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# Supersymmetric States in Anti-de Sitter $D = 3$ Supergravity with Chiral Torsion

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In this Letter, we construct a supersymmetric model, obtained by deforming  $\mathcal{N} = 2$  anti-de Sitter  $D = 3$  supergravity through a chiral vector component of the torsion. Moreover, we study the existence of supersymmetric states of such theory by inspecting the presence of Killing spinors on a specific bosonic solution.

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**Introduction.**—Pure anti-de Sitter Einstein gravity in  $(2 + 1)$  spacetime dimensions ( $\text{AdS}_3$ ) has attracted considerable attention from physicists due to its perturbative equivalence to a Chern-Simons theory [1–3]. As gravity here does not describe propagating local degrees of freedom, its richness is encoded in its global properties [4]. The situation changes when matter sources are included, since they in general spoil the topological properties of the pure Einstein theory and possibly introduce local dynamics. A notable way of including matter in the model, as well as a cosmological constant, is through torsional deformations. Indeed, by properly redefining the torsionful spin connection of the first order formalism, in terms of a torsionless one plus a contorsion, the latter contributes to the energy-momentum tensor in the Einstein equations as a matter source. Torsional deformations in three-dimensional gravity have been extensively studied in the literature, mainly restricted, however, to those choices that do not spoil the local maximal symmetry of the background [5–12]. This is achieved by only considering the (constant) scalar (i.e., Lorentz singlet) component of the torsion tensor, which, in the absence of a

Riemann curvature for the torsionful connection, effectively provides a negative cosmological constant.

It has been recently observed in [13], that, in  $(2 + 1)$  dimensions, one can introduce a vector component of the torsion tensor, defining a nonclosed 1-form, together with the scalar one, provided the two are related by a Beltrami-like equation. The latter is an equation for a massive 1-form in three dimensions, which provides the vector component of the torsion with a chiral quality [14]. For this reason, we shall also refer to such vector component as Beltrami torsion. In the same reference, novel backgrounds featuring these two components of the torsion were constructed and studied. These geometries are not locally  $\text{AdS}_3$  and provide a 1-parameter family of deformations of the Bañados-Teitelboim-Zanelli black hole [15,16]. In fact, these backgrounds are *locally* warped  $\text{AdS}_3$  [17], although globally, to our knowledge, new.

Bosonic spacetimes can sometimes be considered as solutions of supersymmetric models, preserving some amount of supersymmetry. In pure supergravities, maximally symmetric spacetimes admit the maximum number of globally defined Killing vectors and preserve the largest number of global supersymmetries. Anti-de Sitter space in  $(2 + 1)$  dimensions, in particular, is a supersymmetric solution of a Chern-Simons model with a supersymmetric extension of the  $\text{AdS}_3$  isometry group  $\text{SO}(2,2)$  as its structure group, e.g., the Achúcarro-Townsend supergravities with  $\text{OSp}(p|2) \times \text{OSp}(q|2)$  [18]. Supersymmetry of

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locally  $\text{AdS}_3$  solutions have been extensively studied in the literature; see for example [19–21].

The goal of the present investigation is threefold: (1) to construct a supersymmetric model, based on a torsional deformation of an  $\text{AdS}_3$  supergroup; (2) to show that the geometries considered in [13], featuring Beltrami torsion, are solutions to the field equations; and (3) to assess the supersymmetry of such backgrounds.

The last objective will be achieved by studying the existence of Killing spinors on the torsionful backgrounds, and comparing the results with the supersymmetric, undeformed ones. Finding supersymmetric solutions in the presence of torsional deformations that behave as matter sources is somewhat surprising. It can be explained by the fact that the Beltrami torsion defines a new sector, associated with its norm, of which the Bogomol'nyi-Prasad-Sommerfield (BPS) states are expected to be supersymmetric lowest energy configurations.

The Letter is organized as follows: in the next section, we explicitly construct a novel supersymmetric model with four supercharges and review the bosonic solution derived in [13]. In the “Killing Spinors” section that follows, we solve the Killing spinor equation and discuss the global properties of the corresponding solutions, comparing them to the literature [22]. Finally, in “Conclusions,” we comment on the obtained results and discuss possible future developments.

*The geometric setup.*—Let us start by considering the Achucarro-Townsend model [18], based on the supergroup  $\text{OSp}(2|2) \times \text{SO}(1,2)$  and described by the equations [23]

$$\begin{aligned} R^i &\equiv d\omega^i - \frac{1}{2}\epsilon_{jk}^i \omega^j \wedge \omega^k = -i\tau \tilde{\psi}_A \wedge \gamma^i \psi_A, \\ \nabla \psi_A &\equiv d\psi_A + \frac{i}{2}\omega^i \wedge \gamma_i \psi_A + \frac{\tau}{2}A \wedge \epsilon_{AB} \psi^B = 0, \\ dA - \epsilon_{AB} \tilde{\psi}_A \wedge \psi_B &= 0, \\ De^i &\equiv de^i - \epsilon_{jk}^i \omega^j \wedge e^k = \tau \epsilon_{jk}^i e^j \wedge e^k + \frac{i}{2}\tilde{\psi}_A \wedge \gamma^i \psi_A, \end{aligned} \quad (1)$$

