathbb{Q}$ is arbitrary if $5\mid x$, and $\alpha_x = 0$ otherwise. Here $\lfloor\rceil$ denotes rounding to the nearest integer. To check that f is a 1-quasihomomorphism (1) we work mod 5. For simplicity, restrict to the case $n=3$. Then, for $k\in\mathbb{Z}$,

$$\begin{aligned} f(5k) &= (2k,k,\alpha_{5k}), & f(5k+1) &= (2k,k,0),\\ f(5k+2) &= (2k+1,k,0), & f(5k+3) &= (2k+1,k+1,0),\\ f(5k+4) &= (2k+2,k+1,0). \end{aligned}$$

Let $x = 5k+\ell_1$ and $y = 5h+\ell_2$ with $0\leqslant \ell_1\leqslant \ell_2 < 5$. Then we can verify that

$$w_H(f(5(k+h)+(\ell_1+\ell_2)) - f(5k+\ell_1) - f(5h+\ell_2)) \leqslant c = 1$$

in all cases. Roughly speaking, the check boils down to verifying that there are no cases where both $\left\lfloor\frac{x+y}{5}\right\rceil \neq \left\lfloor\frac{x}{5}\right\rceil + \left\lfloor\frac{y}{5}\right\rceil$ and $\left\lfloor\frac{2(x+y)}{5}\right\rceil \neq \left\lfloor\frac{2x}{5}\right\rceil + \left\lfloor\frac{2y}{5}\right\rceil$, and moreover that if $5\mid x+y$, then $f(x+y)-f(x)-f(y) = (0,0,\alpha_x)$ (because in this case, $\frac{x}{5}$ is rounded down if and only if $\frac{y}{5}$ is rounded up). Note that $w_H(f(x)-xf(1))\leqslant 3$ where equality is sometimes achieved (provided there is at least one $x\neq 0$ for which we chose $\alpha_x\neq 0$). However, there also exist 2-approximations of f. For instance, letting $v = (\frac{2}{5},\frac{1}{5},0,\dots,0)\in\mathbb{Q}^n$, one verifies that

$$w_H(f(x)-xv)\leqslant 2 \text{ for all } x\in\mathbb{Z}.$$

In [14], the authors show that for every 1-quasihomomorphism $f:\mathbb{Z}\to\mathbb{Q}^n$, and even for every 1-quasihomomorphism from $\mathbb{Z}$ into the space of *symmetric* $n\times n$-matrices with the rank metric, there is a 2-approximation.

(This result is consistent with the second paragraph of [10], where a proof of the corresponding statement for general matrices is sketched. However, the above example shows that that proof is incomplete: viewing f as a map to the diagonal matrices, and assuming $\alpha_0 = 0$ as is done in that paragraph, we obtain a counterexample to the statement in [10] that there exists either a subspace of codimension 1 living in the kernel of all matrices $f(n+m)-f(n)-f(m)$ or else a subspace of dimension 1 containing all their images.)

On the other hand, the following shows that the best possible approximation of a given quasihomomorphism f is at most twice as close as the homomorphism $x\mapsto xf(1)$.

Remark 1.11. Suppose that a map $f : \mathbb{Z} \to \mathbb{Q}^n$ has a C'-approximation h. Then $h(x) = xv$ for some $v \in \mathbb{Q}^n$, and

$$w_H(f(x) - xv) \leqslant C' \text{ for all } x \in \mathbb{N}.$$

Substituting $x = 1$ yields $w_H(f(1) - v) \leqslant C'$. Thus

$$w_H(f(x) - xf(1)) \leqslant w_H(f(x) - xv) + w_H(xv - xf(1)) \leqslant 2C'.$$

Remark 1.12. A result similar to Theorem 1.6 is easily proven in positive characteristic if we allow the constant C to depend on the characteristic. Let K be a field of characteristic $p > 0$, and let $f : \mathbb{Z} \to K^n$ be a c-quasihomomorphism. Then there exists a constant $C = C(p, c)$ such that $w_H(f(x) - xf(1)) \leqslant C$, for all $x \in \mathbb{Z}$.

