

Quasihomomorphisms from the integers into Hamming metrics

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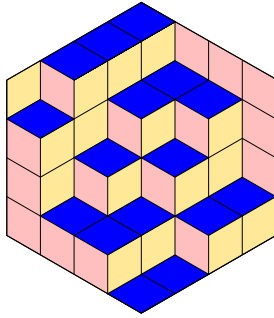
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Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi & Emanuele Ventura

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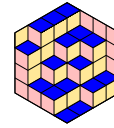


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Quasihomomorphisms from the integers into Hamming metrics

Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi
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ABSTRACT A function $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ is a c -quasihomomorphism if the Hamming distance between $f(x+y)$ and $f(x)+f(y)$ is at most c for all $x, y \in \mathbb{Z}$. We show that any c -quasihomomorphism has distance at most some constant $C(c)$ to an actual group homomorphism; here $C(c)$ depends only on c and not on n or f . This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

1. INTRODUCTION

Let c be a nonnegative real number. A c -quasihomomorphism from a group G to a group H with a left-invariant metric d is a map $f : G \rightarrow H$ such that $d(f(xy), f(x)f(y)) \leq c$ for all x, y in G . A central question in geometric group theory, raised by Ulam in [17, Chapter 6], is whether there exists an actual homomorphism $f' : G \rightarrow H$ such that $d(f(x), f'(x))$ is at most some constant C for all x . (Related questions were studied before Ulam, e.g. by Turing in his work on approximability of groups [16].) Different versions of Ulam's question are of interest: for example, C may be allowed to depend on $c, G, (H, d)$ but not on f ; $G, (H, d)$ may be restricted to certain classes and C is only allowed to depend on c .

A well-known example where the answer to this question is negative is the case where $G = H = \mathbb{Z}$ with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [15, 1], and the answer is yes only for those quasihomomorphisms that correspond to integers. In fact, this construction can be extended to construct completions of fields in general [11].

Much literature in this area focusses on *quasimorphisms*, which are quasihomomorphisms into the real numbers \mathbb{R} with the standard metric; we refer to [12] for a brief introduction. In particular, the concept of a quasimorphism features in bounded cohomology, see [13, 4, 6]. In another branch of the research on quasihomomorphisms H is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan's theorem on ε -representations of amenable groups [9]. For more

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recent results on quasihomomorphisms into nonabelian groups we refer to [7, 8, 5, 2] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [10].

QUESTION 1.1. *Let $c \in \mathbb{N}$. Does there exist a constant $C = C(c)$ such that the following holds: For all $n \in \mathbb{N}$ and all functions $f : \mathbb{Z} \rightarrow \mathbb{C}^{n \times n}$ such that*

$$\text{for all } x, y \in \mathbb{Z}, \text{rk}(f(x + y) - f(x) - f(y)) \leq c,$$

there exists a matrix g such that

$$\text{for all } x \in \mathbb{Z}, \text{rk}(f(x) - xg) \leq C(c)?$$

Here, G equals \mathbb{Z} and H equals $\mathbb{C}^{n \times n}$, both with addition, and the metric on H is defined by $d(A, B) := \text{rk}(A - B)$. In [10, p1], the function $R(\mathbb{Z}, c, \mathbb{C})$ denotes the minimal possible choice of $C(c)$. Our main result is an affirmative answer to Question 1.1 in the special case where all matrices $f(x)$ are assumed to be *diagonal*.

DEFINITION 1.2. *Let $(Q, +)$ be an abelian group. For an element $v \in Q^n$, the Hamming weight $w_H(v)$ is the number of nonzero entries of v . For a pair of elements $u, v \in Q^n$, their Hamming distance is $w_H(v - u)$. This metric is clearly left-invariant, and indeed even bi-invariant.*

DEFINITION 1.3. *Let A be another abelian group. A function $f : A \rightarrow Q^n$ is called a c -quasihomomorphism if*

$$(1) \quad \text{for all } x, y \in A, w_H(f(x + y) - f(x) - f(y)) \leq c.$$

REMARK 1.4. The map $\text{diag} : \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n}$ is an isometric embedding from \mathbb{C}^n with the Hamming metric to $\mathbb{C}^{n \times n}$ with the rank metric. This connects Definition 1.3 to Question 1.1.