where  $i, j, \dots = 0, 1, 2$  are rigid indices in the vector representation of the local Lorentz group  $\text{SO}(1,2) \subset \text{OSp}(2|2)$  and  $A, B, \dots = 1, 2$  are  $\text{SO}(2)$   $R$ -symmetry indices. Here,  $\omega^i = 1/2\epsilon_{jk}^i \omega^{jk}$  is the spin connection of  $\text{SO}(1,2) \subset \text{OSp}(2|2)$ . The connection of the second  $\text{SO}(1,2)$  factor, say  $\omega'^i$ , does not appear in the above equations as it is related to the vielbein  $e^i$  and the spin connection  $\omega^i$  as  $\omega'^i = \omega^i + 2\tau e^i$ . The constant  $\tau$  defines the  $\text{AdS}_3$  radius  $L$  by  $\tau = -\theta/L$ , where  $\theta = \pm 1$ . The sign  $\theta$ , which can be changed by an improper Lorentz transformation, identifies which of the two  $\text{SO}(1,2)$  factors in the  $\text{AdS}_3$  isometry group is contained in  $\text{OSp}(2|2)$ , i.e., it determines whether the  $(p, q)$  Achucarro-Townsend model is  $(2,0)$  or  $(0,2)$ . The Dirac matrices appearing in (1) are written in terms of the Pauli matrices as

$$\gamma^i = \{\sigma_2, i\sigma_1, i\sigma_3\}. \quad (2)$$

In this Letter, we introduce a torsional deformation of the above superalgebra, effected by changing the last of equations (1) into

$$De^i = \tau \epsilon_{jk}^i e^j e^k + \beta e^i + \left( a \epsilon_{ij}^l \beta^l + \frac{i}{2}(1 - 2ib\tau) \delta_j^i \right) \tilde{\psi}_A \gamma^j \psi^A, \quad (3)$$

where  $\beta \equiv \beta_i e^i + \tilde{\psi}_A \lambda^A$  is a 1-form in superspace,  $\lambda_A$  being a Grassmannian field, parametrizing the components of  $\beta$  in the odd directions. Here and in the following the wedge symbol between differential forms is understood. A more general class of deformations and their consistency conditions are described in Appendix A.

The superspace Bianchi identities imply the following relation between the two constants  $a$  and  $b$

$$i + 2\tau(b - 2a) = 0, \quad (4)$$

together with the following constraints on  $\beta_i$  and  $\lambda_A$

$$D_{[p}\beta_{q]} + 2\tau\beta^i \epsilon_{ipq} = 0, \quad (5)$$

$$D_{(i}\beta_{j)} = \beta_i \beta_j + 2\tau^2 \frac{b}{a} \eta_{ij}, \quad (6)$$

$$\lambda_A = \nabla_A \beta_i = 0, \quad (7)$$

where  $\nabla_A \beta_i$  is the component of  $D\beta_i$  along  $\psi^A$ , that is  $D\beta_i = D_j \beta_i e^j + \tilde{\psi}^A \nabla_A \beta_i$ . Relation (5) was named Beltrami equation in [13], in analogy with the description of Beltrami flows in fluid dynamics. In fact, it is the equation describing a chiral (anti-)self-dual massive 1-form in three dimensions (see Ref. [14]), the chirality being related to the sign  $\theta$ . The spacetime projection of the constraint (6) on a bosonic background identifies  $\beta_i$  as a conformal Killing vector, satisfying the condition [24]

$$\mathring{D}\beta^i = 3(\beta_i \beta^i + 2\tau^2 b/a). \quad (8)$$

In the supersymmetric case,  $\mathring{D}$  is the covariant derivative with respect to the “supertorsionless” connection  $\hat{\omega}^i$ , that is,

$$\hat{\omega}^i = \omega^i + \tau e^i - \epsilon_{jk}^i \beta^j e^k, \quad (9)$$

$$\mathring{D}e^i = a(\epsilon_{ij}^l \beta^l + 2\tau \delta_j^i) \tilde{\psi}_A \gamma^j \psi^A. \quad (10)$$

Finally, condition (7) implies that  $\beta_i$  does not transform under supersymmetry, that is

$$\delta_\epsilon \beta_i = i_\epsilon d\beta_i = i_\epsilon (\psi^A \nabla_A \beta_i) = \epsilon^A \nabla_A \beta_i = 0. \quad (11)$$

If  $\beta_i = 0$  and  $b = 0$ , the deformation vanishes and we recover the Achucarro-Townsend model in (1). Notice that, if  $\beta_i \neq 0$ , the  $\text{SO}(2,2)$   $\text{AdS}_3$  symmetry is broken. More specifically, the chirality of  $\beta_i$  breaks the parity invariance  $\theta \rightarrow -\theta$ . In our framework, the sign  $\theta$  accounts for the two inequivalent representations of the Clifford algebra. Since the Lorentz representation of the  $\gamma$  matrices is the same as

the one of the dreibein (the vector irrep), then the parity transformation  $\theta \rightarrow -\theta$  could be equivalently accounted for, on a bosonic background, as acting by  $e^i \rightarrow -e^i$ . In the absence of deformations (that is at  $\beta_i = b = 0$ ), both representations define a same bosonic background. In the presence of a nontrivial  $\beta_i$ , due to the chiral quality of the latter, different signs of  $\theta$  identify different bosonic backgrounds.

