PhD Research Thesis

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Teaching and learning eigentheory in university linear algebra: a multifaceted analysis through different theoretical lenses

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DECLARATIONS

Declaration of previous publication

This thesis includes the following seven original papers, which have been previously published/accepted or submitted for publication in peer reviewed journals or conference proceedings. I certify that I have obtained a written permission from the copyright owners to include the published materials in my thesis. I certify that the material describes work completed during my registration as a doctoral student at the University of Turin.

CH.	Reference	Status
1	Piroi, M. (under review). The emergence of the analytic-structural way of thinking in linear algebra as a blended space.	Under review
2	Piroi, M. (2023). Objectification processes in engineering freshmen while jointly learning eigentheory. In M. Trigueros, B. Barquero, R. Hochmuth, & J. Peters (Eds.), *Proceedings of the fourth conference of the International Network for Didactic Research in University Mathematics* (pp. 313-322). INDRUM2022.	Published 03/2023
3	Piroi, M., & Arzarello, F. (in press). Students' difficulties with eigenvalues and eigenvectors. An exploratory study.	Accepted for publication 17/05/2024
4	Piroi, M. (2023). Introducing eigenspaces: semiotic analysis and didactic engineering. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 2499–2506). Alfréd Rényi Institute of Mathematics and ERME.	Published 02/2024
5	Piroi, M., Barquero, B., & Bosch, M. (2024). Designing and analysing a teaching proposal about linear algebra through the dialogue of two theories. *International Journal of Mathematical Education in Science and Technology*, 1–25.	Published 04/2024

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Extra	Cattabriga, A., Gagliani Caputo, S., Piroi, M. & Vagnoni, D. (under review). Collective documentational genesis for teaching linear algebra: transforming online animated videos into curriculum resources.	Under review

Declaration of co-authorship

I hereby declare that this thesis incorporate material that results from joint research with the following authors:

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The extent of my contribution to the manuscripts as declared hereafter is based on the following scale:
A. The candidate did the majority of the work independently
B. The candidate has made a substantial contribution
C. The candidate has contributed to the work

I will specify the extent of my contribution only for papers with other authors.

Chapter 3. *Students' difficulties with eigenvalues and eigenvectors. An exploratory study.*
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Individual elements	Candidate's contribution
Formulation/identification of the scientific problem	A
Development of the key methods	A
Planning of the experiments and methodology design	A
Conducting the experimental work	A
Conducting the analysis of data	A
Interpretation of the results	A
Writing of the first draft of the manuscript	A
Finalization of the manuscript and submission	A

Chapter 5. *Designing and analysing a teaching proposal about linear algebra through the dialogue of two theories.*
Authors: Candidate, Author 2 & Author 3

Individual elements	Candidate's contribution
Formulation/identification of the scientific problem	A
Development of the key methods	B
Planning of the experiments and methodology design	A
Conducting the experimental work	A
Conducting the analysis of data	B
Interpretation of the results	B
Writing of the first draft of the manuscript	B
Finalization of the manuscript and submission	A

Chapter Extra. *Collective documentational genesis for teaching linear algebra: transforming online animated videos into curriculum resources.*
Authors: Candidate, Author 4, Author 5 & Author 6

Individual elements	Candidate's contribution
Formulation/identification of the scientific problem	A
Development of the key methods	A
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Conducting the experimental work	B
Conducting the analysis of data	B
Interpretation of the results	A
Writing of the first draft of the manuscript	B
Finalization of the manuscript and submission	A

I certify that I have properly acknowledged the contribution of other researchers to my thesis, and have obtained written permission from each of the co-author(s) to include the above materials in my thesis.

We, the undersigned, endorse the above stated contribution of work undertaken for each of the co-authored manuscripts contributing to this thesis.

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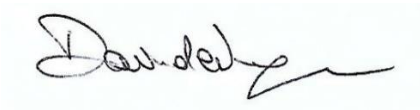
Davide Vagnoni	

General statement

I certify that, with the above qualification, this thesis, and the research to which it refers, is the product of my own work. I declare that, to the best of my knowledge, my thesis does not infringe upon anyone's copyright nor violate any proprietary rights and that any ideas, techniques, quotations, or any other material from the work of other people included in my thesis, published or otherwise, are fully acknowledged in accordance with the standard referencing practices.

Margherita Piroi

ABSTRACT

The aim of this thesis is to explore how geometric representations can be effectively integrated into the teaching and learning of linear algebra to enhance understanding of the underlying algebraic formal structure, rather than just emphasizing geometric properties. Specifically, this research focuses on promoting the learning of concepts related to eigentheory: eigenvectors, eigenvalues, and eigenspaces.

The main focus is on studying the teaching and learning processes of these concepts in linear algebra courses for undergraduate programs in STEM disciplines, such as engineering, where mathematics courses such as calculus and linear algebra are usually taught in the first years.

The first part of the thesis focuses on conducting a detailed analysis of students' understanding of eigenvectors and eigenvalues, highlighting the importance of different semiotic resources and their interrelationships in supporting conceptualisation. Additionally, it explores potential causes of difficulties in understanding these concepts, in particular how certain representations may contribute to common misconceptions. This analysis involves multimodal observation of the different semiotic resources used by students during two phases of a pilot study. Further theoretical constructs are employed to address specific research questions related to the issue under investigation.

Moving on to the second part of the thesis, attention turns to the institutional aspects associated with the problem under study. The compelling results obtained in the pilot study prompted a collaboration between myself and the course instructor involved to adapt the effective aspects of the pilot study activities to the actual teaching context. This context encompasses a linear algebra course for mechanical engineering students at a public Italian university, characterized by constraints that limit inquiry activities and productive class discussions. These constraints are discussed, together with the conditions conducive to the implementation of a teaching sequence on eigentheory, aiming to leverage the positive outcomes observed in the pilot study.

The theoretical framework in the first part of the thesis predominantly focuses on learning processes related to the use of different semiotic resources. The Semiotic Bundle theory serves as a foundational construct, complemented by other specific theoretical frameworks that are suitable for addressing different issues relevant to the overarching research goal. In the second part, the adoption of the Anthropological Theory of the Didactics made it possible to shift the lens of analysis to the institutional aspects that model the actual realization of a teaching sequence on eigentheory, whose design could make use of the observations made in the first part, together with other existing literature on the subject.

The integration of these two frameworks, despite their considerable differences in terms of unit of analysis and analytical tools, seemed imperative. As a result, a significant outcome of this thesis is the proposal of a strategy to coordinate, in the sense suggested by the construct of Networking of theories, the Semiotic Bundle theory with the Anthropological Theory of Didactics, leveraging their complementary nature to deepen an important but still understudied topic.

INTRODUCTION

*Only to an authoritarian mind
can the act of educating
be seen as a dull task.*

Paulo Freire

"Eigenvectors and eigenvalues is one of those topics that a lot of students find particularly unintuitive. Questions like, why are we doing this and what does this actually mean, are too often left just floating away in an unanswered sea of computations".

The quotation provided is the opening of a YouTube video created by the channel 3 Blue 1 Brown, which focuses on teaching higher mathematics from a visual perspective. The channel is highly popular among university students, boasting 6.07 million subscribers as of now (3Blue1Brown, 2015). The introductory segment of the video is part of a playlist titled "the essence of linear algebra" (3Blue1Brown, 2016), addressing an important yet understudied didactic issue related to university students' struggle to grasp the concepts of eigenvalues and eigenvectors. This difficulty is situated within the broader challenge students face in dealing with the formalism of linear algebra (Dorier, 2000).

The research project presented in this thesis stemmed from a dual and complementary interest.

Firstly, it was motivated by the recognition of linear algebra's challenging nature, particularly for first-year students in STEM degree programs at universities, a difficulty well-documented in the literature (Dorier, 2000; Stewart et al., 2019). In addition to this issue being recognized by existing research on the topic, during my experience as a tutor for various linear algebra courses, I was able to directly experience such student difficulties, from which a concrete need to find solutions to support their understanding emerged. A viable solution, as also recognized in the literature (e.g. Gol Tabaghi & Sinclair, 2013; Turgut et al., 2022; Andrews-Larson et al. 2017), seemed to be using dynamic geometric representations of linear algebra concepts via gestures or technological tools such as YouTube videos or GeoGebra applets. However, there is a crucial question of how to utilize these representations effectively to reinforce understanding of the underlying algebraic formal structure, rather than solely focusing on geometric properties. Such a risk, and the consequent need to explore an appropriate use of geometry in the teaching of linear algebra, has been the subject of debate and study for a long time (this issue will be dealt with in section 0.1.1 of this thesis) and, as confirmed also by recent literature, continues to be worthy of investigation.

On the other side, university instruction of mathematical subjects like linear algebra in STEM degree programs faces a notable absence of teaching methodologies informed by active pedagogy, despite the significant advancements made in this area at lower school levels in recent decades (Jaworski et al., 2021). Consequently, there is a pressing need to develop strategies that blend traditional lecture-based teaching with active inquiry-based learning methods, reflecting the evolving educational landscape, taking into account the institutional specificities of university instruction (Barquero et al., 2017; Godino et al., 2016).

In summary, this thesis arose from a dual interest: firstly, in exploring the integration of geometric representations in the study of linear algebra, and secondly, in devising a hybrid teaching sequence that effectively utilizes geometry in the specific context of university linear algebra courses.

I will begin this thesis by laying the groundwork, in Chapter 0, for the research phases and the corresponding results presented in the central chapters (from Chapter 1 to Chapter 6). This will involve describing the context and rationale for the research, outlining the overall theoretical framework and methodology used, and detailing the specific research questions that were raised.

CHAPTER 0 - OVERVIEW OF THE RESEARCH

0.1 Research on University Mathematics Education and on linear algebra teaching and learning

Over the past few decades, there has been a growing recognition of the challenges associated with teaching and learning mathematics at the university level (Durand-Guerrier et al., 2021). This phenomenon is evident in the emergence of a European and international community dedicated to research in University Mathematics Education (UME). This development has been facilitated by the establishment of dedicated thematic working groups at renowned international conferences such as CERME (Congress of the European Society for Research in Mathematics Education; Thematic Working Group 14) and ICME (International Congress on Mathematical Education). The need for fostering a community specifically focused on this area of research has led to the formation of the International Network for Didactic Research in University Mathematics (INDRUM), officially founded at CERME9 in 2015, and organizing conferences every two years since 2016.

Research in this field has become increasingly important. In today's academic landscape, where disciplines are becoming more fluid, mathematics serves as a foundation not only for degree programs specializing in pure and applied mathematics but also for a wide range of disciplines including engineering, computer science, natural sciences, social sciences, economics, and more (Durand-Guerrier et al., 2021). This highlights the growing importance of comprehending students' learning processes specific to the university level, along with the teaching methods that accompany them.

Research in UME, aligning with the trend in research in mathematics education in general, has traditionally pursued two complementary goals: elucidating the processes of learning and teaching mathematics, while concurrently striving to improve them (Reinholz et al., 2020). Nevertheless, teaching mathematics at the university level poses unique challenges compared to lower educational levels. University mathematics courses demand a higher level of abstraction and critical thinking. Students are expected not only to grasp mathematical concepts but also to develop problem-solving skills, mathematical reasoning, and the ability to effectively communicate mathematical ideas. Moreover, university mathematics courses typically cover advanced topics and delve deeper into theoretical aspects, which can be discouraging for some students. In addition to understanding the content itself, university mathematics teachers must also be mindful of the diverse backgrounds and learning styles of their students. This requires them to employ various instructional strategies and resources to accommodate different learning needs and promote student engagement and success. It is also to be taken into account the fact that basic mathematics courses are usually attended by a very high number of students, making it challenging to implement teaching strategies that deviate from traditional frontal lecturing. The advancement of educational technologies can play a crucial role in overcoming these difficulties. Hence, investigating their role is one of the current focuses of UME research.