DEFINITION 1.5. *Let $C \in \mathbb{N}$ and let $f : A \rightarrow Q^n$ be a c -quasihomomorphism. A group homomorphism $h : A \rightarrow Q^n$ is a C -approximation of f if the Hamming distance between f and h satisfies*

$$\text{for all } x \in A, w_H(f(x) - h(x)) \leq C.$$

We are ready to state our main result.

THEOREM 1.6 (Main Theorem). *Let $c \in \mathbb{N}$. Then there exists a constant $C = C(c) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and c -quasihomomorphisms $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$, we have:*

$$\text{for all } x \in \mathbb{Z}, w_H(f(x) - xf(1)) \leq C.$$

Moreover, we can take $C = 28c$.

REMARK 1.7. The coefficient 28 is probably not optimal. However, we certainly have that $C(c) \geq c$. Indeed, any map $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ for which the only nonzero entries of $f(x)$ are among the first c , is automatically a c -quasihomomorphism.

COROLLARY 1.8. *Theorem 1.6 also holds with \mathbb{Q} replaced by any torsion-free abelian group Q , with the same value of $C = C(c)$.*

Proof. Suppose, for a contradiction, that we have a c -quasihomomorphism $f : \mathbb{Z} \rightarrow Q^n$ but $w_H(f(y) - yf(1)) > C$ for some $y \in \mathbb{Z}$. Since Q is torsion-free, the natural map ι from Q into the \mathbb{Q} -vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} Q$ is injective. Consequently, $g := \iota^n \circ f$ is a c -quasihomomorphism $\mathbb{Z} \rightarrow V^n$ with $w_H(g(y) - yg(1)) > C$. Now choose any \mathbb{Q} -linear function $\xi : V \rightarrow \mathbb{Q}$ that is nonzero on the nonzero entries of $g(y) - yg(1)$. Then $h := \xi^n \circ g$ is a c -quasihomomorphism $\mathbb{Z} \rightarrow \mathbb{Q}^n$ with $w_H(h(y) - yh(1)) > C$, a contradiction to Theorem 1.6. \square

REMARK 1.9. As a referee kindly pointed out to us, our result fits in the broader context of \mathcal{G} -stability for a family \mathcal{G} of groups endowed with a bi-invariant metric; this was first introduced in [9] and further studied in [3] under the name of *Ulam stability*. Let \mathcal{G} be the family of groups $\{\mathrm{GL}_n(\mathbb{C})\}_{n \geq 1}$ with the normalized rank metric, i.e. $d(A, B) = \frac{1}{n} \mathrm{rk}(A - B)$. Let \mathcal{G}_d be the subfamily of \mathcal{G} consisting of diagonal matrices. Theorem 1.6 shows that the abelian group \mathbb{Z} is uniformly \mathcal{G}_d -stable with a linear estimate.

Theorem 1.6 shows that for a c -quasihomomorphism $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$, the group homomorphism $\tilde{f} : \mathbb{Z} \rightarrow \mathbb{Q}^n$ defined by $\tilde{f}(x) = xf(1)$ gives a C -approximation for some constant $C \in \mathbb{N}$ independent on n . However, \tilde{f} need not be the homomorphism closest to f , as the next example shows.

EXAMPLE 1.10. Let $c = 1$ and $n \geq 3$. Define $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ to be

$$(2) \quad f(x) = \left(\left\lfloor \frac{2x}{5} \right\rfloor, \left\lfloor \frac{x}{5} \right\rfloor, \alpha_x, 0, \dots, 0 \right),$$

where $\alpha_x \in \mathbb{Q}$ is arbitrary if $5 \mid x$, and $\alpha_x = 0$ otherwise. Here $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer. To check that f is a 1-quasihomomorphism (1) we work mod 5. For simplicity, restrict to the case $n = 3$. Then, for $k \in \mathbb{Z}$,

$$\begin{aligned} f(5k) &= (2k, k, \alpha_{5k}), & f(5k + 1) &= (2k, k, 0), \\ f(5k + 2) &= (2k + 1, k, 0), & f(5k + 3) &= (2k + 1, k + 1, 0), \\ f(5k + 4) &= (2k + 2, k + 1, 0). \end{aligned}$$