We end this section by writing, for the sake of completeness, besides (11), the supersymmetry transformations of the remaining fields:

$$\begin{aligned}\delta_\epsilon e^i &= 2a(\epsilon_{ij}^i \beta^j + 2\tau \delta_j^i) \bar{\epsilon}_A \gamma^j \psi^A, \\ \delta_\epsilon \omega^i &= -2i\tau \bar{\epsilon}_A \gamma^i \psi^A, \\ \delta_\epsilon \psi_A &= \nabla \epsilon_A.\end{aligned}\quad (12)$$

Let us observe that the components  $\beta_i$  of the solution derived in [13] in a nonsupersymmetric setting, remarkably satisfy both conditions (5), (6), which are required for a consistent supersymmetric extension of the field equations. This, together with the fact that  $\beta_i$  does not transform under supersymmetry, suggests that such bosonic solution, to be reviewed below, may preserve some amount of supersymmetry.

The bosonic solution: A bosonic solution to the above defined supersymmetric model was constructed in [13]. It can be expressed, after proper redefinitions, as

$$\begin{aligned}e^0 &= \theta e_{\text{AdS}}^0 - \theta \frac{(1-\xi)}{2\xi} (dt + Ld\phi) f \frac{2r^2 - r_+^2 - r_-^2}{(r_+ + r_-)^2}, \\ e^1 &= \frac{\theta}{\xi} e_{\text{AdS}}^1, \\ e^2 &= \theta e_{\text{AdS}}^2 + \theta \frac{(1-\xi)}{2\xi} (dt + Ld\phi) \frac{2r^2 - r_+^2 - r_-^2}{(r_+ + r_-)^2} \left( \frac{r}{L} - rN \right), \\ \beta &= \frac{2\epsilon}{r_+ + r_-} \sqrt{1-\xi^2} \left( f e^0 + \left( \frac{r}{L} - rN \right) e^2 \right), \\ A &= 0,\end{aligned}\quad (13)$$

with  $\theta$  as defined in the section above,

$$e_{\text{AdS}}^0 = f dt, \quad e_{\text{AdS}}^1 = \frac{dr}{f}, \quad e_{\text{AdS}}^2 = r(d\phi + N dt)$$

and

$$f(r) = \frac{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}{Lr}, \quad N(r) = -\frac{r_+ r_-}{r^2 L}. \quad (14)$$

The sign  $\epsilon = \pm 1$  appearing in (13), embodies a freedom in the definition of the Beltrami torsion  $\beta$ . The ranges of the coordinates  $(t, r, \phi)$  appearing in the above formulas are as follows: the radial coordinate is positive,  $r > 0$ , whereas  $t \in \mathbb{R}$  and  $\phi \in \mathbb{R}$ . The metric in (13) describes a deformation of a locally  $\text{AdS}_3$  geometry which is parametrized by the

non-negative real parameter  $\xi$ , with  $\xi = 1$  corresponding to the undeformed background. The spacetime admits two Killing vectors,  $K_t = \partial_t$  and  $K_\phi = \partial_\phi$ .

The parameters  $r_\pm \in \mathbb{C}$  are integration constants related to the conserved mass and angular momentum of the undeformed geometry by

$$M = \frac{(r_+^2 + r_-^2)}{L^2}, \quad J = \frac{2r_+ r_-}{L}. \quad (15)$$

The quantities in (13), together with the torsionful spin connection given in Appendix B, satisfy the equations

$$De^i = -\frac{\theta}{L} \epsilon^{ijk} e_j \wedge e_k + \beta \wedge e^i, \quad R^i = 0, \quad (16)$$

together with (5) and (6), with parameters  $a = (i/4\tau\xi^2)$  and  $b = (i/2\tau)[(1-\xi^2)/\xi^2]$ . These equations then explicitly read

$$\star d\beta = 2\frac{\theta}{L}\beta, \quad D_{(i}\beta_{j)} = \beta_i \beta_j + 4\tau^2(1-\xi^2)\eta_{ij}. \quad (17)$$

When  $\xi \neq 1$ , the invariance  $\theta \rightarrow -\theta$ , enjoyed by the locally  $\text{AdS}_3$  geometry, is explicitly broken, as the first of (17) is not symmetric under such transformation.