To see this, we observe that for all $u, v \in \mathbb{Z}$ with $u \geqslant 1$, we have

$$w_H(f(uv) - uf(v)) \leqslant (u - 1)c.$$

This follows by repeatedly applying the inequality $w_H(f(uv) - f((u-1)v) - f(v)) \leqslant c$ if $u > 1$; the case $u = 1$ is trivial.

For $x = kp + r$ with $k \in \mathbb{Z}$ and $0 \leqslant r \leqslant p - 1$, we have

$$w_H(f(x) - xf(1)) = w_H(f(kp + r) - rf(1));$$

here we have used that $pf(1) = 0$. We rewrite the latter as

$$w_H(f(kp + r) - f(kp) - f(r) + f(kp) + f(r) - rf(1)).$$

We have $w_H(f(kp+r) - f(kp) - f(r)) \leqslant c$; $w_H(f(kp)) \leqslant (p-1)c$ using our observation with $u = p, v = k$; and also $w_H(f(r) - rf(1)) \leqslant (p - 2)c$ (in the case $r > 0$). In total, this gives $w_H(f(x) - xf(1)) \leqslant 2(p - 1)c$, so we can take $C = 2(p - 1)c$.

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3, we apply this auxiliary result to the component functions of a c-quasihomomorphism $\mathbb{Z} \to \mathbb{Q}^n$ to prove the Main Theorem.

2. Almost homomorphisms are almost zero

Let A be a finite abelian group and let H be a torsion-free abelian group. The only homomorphism $A \to H$ is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

Proposition 2.1. *Let a be a positive integer, A an abelian group of order a, H a torsion-free abelian group, $q \in [0, 1]$, and $f : A \to H$ a map. Suppose that the* zero set

$$Z(f) := \{b \in A \mid f(b) = 0\}$$

has cardinality at most qa. Then the problem set

$$P(f) := \{(b, c) \in A \times A \mid f(b + c) \neq f(b) + f(c)\}$$

has cardinality at least $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$.

The contraposition of this statement says that if $P(f)$ is a small fraction of a^2, so that f can be thought of as an (additive) "almost homomorphism" $A \to H$, then q must be close to 1 so that f is essentially zero.

Proof. Since H is torsion-free, it embeds into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} H$. By basic linear algebra, there exists a $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ such that $\xi(f(b)) \neq 0$ for all $b \notin Z(f)$, so that $Z(\xi \circ f) = Z(f)$. Since $P(\xi \circ f) \subseteq P(f)$, it suffices to prove the proposition for $\xi \circ f$ instead of f. In other words, we may assume from the beginning that $H = \mathbb{Q}$.

Set

$$B := \{b \in A \mid f(b) > 0\}.$$

Let $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$ be the distinct values in $f(B)$, and for each $i = 1, \ldots, k$ set

$$B_i := \{b \in B \mid f(b) = \lambda_i\} \text{ and } n_i := |B_i|;$$

as well as $n := n_1 + \cdots + n_k = |B|$.

Now for each $c \in B_1$ and each $b \in B$ we have

$$f(b) + f(c) = f(b) + \lambda_1 > \lambda_1$$

so that the left-hand side is not in $f(B)$ and in particular not equal to $f(b + c)$. We have thus found $n_1(n_1 + \cdots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_1$.

Next, suppose (b, c) is a pair with $c \in B_2$, $b \in B$, and $(b, c) \notin P(f)$. Then

$$f(b + c) = f(b) + f(c) > f(c) = \lambda_2$$

and hence $b+c \in B_1$. But given c, there are at most n_1 values of b with $b+c \in B_1$. (Note that here we have used that A is a group.) Hence we have at least $n_2(n_2 + \cdots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_2$.

Similarly, we find at least $n_i(n_i + \cdots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_i$. In total, we have therefore found at least

$$\sum_{i=1}^{k} n_i(n_i + \cdots + n_k) \geqslant \frac{n(n+1)}{2} \tag{3}$$

pairs in $P(f)$; see Figure 1.