The recognition of the pivotal role of investigating the institutional peculiarities of teaching and learning mathematics at this level, has produced what Artigue (2021) calls the “socio-cultural” turn. This shift has been characterized also by the widespread use of the macro-theoretical framework of the Anthropological Theory of the Didactics in UME research. As expressed by Marianna Bosch and colleagues (Bosch et al., 2004), adopting such a research perspective represents a significant change in the field, shifting the lens of analysis toward institutional practices that condition and limit, explicitly but mainly also implicitly, students' encounters with mathematical knowledge and their learning experiences. On the other hand, investigating students' understanding of mathematics, across various sectors and levels of depth, remains a significant focus in UME research.

The recently published book “Research and Development in University Mathematics Education”, edited by Viviane Durand-Guerrier, Reinhard Hochmuth, Elena Nardi and Carl Winslow (2021), provides an overview of UME research over the past two or three decades. In its different sections, it covers reflections and reports on: Professional development of university mathematics teachers; Interactions between mathematicians and researchers in UME; Teachers’ and students’ practices at university level; Teaching and learning of specific topics in university mathematics

(Calculus and analysis, discrete mathematics and logic, abstract and linear algebra, and the role of modelling in the use of mathematics in other disciplines).

I have presented a general introduction to existing research regarding UME. As this thesis centers on linear algebra instruction, the following section will offer a more detailed exploration of the evolution of this particular research trend in recent years."

0.1.1 A synthetic literature review on linear algebra teaching and learning

Research in this field began to flourish at the end of the '80s, when a number of scholars started investigating the common problem of students' difficulties with the study of linear algebra. The book edited by Jean-Luc Dorier, "On the Teaching and Learning of Linear Algebra" (Dorier, 2000), can be considered a cornerstone in this strand of research. In the book, the author starts by presenting the historical development of the knowledge area. In the second section, different authors present their theoretical frameworks and/or results of their research on the topic.

Dorier, Robert, Robinet, and Rogalski (Dorier et al., 2000), through the description of various experimental studies conducted between 1987 and 1994, introduce the idea of "the obstacle of formalism in linear algebra". According to the authors, students find the subject particularly challenging due to its epistemic nature and the culmination of formalization efforts. This difficulty is emphasized by students' lack of prior knowledge in logic and elementary set theory, which are essential for understanding and applying linear algebra concepts. Moreover, because of the axiomatic approach used to introduce it, the majority of linear algebra concepts remain in a "concept-object" (p. 93) state, and, besides for solving systems of linear equations, students almost never encounter their actual "tool-like" (ibid.) character. Thus, students' difficulties are likely to be emphasized by the fact that the utility of linear algebra remains hidden to them. Later chapters in the book introduce the problem of the importance and simultaneous difficulty of moving between different representations of linear algebra concepts. In particular, Hillel (2000) distinguishes between different "modes of description" of the basic objects and operations of linear algebra. They include:

• The *abstract mode*, that uses the language and concepts of the general axiomatic theory, as vector spaces, subspaces, linear dependence, span, kernels, etc.

• The *algebraic mode*, using the language and concepts of the theory as specifically related to the vector spaces R^n, as n-tuples, matrices, rank, etc.

• The *geometric mode*, whose language and concepts refer to the 2- and 3-dimensional geometric space, with lines, points, points of intersections, directions, geometric transformations, etc.

Sierpinska (2000) also focuses on the problem of coordinating different representations in linear algebra. She does not refer only to the language or type of representation used but actually distinguishes between what she calls "three modes of thinking": the *synthetic-geometric*, in which a concept is thought about in terms of its geometric properties, the *analytic-arithmetic,* that focuses on the numerical components of the elements of vector spaces and to algebraic ways to compute them, and the *analytic-structural,* in which structural entities are considered and formal language is used. According to the author, while these three modes appeared sequentially in the historical development of mathematics, they should not be regarded as sequential steps or stages in students' development of algebraic thought. Oppositely, they should all be presented in the teaching of linear algebra and considered as equally useful for students' understanding, particularly when they are in interaction.

In the last decade, the study of linear algebra teaching instruction has notably advanced. The growing interest in the subject led to a related thematic discussion at the 13th International Congress in Mathematics Education (ICME 13), which in turn resulted in the publication of the ICME 13 series volume "Challenges and Strategies in Teaching Linear Algebra" (Stewart et al., 2019). The book involved the collaboration of 18 authors from nine different countries. Another publication showing the international collaborative effort in advancing the field is the ZDM special issue titled "Research on teaching and learning in linear algebra", published in 2019. This issue comprises 16 research-based papers, preceded by a literature review on the topic up to that year (Stewart et al., 2019).

Although published papers on the topic cover a wide range of specific topics and theoretical frameworks employed, there are a few theoretical frameworks that have been particularly used to investigate linear-algebra related didactical issues. For example, a significant number of researches draw on APOS[1] theory to examine students' dealing with linear algebra concepts (e. g., Trigueros, 2018; Altieri and Schirmer, 2019. Other works are reported in Stewart et al., 2019). An American team of researchers has widely used the Realistic Mathematics Education framework (Gravemeijer, 1999) to design tasks and teaching scenarios and to evaluate their impact on students' understanding of different concepts (e. g., Wawro et al., 2012; Zandieh et al., 2017; Mauntel et al., 2024). These studies have been conducted in the context of the project "IOLA", which stands for "Inquiry-Oriented Linear Algebra". The project, a collaboration between Virginia Tech, San Diego State University, Arizona State University and The Florida State University, focuses on developing teaching materials[2] (Wawro et al., 2013) consisting of task sequences aimed at allowing an inquiry-oriented approach to the teaching and learning of linear algebra. Drawing on RME, the tasks have been designed to facilitate students' engagement in such a way that their autonomous mathematical activity can serve as a basis for the development of formal mathematical knowledge.

Besides experimental research on the teaching and learning of specific concepts, there are transversal issues that have been studied. For instance, a widely discussed topic in this area of research is whether and to what extent the use of geometric and visual representations can support students' understanding of linear algebra concepts. This debate has ancient roots and a long history, involving not only theoretical research in mathematics education but also significant school reforms (Gueudet-Chartier, 2006). In France, during the 1960, extensive discussions occurred regarding the incorporation of geometry into the teaching of linear algebra. Two opposing views emerged: one suggesting that geometry should be introduced by means of an axiomatic system and only then used as an intuitive introduction to linear algebra, and the other,

[1] The theory has been developed by Dubinsky (Cottrill et al., 1996). The acronym stands for "Action, Processes, Objects and Schemas".
[2] Materials available at http://iola.math.vt.edu.

supported by Dieudonné (Dieudonné, 1964), one of the leading exponents of the Bourbaki group, proposing that linear algebra should be introduced immediately, as geometry is simply an application of it. During the "modern mathematics" reform, the latter position was accepted, guiding the revision of the curriculum to include linear algebra already in high school. Both Gueudet-Chartier (2004) and Harel (2019) report that researchers that have studied the use of geometry for teaching linear algebra have demonstrated both its usefulness in aiding students and the specific difficulties it can generate.

Confirming the importance of exploring the topic further, Harel (2019) distinguished a variety of ways in which geometry can be used in the teaching of linear algebra, with the hope that this classification would facilitate the examination of the benefits and risks of each approach. Among these ways, Harel considers *generalization*, that refers to cases in which geometric content is used in linear algebra instruction to generalize concepts. In this case, a question of critical importance is whether students are able to carry out the generalization from a geometric context to a linear-algebraic concept. “If a student does not successfully make the targeted generalization (i.e., the generalization intended by the instructor), then her or his understanding of the respective linear-algebraic concept is likely to remain confined within the geometric setting” (Harel, 2019, p. 1034). *Reduction* occurs when a newly introduced general concept or theorem is applied in a geometric context, often within the coordinate plane or space. A common risk with reduction is that students are able to follow the description in the geometric model, but their understanding of it remains restricted to the given model. Other specific ways of employing geometric representations in linear algebra instruction, as outlined in Harel’s work, will not be detailed here, but their names suggest their distinctiveness: application, metaphorical representation, literal representation and *spatially-based terminology*. On a more general level, Harel adds also s*elf-contained investigation*, which refers to the organization of a full program for linear algebra presenting content mainly in the context of two- and three-dimensional Euclidean spaces.

Several recent papers by various authors have indicated a positive effect of using geometry, particularly dynamic geometry, on student learning. For instance, Gol Tabaghi and Sinclair (2013) demonstrated the beneficial effect of using dynamic geometry software to explore the concepts of eigenvalue and eigenvector. Additionally, Thomas and Stewart (2011), Turgut et al. (2022), and Andrews-Larson et al. (2017), among others, have found that emphasizing the visualization of linear algebra concepts, also through the use of dynamic geometry software, enhances students' understanding of the subject. Indeed, the debate on the role of geometry in the teaching of linear algebra has been revitalized by an increasing number of studies analyzing the role of digital technologies in providing dynamic geometric representations. Nonetheless, the role of geometry and its potential for enhancing the understanding of linear algebra concepts without compromising their abstract nature still remain key points deserving further investigation.

0.2 Rationale, aim of the study and main research questions

The interest in contributing to the study of the specific issue of how to integrate geometrical representations in linear algebra instruction in a way that can support students' structural understanding of the concepts, guided my research. The wider aim can be then described as investigating which role the use of geometry can have, and particularly of dynamic visual representations, in the teaching and learning of linear algebra.

Recognizing the complexity and the size of the research problem, I decided to limit it to the study of a single topic: eigentheory. Eigentheory, a fundamental yet challenging concept in linear algebra, holds significant relevance across STEM disciplines. Given its complexity and the common struggle students face in grasping it (that will be deepened much more in Section 0.3) investigating its teaching and learning processes, specifically with respect to the issue of the utility and limits of using geometric representations, becomes particularly interesting.

The primary general research question guiding this project is:

In what ways can the use of geometric representations of eigenvectors and eigenvalues support students' structural understanding of these and related concepts?

(GQ1)

With "structural", I intend that we are interested in students' understanding of the abstract properties of concepts, that is those that are shared by all vector spaces. Thus we are interested in studying the integration of geometry in a way that can actually support an abstract understanding of eigenvectors and eigenvalues, avoiding the risk that students take them as prototypical examples, without distinguishing the mathematical relevant features. As Sierpinska (2000) points out, the risk is concrete that "the visualizations provided by the teacher are taken holistically by the students without their trying to discriminate between the mathematically relevant features and those contingent on the technical support used to create them" (p. 229).

The project started with a pilot study aimed at addressing this question, comprising two phases. These phases have been analyzed using different theoretical lenses, each contributing to the exploration of GQ1. In Section 0.5 and in Chapters 1, 2 and 3 the pilot study and the different results will be presented.

Building upon the pilot study results and insights gleaned from existing literature (see Section 0.3.3), we developed a teaching sequence aimed at investigating in what way these findings could have an impact on the actual teaching of eigentheory in a linear algebra course for an engineering degrees, considering specific contextual conditions and constraints. Consequently, the research scope expanded to encompass institutional aspects related to GQ1, leading to a second general research question:

What teaching scenario can support a meaningful integration of geometry in the teaching of eigentheory in a linear algebra course for engineering? What conditions can support and what constraints can hinder such implementation?

(GQ2)

With the terms "meaningful integration" we intend "the integration of geometry in a way that can actually support an abstract understanding of eigenvectors and eigenvalues, avoiding the risk that students take them as prototypical examples, without distinguishing the mathematical relevant features", as stated above.

What has been presented here, is only a synthetic overview of the research aim and general research questions. In Section 0.4 I will describe in more detail the structure of the research, the theoretical frameworks used and the methodology employed. For each step of the research (described in different chapters of this thesis), I will then specify the theoretical frameworks adopted and, in relation to these, the specific research questions addressed in each chapter.

Before moving on to this theoretical and methodological section, it is helpful to contextualize the study with another preliminary introduction of eigentheory and its epistemological and didactic characteristics.

0.3 Preliminary analysis of eigentheory teaching and learning

Eigenvalues are commonly first encountered within the realm of linear algebra or matrix theory. Nevertheless, their origins trace back historically to investigations into quadratic forms and differential equations. It is specifically in the field of celestial mechanics that eigentheory saw its emergence.