Equation (17), when rewritten in terms of the bosonic restriction of the torsionless spin connection  $\hat{\omega}_i$  in (9) reads

$$\hat{D}_{[i}\beta_{j]} = \frac{\theta}{L} \epsilon_{ijk} \beta^k, \quad \hat{D}_{(i}\beta_{j)} = 0, \quad (18)$$

and identifies a proper Killing vector among the admissible conformal ones. In fact, one can verify that the vector  $\beta^\mu \partial_\mu$ , where  $\beta^\mu \equiv g^{\mu\nu} \beta_\nu$ , reads

$$\beta^\mu \partial_\mu = 2\theta \epsilon \frac{\sqrt{1-\xi^2}}{(r_+ + r_-)} \left( K_t - \frac{1}{L} K_\phi \right). \quad (19)$$

Using (18), (19), it immediately follows that its norm is constant

$$\partial_i \beta^2 = 2\frac{\theta}{L} \beta^j \beta^k \epsilon_{ijk} = 0, \quad (20)$$

its value being  $\beta^2 = -4(1-\xi^2)/L^2$ . Therefore  $\beta^2$  is a characteristic constant of the background, associated with the deformation from a locally  $\text{AdS}_3$  solution. It can be viewed as an order parameter, selecting a distinct sector of the space of solutions. These properties identify the local geometry of the spacetime as an instance of Lorentzian  $K$ -contact,  $\eta$ -Einstein manifold [26,27].

The Riemann curvature for the torsionless spin connection is

$$\hat{R}_0 = \frac{1}{L^2} e^1 e^2, \quad \hat{R}_1 = -\frac{1}{L^2} e^0 e^2, \quad \hat{R}_2 = \frac{4\xi^2 - 3}{L^2} e^0 e^1. \quad (21)$$

The geometry described above may admit the identification  $\phi \sim \phi + 2\pi$ , corresponding to a quotient along the Killing

vector  $K_\phi$ . The quotient is well-defined whenever it does not introduce closed timelike curves. Such causal singularities can be studied by inspecting the reality of the zeroes of the metric component  $g_{\phi\phi}(r)$

$$g_{\phi\phi} = -\frac{(1-\xi^2)}{\xi^2(r_+ + r_-)^2}(r^2 - \hat{r}_+^2)(r^2 - \hat{r}_-^2), \quad (22)$$

depending on the various parameters. For  $\xi \neq 1$ , the causal singularities are formally located at  $r = \hat{r}_\pm$ , where

$$\hat{r}_\pm^2 = \frac{(1-\xi^2)(r_+^2 + r_-^2) - \xi(r_+ + r_-)^2 \pm \xi\sqrt{\Delta}}{2(1-\xi^2)}, \quad (23)$$

with

$$\Delta = (r_+ + r_-)^2(2\xi^2(r_+^2 + r_-^2) - (r_+ - r_-)^2), \quad (24)$$

while for  $\xi = 1$  there is a single zero at  $r = 0$ . When either  $\hat{r}_+$  or  $\hat{r}_-$  are real and covered by a horizon ( $r_+ > \hat{r}_+$ ), the spacetime is a black hole; if not it has a naked singularity. When singularities are absent, that is for  $\hat{r}_\pm \notin \mathbb{R}_{\geq 0}$ , the corresponding configurations are regular, albeit not always well defined in the given coordinates (13).

Below we discuss the various cases in which  $M$  and  $J$  are real, together with the corresponding behavior of vielbein, metric, and torsion.

Case A:  $r_\pm \in \mathbb{R}$ ,  $0 < \xi \leq 1$ ,  $r_+ \geq |r_-| \Leftrightarrow M \geq |J|/L$ . When  $|r_-| < r < r_+$ ,  $e^0, e^1$  become imaginary. This can easily be avoided by considering a complex Lorentz rotation

$$\Lambda = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (25)$$

which makes the whole solution real and well-defined. We can further distinguish three cases: (A1)  $r_+ > |r_-| \geq 0 \Leftrightarrow M > |J|/L$ . For  $\xi \neq 1$ , these solutions, for strictly positive  $r_-$ , which means  $J > 0$ , are black holes. (A2)  $r_+ = |r_-| > 0 \Leftrightarrow M = |J|/L$ . Extremal black hole solutions only exist for strictly positive angular momentum. For  $\xi \neq 1$ ,  $r_+ = -r_-$  leads to ill-defined quantities, since it would correspond to a divergent metric and torsion. (A3)  $r_+ = r_- = 0 \Leftrightarrow M = J/L = 0$ . When  $\xi \neq 1$  the solution is not well-defined in the chosen coordinates.

Case B:  $r_\pm \in i\mathbb{R}$ ,  $\xi \geq 1$ ,  $\text{Im}(r_+) \geq |\text{Im}(r_-)| \Leftrightarrow M \leq -|J|/L$ . The condition  $\xi \geq 1$  is required by the reality of  $\beta$ . The above conditions can further be distinguished into (B1)  $\text{Im}(r_+) > |\text{Im}(r_-)| \Leftrightarrow M < -|J|/L$ . Causal singularities continuously connected to the solution at  $\xi = 1$ , only exist for  $\text{Im}(r_-) \geq 0$ , that is  $J \leq 0$ . For  $J > 0$ , i.e., for  $\text{Im}(r_-) < 0$ , causal singularities only occur for  $\xi > \xi_0 > 1$ , where  $\xi_0$  has the following expression in terms of  $r_\pm$ :

$$\xi_0 = \frac{r_+^2 + r_-^2}{r_+^2 + 4r_+r_- + r_-^2}. \quad (26)$$

In this case, the singular solutions are not continuously connected to the locally AdS ones. All these singularities, for  $M < 0$ , are naked, just as for the undeformed solutions. (B2)  $\text{Im}(r_+) = |\text{Im}(r_-)| > 0 \Leftrightarrow M = -|J|/L$ . When  $\xi \neq 1$ , causal singularities do not exist and, as before,  $r_+ = -r_-$  is not well-defined in the chosen coordinates.