Let $x = 5k + \ell_1$ and $y = 5h + \ell_2$ with $0 \leq \ell_1 \leq \ell_2 < 5$. Then we can verify that

$$w_H(f(5(k + h) + (\ell_1 + \ell_2)) - f(5k + \ell_1) - f(5h + \ell_2)) \leq c = 1$$

in all cases. Roughly speaking, the check boils down to verifying that there are no cases where both $\lfloor \frac{x+y}{5} \rfloor \neq \lfloor \frac{x}{5} \rfloor + \lfloor \frac{y}{5} \rfloor$ and $\lfloor \frac{2(x+y)}{5} \rfloor \neq \lfloor \frac{2x}{5} \rfloor + \lfloor \frac{2y}{5} \rfloor$, and moreover that if $5 \mid x + y$, then $f(x + y) - f(x) - f(y) = (0, 0, \alpha_x)$ (because in this case, $\frac{x}{5}$ is rounded down if and only if $\frac{y}{5}$ is rounded up). Note that $w_H(f(x) - xf(1)) \leq 3$ where equality is sometimes achieved (provided there is at least one $x \neq 0$ for which we chose $\alpha_x \neq 0$). However, there also exist 2-approximations of f . For instance, letting $v = (\frac{2}{5}, \frac{1}{5}, 0, \dots, 0) \in \mathbb{Q}^n$, one verifies that

$$w_H(f(x) - xv) \leq 2 \text{ for all } x \in \mathbb{Z}.$$

In [14], the authors show that for every 1-quasihomomorphism $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$, and even for every 1-quasihomomorphism from \mathbb{Z} into the space of symmetric $n \times n$ -matrices with the rank metric, there is a 2-approximation.

(This result is consistent with the second paragraph of [10], where a proof of the corresponding statement for general matrices is sketched. However, the above example shows that that proof is incomplete: viewing f as a map to the diagonal matrices, and assuming $\alpha_0 = 0$ as is done in that paragraph, we obtain a counterexample to the statement in [10] that there exists either a subspace of codimension 1 living in the kernel of all matrices $f(n + m) - f(n) - f(m)$ or else a subspace of dimension 1 containing all their images.)

On the other hand, the following shows that the best possible approximation of a given quasihomomorphism f is at most twice as close as the homomorphism $x \mapsto xf(1)$.

REMARK 1.11. Suppose that a map $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ has a C' -approximation h . Then $h(x) = xv$ for some $v \in \mathbb{Q}^n$, and

$$w_H(f(x) - xv) \leq C' \text{ for all } x \in \mathbb{N}.$$

Substituting $x = 1$ yields $w_H(f(1) - v) \leq C'$. Thus

$$w_H(f(x) - xf(1)) \leq w_H(f(x) - xv) + w_H(xv - xf(1)) \leq 2C'.$$

REMARK 1.12. A result similar to Theorem 1.6 is easily proven in positive characteristic if we allow the constant C to depend on the characteristic. Let K be a field of characteristic $p > 0$, and let $f : \mathbb{Z} \rightarrow K^n$ be a c -quasihomomorphism. Then there exists a constant $C = C(p, c)$ such that $w_H(f(x) - xf(1)) \leq C$, for all $x \in \mathbb{Z}$.

To see this, we observe that for all $u, v \in \mathbb{Z}$ with $u \geq 1$, we have

$$w_H(f(uv) - uf(v)) \leq (u - 1)c.$$

This follows by repeatedly applying the inequality $w_H(f(uv) - f((u-1)v) - f(v)) \leq c$ if $u > 1$; the case $u = 1$ is trivial.