Let $B' := \{b' \in A \mid f(b') < 0\}$ and $n' := |B'|$. Repeating the same argument above with B' and n', we find at least $n'(n'+1)/2$ further pairs in $P(f)$, disjoint from those found above. Since $|Z(f)| \leqslant qa$, we have $n + n' \geqslant a(1 - q)$. Therefore

$$|P(f)| \geqslant \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2 + n'^2}{2} + \frac{n + n'}{2} \geqslant \left(\frac{n + n'}{2}\right)^2 + \frac{n + n'}{2},$$

where the second inequality is the Cauchy-Schwarz inequality

$$\left(n^2 + n'^2\right)\left(\frac{1}{2^2} + \frac{1}{2^2}\right) \geqslant \left(\frac{n}{2} + \frac{n'}{2}\right)^2.$$

Since $n + n' \geqslant a(1 - q)$, we conclude that

$$|P(f)| \geqslant \left(\frac{a - qa}{2}\right)\left(\frac{a - qa}{2} + 1\right).$$ □

Remark 2.2. The lower bound in Proposition 2.1 is sharp. Let $a = 2k + 1 \in \mathbb{Z}$, consider $A := \mathbb{Z}/a\mathbb{Z}$ and define $f : A \to \mathbb{Z}$ as $f(x) :=$the representative of $x + a\mathbb{Z}$ in $\{-k, \ldots, 0, \ldots, k\}$. We take $q = \frac{Z(f)}{a} = \frac{1}{2k+1}$. Then $f(x+y) = f(x)+f(y)$ if and only if the right-hand side is still inside the interval $\{-k, \ldots, k\}$, and a straightforward count shows that this is the case for $3k^2 + 3k + 1$ pairs $(x, y) \in A^2$. Hence $P(f)$ has size $k(k + 1)$, which equals $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$.

A similar construction for $a = 2k$ yields a problem set of size $\frac{a^2}{4} = k^2$, which equals the ceiling of the lower bound $\frac{a^2}{4} - \frac{1}{4}$.

Below, we will use the following strengthening of Proposition 2.1:

Proposition 2.3. *Let a, A, H, q and f be as in Proposition 2.1. Furthermore, let $p \in [0, \frac{1-q}{2})$ and let $S \subseteq A$ be a subset of cardinality at most pa. Then the set*

$$P_S(f) := \{(b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \text{ and } b + c \notin S\}.$$

has cardinality at least $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$.

Proof. Keep the notation from the proof of Proposition 2.1. Recall $n = |B|$ and $n' = |B'|$. Note that for a fixed b, there can be at most pa choices of c with $b + c \in S$. We then find at least $n_i(n_i + \cdots + n_k - pa)$ pairs $(b,c) \in P_S(f)$ with $b \in B_i$. Letting $k' \leqslant k$ be the largest index for which the second factor $(n_{k'} + \cdots + n_k - pa)$ is nonnegative, as in the proof of Proposition 2.1, we find that B contributes at least

$$\begin{aligned} \sum_{i=1}^{k'} n_i(n_i + \cdots + n_k - pa) &= \sum_{i=1}^{k'} n_i(n - n_1 - \cdots - n_{i-1} - pa) \\ &\geqslant (n - pa)(n - pa + 1)/2 \end{aligned} \tag{4}$$

to $P_S(f)$; see Figure 1. Similarly, B' contributes at least $(n' - pa)(n' - pa + 1)/2$, and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with n, n' replaced by $n - pa, n' - pa$. □

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write $[a]$ for the set $\{1, 2, \ldots, a\}$.