Case C: In this case,  $r_\pm$  are complex conjugate to each other, up to a sign. We then have (C1)  $r_+ = r_-^*$ ,  $0 < \xi \leq 1$ ,  $\Leftrightarrow -J/L < M < J/L$ ,  $J > 0$ . When  $\xi < 1$ , causal singularities continuously connected with the solution at  $\xi = 1$  may exist for  $|\text{Re}(r_+)| > |\text{Im}(r_+)|$ , which corresponds to  $M > 0$ . (C2)  $r_+ = -r_-^*$ ,  $\xi \geq 1$ ,  $\Leftrightarrow J/L < M < -J/L$ ,  $J < 0$ . In the  $\xi > 1$  case, causal singularities exist only for  $|\text{Re}(r_+)| < |\text{Im}(r_+)|$ , that is for  $M < 0$ .

The causal singularities of the metric, due to compactification, are represented in Fig. 1.

Let us end this section with a comment. In the undeformed case, causal singularities always exist, at  $r = 0$ , for any value of  $M$  and  $J$ . By contrast, the above analysis shows that, in the presence of  $\beta$ , they only exist for specific values of  $J$ , thus spoiling the symmetry  $J \rightarrow -J$ . More precisely, given a solution, the one with opposite sign of  $J$  corresponds to that of the alternative choice of  $\theta$  in (13).

Such feature indicates that  $\beta$  is capable of regularizing the causal structure of the  $(2+1)$ -dimensional spacetime, as a consequence of its chiral nature [see the discussion below Eq. (17)]. This asymmetry was already encountered in [13], where two inequivalent, deformed solutions with spacelike  $\beta$ , satisfying (13), were considered,

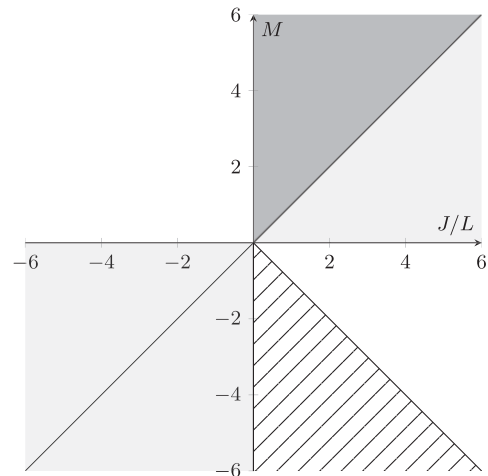


FIG. 1. We depict here the behavior of the deformed geometry after compactification of  $\phi$ . White regions correspond to regular geometries, while dark gray regions are black holes. Finally, the light gray region denotes naked singularities, while the white-dashed area corresponds to naked singularities that are not continuously connected with  $\xi = 1$ .



corresponding to positive and negative angular momentum in Case A above. In one case singularities were present, while the other defined a regular solution.

*Killing spinors.*—In this section, we focus on the solution of the Killing spinor equation, in the background shown in (13), that is on the spinors  $\varepsilon_A$  satisfying the requirement  $\delta\psi_A = 0$  in (12). The equation can be expressed in terms of the Dirac-spinor parameter  $\eta \equiv \varepsilon_1 + i\varepsilon_2$  as

$$d\eta + \frac{i}{2}\omega_i\gamma^i\eta = 0. \quad (27)$$

The strategy for solving the above equation follows the one shown in [22], which we will briefly outline here. If we define

$$x^\pm = t \pm L\phi, \quad \partial_\pm = \frac{1}{2}\left(\partial_t \pm \frac{\partial_\phi}{L}\right), \quad (28)$$

we realize that the spin connection has only components along  $dx^+$  and  $dr$ ; see Appendix B. Therefore the solution will not depend on  $x^-$ , that is  $\eta = \eta(x^+, r)$ . The equation along  $dx^+$  reduces to a single algebraic equation for one of the two components of  $\eta$ , provided that the spinor is written in terms of independent eigenfunctions of  $\partial_+$ . Finally, the equation along  $dr$  fixes the remaining function.

In [22], the total number of Killing spinors in the undeformed geometry was computed, considering both the two possible inequivalent representations of the three-dimensional Clifford algebra:  $\theta\gamma^i$ , with  $\theta = \pm 1$ . We remark that this freedom is the same that we have encoded in our dreibein definition in (13), with the same meaning for  $\theta$ . In the deformed case, due to the chirality of  $\beta$ , only one of the possible values of  $\theta$  is allowed, implying, in general, a lower number of Killing spinors with respect to [22].