For $x = kp + r$ with $k \in \mathbb{Z}$ and $0 \leq r \leq p - 1$, we have

$$w_H(f(x) - xf(1)) = w_H(f(kp + r) - rf(1));$$

here we have used that $pf(1) = 0$. We rewrite the latter as

$$w_H(f(kp + r) - f(kp) - f(r) + f(kp) + f(r) - rf(1)).$$

We have $w_H(f(kp+r) - f(kp) - f(r)) \leq c$; $w_H(f(kp)) \leq (p-1)c$ using our observation with $u = p, v = k$; and also $w_H(f(r) - rf(1)) \leq (p-2)c$ (in the case $r > 0$). In total, this gives $w_H(f(x) - xf(1)) \leq 2(p-1)c$, so we can take $C = 2(p-1)c$.

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3, we apply this auxiliary result to the component functions of a c -quasihomomorphism $\mathbb{Z} \rightarrow \mathbb{Q}^n$ to prove the Main Theorem.

2. ALMOST HOMOMORPHISMS ARE ALMOST ZERO

Let A be a finite abelian group and let H be a torsion-free abelian group. The only homomorphism $A \rightarrow H$ is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

PROPOSITION 2.1. *Let a be a positive integer, A an abelian group of order a , H a torsion-free abelian group, $q \in [0, 1]$, and $f : A \rightarrow H$ a map. Suppose that the zero set*

$$Z(f) := \{b \in A \mid f(b) = 0\}$$

has cardinality at most qa . Then the problem set

$$P(f) := \{(b, c) \in A \times A \mid f(b+c) \neq f(b) + f(c)\}$$

has cardinality at least $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$.

The contraposition of this statement says that if $P(f)$ is a small fraction of a^2 , so that f can be thought of as an (additive) ‘‘almost homomorphism’’ $A \rightarrow H$, then q must be close to 1 so that f is essentially zero.

Proof. Since H is torsion-free, it embeds into the \mathbb{Q} -vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} H$. By basic linear algebra, there exists a \mathbb{Q} -linear function $\xi : V \rightarrow \mathbb{Q}$ such that $\xi(f(b)) \neq 0$ for all $b \notin Z(f)$, so that $Z(\xi \circ f) = Z(f)$. Since $P(\xi \circ f) \subseteq P(f)$, it suffices to prove the proposition for $\xi \circ f$ instead of f . In other words, we may assume from the beginning that $H = \mathbb{Q}$.

Set

$$B := \{b \in A \mid f(b) > 0\}.$$

Let $\lambda_1 > \lambda_2 > \dots > \lambda_k > 0$ be the distinct values in $f(B)$, and for each $i = 1, \dots, k$ set

$$B_i := \{b \in B \mid f(b) = \lambda_i\} \text{ and } n_i := |B_i|;$$

as well as $n := n_1 + \dots + n_k = |B|$.

Now for each $c \in B_1$ and each $b \in B$ we have

$$f(b) + f(c) = f(b) + \lambda_1 > \lambda_1$$

so that the left-hand side is not in $f(B)$ and in particular not equal to $f(b + c)$. We have thus found $n_1(n_1 + \dots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_1$.

Next, suppose (b, c) is a pair with $c \in B_2$, $b \in B$, and $(b, c) \notin P(f)$. Then

$$f(b + c) = f(b) + f(c) > f(c) = \lambda_2$$

and hence $b + c \in B_1$. But given c , there are at most n_1 values of b with $b + c \in B_1$. (Note that here we have used that A is a group.) Hence we have at least $n_2(n_2 + \dots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_2$.

Similarly, we find at least $n_i(n_i + \dots + n_k)$ pairs $(b, c) \in P(f)$ with $c \in B_i$. In total, we have therefore found at least

$$(3) \quad \sum_{i=1}^k n_i(n_i + \dots + n_k) \geq \frac{n(n+1)}{2}$$

pairs in $P(f)$; see Figure 1.