In the 18th century, various scientists, including Euler (Euler, 1871), were deeply involved in the study of the rotational motion of rigid bodies, leading to the discovery of the importance of the principal axes of rotation. In the following century, Augustin-Louis Cauchy continued these studies and aimed to show, in modern terms, that a symmetric matrix has real eigenvalues. In a note to the Paris Academy of Science, Cauchy wrote:

> It is known that the determination of the axes of a surface of the second degree or of the principal axes and moments of inertia of a solid body depend on an equation of the third degree, the three roots of which are necessarily real. However, geometers have succeeded in demonstrating the reality of the three roots by indirect means only... The question that I proposed to myself consists in establishing the reality of the roots directly... [Hawkins, 1975, p. 20]

Three years later, in 1829, in the paper "Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planétes", he succeeded in a direct proof of this fact (Tou, 2018).

In "Mémoire sur l'integration des équations linéaires" (Cauchy, 1840), Cauchy presented his method for solving a system of linear, first-order differential equations with constant coefficients, equivalent to what is today called an eigenvalue problem. In this work, Cauchy introduced the term “characteristic equation”. As for what we now call “eigenvalues”, Cauchy referred to them as “valeurs propres” (proper values).

The first to use the prefix “*eigen*” was David Hilbert. The term appeared in his 1904 paper "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen" ("Principles of a general theory of linear integral equations"), aimed at providing a general mathematical theory for systems of linear equations, largely separated from the theory of celestial mechanics (Tou, 2018). Hilbert’s work on general relativity and quantum mechanics was highly influential. The prefix “*eigen*” continued appearing in his later works. In "Die Grundlagen der Physik" ("Foundations of Physics"), he coined the term “Eigenzeit” (literally “eigentime”) to refer to eigenvalues of a particular matrix. In "Über die Grundlagen der Quantenmechanik" ("Foundations of Quantum Mechanics"), in the analysis of the Schrödinger wave equation, he used the terms “Eigenwert” (eigenvalue) and “Eigenfunktion” (Eigenfunction). Hilbert’s student, John von Neumann, translated the prefix *eigen* into “proper”. For a while, before the terms became definitely as they are today, eigenvalues were “*proper values*”, eigenfunctions “*proper functions*” and so on.

0.3.1 Definitions, properties and use of eigentheory in STEM domains

The context and rationale for the emergence of eigentheory have undergone significant reduction in its current didactical transposition. Eigenvalues and eigenvectors are commonly introduced in linear algebra courses, typically after covering other preliminary concepts such as linear dependence, basis, linear transformations, matrices of linear transformations and determinants.

As a brief recapitulation, given an endomorphism T (a linear transformation where the domain and codomain match) of R^n, if there exists a nonzero vector v for which there exists a value v such that T$v = \lambda v$, then v is an eigenvector for T and λ the eigenvalue associated with it.

Narrowing our focus to n-dimensional finite spaces, T can be represented by an $n\ x\ n$ matrix M. Consequently, $Tv = \lambda v$ can be rewritten as $Mv = \lambda v$. This equation holds if and only if $(M - \lambda I)v = 0$ has non trivial solutions (where I is the $n\ x\ n$ identity matrix), which occurs when the determinant of $M - \lambda$I is zero. This leads to a polynomial in λ, called the *characteristic polynomial*, defined by $det(M - \lambda \mathrm{I}) = 0$. The roots of this polynomial are the eigenvalues λ of the matrix M (and thus of the endomorphism T).

In a traditional teaching sequence, eigenvalues, eigenvectors and how to find them are usually introduced in this manner. Eigenspaces are then defined as vector spaces spanned by independent eigenvectors associated with a specific eigenvalue. Subsequently, the process of finding eigenvectors, and eigenspaces of a matrix M is explained to show how they can be used to diagonalize M, i.e., to find a new diagonal matrix D similar to M.

This is the common sequence in which eigentheory is usually introduced, but one could reverse this approach by starting with finding a way to diagonalize a matrix M representative of a linear transformation T. This approach emphasizes the need to find a basis of vectors $(v_1, v_2, \dots, v_n)$ such that M with respect to that basis is diagonal. Each v_i must satisfy $T(v_i)$ being equal to a multiple of v_i. From here, the need to find vectors v such that there exists a λ such that $Tv = \lambda v$ and this leads back to the problem of finding eigenvectors and eigenvalues.

The topic of eigenvectors could be also approached from a geometric perspective. Considering a two or three-dimensional geometric transformation, a vector is an eigenvector for that transformation if, under its effect, it gets stretched or shrunk while maintaining the same direction. The associated eigenvalue is the ratio between the image vector and the starting vector.

If it is true that a comprehensive formulation of eigentheory, with its main definition and theorems, is typically presented in the linear algebra courses taught in the first years of undergraduate STEM programs, it is in other domain-specific disciplines that this theory finds its applications (Wawro et al., 2019). For instance, within mathematics, eigentheory is applied in the study of stochastic processes, Markov chains, predator-prey models, and the connectivity of graphs and digraphs. In civil and mechanical engineering, eigenvectors are utilized to describe the modes of vibration and the propagation of waves in physical systems, such as musical instruments or electromagnetic fields, as well as physical structures. Eigentheory also plays a significant role in quantum mechanics, where it is used to determine the possible measurements of observable spin or energy systems (Wawro et al., 2020). In the context of computer science and data analysis, eigenvectors are employed for Principal Component Analysis (PCA) to compress data while preserving the most significant information (Paul et al., 2013).

0.3.2 Epistemological analysis and didactical phenomena related to eigenvectors and eigenvalues

There are some aspects in the typical presentation of this topic in a linear algebra course that, in our opinion (guided also by direct experience), are crucial for an epistemological analysis of the topic:

- Firstly, it is noteworthy that eigenvalues are usually presented before eigenvectors. This is particularly surprising, as it would seem more natural to first consider those vectors whose image is their multiple (eigenvectors) and only then consider the factor of multiplication (the eigenvalue). Thus, students are asked to find eigenvalues without knowing for which eigenvectors they are searching them, or even if such eigenvectors exist. Borrowing terminology from Game Theory, finding the eigenvalues of a linear transformation can be considered a “game of imperfect information” (Ross, 2024). This is especially evident when considering the geometric interpretation of eigenvectors and eigenvalues.

- In the formal definition, the formula $Tv = \lambda v$ appears particularly "static". This means, roughly speaking, that it is likely to be interpreted as "Tv must be λv". Consequently, the "search" for vectors such that their image is a multiple of them is not thus intended.

- The high level of formalism in the definition, including logical operators such as $\exists$, further complicates the correct interpretation of the definition (these aspects will be described in detail in Chapter 3). In Figure 0.1, an example of classical definition retrieved from a linear algebra book is depicted. Note that existence is explicitly required only for v, but as a matter of fact it is implicitly required also for λ. This is likely to make the interpretation of the definition even more complicated.

  5.5 **Definition** ***eigenvalue***

  Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an ***eigenvalue*** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

  Figure 0.1. Definition of eigenvalue as presented in (Axler, 2015)

- Considering the geometric interpretation of eigenvector and eigenvalues, it is important to mention that different representations of linear transformation convey different information. Static images lack much information compared to dynamic ones. Regarding the latter, there are different possibilities for representing linear transformation and consequently of visualizing eigenvectors and eigenvalues. We provide examples of two different dynamic representations for the two possibilities in R^2.

  The first possibility is to represent a vector u and its image $B(u)$ simultaneously, as demonstrated in the GeoGebra applet in Figure 0.2. With the movement of the vector, the image moves accordingly. In the general position (Fig. 0.2a), a vector and its image are not aligned. In the case the vector and the image overlap, then v is an eigenvector (Fig. 0.2b).

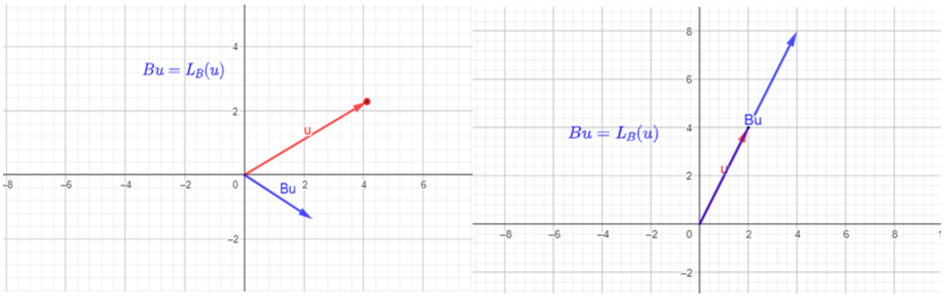


Figure 0.2 (a and b). First possibility of dynamic visualization of eigenvectors

The second possibility is to represent a linear transformation as a movement with a time component. This means that the representation shows the vectors and then illustrates them morphing and moving as they reach the position of their image. This strategy is employed by Grant Sanderson, the creator of the YouTube channel "3Blue1Brown". In his video playlist "The essence of linear algebra" he illustrates key concepts of linear algebra. For linear transformations, he uses a grid where he represents the vectors. To illustrate the effects of the linear transformation, he shows how the grid stretches and how the vectors morph and move to become their image. In this case we see general vectors changing size and direction, while eigenvectors remain in the same direction but can be stretched or shrunk (Fig. 0.3). This method of visualizing linear transformations uses a temporally dynamic situation to describe an abstract concept that inherently lacks spatial and temporal features. Talmy (1996) refers to this phenomenon as "fictive motion", a cognitive mechanism through which a static concept is conceptualized in terms of a dynamic entity. A classic example are sentences such as "The road runs through the forest", where the term "runs" has a metaphorical connotation. Fictive motion underlies numerous mathematical concepts (Lakoff & Núñez, 2000). It would be certainly interesting to further analyze the phenomenon of fictive motion with respect to the way linear transformations are represented and imagined, and its impact on the conceptualization of eigenvectors.

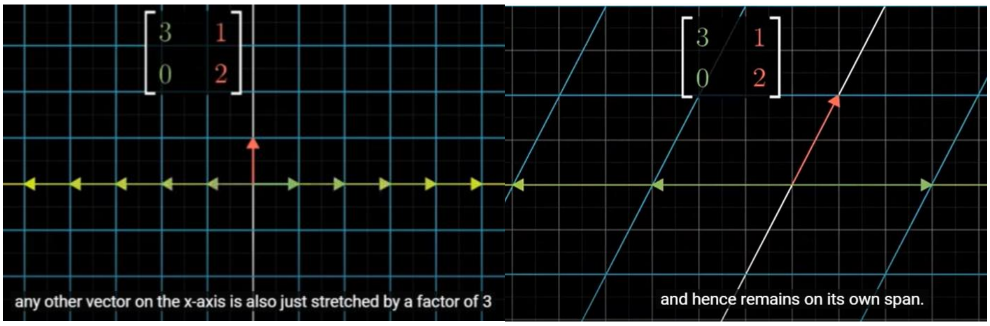


Figure 0.3 (a and b). Second possibility of dynamic visualization of eigenvectors, as in 3Blue1Brown YouTube videos.

0.3.3 Existing studies on the topic: a literature review

While interest in the teaching and learning of eigentheory is relatively recent, the topic has garnered significant attention among researchers studying the didactics of linear algebra, leading to numerous studies investigating related issues.

Stewart and Thomas (2011) used Tall's 'three worlds' to explore students' understanding of eigentheory. Extending the APOS (Action, Process, Object, Schema) theory of Dubinsky and others (Cottrill et al., 1996; Dubinsky & McDonald, 2001), Tall (2008) describes learning as taking place in three worlds: the *embodied* world, where we use physical attributes of concepts and what our sensory experience can capture; the *symbolic* world, where symbolic representations of concepts are used and manipulated; the *formal* world, where properties of objects are described axiomatically.

Stewart and Thomas point out that if it is true that eigenvalues and eigenvectors are introduced into the formal world via their definition with the formula $Tv = \lambda v$, then their embedded symbolic form allows students to move quickly to manipulations in the symbolic world without the need to fully grasp the meaning of the definition. In this way, "the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust" (Stewart & Thomas, 2011, p. 280).