The solution reads as follows:

$$\begin{aligned} \eta(x^+, r) &= e^{-\frac{1}{2L^2}\xi(r_+-r_-)x^+} \begin{pmatrix} F_+(r; r_-, r_+, \xi) \\ F_-(r; r_-, r_+, \xi) \end{pmatrix} \eta_0 + e^{\frac{1}{2L^2}\xi(r_+-r_-)x^+} \begin{pmatrix} F_+(r; r_+, r_-, \xi) \\ F_-(r; r_+, r_-, \xi) \end{pmatrix} \eta_1, \\ F_\pm(r; x, y, \xi) &= -\sqrt{\frac{(r \pm y)(r \mp x)}{r}} \left( r \pm x - \frac{\xi(r \mp y)}{1 + \epsilon\theta\sqrt{1 - \xi^2}} \right). \end{aligned} \quad (29)$$

The above expression, for  $\xi = 1$ , reproduces the result of [22] [see Eq. (3.39) of [22], with  $\theta = -1$ ]. Notice that, for  $\xi = 1$ , the function  $F_\pm(r; x, y, 1) \propto r^{\frac{1}{2}}$  as  $r \rightarrow \infty$ . The Beltrami torsion, in this Lorentz frame, changes the asymptotic power-law behavior into  $r^{\frac{3}{2}}$ .

In the following, we shall consider the properties of the above Killing spinors, depending on the parameters  $r_\pm, \xi$ , according to the cases A, B, and C listed in the previous section.

The solution (29) is globally well-defined, for all considered ranges of coordinates. However, we shall focus on the case in which the coordinate  $\phi$  is compact, in the interval  $[0, 2\pi)$ . Then, besides ensuring the absence of closed timelike curves, we have to impose that the Killing spinor satisfies periodic or antiperiodic boundary conditions along the compact direction.

**Case A:** For the Killing spinor to satisfy periodic or antiperiodic boundary conditions along the compact direction, due to the reality of the exponentials, we need to impose the vanishing of the exponents, which requires the extremal limit  $r_+ = r_- > 0 \Leftrightarrow M = J/L$ . This corresponds to case A2 for positive angular momentum.

In this case, the background preserves only one supercharge, i.e., it is 1/4 BPS, since the Killing spinor only features one independent parameter,  $\eta_0 + \eta_1$  [28].

**Case B:** Compatibility of the Killing spinor solution with the quotient along  $K_\phi$  requires imposing a quantization condition on the exponents, which in this case are purely imaginary,

$$\xi(\text{Im}(r_+) - \text{Im}(r_-)) = \xi \sqrt{-\left(M - \frac{J}{L}\right)} = kL, \quad (30)$$

with  $k \in \mathbb{Z}_{\geq 0}$ . The above equation resembles the Dirac quantization condition on the magnetic monopole charge: in this analogy, the torsion parameter  $\xi$  can be seen as the

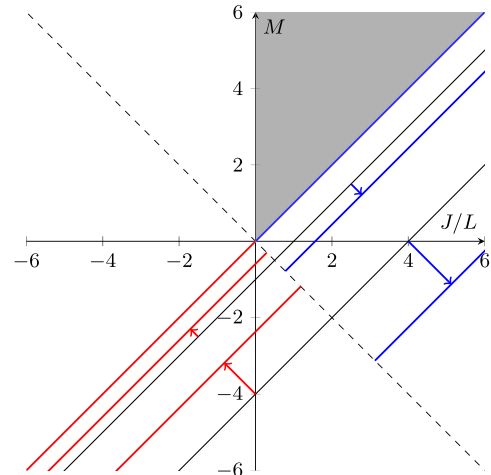


FIG. 2. We depict here the globally well-defined Killing spinors in the quotient geometries. Diagonal red and blue lines correspond to  $\xi = 1.3$ ,  $L = 1$  and  $\xi = 0.8$ ,  $L = 1$ , respectively. The arrows highlight the difference between the deformed and  $\xi = 1$  backgrounds.

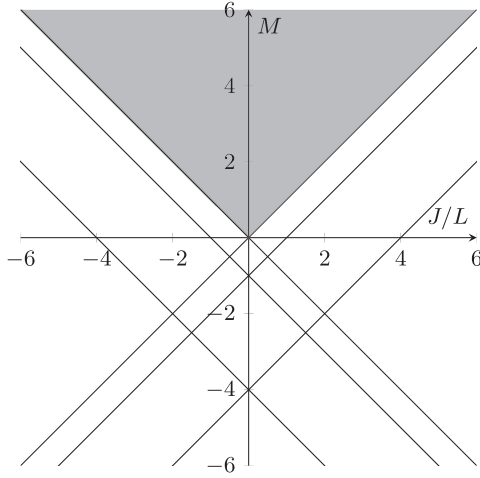


FIG. 3. Diagonal lines correspond here to the global Killing spinors of the undeformed geometry, which, due to the parity symmetry  $\theta \rightarrow -\theta$ , also admits lines with negative slope.

dual to the combination  $M - J/L$ . Equation (30) can be rewritten as

$$M = \frac{J}{L} - \frac{k^2 L^2}{\xi^2}. \quad (31)$$

In particular, since  $\xi \geq 1$ , the distance between consecutive lines in the  $(M, J)$  plane decreases with the increasing of  $\xi$ . This corresponds to cases B1 and B2, the latter only for negative angular momentum. The bosonic solution preserves two supercharges, corresponding to  $\eta_0$  and  $\eta_1$ , i.e., it is 1/2 BPS.