Let $B' := \{b' \in A \mid f(b') < 0\}$ and $n' := |B'|$. Repeating the same argument above with B' and n' , we find at least $n'(n'+1)/2$ further pairs in $P(f)$, disjoint from those found above. Since $|Z(f)| \leq qa$, we have $n + n' \geq a(1 - q)$. Therefore

$$|P(f)| \geq \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2 + n'^2}{2} + \frac{n + n'}{2} \geq \left(\frac{n + n'}{2}\right)^2 + \frac{n + n'}{2},$$

where the second inequality is the Cauchy-Schwarz inequality

$$(n^2 + n'^2) \left(\frac{1}{2^2} + \frac{1}{2^2}\right) \geq \left(\frac{n}{2} + \frac{n'}{2}\right)^2.$$

Since $n + n' \geq a(1 - q)$, we conclude that

$$|P(f)| \geq \left(\frac{a - qa}{2}\right) \left(\frac{a - qa}{2} + 1\right). \quad \square$$

REMARK 2.2. The lower bound in Proposition 2.1 is sharp. Let $a = 2k + 1 \in \mathbb{Z}$, consider $A := \mathbb{Z}/a\mathbb{Z}$ and define $f : A \rightarrow \mathbb{Z}$ as $f(x) :=$ the representative of $x + a\mathbb{Z}$ in $\{-k, \dots, 0, \dots, k\}$. We take $q = \frac{Z(f)}{a} = \frac{1}{2k+1}$. Then $f(x+y) = f(x) + f(y)$ if and only if the right-hand side is still inside the interval $\{-k, \dots, k\}$, and a straightforward count shows that this is the case for $3k^2 + 3k + 1$ pairs $(x, y) \in A^2$. Hence $P(f)$ has size $k(k+1)$, which equals $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$.

A similar construction for $a = 2k$ yields a problem set of size $\frac{a^2}{4} = k^2$, which equals the ceiling of the lower bound $\frac{a^2}{4} - \frac{1}{4}$.

Below, we will use the following strengthening of Proposition 2.1:

PROPOSITION 2.3. *Let a, A, H, q and f be as in Proposition 2.1. Furthermore, let $p \in [0, \frac{1-q}{2})$ and let $S \subseteq A$ be a subset of cardinality at most pa . Then the set*

$$P_S(f) := \{(b, c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \text{ and } b+c \notin S\}.$$

has cardinality at least $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$.

Proof. Keep the notation from the proof of Proposition 2.1. Recall $n = |B|$ and $n' = |B'|$. Note that for a fixed b , there can be at most pa choices of c with $b+c \in S$. We then find at least $n_i(n_i + \dots + n_k - pa)$ pairs $(b, c) \in P_S(f)$ with $b \in B_i$. Letting $k' \leq k$ be the largest index for which the second factor $(n_{k'} + \dots + n_k - pa)$ is nonnegative, as in the proof of Proposition 2.1, we find that B contributes at least

$$\begin{aligned} \sum_{i=1}^{k'} n_i(n_i + \dots + n_k - pa) &= \sum_{i=1}^{k'} n_i(n - n_1 - \dots - n_{i-1} - pa) \\ (4) \qquad \qquad \qquad &\geq (n - pa)(n - pa + 1)/2 \end{aligned}$$

to $P_S(f)$; see Figure 1. Similarly, B' contributes at least $(n' - pa)(n' - pa + 1)/2$, and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with n, n' replaced by $n - pa, n' - pa$. \square

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write $[a]$ for the set $\{1, 2, \dots, a\}$.

COROLLARY 2.4. *Let $p, q \in [0, 1]$ such that $p < \frac{1-q}{2}$. Let $f : [2a] \rightarrow \mathbb{Q}$ such that:*

- (1) $|Z_a(f)| \leq qa$, where $Z_a(f) := \{x \in [a] \mid f(x) = 0\}$ is the zero set of $f|_{[a]}$.
- (2) $|NP(f)| \leq pa$, where $NP(f) := \{x \in [a] \mid f(x+a) \neq f(x)\}$ is the nonperiodicity set.

