Doctoral Dissertation
Doctoral Program in Mathematics (36^{th}cycle)

Walking around quasi-orders on graphs, spaces, and their subsets

By

Salvatore Scamperti

******

Supervisors:
Prof. Raphaël Carroy
Prof. Luca Motto Ros

Doctoral Examination Committee:
Prof. Dominique Lecomte, Referee, Sorbonne Université (Paris, France)
Prof. Alberto Giulio Marcone , Referee, Università degli Studi di Udine (Udine, Italy)

Politecnico di Torino
2024

Introduction

The thesis consists of three distinct parts:

- Chapter 1 is a preprint written in collaboration with Luca Motto Ros;
- Chapter 2 is the published article [CMRS23];
- Chapter 3 is a work in progress.

Each part is characterized by its own motivations and conceptual frameworks. In this concise introduction, we provide a broad overview for each chapter. The reader refers to the first section of each of them for a thorough description of the framework and a more detailed exposition of the results obtained.

Chapter 1. The focus of this part is graph homomorphism on undirected graphs. A graph G is a pair (V, E), where V is the set of vertices and E is the set of edges. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a graph homomorphism is a function $h : V_1 \to V_2$ such that if $x \; E_1 \; y$ then $h(x) \; E_2 \; h(y)$. The graph homomorphism relation $\preccurlyeq$ is an extensively studied quasi-order. It finds application in several fields due to the possibility of encoding objects and their relationships as graphs and homomorphisms between them. Independently of the field one is working in, a recurring theme is that the graph homomorphism quasi-order is very complicated, or, in other words, it can encode "a lot"; in such cases, one usually says that graph homomorphism is universal. However, the proofs of those universality results are often ad hoc constructions which depend on the framework at hand, and cannot be used in other ones. The primary goal of this chapter is to unify these different perspectives, designing a standard operation based on connected sums which allows us to reprove at once many existing universality results, as well as to discover several new instances of this phenomenon.

Chapter 2. The goal is to classify the Wadge Hierarchies on arbitrary zero-dimensional Polish spaces, up to order-isomorphism. Given a topological space X, a continuous reduction between two subsets A and B is a continuous function $f : X \to X$ such that $f^{-1}(B) = A$. When such a function exists, we write $A \leq^X_{\mathsf{W}} B$. The relation $\leq^X_{\mathsf{W}}$ is a quasi-order and we say that $A, B \subseteq X$ are Wadge equivalent, and we write $A \equiv^X_{\mathsf{W}} B$, if $A \leq^X_{\mathsf{W}} B$ and $B \leq^X_{\mathsf{W}} A$. The quasi-order $\leq^X_{\mathsf{W}}$ induces a partial order on the quotient $X/\equiv^X_{\mathsf{W}}$, called Wadge Hierarchy. If we let X be the Baire or the Cantor space, then the properties of the corresponding Wadge Hierarchies are well-known and we have a full description for them. Much less is known for the other zero-dimensional Polish spaces Z. Guided by the Wadge Hierarchies on the Baire and the Cantor spaces, we establish when a Wadge Hierarchy on an arbitrary zero-dimensional space Z is order-isomorphic to the one on the Baire space, on the Cantor space, or neither, and also how difficult it is to classify such Wadge Hierarchies up to isomorphism.

Chapter 3. The last chapter focuses on the analysis of some new well quasi-orders.[1] Let Q be a set. A quasi-order $\leq_Q$ on Q is a well quasi-order (wqo) if it is well-founded and each infinite subset of Q has at least two elements that are in relation with respect to $\leq_Q$. We begin with a notion related to Wadge reducibility, the injective continuous reducibility (or injective Wadge reducibility). Let X be a topological space. An injective continuous reduction between two sets $A, B \subseteq X$ is an injective continuous function $f : X \to X$ such that $f^{-1}(B) = A$. When such a reduction exists, we write $A \leq^X_{\mathsf{iW}} B$. Few things are known about the injective Wadge reduction and its behaviour within the Borel hierarchy still remains an open problem. However, we can prove that in an arbitrary zero-dimensional Polish space Z, $\leq^Z_{\mathsf{iW}}$ is a wqo on the union of the first ω levels of the difference hierarchy on $\mathbf{\Pi}^0_2$ sets. Moreover, we prove that if Z satisfies some reasonable constraints, then SLO does not hold with respect to $\leq^Z_{\mathsf{iW}}$. We then move to a more classical quasi-order, that is embeddability on the class of zero-dimensional separable metrizable topological spaces which are either countable or contain a copy of the Cantor space. We prove that this is again a wqo. Finally, in the last section, we study continuous injectability. Fix a topological space X and $A, B \subseteq X$: then $A \leq_{\mathsf{inj}} B$ if and only if there exists an injective continuous function $f : A \to B$. We prove that continuous injectability on analytic subsets of $\mathbb{R}$ is a linear well quasi-order of length $\omega_1 + 3$. It is already known that the continuous injectability on $\mathbb{R}^2$ is an analytic complete quasi-order. If we change the underlying topological space to the Cantor fence $2^{\mathbb{N}} \times [0, 1]$ (a natural space in between $\mathbb{R}$ and $\mathbb{R}^2$, up to homeomorphism), the situation is radically different. In particular, we identify some special points, called trouble witnesses, and show that continuous injectability is a wqo on closed subsets of the Cantor fence that do not have trouble witnesses.

[1] Actually, we prove that all the quasi-orders presented in the chapter are better quasi-order (bqo). However, for the sake of clarity, in this introduction we only speak of wqo. Expert readers may simply substitute all instances of wqo with bqo.

References

[CMRS23] R. Carroy, L. Motto Ros, and S. Scamperti. A classification of the wadge hierarchies on zero-dimensional polish spaces. *The Journal of Symbolic Logic*, page 1–31, 2023.