The two authors confirm that phenomena associated with the elements previously highlighted in the epistemological analysis can hinder students' understanding of these concepts. For example, they note that the explanation of what an eigenvector is can start with it as an object and then explain the effects of applying a transformation to it, with the result that its image keeps the same direction. They also note that the fact that eigenvalues are usually presented before eigenvectors can be a problem for students, since “to find the eigenvector one must first find its associated eigenvalue, holding in abeyance any action to be performed upon the eigenvector until it’s found” (p. 280). In summary, the study shows that the students involved, obviously influenced by the way the subject was presented to them, tend to think about the concepts of eigenvector and eigenvalue primarily in a symbolic way. As a result, they are unable to conceptualize these notions and to apply them even in simple situations. They also show difficulty which is very common among students, that is they do not understand the algebraic progression from $Tv = \lambda v$ to $(T - \lambda I)v = 0$.

Sinclair and Gol Tabaghi (2010) investigated how the meaning of eigenvectors and eigenvalues is constructed through communicative activities involving talk, gestures and diagrams. They focus on contexts that are not pedagogically intentional. Specifically, they conduct and analyze interviews with six mathematicians in the mathematics department of a Canadian university who are interested in both pure and applied mathematics. The authors show how the meaning of eigenvectors, as communicated by the interviewed mathematicians through language and gestures, depends strongly on both temporal and motional dimensions of thinking that are absent from written, formal definitions of the concept. They therefore extend their interest in investigating the relationships between the ways in which mathematicians communicate through speech and gesture and the written diagrams they adopt, adopting Chatelet's vision of gesture as the central mediating link between the body and diagrams. Recognizing the limitations of static diagrams in representing time and motion, they suggest that in the absence of more dynamic representations, teachers should pay more attention to providing different and more fruitful diagrams. At the same time, the authors point out that digital technologies are increasingly available and that, as their findings suggest, engagement with dynamic representations of geometry could lead to the use of many more and qualitatively different forms of gesture, presumably enhancing an embodied understanding of the concepts.

Indeed, the two authors then shifted their investigation to the role of dynamic geometry software (DGS) in the dynamic representation of eigenvectors. In Gol

Tabaghi and Sinclair (2013), they analyze the thinking of undergraduate mathematics students as they explore eigenvectors and eigenvalues in interaction with the DGS sketchpad. Using Sierpinska's framework of three modes of thinking in linear algebra, the authors show that the use of the 'eigen-sketch', together with interaction with the interviewer and appropriately designed tasks, enabled students to develop a degree of flexibility between the synthetic geometric and analytic-arithmetic modes of thinking. In particular, the dynamic features of the software used enabled the students to reconstruct mental objects by imposing movement on them, even when they were not using the software. For this reason, the two authors specifically characterized this mode of thinking, supported by dynamic geometric representations, as *dynamic-synthetic-geometric.*

Other researches (Çağlayan, 2015; Beltran-Meneu et al., 2017; Orozco-Santiago et al., 2019) have shown how exploration of eigentheory through dynamic geometry software can help students develop a robust geometric understanding of eigenvectors and eigenvalues.

Çağlayan (2015) presents a qualitative case study wherein mathematics majors were interviewed to explore their capacity for visualizing the properties of vector spaces and eigenvalue-eigenvector relationships. During the interviews, students were asked to utilize GeoGebra to determine the eigenvalues and eigenvectors of given 2×2 matrices and to articulate their problem-solving procedures. Students participating in the study autonomously chose to examine multiple representations of the eigenvalue problem by selecting 2x2 matrices with different properties. They traced the trajectory of the matrix-applied product vector using the GeoGebra tool, enabling trace functionality, and categorized the behavior of the trajectory (ellipse, circle, line) in coordination with the matrix type under consideration. The analysis employed Sierpinska’s framework on the three modes of thinking in linear algebra (refer to Section 0.1.1) to examine the interplay of these modes in students’ process of solving the eigenvalue problem. The study reveals that when students primarily relied on the synthetic-geometric approach, they needed to integrate analytic thinking to solve problems involving special matrices. For instance, when confronted with a singular matrix, students initially did not recognize or expect that one of the eigenvalues would be zero using only the synthetic-geometric approach. However, their successful analytic approaches complemented their visualization. The study thus supports the idea that synthetic–analytic interactions are fundamental to understanding the concepts of eigenvalue and eigenvector, and that the utilization of dynamic environments such as GeoGebra facilitates such interactions.

Beltran-Meneu and colleagues (2017) present a teaching proposal aimed at architecture students, designed to enhance their geometric understanding of eigenvalues and eigenvectors by demonstrating physical applications of these concepts and integrating dynamic representations through GeoGebra. Following the implementation of the teaching proposal, all students underwent a test to assess whether it had a positive impact on their geometric comprehension of eigenvectors and eigenvalues. It was observed that students who participated in the teaching proposal demonstrated a greater ability to integrate a geometric perspective on eigenvectors and eigenvalues, concepts that are typically introduced algebraically. The authors report that this teaching approach has proven effective in helping students establish connections between the three worlds of mathematical thinking as outlined by Tall (2008).

Salgado and Trigueros (2015) adopted a different perspective to introduce eigentheory. They present a classroom experience in which they introduced eigenvectors, eigenvalues and eigenspaces through a modelling problem and activities designed on the basis of APOS theory. Specifically, the authors designed an employment-unemployment problem in an economic context that requires the use of eigenvectors and eigenvalues for its solution. The study showed that the introduction of an appropriate modelling problem got students involved in its solution and developed strategies that promoted the use of previously constructed linear algebra concepts. The design of the proposed activities was based on the genetic decomposition in Figure 0.4, with the steps that the authors believe are necessary to construct the concepts involved.

Genetic decomposition is a model employed by APOS theory to predict how a particular concept is constructed. It comprises a detailed description of the actions and processes necessary for constructing the concept of interest, along with coordinating these processes into new ones or encapsulating[3] them in an object. Its design may stem from an epistemological or historical analysis of the mathematics involved in learning the concept, an analysis of the literature on the concept, the researchers' teaching experience, or a combination of these factors.

In the genetic decomposition proposed by Salgado and Trigueros, the prior knowledge required to construct the basic concepts of eigentheory is outlined in the second line: span, vectors, matrix, and systems of equations. Actions are performed on a matrix and a vector to obtain their product (Av and λv) and to derive the resulting new vector. These actions are internalized into a process wherein students view the outcomes as transforming a vector into a new parallel vector, associating the scalar of the product with the vector's direction (third line of Figure 0.4). Both processes must be coordinated to achieve the equality ($Av = \lambda v$). The resulting equation is encapsulated as an object, allowing the scalar and vector within it to be named as eigenvalue and eigenvector, respectively (fourth line of Figure 0.4). Students engage in actions on the resulting equation to determine the conditions the scalar must satisfy for the equation to have solutions. The process concludes with the encapsulation of the concept of eigenspace using the concept of span.

[3] According to APOS theory, encapsulation occurs when students apply actions to a process and are able to conceive of it as a whole. In this case, they encapsulate the process and construct a cognitive object.

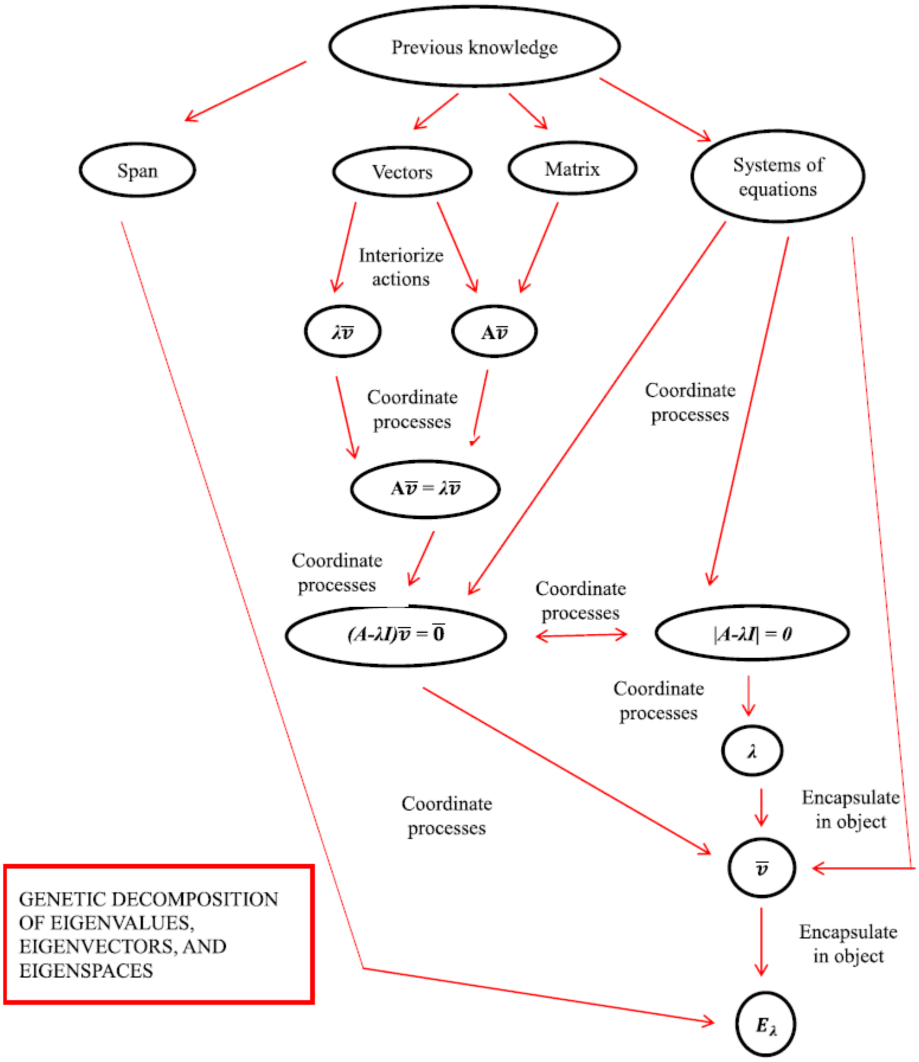


Figure 0.4. Genetic decomposition of eigenvalues, eigenvectors and eigenspaces (Salgado & Trigueros, 2015)

In the context of the aforementioned IOLA project, which is rooted in the curriculum and task design guidelines of the Realistic Mathematics Education theory (Gravemeijer, 1999), Plaxco and colleagues (Plaxco et al., 2019) present and analyze students' engagement in an instructional sequence designed to support students' development of the concepts of eigenvectors, eigenvalues and eigenspaces. The sequence followed the implementation of the unit on linear transformations, where students completed a series of tasks to determine matrices for various transformations based on descriptions of their effects on specific inputs. In the sequence for introducing eigenvectors, students were presented with a series of problems (see Figure 0.5) that required them to coordinate algebraic representations of matrices with information regarding stretching directions and factors of the relative geometric transformations. As students worked through these problems, they could reinvent strategies for determining eigenvectors, eigenvalues, and the characteristic polynomial.

1. The transformation defined by the matrix $A = \begin{bmatrix} 1 & -8 \\ -4 & 5 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in the directions $y = \frac{1}{2}x$ and $y = -x$. Figure out the factor by which anything in the $y = \frac{1}{2}x$ direction is stretched and the factor by which anything in the $y = -x$ direction is stretched.
2. The transformation defined by the matrix $B = \begin{bmatrix} -8 & 2 \\ -55 & 13 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in one direction by a factor of 3 and some other direction by a factor of 2. Figure out what direction gets stretched by a factor of 3 and what direction gets stretched by a factor of 2.
3. The transformation defined by the matrix $C = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in two directions. Find the directions and the factors by which it stretches in those directions.

Figure 0.5. Problems proposed to students by (Plaxco et al., 2019)

In recent years, the study of eigentheory teaching and learning has continued to attract diverse research, with the focus also shifting to other aspects, such as the properties of eigentheory-related concepts, teachers' practices, and the use of different digital environments, rather than just DGSs. For example, Wawro and colleagues (Wawro et al., 2019) have investigated students' reasoning about eigenspaces. Stewart and colleagues (Stewart et al., 2019) have focused on teachers' didactic choices related to the use and the interplay of Tall's three worlds in their teaching by analyzing a mathematician's teaching journal. Regarding the use of digital technologies, Albano and colleagues (Albano et al., 2019) have analyzed how the use of a Matlab LiveScript in a linear algebra course for an engineering degree can promote students' reasoning and construction of conceptual structures when solving mathematical tasks related to the diagonalization of matrices.