Then

$$|P(f)| \geq \frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a,$$

where

$$P(f) = \{(x, y) \in [a] \times [a] \mid f(x+y) \neq f(x) + f(y)\}.$$

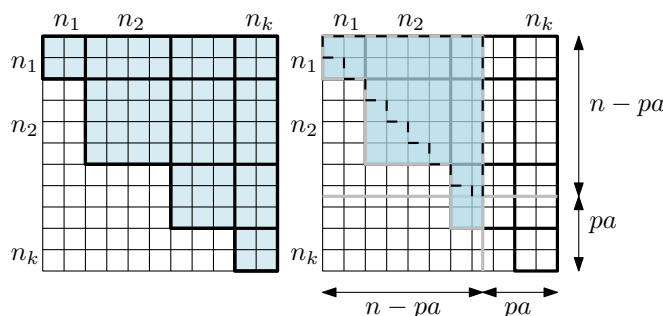


FIGURE 1. On the left, a graphical proof of the inequality (3): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (4): the two expressions on top represent the area of the shaded region, while the bottom expression represents the area enclosed by the dashed line.

Proof. Let \tilde{f} be the restriction of f to the interval $[a]$, and identify $\mathbb{Z}/a\mathbb{Z}$ with $[a]$ with the group operation \star defined by $x \star y := x + y \pmod{a}$.

Let $S = NP(f)$, and apply Proposition 2.3 to \tilde{f} . We find that

$$P_S(\tilde{f}) = \{(b, c) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \mid \tilde{f}(b \star c) \neq \tilde{f}(b) + \tilde{f}(c) \text{ and } b \star c \notin S\}$$

has cardinality at least $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$. Since $b \star c \notin S$ implies that $\tilde{f}(b \star c) = f(b + c)$, this set is contained in the problem set $P(f)$. \square

3. PROOF OF THE MAIN THEOREM

The main goal of this section is to prove Theorem 1.6. We start with some definitions.

DEFINITION 3.1. Let $1 < a$, and $f : [2a] \rightarrow \mathbb{Q}$. We define the following problem sets of f :

$$P(f) := \{(x, y) \in [a] \times [a] \mid f(x + y) \neq f(x) + f(y)\},$$

and

$$P_1(f) := \{x \in [a] \mid f(x + 1) \neq f(x) + f(1)\},$$

and

$$P_a(f) := \{x \in [a] \mid f(x + a) \neq f(x) + f(a)\}.$$

Furthermore, we recall that $Z_a(f)$ denotes the zero set of $f|_{[a]}$:

$$Z_a(f) := \{x \in [a] \mid f(x) = 0\}.$$

The following proposition says that $P_1(f), P_a(f), P(f)$ cannot be simultaneously small.

PROPOSITION 3.2. Let $p, q \in (0, 1)$ such that $1 - q - 2p > 0$, $a \in \mathbb{N}$ with $1 < a$, and let $f : [2a] \rightarrow \mathbb{Q}$ such that $f(a) \neq af(1)$. Then at least one of the following holds:

- (i) $|P_1(f)| > qa$,
- (ii) $|P_a(f)| > pa$,
- (iii) $|P(f)| \geq F(p, q)a^2$,

where

$$F(p, q) := \frac{(1 - q - 2p)^2}{4}.$$

Proof. Without loss of generality we can assume $f(a) = 0$ and hence $f(1) \neq 0$. Indeed, suppose we have shown the statement for every \tilde{f} with $\tilde{f}(a) = 0$. Then for any $f : [2a] \rightarrow \mathbb{Q}$ with $f(a) \neq af(1)$, we take $\tilde{f} : [2a] \rightarrow \mathbb{Q}$ to be $\tilde{f}(x) = af(x) - xf(a)$. Now we observe that $\tilde{f}(a) = 0 \neq a\tilde{f}(1)$, and that $P(f) = P(\tilde{f})$, $P_1(f) = P_1(\tilde{f})$, $P_a(f) = P_a(\tilde{f})$.

To prove the proposition we will assume that ((i)) and ((ii)) are false, and prove that then ((iii)) must hold. Write $Z_a(f) = \{x_1, \dots, x_m\}$, where $x_1 < \dots < x_m$. Note that for $1 \leq i < m$, one of the elements $x_i, x_i + 1, \dots, x_{i+1} - 1$ needs to be in $P_1(f)$ since $f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i)f(1)$. Likewise, at least one of the elements $1, 2, \dots, x_1 - 1$ needs to be in $P_1(f)$. Thus we have

$$|Z_a(f)| \leq |P_1(f)| \leq qa,$$

and by assumption we have $|NP(f)| = |P_a(f)| \leq pa$. Now we can apply Corollary 2.4 to conclude. \square

We now prove Theorem 1.6.