Corollary 2.4. *Let $p, q \in [0,1]$ such that $p < \frac{1-q}{2}$. Let $f : [2a] \to \mathbb{Q}$ such that:*

(1) *$|Z_a(f)| \leqslant qa$, where $Z_a(f) := \{x \in [a] \mid f(x) = 0\}$ is the zero set of $f\big|_{[a]}$.*
(2) *$|NP(f)| \leqslant pa$, where $NP(f) := \{x \in [a] \mid f(x+a) \neq f(x)\}$ is the nonperiodicity set.*

Then

$$|P(f)| \geqslant \frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a,$$

where

$$P(f) = \{(x,y) \in [a] \times [a] \mid f(x+y) \neq f(x) + f(y)\}.$$

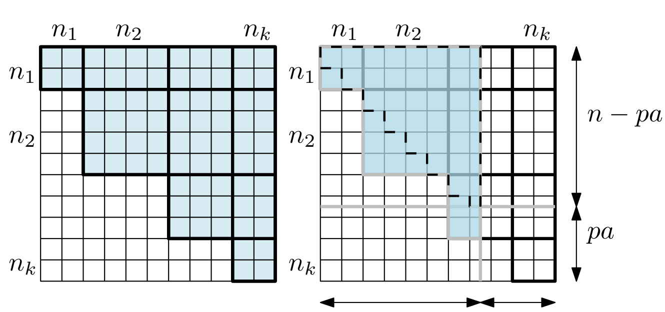

Figure 1. On the left, a graphical proof of the inequality (3): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (4): the two expressions on top represent the area of the shaded region, while the bottom expression represents the area enclosed by the dashed line.

Proof. Let $\tilde{f}$ be the restriction of f to the interval $[a]$, and identify $\mathbb{Z}/a\mathbb{Z}$ with $[a]$ with the group operation $\star$ defined by $x \star y := x + y \pmod{a}$.

Let $S = NP(f)$, and apply Proposition 2.3 to $\tilde{f}$. We find that

$$P_S(\tilde{f}) = \{(b,c) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \mid \tilde{f}(b \star c) \neq \tilde{f}(b) + \tilde{f}(c) \text{ and } b \star c \notin S\}$$

has cardinality at least $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$. Since $b \star c \notin S$ implies that $\tilde{f}(b \star c) = f(b+c)$, this set is contained in the problem set $P(f)$. □

3. Proof of the main theorem

The main goal of this section is to prove Theorem 1.6. We start with some definitions.

Definition 3.1. *Let $1 < a$, and $f : [2a] \to \mathbb{Q}$. We define the following* problem sets *of f:*

$$P(f) := \{(x,y) \in [a] \times [a] \mid f(x+y) \neq f(x) + f(y)\},$$

and

$$P_1(f) := \{x \in [a] \mid f(x+1) \neq f(x) + f(1)\},$$

and

$$P_a(f) := \{x \in [a] \mid f(x+a) \neq f(x) + f(a)\}.$$

Furthermore, we recall that $Z_a(f)$ denotes the zero set of $f\big|_{[a]}$:

$$Z_a(f) := \{x \in [a] \mid f(x) = 0\}.$$

The following proposition says that $P_1(f), P_a(f), P(f)$ cannot be simultaneously small.

Proposition 3.2. *Let $p, q \in (0,1)$ such that $1 - q - 2p > 0$, $a \in \mathbb{N}$ with $1 < a$, and let $f : [2a] \to \mathbb{Q}$ such that $f(a) \neq af(1)$. Then at least one of the following holds:*

(i) $|P_1(f)| > qa$,
(ii) $|P_a(f)| > pa$,
(iii) $|P(f)| \geqslant F(p,q)a^2$,

where

$$F(p,q) := \frac{(1-q-2p)^2}{4}.$$

Proof. Without loss of generality we can assume $f(a) = 0$ and hence $f(1) \neq 0$. Indeed, suppose we have shown the statement for every $\tilde{f}$ with $\tilde{f}(a) = 0$. Then for any $f : [2a] \to \mathbb{Q}$ with $f(a) \neq af(1)$, we take $\tilde{f} : [2a] \to \mathbb{Q}$ to be $\tilde{f}(x) = af(x) - xf(a)$. Now we observe that $\tilde{f}(a) = 0 \neq a\tilde{f}(1)$, and that $P(f) = P(\tilde{f})$, $P_1(f) = P_1(\tilde{f})$, $P_a(f) = P_a(\tilde{f})$.

