Generalized additivity of masses in special relativity

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Starting from the literature about the invariant mass, we will discuss the expression of the generalized additivity of masses in the special relativity for a system of particles.

In special relativity the mass is an invariant quantity and therefore is the same for all observers in all reference frames. Besides this invariant, we find in literature the "relativistic mass", which is defined as dependent on the velocity. We can find the relativistic mass given as:

$$m_{rel} = \frac{m}{\sqrt{1 - v^2/c^2}}$$

It is fundamental to remember that energy and momentum are linked to the invariant mass by:

$$E^2 - (pc)^2 = (mc^2)^2$$

Let us consider some literature.

Hecht, 2009, discusses mass and energy in Einstein's thought. "Early on, Einstein embraced the idea of a speed-dependent mass but changed his mind in 1906 and thereafter carefully avoided that notion entirely. He shunned, and explicitly rejected, what later came to be known as 'relativistic mass'. Nonetheless many textbooks and articles credit him with the relation $E=mc^2$, where E is the total energy, m is the relativistic mass, and **c** is the vacuum speed of light. Einstein never derived this relation, at least not with that understanding of the meaning of its terms. He consistently related the 'rest energy' of a system to its invariant inertial mass." This is what we can find in the abstract by Hecht, 2009. It is therefore required to read something more from the article.

"Einstein's first paper on relativity appeared when the concept of a speed-dependent electromagnetic mass had already become a topic of considerable interest. He accepted this idea but changed his mind after being confronted by a far more compelling

insight". Hecht shows "that after reading Planck's 1906 article in which the concept of relativistic momentum was introduced, Einstein came to realize that it was the relativistic equations for energy and momentum that were primary. From that perspective it became clear that the inertial mass *m* was invariant, and he never again spoke of mass as being speed dependent. Over the next several years, no doubt unaware of Einstein's change of mind, a number of researchers continued to elaborate on the idea that inertial mass varied with relative speed *v*. For them Newtonian mass had to be replaced by the idea of "relativistic mass" $m_{rel}(v)$, where:

$$m_{rel} = m_0(1 - v^2/c^2)^{-1/2}$$

Here m_0 is the object's rest mass, the inertial mass when *v*=0. It was already commonplace to represent $(1 - v^2/c^2)^{-1/2}$ by γ, whereupon $m_{rel} = m_0\gamma$ or just $m_{rel} = m\gamma$. After 1908 there were two conflicting interpretations of relativistic dynamics: Einsteins's invariant-mass perspective and the relativistic mass formulation. Meanwhile Einstein had shown that the energy of a system at rest was proportional to its inertial mass. Over the decades that followed, this extremely significant discovery took on the symbolic form *E=mc²*, wherein *E* is the total energy and *m* is the relativistic mass. Surprisingly, Einstein never derived nor ever accepted this relation. As *E=mc²* was becoming the most widely recognized symbol of the Atomic Age, Einstein maintained that this general statement was formulated 'somewhat inexactly' " (Hecht, 2009, Einstein, 1976). Among many others, Hecht is referring to Lev B. Okun, 1989, and his "concept of mass (mass, energy, relativity)".

Okun hopes that he will "succeed to convince the reader that the term "rest mass" m_0 is superfluous, that instead of speaking of the "rest mass" m_0 one should speak of the mass *m* of a body which for ordinary bodies is the same, in the theory of relativity and in Newtonian mechanics, that in both theories the mass *m* does not depend on the reference frame, that the concept of mass dependent on velocity arose at the beginning of the twentieth century as a result of an unjustified extension of the Newtonian relation between momentum and velocity to the range of velocities comparable to the velocity of light in which it is invalid, and that at the end of the twentieth century one should bid a final farewell to the concept of mass dependent on velocity" (Okun, 1989).

"The fundamental relations of the theory of relativity for a freely moving particle (system of particle, body) are: $E^2 - (pc)^2 = (mc^2)^2$ (5.1) , $\vec{p} = \vec{v} E/c^2$ (5.2) , where *E* is the energy, ***p*** the momentum, *m* is the mass, and ***v*** the velocity of the particle (or system of particles, or body). It should be emphasized that the mass *m* and the velocity ***v*** for a particle or a body are the same quantities with which we deal in Newtonian mechanics. Like the four-dimensional coordinates *t* and ***r***, the energy *E* and the momentum ***p*** are the components of a four-dimensional vector. They change on the transition from one inertial system to another in accordance with the Lorentz

transformation. The mass, however, is not changed - it is a Lorentz invariant" (Okun, 1989).