Case C: A Killing spinor solution, satisfying correct boundary conditions, can only be obtained in case C1, where the exponents are complex. This leads to the quantization rule

$$2\xi \text{Im}(r_+) = \xi \sqrt{-\left(M - \frac{J}{L}\right)} = kL, \quad (32)$$

with  $k \in \mathbb{Z}_{>0}$ . The background preserves two supercharges,  $\eta_0$  and  $\eta_1$ , i.e., it is 1/2 BPS.

We sum up the results of above cases A, B, C in Fig. 2, whereas the undeformed configuration is pictured in Fig. 3.

**Conclusions.**—In this Letter, we have constructed a supersymmetric model by suitably altering the structure of  $\text{OSp}(2|2) \times \text{SO}(1, 2)$  Achucarro-Townsend supergravity. Such deformation is induced by the vector component of the torsion  $\beta_i$ , which is required to be a conformal Killing vector satisfying the Beltrami equation, besides being a singlet under supersymmetry. The resulting equations are covariant under  $\text{OSp}(2|2)$ , while the second  $\text{SO}(1, 2)$  symmetry is broken by  $\beta_i$ , thus spoiling the  $\text{AdS}_3$  isometry group.

The backgrounds constructed in [13] are shown to be solution of this supersymmetric model. These geometries

are deformations of  $\text{AdS}_3$ , through Killing vectors of the latter. Indeed, the Beltrami torsion happens to define a proper Killing vector  $\beta^\mu \partial_\mu$  of both  $\text{AdS}_3$  and of the deformed background; see Eq. (19). It comes with no surprise that  $\beta_\mu$  satisfies the Beltrami-like equation, since this is a property of all  $\text{AdS}_3$  Killing vectors [29]. The geometries constructed in this way are shown to preserve, after compactification along  $K_\phi$ , half the amount of supersymmetry of the undeformed ones, due to the chiral nature of the Beltrami torsion. In particular, the only geometries with protected causal singularities that preserve some supersymmetry are extremal black holes with  $J > 0$  (they are 1/4 BPS). Extremal black holes with negative  $J$  of the undeformed theory are regularized by the presence of  $\beta_i$  and the resulting geometries do not preserve any amount of supersymmetry.

The deformation of more general  $(p, q)$  Achucarro-Townsend supergravities, as well as the embedding of the constructed supersymmetric geometries in higher dimensional supergravity have not been considered here and they would deserve a detailed investigation.

Moreover, let us observe that the deformation of both  $\text{AdS}_3$  gravity and supergravity induced by  $\beta_i$  was only introduced on-shell, that is, at the level of the Maurer-Cartan equations of the superalgebra. In several contexts, which include the  $\text{AdS}_3/\text{CFT}_2$  correspondence, an action principle would be desirable. This would allow a concrete study of the two-dimensional quantum theory, dual to the bulk gravity model. This is expected to be a deformation of a conformal field theory, parametrized by the Beltrami torsion  $\beta_i$ , which is shown to break conformal symmetry of the locally  $\text{AdS}_3$  background. Moreover, an action principle would allow a consistent definition of finite conserved charges, associated with the existing Killing vectors  $K_i$  and  $K_\phi$ , through renormalization, as in [30–32]. This would ultimately allow for a better understanding of the thermodynamics of the susy-preserving extremal black holes mentioned above, as well as of the naked singularities occurring for  $M < |J|/L$ . This would nicely complement the results discussed in [33].

Another interesting line of future research is the possible relationship between the supersymmetric model of gravity with torsion constructed here and models of topological and new massive (super)gravities [34,35].

Finally, as pointed out above, the geometries discussed here feature a Beltrami torsion that is a proper Killing vector, but the supersymmetric model obtained in the second section seems to admit more general deformations that satisfy the conformal Killing equation. It would then be interesting to look for more general bosonic solutions with this property.

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*Appendix A: A general class of deformations.*—Let us consider here, in place of (3), the following more general extension of the Achúcarro-Townsend theory:

$$De^i = \tau \epsilon_{jk}^i e^j e^k + \frac{i}{2} \bar{\psi}_A \gamma^i \psi_A + B^{(2)i},$$

with its integrability condition reading

$$DB^{(2)i} + 2\tau \epsilon_{jk}^i B^{(2)j} e^k = 0. \quad (A1)$$

Since  $B^{(2)i}$  is a 2-form in superspace, we can postulate the following parametrization:

$$B^{(2)i} = \beta e^i + \bar{\psi}^A B_{Aj}^i e^j + B_j^i \bar{\psi}_A \gamma^j \psi^A, \quad (A2)$$

where  $B_{Aj}^i$  and  $B_j^i$  are, in principle, additional fields, contributing to the deformation of the  $\text{OSp}(2|2) \times \text{SO}(1,2)$  model. The parametrization chosen in the main text is instead *rheonomic*, in the sense that it does not introduce additional fields, besides the “spacetime” component  $\beta_i$ . It corresponds to the choice  $B_{Aj}^i = 0$ ,  $B_j^i = a \epsilon_{lj}^i \beta^l + b \tau \delta_j^i$ .