To prove the proposition we will assume that ((i)) and ((ii)) are false, and prove that then ((iii)) must hold. Write $Z_a(f) = \{x_1, \ldots, x_m\}$, where $x_1 < \cdots < x_m$. Note that for $1 \leqslant i < m$, one of the elements $x_i, x_i + 1, \ldots, x_{i+1} - 1$ needs to be in $P_1(f)$ since $f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i)f(1)$. Likewise, at least one of the elements $1, 2, \ldots, x_1 - 1$ needs to be in $P_1(f)$. Thus we have

$$|Z_a(f)| \leqslant |P_1(f)| \leqslant qa,$$

and by assumption we have $|NP(f)| = |P_a(f)| \leqslant pa$. Now we can apply Corollary 2.4 to conclude. □

We now prove Theorem 1.6.

Proof of the Main Theorem. Consider a c-quasihomomorphism $f = (f_1, \dots, f_n) : \mathbb{Z} \to \mathbb{Q}^n$. Our goal is to show that for every $a \in \mathbb{Z}$ we have $w_H(f(a) - af(1)) \leqslant C$ for some constant C depending only on c. We start with the case $a > 0$.

Write $I_a := \{i \in [a] \mid f_i(a) \neq af_i(1)\}$, and note that $|I_a| = w_H(f(a) - af(1))$. We will show that $|I_a| \leqslant C'$ for some constant C' depending on c only. To this end, fix small parameters $p, q \in (0,1)$ (to be optimized over later) and write $f_i^a := f_i\big|_{[2a]}$ for the restriction of f_i to $[2a]$. By Proposition 3.2, for every $i \in I_a$, we have

(i) $|P_1(f_i^a)| > qa$, or
(ii) $|P_a(f_i^a)| > pa$, or
(iii) $|P(f_i^a)| \geqslant F(p,q)a^2$.

Let m_0 be the number of coordinates $i \in I_a$ such that (iii) holds. We define m_1 and m_2 analogously, for (i) and (ii) respectively.

By counting the number of triples $(i, x, y) \in [n] \times [a] \times [a]$ such that $f_i(x + y) - f_i(x) - f_i(y) \neq 0$ in two ways, we see that

$$\sum_{x=1}^{a}\sum_{y=1}^{a} w_H\left(f(x+y) - f(x) - f(y)\right) = \sum_{i=1}^{n} |P(f_i^a)| \geqslant \sum_{i \in I_a} |P(f_i^a)|.$$

Because f is a c-quasihomomorphism, the very left-hand side is at most a^2c. On the other hand, the very right-hand side is at least $m_0F(p,q)a^2$, so

$$a^2c \geqslant \sum_{x=1}^{a}\sum_{y=1}^{a} w_H\left(f(x+y) - f(x) - f(y)\right) \geqslant \sum_{i \in I_a} |P(f_i^a)| \geqslant m_0F(p,q)a^2.$$

So we obtain $m_0 \leqslant \frac{c}{F(p,q)}$. Similarly we find

$$ac \geqslant \sum_{x=1}^{a} w_H\left(f(x+1) - f(x) - f(1)\right) = \sum_{i=1}^{n} |P_1(f_i^a)| \geqslant \sum_{i \in I_a} |P_1(f_i^a)| > m_1qa,$$

so that $m_1 < \frac{c}{q}$. Finally,

$$ac \geqslant \sum_{x=1}^{a} w_H\left(f(x+a) - f(x) - f(a)\right) = \sum_{i=1}^{n} |P_a(f_i^a)| \geqslant \sum_{i \in I_a} |P_a(f_i^a)| > m_2pa.$$

So $m_2 < \frac{c}{p}$. But now $|I_a| \leqslant m_0 + m_1 + m_2 < c(\frac{1}{F(p,q)} + \frac{1}{q} + \frac{1}{p}) =: C'$.