In addition, as in Newtonian mechanics, the energy and momentum are additive — the total energy and total momentum of n free particles are, respectively, $E = \sum_{i=1}^{n} E_i$, $\vec{p} = \sum_{i=1}^{n} \vec{p_i}$ (5.3). With regard to the mass, in theory of relativity the mass of an isolated system is conserved (does not change with the time), but does not possess the property of additivity" (Okun, 1989). However, it is necessary "to include among the bodies not only "matter," say atoms, but also "radiation" (photons)." (Okun, 1989).

"For massive particles (as we shall call all particles with nonzero mass, even if they are very light) the relations for the energy and momentum can be conveniently expressed in terms of the mass and velocity. For this we substitute (5.2) in (5.1): $E^2(1 - v^2/c^2) = m^2 c^4$ (6.2) and, taking the square root, we obtain $E = mc^2(1 - v^2/c^2)^{-1/2}$. (6.3) Substituting (6.3) in (5.2), we obtain $\vec{p} = m\vec{v}(1 - v^2/c^2)^{-1/2}$. (6.4). It is obvious from (6.3) and (6.4) that a massive body (with $m \neq 0$) cannot move with the speed of light, since then the energy and momentum of the body would have to be infinite" (Okun, 1989).

"In the theory of relativity the mass of a system is not equal to the mass of the bodies that make up the system. This assertion can be illustrated by several examples." (Okun, 1989). Here a case: "Consider two photons moving in opposite directions with equal energies E. The total momentum of such a system is zero, and the total energy (it is the rest energy of the system of the two photons) is $2E$. Therefore, the *mass* of this system is $2E/c^2$. It is easy to show that a system of two photons will have zero mass only when they move in the same direction" (Okun. 1989).

Let us consider a system consisting of n particles. "The mass of this system is determined by:

$$m = \left[\left(\sum_{i=1}^{n} \frac{E_i}{c^2} \right)^2 - \left(\sum_{i=1}^{n} \frac{\vec{p_i}}{c} \right)^2 \right]^{1/2}$$

(9.1)

where $\sum E_i$ is the sum of the energies of these bodies, and $\sum \vec{p_i}$ is the vector sum of their momenta" (Okun. 1989). For further discussions, see please Landau and Lifshitz, 1973.

As an example, let us pass to the "Notes on Relativistic Dynamics" by Styer, 2021. "Two-particle system. Two particles move on the x-axis. Particle A has mass m_A and velocity (relative to frame F) v_A, particle B has mass m_B and velocity (relative to frame F) v_B. a) Show that the two-particle system has mass M, where

$$M^2 = m_A^2 + m_B^2 + 2m_A m_B \frac{1 - v_A v_B/c^2}{\sqrt{(1 - (v_A/c)^2)(1 - (v_B/c)^2)}}$$

$$(4.3)$$

Frame F' moves relative to frame F at velocity V, so in this frame the two particles have velocities

$$v'_A = \frac{v_A - V}{1 - v_A V/c^2} \quad \text{and} \quad v'_B = \frac{v_B - V}{1 - v_B V/c^2}$$

$$(4.4)$$

b) Show that in the frame F', the system has the same mass M given above" (Styer, 2021).

Note that, in (4.3), when the velocities are all equals, the additivity is the usual one.

For b), let us use WolframAlpha and calculate:

$$1 - \frac{(v_A - V)(v_B - V)/c^2}{(1 - v_A V/c^2)(1 - v_B V/c^2)}$$

It is equal to:

$$A = \frac{(V^2 - c^2)(v_A v_B - c^2)}{(v_A V - c^2)(v_B V - c^2)}$$

And

$$\sqrt{\left(1 - \frac{((v_A - V)/c)^2}{(1 - v_A V/c^2)^2}\right)\left(1 - \frac{((v_B - V)/c)^2}{(1 - v_B V/c^2)^2}\right)}$$

It is equal to

$$B = \sqrt{\frac{(v_A^2 - c^2)(v_B^2 - c^2)(c^2 - V^2)^2}{(v_A V - c^2)^2(v_B V - c^2)^2}}$$