In the next section, we will delve into the theoretical and methodological apparatus employed for design and analysis in the overall project.

0.4 Theoretical framework and methodology

As described in Section 0.2, the research project started with the intention of exploring an educational issue: that of eigentheory teaching and learning in university STEM courses. Although the literature on this subject is growing, it remains limited. Consequently, my project began with a pilot study, aiming to investigate further students' difficulties in comprehending eigenvectors, eigenvalues, and related concepts, specifically within the context of mathematics courses in Italian universities. The purpose of the pilot study was to enrich the knowledge already present in the literature on this subject, and then, on the basis of this, to implement a suitably designed teaching intervention. The research as a whole fits into the broad framework of *design research*, that is, in the definition given by Henrick et al. (2015): "a family of methodological approaches in which research and the design of supports for learning are interdependent" (p. 497). Specifically, I used the methodology of *didactical engineering* (Artigue, 2015), that I will introduce in this section.

As just mentioned, the initial phase of the project aimed to deepen our understanding about the teaching and learning of eigentheory (see GQ1). Specifically, I was interested in investigating the role of the semiotic resources used in these processes. Embracing a multimodal perspective, I explored different kinds of resources used in teaching and learning and considered their interdependence. The theoretical framework that was chosen prior to data collection is the Semiotic Bundle Theory (Arzarello, 2006). This framework guided the pilot study, influencing the selection of data to be collected and analyzed in the initial phase. Unexpected aspects surfaced from these data, prompting the consideration of enlarging the theoretical framework, including theoretical lenses different from those initially envisioned. Analyses conducted through these alternate lenses not only confirmed but also deepened the findings present in the literature. These insights, along with a comprehensive institutional and epistemological analysis, facilitated the co-design of an intervention with the teacher of a linear algebra course in the first year of a mechanical engineering degree program at a public Italian university. Recognition of the real context of the study, with its inherent conditions and constraints, was crucial to both the design and subsequent analysis of the results. Consequently, the Anthropological Theory of Didactics (Chevallard, 2015) – from now on referred to as ATD - was introduced as an additional theoretical framework. In the final analysis, coordinating the lenses of the Semiotic Bundle and of the Anthropological Theory of the Didactics proved particularly helpful in interpreting the observed phenomena.

This section will first provide a detailed presentation of the didactical engineering methodology across its various phases. Following that, I will introduce the different theoretical frameworks adopted in the pilot study, as well as key concepts of ATD. Subsequently, I will describe the methodology of the networking of theories that will be useful for describing how the different theoretical approaches are combined in the analysis. Lastly, I will outline the flow of the entire project, specifying the theoretical lenses used for analysis at each step in didactical engineering and introducing the structure of the central chapters of this thesis.

0.4.1 Description of the Didactical Engineering methodology

Didactical Engineering (DE) has its roots in the context of what has been often called the "French didactical culture" (Artigue, 2014), in which design has always played an important role. Specifically, DE emerged as a research methodology in the early '80s, closely connected with the Theory of Didactical Situations (TDS) initiated by Brousseau (1997). The development of DE was driven by the desire among French didacticians to establish a methodology aligned with the specific purpose of the didactics of mathematics (according to TDS): studying the intentional dissemination of mathematical knowledge through purposefully designed didactical systems.

In essence, DE was introduced to address the need for a methodology that could facilitate productive relationships between research and practice, by enabling researchers to examine didactical systems in their concrete functioning. Michèle Artigue specifies this role of DE as an intermediary between the science of didactics and the reality of the classroom, in the Encyclopedia of Mathematics Education (Artigue, 2014):

> The idea of didactical engineering (DE) […] contributed to firmly establish the place of design in mathematics education research. Foundational texts regarding DE such as (Chevallard, 1982) make clear that the ambition of didactic research of understanding and improving the functioning of didactic systems where the teaching and learning of mathematics takes place cannot be achieved without considering these systems in their concrete functioning, paying the necessary attention to the different constraints and forces acting on them. Controlled realizations in classrooms should thus be given a prominent role in research methodologies for identifying, producing and re-producing didactic phenomena, for testing didactic constructions. (p. 4)

The methodology of DE is structured into different phases: preliminary analyses; conception and a priori analysis; realization, observation and data collection; a posteriori analysis and validation.

Preliminary analysis lays the foundation for the whole design and takes into consideration different aspects that together contribute to the choices shaping the activity conception. The principal dimensions studied at this preliminary stage are the epistemological, the institutional, and the didactical ones.

- The *epistemological* questioning of the mathematical content involved is a pivotal aspect in both TDS and ATD. Indeed in the preliminary stage of a DE, the quest for a rationale behind each piece of knowledge to be taught is a fundamental step helping fix the precise goals of the DE and identify the possible epistemological obstacles that students might encounter. Typically, this stage involves analyzing the historical development of the concepts at hand. The epistemological analysis enables researchers to adopt a reflective stance and establish distance from the educational environment in which they are immersed. Specifically, it aids in constructing a *Reference Epistemological Model*, a notion widely employed in ATD and that will be further elucidated in Section 0.4.2. As highlighted by González-Martín et al. (2014), the epistemological analysis appears as a particularly helpful tool for DEs concerning university mathematics education. Indeed, during the transition from secondary school to university, there is a substantial increase in the level of abstraction and complexity within the field of mathematics. Conducting an epistemological analysis becomes crucial to guarantee that essential aspects of the mathematical topics to be addressed are incorporated into the design and implementation of activities.

- The aim of the *institutional* analysis is to identify the specificities of the Institution in which the DE is implemented. This involves highlighting the conditions and constraints to its implementation. In the context of ATD, also pertaining to the French didactics tradition, this is referred to as the ecological dimension of the didactics. The conditions and constraints examined in the institutional analysis can span various elements within the educational setting. These may include the physical layout of classrooms, the availability of resources, the particularities of curricular choices, associated teaching practices, and even more general pedagogical decisions adopted within the institution. In the terminology of ATD, these conditions and constraints may manifest at different levels of the *scale of levels of didactical codeterminacy* (Chevallard, 2002). I will develop this concept in Section 0.4.2.
- Lastly, the *didactical* analysis helps investigate what information previous research has provided about the teaching and learning of the concept at stake. This last dimension is substantially cognitive, and in order to study it, different theoretical frameworks can be integrated into the methodology of DE.

The preliminary analysis guides the phase of **conception and a priori analysis**, where research hypotheses are made explicit and employed in the design of the activities to implement. Design choices taken in the conception phase can regard the macro-level of the design, involving the global project, or the micro-level, concerning specific didactical variables. The a priori analysis clarifies how the choices made in the conception relate to the preliminary analysis, taking into account its different dimensions. Moreover, it is used to anticipate what the situation a priori can offer in terms of learning of the mathematical knowledge at stake. Substantially, it creates a reference with which the actual implementation in the classroom will be compared and eventually contrasted.

In the **realization** phase, the activities are implemented, under the researchers' observations, and data are collected for the analysis a posteriori. The type of data collected strongly depends on the goals of the DE, on the hypotheses made in the a priori analysis and on the theoretical framework accompanying the DE. During this stage, researchers typically conduct an 'in vivo' analysis, attempting to interpret in real-time (or shortly after) the events unfolding in the classroom. It is likely that the project will not develop precisely as envisioned during the conception phase. Consequently, based on the insights gained from the in-vivo analysis, certain teaching variables or choices may be modified throughout the project. Any adaptations should be thoroughly documented alongside the overall analysis.

Following the implementation, **a posteriori** analysis of the collected data takes place. Depending on the theoretical framework underpinning the DE, the specific research questions posed and the nature of the collected data, various methodological tools can be employed for a posteriori analysis. A crucial feature of this design research methodology is that its **validation** is internal and achieved through a comparison between the a priori and a posteriori analyses. This comparison allows the research hypotheses to be put to the test.

0.4.2 Overview of the theoretical lenses adopted

In the following sections, I introduce the main theoretical frameworks utilized for analyzing the data collected throughout the project. I will not present here the other theories used in the different chapters, as each of these frameworks is presented in detail in the respective chapters where they are applied, highlighting the key features that are essential for understanding the analysis. Here, we provide an overview of the main aspects of each framework and of the construct of Networking of Theories.

The Semiotic Bundle theory

A large branch of research in mathematics education investigates how the study of the semiotic resources used by teachers and students can enhance the understanding of processes that are part of the teaching and learning of mathematics (Presmeg et al., 2016). The relevance of a semiotic approach in mathematics education research lies in the fact that mathematical objects are abstract and necessitate signs for being represented. Duval's assertion, “there is no noetic without semiotic” (Duval, 1995), emphasizes this connection. Traditionally, the study of semiotics in mathematics education focused mainly on oral and written representations, such as speech, written languages, and the algebraic register. However, since the early 2000s, increased attention has been given to considering a broader range of semiotic resources. The Semiotic Bundle (SB) theory (Arzarello, 2006) frames mathematical learning processes within the multimodal paradigm, acknowledging the use of various semiotic resources by students and teachers. This includes not only written and oral resources but also extra-linguistic modes of expression, such as gestures, glances, actions, produced also to interact with various instruments, from pencils to sophisticated ICT devices. Arzarello developed this semiotic tool to analyze the variety of semiotic resources and their relationships and evolution in students' and teachers' productions and interactions, forming what is termed the 'semiotic bundle.' This bundle comprises a collection of semiotic sets and a set of relationships between these sets. Importantly, a “semiotic set” enlarges the notion of a “semiotic system”. While a semiotic system, for example in the definition by Ernest (Ernest, 2006), consists of a set of signs, rules of sign production, and relationships between the signs, it typically considers only compositional systems like formal languages. In contrast, Arzarello's concept of a semiotic set encompasses open sets of signs, including informal drawings, sketches, and gestures. A semiotic bundle, therefore, is a system of signs, including written words, diagrams, spoken words, gestures, and actions, produced by one or more interacting subjects, evolving over time. Both the collection of sets within the bundle and the relationships between them can change as the subject or interacting subjects produce them.

These relationships are analyzed using two types of lenses within this framework: a synchronic analysis allows to study the relationships among signs produced simultaneously, while diachronic analysis examines relationships among semiotic sets activated by the subject (or subjects) in successive moments, capturing their evolution from one semiotic set to the other. This last kind of analysis has been used by other theories, but the element of novelty brought by the theory of the semiotic bundle is the opportunity to use it to observe phenomena considering semiotic sets instead of the more restricted semiotic systems. For instance, the dynamic relationship between gestures and speech can be analyzed, providing crucial information about the subject's ideas and thinking. In this case, gestures and language form a semiotic bundle, composed of two deeply intertwined semiotic sets, with the second representing also a semiotic system. Obviously, these elements must be analyzed also in relation to other semiotic resources employed, as written signs.

A disclaimer: Arzarello and Sabena (2014) introduced the theoretical approach of Action, Production, Communication (APC). The APC approach focuses on the classroom processes of teaching and learning mathematics, at both cognitive and didactic levels. The three components of APC are seen as mutually enriching and inseparable, and are analysed through the lens of the semiotic bundle. In this thesis we will generally refer to the semiotic bundle, but in the paper in Chapter 5 we referred to the APC. However, in this thesis we do not distinguish between the two, as the focus is always on the analytical lens, even in Chapter 5. Therefore, in Chapter 0 and in Section 7 'Discussion and conclusions' we generalise by using the term “semiotic bundle”.

The Anthropological Theory of the Didactics

Yves Chevallard coined the term "Anthropological Theory of the Didactic" (Chevallard, 1991) to describe the research framework in mathematics education that emerged in the 1980s with the first investigations of didactic transposition (Bosch et al., 2020). In relation to this research program, Josep Gascon proposed to talk about the “epistemological paradigm” (Gascon, 1993) in mathematics education to emphasize its distinctiveness from the prevalent research paradigm of that time, which predominantly focused on the cognitive aspects of mathematics learning. The anthropological approach to education is distinguished in having a common view of all types of activities, including didactical activities. To enable the study of a multiplicity of processes from a unified perspective, ATD describes every human activity as a "praxeology," or as the union of several praxeologies.