Proof of the Main Theorem. Consider a c -quasihomomorphism $f = (f_1, \dots, f_n) : \mathbb{Z} \rightarrow \mathbb{Q}^n$. Our goal is to show that for every $a \in \mathbb{Z}$ we have $w_H(f(a) - af(1)) \leq C$ for some constant C depending only on c . We start with the case $a > 0$.

Write $I_a := \{i \in [a] \mid f_i(a) \neq af_i(1)\}$, and note that $|I_a| = w_H(f(a) - af(1))$. We will show that $|I_a| \leq C'$ for some constant C' depending on c only. To this end, fix small parameters $p, q \in (0, 1)$ (to be optimized over later) and write $f_i^a := f_i|_{[2a]}$ for the restriction of f_i to $[2a]$. By Proposition 3.2, for every $i \in I_a$, we have

- (i) $|P_1(f_i^a)| > qa$, or
- (ii) $|P_a(f_i^a)| > pa$, or
- (iii) $|P(f_i^a)| \geq F(p, q)a^2$.

Let m_0 be the number of coordinates $i \in I_a$ such that (iii) holds. We define m_1 and m_2 analogously, for (i) and (ii) respectively.

By counting the number of triples $(i, x, y) \in [n] \times [a] \times [a]$ such that $f_i(x + y) - f_i(x) - f_i(y) \neq 0$ in two ways, we see that

$$\sum_{x=1}^a \sum_{y=1}^a w_H(f(x + y) - f(x) - f(y)) = \sum_{i=1}^n |P(f_i^a)| \geq \sum_{i \in I_a} |P(f_i^a)|.$$

Because f is a c -quasihomomorphism, the very left-hand side is at most a^2c . On the other hand, the very right-hand side is at least $m_0F(p, q)a^2$, so

$$a^2c \geq \sum_{x=1}^a \sum_{y=1}^a w_H(f(x + y) - f(x) - f(y)) \geq \sum_{i \in I_a} |P(f_i^a)| \geq m_0F(p, q)a^2.$$

So we obtain $m_0 \leq \frac{c}{F(p, q)}$. Similarly we find

$$ac \geq \sum_{x=1}^a w_H(f(x + 1) - f(x) - f(1)) = \sum_{i=1}^n |P_1(f_i^a)| \geq \sum_{i \in I_a} |P_1(f_i^a)| > m_1qa,$$

so that $m_1 < \frac{c}{q}$. Finally,

$$ac \geq \sum_{x=1}^a w_H(f(x + a) - f(x) - f(a)) = \sum_{i=1}^n |P_a(f_i^a)| \geq \sum_{i \in I_a} |P_a(f_i^a)| > m_2pa.$$

So $m_2 < \frac{c}{p}$. But now $|I_a| \leq m_0 + m_1 + m_2 < c(\frac{1}{F(p, q)} + \frac{1}{q} + \frac{1}{p}) =: C'$.

The case $a = 0$ is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \leq c.$$

Finally, let us consider the case $a < 0$. Then

$$\begin{aligned} w_H(f(a) - af(1)) &\leq w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) \\ &\quad + w_H(f(-a) - (-a)f(1)) \leq 2c + C' =: C. \end{aligned}$$

This completes the proof of the qualitative part of the Main Theorem. To obtain the explicit bound $28c$, we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p, q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1 - q - 2p)^2}.$$

This function is strictly convex for $(p, q) \in \mathbb{R}_{>0}^2$, so it has at most one minimum in the positive orthant. We find this by setting the partial derivatives to zero and solving for p, q . The minimum is ≈ 27.6817 and attained at $(p, q) \approx (0.1167, 0.16500)$. \square