The constraint equation (A1) is a 3-form in superspace, which, once decomposed along the supervielbein basis, yields the following conditions:

$$D_{[p} \beta_{q]} + 2\tau \beta^i \epsilon_{ipq} = 0, \quad (A3)$$

$$\begin{aligned} \nabla_A \beta_{[l} \delta_{m]}^i - 2\beta_j B_{A[l}^i \delta_{m]}^j - D_{[l} B_{Am]}^i \\ - \tau B_{Aj}^i \epsilon_{lm}^j + B_{A[l}^i \beta_{m]} - 2\tau \epsilon_{j[l}^i B_{Am]}^j = 0, \end{aligned} \quad (A4)$$

$$\text{Tr}(\nabla_A B_{Bk}^i) \epsilon^{AB} + (\bar{B}_{Aj}^i B_{Bk}^j) \epsilon^{AB} = 0, \quad (A5)$$

$$\text{Tr}(\nabla_{(A} B_{B)0k}^i \gamma_l) - (\bar{B}_{Aj}^i \gamma_l B_{B)0k}^j) = 0, \quad (A6)$$

$$\begin{aligned} D_k B_l^i - i\beta_j \delta_l^i \delta_k^j - 2\beta_j B_l^i \delta_k^j + \frac{1}{2} \text{Tr}(\nabla_A B_k^{Ai} \gamma_l) \\ - \frac{1}{2} (\bar{B}_{Aj}^i \gamma_l B_k^{Aj}) + 2\tau \epsilon_{jk}^i B_l^j = 0, \end{aligned} \quad (A7)$$

$$\nabla_A B_k^i = B_{Aj}^i \left( B_k^j + \frac{i}{2} \delta_k^j \right). \quad (A8)$$

Here,  $\text{Tr}$  denotes a trace along the spinorial indices, whereas  $(\ )_0$  indicates symmetric-traceless indices. The above derivation relies on the Fierz identity

$$\psi^A \bar{\psi}^B = -\frac{1}{4} \epsilon^{AB} \bar{\psi}_C \psi_D \epsilon^{CD} + \frac{1}{2} \gamma_l \bar{\psi}^B \gamma^l \psi^A, \quad (A9)$$

where  $\epsilon_{12} = \epsilon^{12} = 1$ . Notice that supersymmetry requires the parametrization of  $B^{(2)i}$  to contain at least one of  $B_{Aj}^i$  and  $B_j^i$ .

These relations gracefully reduce to only three conditions, (5), (6), and (7), in the case of the *rheonomic* parametrization (3).

*Appendix B: The torsionful spin connection.*—The spin connection of the bosonic solution (13) reads

$$\begin{aligned} \omega_0 &= \omega_{0\text{AdS}} + \frac{f}{L} \frac{(1-\xi)(2r^2 - r_+^2 - r_-^2)}{(r_+ + r_-)^2} (dt + Ld\phi) + \frac{2\theta\epsilon\sqrt{1-\xi^2}(1-LN)}{fL\xi(r_+ + r_-)} r dr, \\ \omega_1 &= \omega_{1\text{AdS}} - f \frac{2\theta\epsilon\sqrt{1-\xi^2}r}{L(r_+ + r_-)} (dt + Ld\phi) - \frac{2(1-\xi)}{fL\xi} dr, \\ \omega_2 &= \omega_{2\text{AdS}} + \frac{2\theta\epsilon\sqrt{1-\xi^2}}{\xi(r_+ + r_-)} dr + \frac{r}{L} \frac{(1-\xi)(2r^2 - r_+^2 - r_-^2)}{(r_+ + r_-)^2} \left( \frac{1}{L} - N \right) (dt + Ld\phi), \\ \omega_{0\text{AdS}} &= \frac{f}{L} (dt + Ld\phi), \quad \omega_{1\text{AdS}} = \left( \frac{1}{L} - N \right) \frac{dr}{f}, \quad \omega_{2\text{AdS}} = -\frac{r}{L} \left( \frac{1}{L} + N \right) (dt + Ld\phi). \end{aligned} \quad (B1)$$

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$$S = - \begin{pmatrix} e^{i\frac{\pi}{8}} & 0 \\ 0 & e^{-i\frac{\pi}{8}} \end{pmatrix}, \quad (33)$$

which is the spinorial transformation induced by (25).

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$$\partial_{[\mu} K_{(\pm)x|\nu]} = \pm \frac{1}{L} \sqrt{|g|} \epsilon_{\mu\nu\rho} K_{(\pm)x}^\rho.$$

In our solution  $\beta^\mu \propto \sqrt{1 - \xi^2} K_{(\theta)2}^\mu$ , where  $K_{(\theta)2}$  is a combination of  $K_t$  and  $K_\phi$  only.

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