A praxeology is composed of two complementary elements or blocks: a praxis and a logos. The *praxis* block represents the practical aspect of an activity and consists of a specific *type of task* T_i along with the *technique* τ_i or set of techniques used to execute the task. It pertains to the "know-how" of the activity. On the other hand, the *logos* component deals with the description and justification of the technique, addressing the "know-why" behind the application of a technique to solve a particular type of task. It encompasses a discourse about the technique itself, termed as *technology* θ_i, as well as a broader *theory* Θ_i, upon which the techniques are founded.

The definition of praxeologies is underpinned by two fundamental principles. Firstly, they are dynamic entities that undergo evolution over time, a crucial aspect in the analysis of teaching and learning processes. Secondly, praxeologies have an institutional relativeness, implying that their characteristics are heavily influenced by the institution in which they are employed. Consequently, they can vary when transferred from one institution to another. According to ATD, this variability is exemplified in the process of *didactic transposition*. A piece of (in this case mathematical) knowledge, originates in the context of the so-called "*scholarly knowledge*". Then it undergoes a process of selection and adaptation for teaching purposes, as it moves to the school institution, being transformed into "*knowledge to be taught*". This intentional knowledge, intended to be taught, encounters several constraints when it comes to its actual implementation in the classroom. As a result of these transposition processes, the *taught knowledge* eventually diverges significantly from scholarly knowledge.

The set of conditions and constraints to the implementation of an intended didactical proposal, is indeed another key aspect studied by ATD. It is referred to as the *ecological dimension* of teaching processes. These conditions and constraints can manifest across various levels within what is termed the *scale of levels of didactic codeterminacy*. This hierarchical scale (Fig. 0.6) ranges from the broadest level of humankind and societies to narrower levels such as disciplines, subjects, and specific questions. It emphasizes the reciprocal influence between mathematical content and teaching organization, evident in both didactic systems and societal structures. The scale provides a useful tool for analyzing the ecology of mathematical practices within educational institutions, aiding in identifying at what level a constraint might appear, and at what level specific conditions can be leveraged or changed to enhance specific didactic approaches.

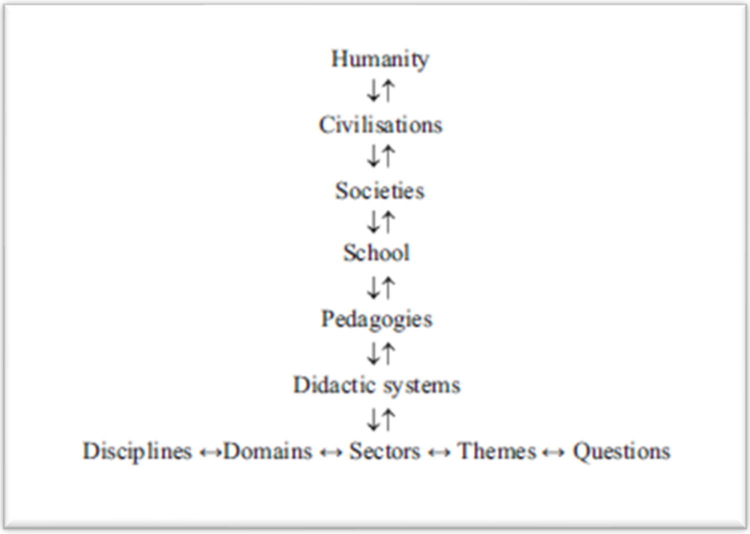


Figure 0.6. Scale of levels of didactic codetermination (retrieved from Gascón & Nicolás, 2022)

0.4.3 Networking of theories

The idea of networking of theories in mathematics education has been investigated since 2005. This term denotes a collection of "research practices that aim[s] at creating a dialogue and establishing relationships between parts of theoretical approaches while respecting the identity of the different approaches" (Bikner-Ahsbahs & Prediger, 2014, p. 118). This networking of theories approach is based on the idea that the diversity of theories in our field should be regarded as a source of richness and that to make this diversity fruitful, different theories should come into interaction (Prediger et al., 2008). It is crucial to note that not only is there a multitude of theories within our field, but there is also not a universally agreed-upon definition of "theory" among mathematics education researchers (ibid). Assuming Radford's perspective (Radford, 2008), a theory can be seen as a way to produce understanding and modes of actions based on a set of principles (P), a methodology (M) coherent with the principles and a set of paradigmatic questions (Q) related to the principles and the methodology. Hence theories can be identified as a triplet (P, M, Q) and connections between different theories can be drawn between the three elements P, M and Q.

A wide range of networking strategies exists, as depicted in Fig. 0.7, spanning from the extreme of "ignoring other theories" to the opposite extreme of "globally unifying."

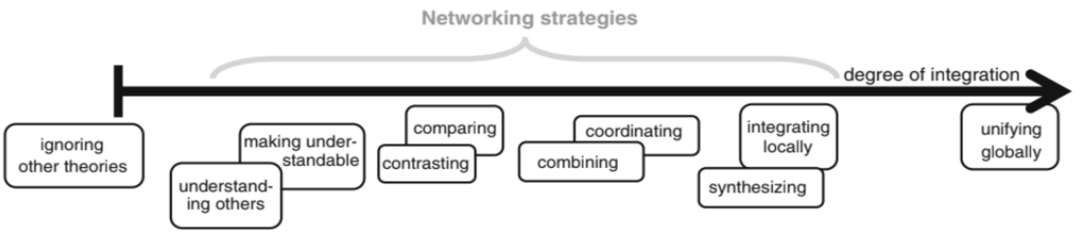


Figure 0.7. A landscape of strategies for connecting theoretical approaches, taken from (Prediger et al., 2008)

Between such different strategies, those of "coordinating and combining" are mostly used for a networked understanding of empirical data. While combining refers to the idea of juxtaposing different theories to analyze a phenomenon, coordinating includes a careful analysis of the mutual relationship between the different elements of the theories. This networking strategy is particularly fruitful when the empirical components of the theories (as their analytical lenses, typical research questions, etc.) are complementary, i.e. when they highlight different aspects of the phenomena under study.

This is indeed the case of ATD and the semiotic bundle theory. The two theoretical lenses meet their complementarity in the different units of analysis: the first approach aims at studying the emergence and development of praxeologies in different institutions, while the latter uses a fine-grained lens to study cognitive phenomena that appear in small fragments of mathematical activities. Arzarello, Bosch, Gascon and Sabena (Arzarello et al., 2008) have initiated a dialogue on the complementarity of the two theories with respect to the study of the role of ostensives (or semiotic resources). In this thesis, I propose a novel methodological approach to network the two frameworks (integrating the additional cited frameworks in the analysis of cognitive aspects, together with the Semiotic Bundle lens). This approach is grounded in the structure of didactical engineering. The networking strategy is extensively detailed in Chapter 5. In the next subsection, I will outline the steps of the didactical engineering that shape this research project, offering insight into how networking has been achieved through the coordination of the different theories employed for analysis at different stages of the DE.

0.5 Structure of the thesis based on the phases of didactical engineering and specific research questions

As mentioned earlier, this project adheres to the steps of the didactical engineering methodology. The initial phase, termed the **preliminary** analysis, comprises epistemological, institutional, and didactical analyses. The first two aspects have already been presented in Section 0.3. The didactical analysis, the third component, draws from both existing literature, which has also been synthesized in the same section, and the findings of a pilot study conducted by myself.

The initial three chapters will detail this pilot study and present its various outcomes. In essence, the pilot study involved a series of task-based interviews with engineering students who had completed the entire linear algebra course in the previous year. Additionally, it encompassed the implementation and analysis of an optional small-group activity offered to students currently enrolled in the linear algebra course during tutoring sessions shortly after the topic had been taught in class by the Teacher of the course.

Chapter 1 analyses one of the interviews, using the semiotic bundle theory and the conceptual bending theory. The second and the third chapters analyze the activities in groups. Chapter 2, using the theory of Objectification, analyses students' attempts to understand eigenvectors and eigenvalues describing them in terms of "objectification processes". Chapter 3 analyzes the same activity through a different theoretical lens: Sfard's theory on the process/object duality and the concept of commognitive conflict. In these last two chapters, in particular, difficulties encountered by students when encountering and using the concepts of eigenvectors and eigenvalues are described and analyzed.

The insights gathered from the preliminary analysis have guided the conception phase of this project. Collaboratively undertaken by myself and the instructor of a linear algebra course within the Mechanical Engineering program at the University of Bologna, this phase aimed to redesign the course, which she had been teaching for seven years. During this stage, the institutional and epistemological analyses emerged as essential components, alongside the didactical analysis, to properly design a didactical situation aimed at addressing the issues emerged in the pilot study.

In the **conception** phase, design choices were made both at a macro level and at a micro level. The first action affected the entire course, and it concerned the use of videos retrieved from YouTube as official teaching resources.

In previous years, the teacher had been suggesting unofficial additional resources for the course, including some videos from a YouTube channel called 3B1B. This channel offers videos on advanced mathematical concepts, using sophisticated animations to illustrate them. It features a playlist titled 'The Essence of Linear Algebra', comprising various videos designed to visually explain the geometric intuitions underlying numerous topics covered in a typical linear algebra course. Our pilot study confirmed that these resources were particularly beneficial in helping students understand linear algebra concepts (see Chapter 8 - Extra). As a result, we decided to make these resources official within the course. We achieved this by creating a document[4] (in the terminology of the Documentational Approach to Didactics Gueudet and Trouche (2009), a document is a set of resources with specific usage schemes) composed of the videos themselves and other technological and didactical resources. Specifically, we utilized a social annotation platform called Perusall (https://www.perusall.com/), which allows for the sharing of various types of resources (images, texts, videos, etc.) within a virtual course, and enables instructors to require specific actions in relation to these resources. For example, students can highlight, comment on, or respond to assignments posed in relation to the resources. In our case, we organized a parallel virtual course on Perusall, consisting of eight different assignments. Each assignment required students to watch one of the videos from the aforementioned playlist and potentially comment on interesting or unclear parts. An assignment was linked to each video, and students were required to complete it before the lecture during which the topic covered in the video would be presented by the teacher. This approach allowed the video to serve as an introduction to the topic, with the related assignment serving as students' initial attempt to engage with that concept. Subscription to and use of Perusall were not mandatory for students. This thesis will focus on the micro-didactic choice, which still has to be introduced. Therefore, I have placed the chapter describing the design of activities with Perusall and the videos, along with their analysis according to the Documentational Approach to Didactics, in a separate final 'Extra' chapter.

Returning to the overall design, as previously mentioned, in addition to the macro choice involving the entire course, we also decided to intervene at a micro level in the redesign of the course module regarding eigentheory. Our goal was to transpose the positive aspects resulting from the pilot study into the regular course setting. However, this desire was met with several institutional constraints. Some of these included the high number of students attending classes (more than 200), classroom arrangements unsuitable for group activities, and time constraints related to the

[4] In the terminology of the Documentational Approach to Didactics (Gueudet and Trouche, 2009), a document is a set of resources with specific usage schemes.

extensive course curriculum, covering a vast number of topics. At this stage, institutional and ecological analysis became fundamental.

To address some of these constraints, we introduced a technological element: a Padlet, an online noticeboard. This tool served to magnify the classroom discussions, otherwise difficult to realize in the given setting, and fostered interaction between teachers and students. The four classes on eigentheory were structured as alternating moments of student activity and frontal lectures. Each lecture began with a small group activity, where students, grouped in pairs or groups of three (as permitted by the desk arrangement), had to solve a task or answer a question. At the end of their allotted time, they could anonymously post a picture of their solution or attempts on the Padlet. This allowed the teacher to immediately assess students' responses and identify mistakes or interesting ideas to be addressed in the subsequent explanation of the topic.

The design of the tasks and questions posed, as well as their **analysis a priori**, was based on the results of the pilot study. Chapter 4 presents how these results guided the selection of tasks to be posed. I won't delve into details here, as the designed and implemented sequence will be described in Chapters 4 and 5.