REFERENCES

- [1] N. A'Campo, *A natural construction for the real numbers*, Elem. Math. **76** (2021), no. 3, 89–105, <https://doi.org/10.4171/em/455>.
- [2] Michael Brandenbursky and Misha Verbitsky, *Non-commutative Barge-Ghys quasimorphisms*, 2023, <https://arxiv.org/abs/2212.12958>.
- [3] M. Burger, N. Ozawa, and A. Thom, *On Ulam stability*, Israel J. Math. **193** (2013), no. 1, 109–129, <https://doi.org/10.1007/s11856-012-0050-z>.
- [4] Danny Calegari, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009, <https://doi.org/10.1142/e018>.
- [5] Marcus De Chiffre, Narutaka Ozawa, and Andreas Thom, *Operator algebraic approach to inverse and stability theorems for amenable groups*, Mathematika **65** (2019), no. 1, 98–118, <https://doi.org/10.1112/s0025579318000335>.
- [6] Roberto Frigerio, *Bounded cohomology of discrete groups*, Mathematical Surveys and Monographs, vol. 227, American Mathematical Society, Providence, RI, 2017, <https://doi.org/10.1090/surv/227>.
- [7] K. Fujiwara and M. Kapovich, *On quasihomomorphisms with noncommutative targets*, Geom. Funct. Anal. **26** (2016), no. 2, 478–519, <https://doi.org/10.1007/s00039-016-0364-9>.
- [8] William Timothy Gowers and Omid Hatami, *Inverse and stability theorems for approximate representations of finite groups*, Sbornik: Mathematics **208** (2017), no. 12, 1784.
- [9] D. Kazhdan, *On ε -representations*, Israel J. Math. **43** (1982), no. 4, 315–323, <https://doi.org/10.1007/BF02761236>.
- [10] D. Kazhdan and T. Ziegler, *Approximate cohomology*, Selecta Math. (N.S.) **24** (2018), no. 1, 499–509, <https://doi.org/10.1007/s00029-017-0335-5>.
- [11] Steffen Kionke, *Constructing the completion of a field using Quasimorphisms*, p-Adic Numbers Ultrametric Anal. Appl. **11** (2019), no. 4, 335–337, <https://doi.org/10.1134/S2070046619040083>.
- [12] D. Kotschick, *What is... a quasi-morphism?*, Notices Amer. Math. Soc. **51** (2004), no. 2, 208–209.
- [13] Nicolas Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, vol. 1758, Springer-Verlag, Berlin, 2001, <https://doi.org/10.1007/b80626>.
- [14] Tim Seynnaeve, Nafie Tairi, and Alejandro Vargas, *One-quasihomomorphisms from the integers into symmetric matrices*, 2023, <https://arxiv.org/abs/2302.01611>.
- [15] Ross Street, *An efficient construction of real numbers*, Austral. Math. Soc. Gaz. **12** (1985), no. 3, 57–58.
- [16] A. M. Turing, *Finite approximations to Lie groups*, Ann. of Math. (2) **39** (1938), no. 1, 105–111, <https://doi.org/10.2307/1968716>.
- [17] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.

JAN DRAISMA, Mathematical Institute, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland;
and Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600MB, Eindhoven, the Netherlands
E-mail : jan.draisma@unibe.ch

ROB H. EGGERMONT, Department of Mathematics and Computer Science, Eindhoven University of
Technology, P.O. Box 513, 5600MB, Eindhoven, the Netherlands
E-mail : r.h.eggermont@tue.nl

TIM SEYNNAEVE, Department of Computer Science, KU Leuven, Celestijnenlaan 200A, 3001 Leuven,
Belgium
E-mail : tim.seynnaeve@kuleuven.be

NAFIE TAIRI, Mathematical Institute, University of Bern, Alpeneggstrasse 22, 3012 Bern, Switzerland
E-mail : nafie.tairi@unibe.ch

EMANUELE VENTURA, Politecnico di Torino, Dipartimento di Scienze Matematiche “G.L. Lagrange”,
Corso Duca degli Abruzzi 24 10129 Torino, Italy
E-mail : emanuele.ventura@polito.it