Therefore:

$$A/B = \frac{\dfrac{-(c^2 - V^2)(v_A v_B - c^2)}{(v_A V - c^2)(v_B V - c^2)}}{\sqrt{\dfrac{(v_A^2 - c^2)(v_B^2 - c^2)(c^2 - V^2)^2}{(v_A V - c^2)^2(v_B V - c^2)^2}}} = \frac{-(v_A v_B - c^2)}{\sqrt{(-v_A^2 + c^2)(-v_B^2 + c^2)}}$$

$$= \frac{1 - v_a v_b/c^2}{\sqrt{(1 - v_A^2/c^2)(1 - v_B^2/c^2)}}$$

Accordingly, we have invariance.

Let us pass to point a), using (9.1) $m = \left[\left(\sum_{i=1}^{n} \frac{E_i}{c^2}\right)^2 - \left(\sum_{i=1}^{n} \frac{\vec{p_i}}{c}\right)^2\right]^{1/2}$.

$$M = \sqrt{(E_A/c^2 - E_B/c^2) - (p_A/c + p_B/c)^2}$$

$$M^2 = \left(m_A(1 - v_A^2/c^2)^{-1/2} + m_B(1 - v_B^2/c^2)^{-1/2}\right)^2$$
$$- \left(m_A(v_A/c)(1 - v_A^2/c^2)^{-1/2} + m_B(v_B/c)(1 - v_B^2/c^2)^{-1/2}\right)^2$$

$$M^2 = m_A^2(1 - v_A^2/c^2)^{-1} + m_B^2(1 - v_B^2/c^2)^{-1} + 2m_A m_B(1 - v_A^2/c^2)^{-1/2}(1 - v_B^2/c^2)^{-1/2} - m_A^2(v_A/c)^2(1 - v_A^2/c^2)^{-1} - m_B^2(v_B/c)^2(1 - v_B^2/c^2)^{-1} - 2m_A m_B(v_A v_B/c^2)(1 - v_A^2/c^2)^{-1/2}(1 - v_B^2/c^2)^{-1/2}$$

Therefore

$$M^2 = m_A^2 + m_B^2 + 2m_A m_B(1 - (v_A v_B/c)^2)(1 - v_A^2/c^2)^{-1/2}(1 - v_B^2/c^2)^{-1/2}$$

that is

$$M^2 = m_A^2 + m_B^2 + 2m_A m_B \frac{1 - v_A v_B/c^2}{\sqrt{(1 - (v_A/c)^2)(1 - (v_B/c)^2)}}$$

This is a "generalized" additivity of masses in special relativity. The meaning is the following: this expression is making the "additivity" applicable in a wider manner. Generalization of additivity for integer numbers has been proposed in 2019 by Sparavigna.

In the case that we have three particles:

$$M^2 = m_A^2 + m_B^2 + m_C^2 + 2m_A m_B \frac{1 - v_A v_B/c^2}{\sqrt{(1 - (v_A/c)^2)(1 - (v_B/c)^2)}}$$

$$+2m_A m_C \frac{1 - v_A v_C/c^2}{\sqrt{(1 - (v_A/c)^2)(1 - (v_C/c)^2)}} + 2m_B m_C \frac{1 - v_B v_C/c^2}{\sqrt{(1 - (v_B/c)^2)(1 - (v_C/c)^2)}}$$

In general:

$$M^2 = \sum_{i=1}^{n} m_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} m_i m_j \frac{1 - v_i v_j/c^2}{\sqrt{(1 - (v_i/c)^2)\left(1 - (v_j/c)^2\right)}}$$

That is:

$$M^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j \frac{1 - v_i v_j/c^2}{\sqrt{(1 - (v_i/c)^2)\left(1 - (v_j/c)^2\right)}}$$

Let us introduce $v/c = \tanh\beta$. Then $\gamma = 1/\sqrt{1 - \tanh^2\beta} = \cosh\beta$ (Dray, 2012):

$$M^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j \cosh\beta_i \cosh\beta_j \left(1 - \tanh\beta_i \tanh\beta_j\right)$$

$$M^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j \left(\cosh\beta_i \cosh\beta_j - \sinh\beta_i \sinh\beta_j\right)$$

Consequently, when $\beta_i = \beta_j = \beta$, we find immediately $M^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j$.

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