The sequence so designed was implemented in the autumn term of 2022. Throughout this **realization phase**, I audio and video recorded recorded all four two-hour lessons. Additionally, I collected all the images posted on the Padlet during the proposed activities, as well as audio and video recordings of three randomly selected small groups during the activities. Since the investigation began with a focus on the use of semiotic resources within the multimodal paradigm adopted by the Semiotic Bundle Theory, we continued to observe them in this second phase of the research. Therefore, it was essential to collect data that could demonstrate the different semiotic resources used by students in their activities, not just those traceable on paper.

At the conclusion of the eigentheory module, we administered a questionnaire to the students to gather their opinions and feedback on the unconventional teaching sequence. The questionnaire is described in detail in Chapter 6.

The analysis **a posteriori** encompassed various dimensions. The main one regards the reanalysis of the whole process through the lens of the Anthropological Theory of the Didactics. The decision to adapt the pilot activities to the traditional classroom setting underscored the need for an institutional perspective in the analysis. ATD provided a suitable framework for this purpose.

As a first step, we (this work has been done together with Marianna Bosch and Berta Barquero from the University of Barcelona, Spain) revisited the steps of the didactic engineering process to reanalyze them through this new lens. This involved reexamining the outcomes of the pilot study in terms of praxeologies and deepening the preliminary epistemological analysis by constructing a Reference Epistemological Model (REM). REMs are widely used in ATD to analyze praxeological organizations, articulating the researcher's vision of the mathematical knowledge under construction and how it should be translated into educational practice.

Subsequently, we analyzed the implementation of the eigentheory sequence through a praxeological lens, identifying the conditions that facilitated or hindered the presentation of knowledge in alignment with the constructed Reference Epistemological Model. The praxeological analysis of the didactic engineering process is detailed in Chapter 5. While this chapter primarily focuses on praxeological analysis, it also emphasizes the ostensive dimension of mathematical activity as theorized by ATD. This work aims to establish the groundwork for operationalizing the dialogue between the Semiotic Bundle Theory and ATD.

Adopting the ATD perspective, particularly through ecological analysis, helped identify numerous constraints present at various levels of didactic codeterminacy. These constraints were not solely material but were also closely tied to higher levels of didactic codeterminacy, such as school organization and pedagogy. It was hypothesized that students accustomed to a specific dominant pedagogical model might not fully embrace the new lesson structure. This hypothesis guided the analysis of the questionnaire administered to students at the end of the eigentheory module. The a priori analysis of students' attitudes, along with the analysis of their questionnaire responses, is presented in Chapter 6.

Furthermore, the data collected enabled a detailed a posteriori analysis of the use of semiotic resources and their intertwining and evolution in time, during the small group activities proposed within the sequence. This analysis serves to validate the hypotheses formulated during the conception and a priori analysis phase, as outlined in Chapter 4. This represents a significant step forward in bridging the gap between the Semiotic Bundle theory and ATD. We have begun analyzing these aspects, and an ongoing work is currently focusing on examining students' dynamic utilization of semiotic resources in one of the proposed activities. However, as they are still in an embryonic stage, detailed results of this analysis are not presented in this thesis.

0.5.1 Schematic synthesis of the structure of the thesis

I present here, in a schematic manner, the structure of the thesis to guide the reader through the different chapters and to provide a clearer overview of the flow of the stages of the Didactical Engineering adopted, along with the theoretical frameworks employed for analysis in each step.

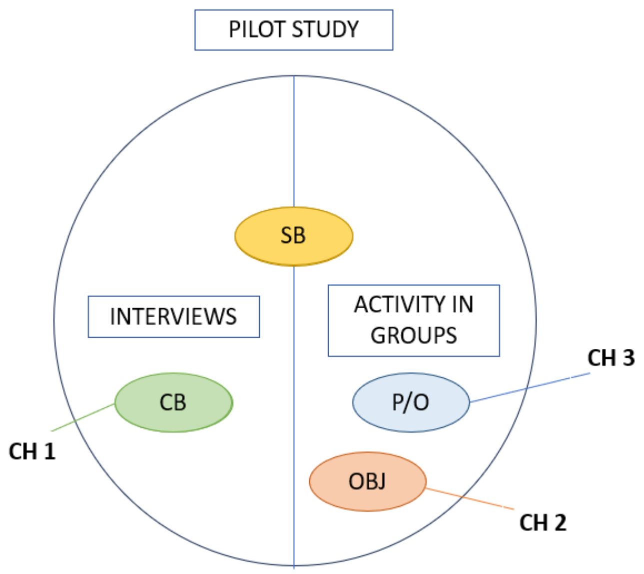


Figure 0.8. Scheme of the pilot study

The pilot study (Fig. 0.8) is presented in Chapter 1, Chapter 2 and Chapter 3. As aforementioned, the pilot study consisted of two different researches. The first one, consisting of interviews of students that had already completed the linear algebra course, is described in Chapter 1 and analyzed through the Semiotic Bundle theory, and Conceptual Blending theory (Fauconnier and Turner, 2002). The second research, consisting of the implementation of optional small-group activities, is presented in Chapters 2 and 3, and analyzed respectively with the Process/Object construct by Sfard (1992) and the Theory of Objectification by Radford (2021). Also in these two chapters, the Semiotic Bundle accompanies the other employed frameworks.

As shown in the schema in Figure 0.9, the results of the pilot study, contribute together with the analyses presented in Section 0.3 to the preliminary analysis for the Didactical engineering.

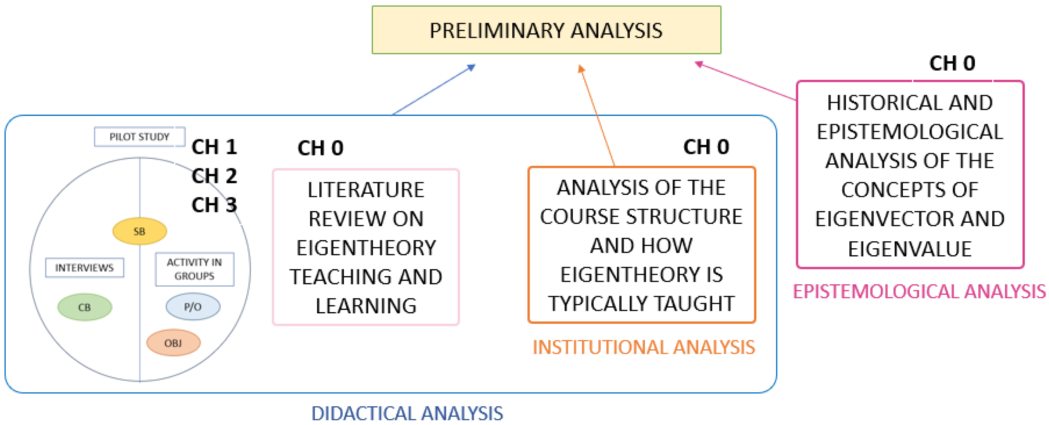


Figure 0.9. Schematic description of the preliminary analysis

Figure 0.10 illustrates the different stages of the Didactical Engineering process. As visible, Chapter 4 describes the conception phase and a priori analysis, adopting the Semiotic Bundle theory, and using some constructs and ideas pertaining to the Anthropological Theory of the Didactic. Chapter 6 presents some partial results of a posteriori analysis, focusing on students' attitude towards the proposed teaching sequence, analyzed through the lens of ATD, specifically using the construct of the scale of levels of didactic codetermination.

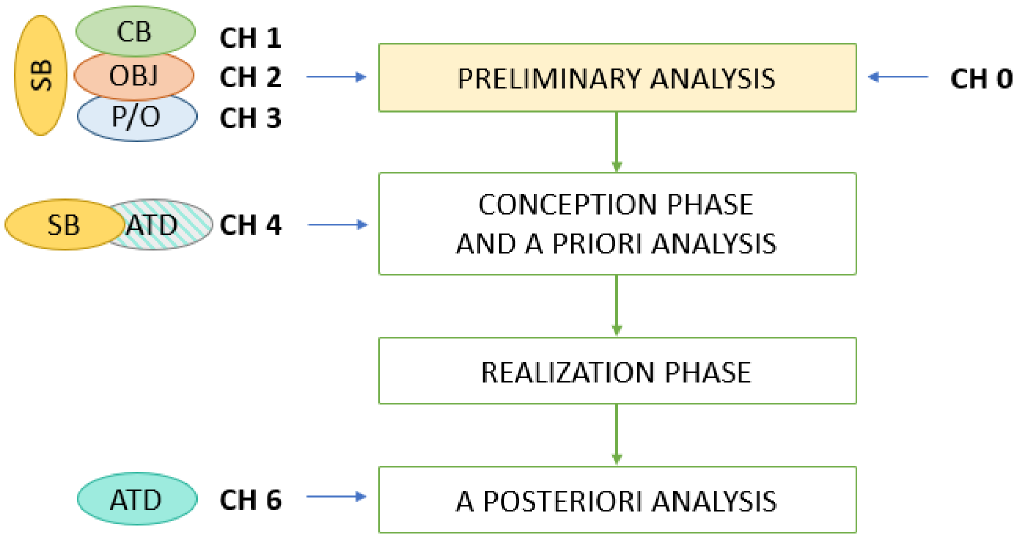


Figure 0.10. Different steps of the DE and positioning of Chapters 0, 1, 2, 3, 4 and 6

Chapter 5 necessitates a separate scheme (Figure 0.11). As previously discussed, this chapter adopts the lens of ATD to revisit the earlier stages of the DE. In doing so, it guides a posteriori analysis in light of this new perspective. Moreover, in Chapter 5, a first attempt to coordinate the Semiotic Bundle theory with ATD via the networking of theories is presented.

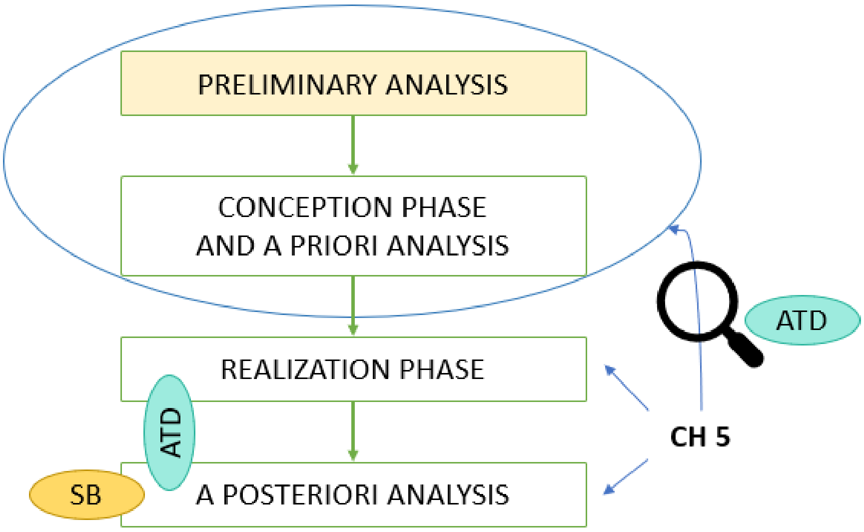


Figure 0.11. Schematic representation of Chapter 5 with respect to the DE

In conclusion to this introductory chapter, the tables (0.1 to 0.6) provided below offer a summary of the six chapters, ranging from Chapter 1 to Chapter 6. Each chapter corresponds to a paper with the same title and outlines its focus, adopted theoretical framework, specific research questions, and a synopsis of its contents.

Table 0.1. Schematic description of Chapter 1

CHAPTER 1 **The emergence of the analytic-structural way of thinking in linear algebra as a blended space**	
Research focus	Fine-grained analysis of the role of geometric representations of eigenvectors in the construction of formal thinking.
Theoretical framework	Three modes of thinking in linear algebra (Sierpinska, 2000);

	Semiotic bundle (Arzarello, 2006); Conceptual blending (Fauconnier & Turner, 2002).
Research questions	1.1 How can the emergence of the AS way of thinking be described as a blending between the SG and the AA way of thinking?
Chapter 1 aims to deepen the understanding of how geometric representations of eigenvectors can enhance the formal structural comprehension of the concept. By elaborating Sierpinska's theoretical framework (Sierpinska, 2000)- that delineates three modes of thinking in linear algebra: analytic-arithmetic, synthetic-geometric, and analytic-structural - it examines and interprets a collection of signs generated by an engineering student during an interview, first phase of the pilot study of this project, where she was prompted to recall linear algebra topics from the previous year. The provided case study serves as a model for showing a possible path for the emergence of the analytic-structural mode. Employing the theory of conceptual blending (Fauconnier and Turner, 2002), I show how the progression of utilized signs and their interrelations can reveal the emergence of the analytic-structural mode as a blend of the other two modes of thinking.	