The case $a = 0$ is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \leqslant c.$$

Finally, let us consider the case $a < 0$. Then

$$\begin{aligned} w_H(f(a) - af(1)) \leqslant & w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) \\ & + w_H(f(-a) - (-a)f(1)) \leqslant 2c + C' =: C. \end{aligned}$$

This completes the proof of the qualitative part of the Main Theorem. To obtain the explicit bound $28c$, we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p,q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1-q-2p)^2}.$$

This function is strictly convex for $(p,q) \in \mathbb{R}^2_{>0}$, so it has at most one minimum in the positive orthant. We find this by setting the partial derivatives to zero and solving for p, q. The minimum is ≈ 27.6817 and attained at $(p,q) \approx (0.1167, 0.16500)$. ☐

References

[1] N. A'Campo, *A natural construction for the real numbers*, Elem. Math. **76** (2021), no. 3, 89–105, https://doi.org/10.4171/em/455.

[2] Michael Brandenbursky and Misha Verbitsky, *Non-commutative Barge-Ghys quasimorphisms*, 2023, https://arxiv.org/abs/2212.12958.

[3] M. Burger, N. Ozawa, and A. Thom, *On Ulam stability*, Israel J. Math. **193** (2013), no. 1, 109–129, https://doi.org/10.1007/s11856-012-0050-z.

[4] Danny Calegari, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009, https://doi.org/10.1142/e018.

[5] Marcus De Chiffre, Narutaka Ozawa, and Andreas Thom, *Operator algebraic approach to inverse and stability theorems for amenable groups*, Mathematika **65** (2019), no. 1, 98–118, https://doi.org/10.1112/s0025579318000335.

[6] Roberto Frigerio, *Bounded cohomology of discrete groups*, Mathematical Surveys and Monographs, vol. 227, American Mathematical Society, Providence, RI, 2017, https://doi.org/10.1090/surv/227.

[7] K. Fujiwara and M. Kapovich, *On quasihomomorphisms with noncommutative targets*, Geom. Funct. Anal. **26** (2016), no. 2, 478–519, https://doi.org/10.1007/s00039-016-0364-9.

[8] William Timothy Gowers and Omid Hatami, *Inverse and stability theorems for approximate representations of finite groups*, Sbornik: Mathematics **208** (2017), no. 12, 1784.

[9] D. Kazhdan, *On ε-representations*, Israel J. Math. **43** (1982), no. 4, 315–323, https://doi.org/10.1007/BF02761236.

[10] D. Kazhdan and T. Ziegler, *Approximate cohomology*, Selecta Math. (N.S.) **24** (2018), no. 1, 499–509, https://doi.org/10.1007/s00029-017-0335-5.

[11] Steffen Kionke, *Constructing the completion of a field using Quasimorphisms*, p-Adic Numbers Ultrametric Anal. Appl. **11** (2019), no. 4, 335–337, https://doi.org/10.1134/S2070046619040083.

[12] D. Kotschick, *What is... a quasi-morphism?*, Notices Amer. Math. Soc. **51** (2004), no. 2, 208–209.

[13] Nicolas Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, vol. 1758, Springer-Verlag, Berlin, 2001, https://doi.org/10.1007/b80626.

[14] Tim Seynnaeve, Nafie Tairi, and Alejandro Vargas, *One-quasihomomorphisms from the integers into symmetric matrices*, 2023, https://arxiv.org/abs/2302.01611.

[15] Ross Street, *An efficient construction of real numbers*, Austral. Math. Soc. Gaz. **12** (1985), no. 3, 57–58.

[16] A. M. Turing, *Finite approximations to Lie groups*, Ann. of Math. (2) **39** (1938), no. 1, 105–111, https://doi.org/10.2307/1968716.

[17] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.

Jan Draisma, Mathematical Institute, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland; and Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600MB, Eindhoven, the Netherlands
E-mail : jan.draisma@unibe.ch

Rob H. Eggermont, Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600MB, Eindhoven, the Netherlands
E-mail : r.h.eggermont@tue.nl

Tim Seynnaeve, Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Leuven, Belgium
E-mail : tim.seynnaeve@kuleuven.be

Nafie Tairi, Mathematical Institute, University of Bern, Alpeneggstrasse 22, 3012 Bern, Switzerland
E-mail : nafie.tairi@unibe.ch

Emanuele Ventura, Politecnico di Torino, Dipartimento di Scienze Matematiche "G.L. Lagrange", Corso Duca degli Abruzzi 24 10129 Torino, Italy
E-mail : emanuele.ventura@polito.it