Table 0.2. Schematic description of Chapter 2

CHAPTER 2 **Objectification processes in engineering freshmen while jointly learning eigentheory**	
Research focus	Deepening students' understanding of eigenvectors and eigenvalues, through the analysis of the semiotic means of objectification used in a collective activity.
Theoretical framework	Theory of Objectification (Radford, 2021);

	Semiotic bundle (Arzarello, 2006).
Research questions	2.1 Can our designed activity trigger and support first year university students' objectification process of eigenvector and eigenvalue concepts, and if so, how? 2.2 What information can the analysis of the evolution in time of the semiotic means of objectification mobilized by students give about these objectification processes?
In Chapter 2, I present the first results of the second phase of the pilot study. I describe the small-group activity proposed to students, where university engineering freshmen, working in small groups, are prompted to jointly reconceptualize eigentheory notions and rules and to solve some problems. Drawing on a sociocultural theory, namely the Theory of Objectification (Radford, 2021), I analyze students' collective meaning-making processes. Specifically, a few excerpts of one small group's work are presented and analyzed with a focus on students' use of different semiotic resources, their mutual relationship and evolution. Specifically, in the first example provided, students' difficulties with the formula $Av = \lambda v$, described also by Stewart and Thomas (2011) are deepened. The second example shows how the embodied geometric representation of eigenvectors enhances students' comprehension of them. In the third one, it is shown how students' autonomously conceptualize the notion of eigenspace.	

Table 0.3. Schematic description of Chapter 3

CHAPTER 3 **Students' difficulties with eigenvalues and eigenvectors.** **An exploratory study**

Research focus	Deepening causes for students' difficulties in understanding eigenvectors and eigenvalues, and specifically the formula $f(v) = \lambda v$.
Theoretical framework	Process/object duality (Linchevski & Sfard, 1991; Sfard, 1992); Commognitive conflict (Sfard, 2008).
Research questions	3.1 In what ways is a commognitive conflict likely to arise in a discourse on eigenvectors, when newcomers participate in that specific discourse? 3.2 What aspects in the formulation of the equation "f(v) = λv" may hinder the development of a structural view of eigenvectors and eigenvalues?
Chapter 3 extends the analysis of the activities conducted during the second phase of the pilot study. Here, our focus shifts to the challenges students encounter in comprehending the concepts of eigenvectors and eigenvalues, particularly stemming from their conventional definition via the formula f(v) = λv. Employing Sfard's framework of process/object (Linchevski & Sfard, 1991; Sfard, 1992), we closely examine students' verbal and written expressions to identify elements within the formal definition of eigenvectors and eigenvalues that might impede their understanding. Additionally, we explore how commognitive conflicts (Sfard, 2008), evident in students' discourse with their peers, may exacerbate difficulties in grasping the essence of these mathematical concepts.	

Table 0.4. Schematic description of Chapter 4

CHAPTER 4 **Introducing eigenspaces: semiotic analysis and didactic engineering**

Research focus	Description of didactical choices for the design of the second part of the research project, based on the results of the pilot study.
Theoretical framework	Didactic engineering (Artigue, 2015); Semiotic bundle (Arzarello, 2006).
Research questions	4.1 What kind of mathematical-didactic proposal, compatible with the learning context described with constraints such as the very high number of students attending the classes, can be implemented to foster the understanding of eigentheory?
Chapter 4 serves as a bridge between the preliminary findings outlined in Chapters 1, 2, and 3, and the subsequent implementation detailed in Chapters 5 and 6. Here, I elucidate how the results from the pilot study, derived from multimodal semiotic, along with a comprehensive epistemological, didactical, and ecological examination of the mathematical knowledge and learning environment, have influenced the conception and preliminary analysis of activities and methodologies within the didactic engineering framework. I introduce the teaching sequence implemented in the latter phase of the research project and present a multimodal analysis of a segment of student activity during the course implementation to validate the hypotheses underpinning its design. Specifically, the analysis of this excerpt allowed us to observe how students' mobilization of multimodal, dynamical, resources, such as their gestures, enabled them to visualize that the set of eigenvectors related to the same eigenvalue form a vector subspace.	

Table 0.5. Schematic description of Chapter 5

CHAPTER 5

Integrating semiotic and praxeological analyses: the case of eigentheory for engineers	
Research focus	Reanalysis of the design of the implemented sequence for introducing eigentheory in terms of praxeologies and proposal for coordinating ATD and SB theory and promoting the dialogue between the two theories.
Theoretical framework	Didactic engineering (Artigue, 2015); Anthropological Theory of the Didactics (Chevallard, 2015; Bosch & Chevallard, 1999); Semiotic bundle (Arzarello, 2006).
Research questions	5.1 What kind of mathematical-didactic proposal, compatible with a traditional first-year university lecturing linear algebra course (groups with many students, 2-hour lectures per week with only one lecturer, etc.), can be implemented to foster the emergence of geometrical praxeologies and their articulation with the more dominant algebraic ones about eigentheory? 5.2 How to analyse the instruction proposal and its effects on students' activities on eigentheory linking the semiotic and the praxeological analyses to show their interdependence, that is, how the variety of semiotic registers used by students and the praxeologies they encounter depend on each other.
Chapter 5 begins with an introduction to the theoretical frameworks of the Anthropological Theory of Didactics (ATD) and the Semiotic Bundle (SB), presenting how a first attempt to coordinate the two theories with a focus on the	

use of ostensive objects (i.e. semiotic resources) was proposed by Arzarello et al. (2008). The first stages of Didactic Engineering (DE) are then reinterpreted through the lens of ATD and using the concept of praxeology. In particular, an ecological analysis is carried out in order to highlight the conditions that allow and the constraints that hinder the realization of the teaching sequence on eigentheory designed and implemented as the second stage of the research project. Furthermore, a reference epistemological model (Bosch & Gascón, 2006) for the study of eigentheory is created in the form of a question-and-answer map, which serves as a reference to be compared with the actual way in which the topic is encountered by students in the implemented sequence. Finally, it is shown how the way the analysis is articulated through the different lenses in the different stages of DE allows a fruitful coordination between ATD and SB theory.

Table 0.6. Schematic description of Chapter 6

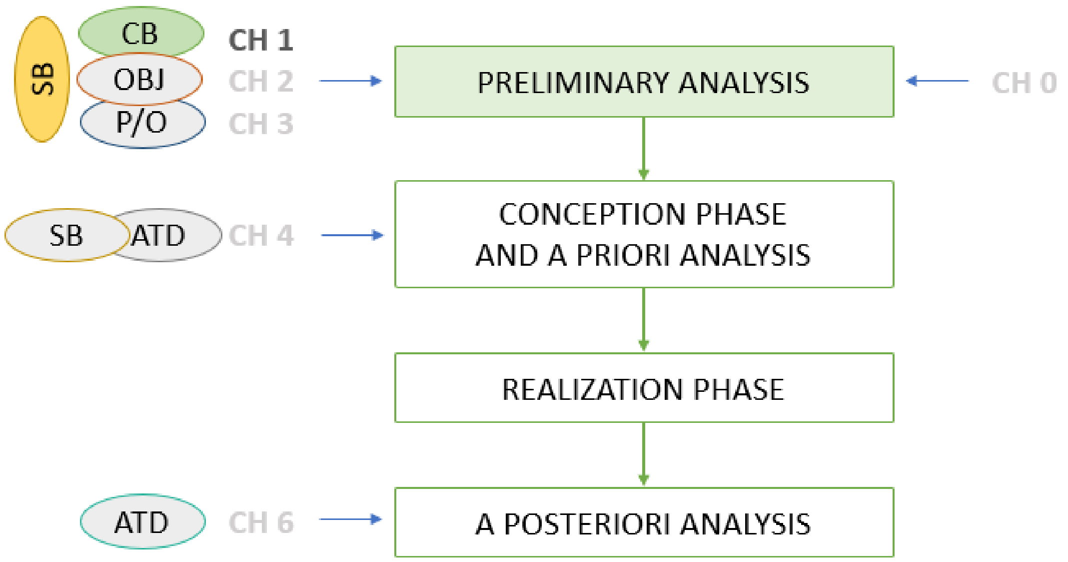
CHAPTER 6 **University students' attitude towards a non-standard instructional proposal for introducing eigentheory**	
Research focus	Analysis of students' attitude towards the implemented sequence.
Theoretical framework	Anthropological Theory of the Didactics (Chevallard, 2019).
Research questions	6.1 What is the attitude of first-year engineering students toward a nonstandard educational proposal that involves their active participation and cooperative work? 6.2 To what extent can students' expectations of mathematics teaching in undergraduate STEM courses

	be a constraint to the effective implementation of innovative teaching proposals at this school level?
In Chapter 6, I present part of the a posteriori analysis regarding students' attitude towards the proposed teaching sequence. This analysis involves examining students' responses to the questionnaire administered at the conclusion of the four classes comprising the sequence. Via inductive content quality analysis (Mayring's (2000), and using the notion of the scale of levels of didactic codeterminacy (Chevallard, 2019), I show how recurring themes in students' responses likely stem from higher levels of the scale, such as those associated with school and pedagogy.	

CHAPTER 1 - THE EMERGENCE OF THE ANALYTIC-STRUCTURAL WAY OF THINKING IN LINEAR ALGEBRA AS A BLENDED SPACE

This is the author's original manuscript of an article submitted for publication to *Educational Studies in Mathematics.*

THE CHAPTER IN THE RESEARCH PROJECT



Abstract

This study aims at elaborating a well-established theoretical framework that distinguishes three modes of thinking in linear algebra: the analytic-arithmetic, the synthetic-geometric, and the analytic-structural mode. It describes and analyzes the bundle of signs produced by an engineering student during an interview, where she was asked to recall linear algebra topics she had learned the previous year. Following the theory of conceptual blending, the presented case study is used as a model for the description of the emergence of the analytic-structural mode. It shows how the evolution of the used signs and their relationships suggests that the analytic-structural mode emerges as a blend between the other two ways of thinking.

Keywords: Linear algebra, conceptual blending, semiotic bundle, analytical-structural mode of thinking, gestures

1.1 Introduction and rationale

Linear algebra is a fundamental subject for STEM disciplines. Usually taught at the first year of university in courses as mathematics, computer science, physics or engineering, it is well known to represent a big obstacle for many students worldwide. In the last decades a growing body of literature has been focusing on the teaching and learning of this subject (Stewart et al., 2018; Stewart et al., 2019). Most of these studies draw on existing theoretical frameworks in mathematics education to study linear algebra teaching and learning related phenomena. Stewart and Thomas, for example (Stewart, 2008; Thomas & Stewart, 2011), combine APOS theory and Tall's three worlds framework (Tall, 2008) to study students' difficulties with some specific linear algebra notions. APOS theory and the theoretical tool of genetic decomposition (Dubinsly & McDonald, 2001) was utilized also in other works investigating students' conceptions of linear algebra ideas (Oktaç, 2019; Salgado & Trigueros, 2015). Various other authors involved in the research in this specific area of study, widely used Realistic Mathematics Education (Gravemeijer, 1999) to design tasks and studying students' reinvention processes of linear algebra notions (Plaxco et al., 2018; Zandieh et al., 2017). Many researchers (Bagley & Rabin, 2016; Caglayan, 2015; Dogan-Dunlap, 2010; Gol Tabaghi & Sinclair, 2013), though, in order to investigate students' understanding and difficulties with this subject, have made used of a theoretical framework specifically developed by Sierpinska (2000) for the study of linear algebra teaching and learning processes and published in what could be regarded at as the foundational work for research in this strand of mathematics education: "On the teaching of linear algebra", edited by J.L. Dorier (Dorier, 2000).

Whitin her framework, Sierpinska (2000) distinguishes between three modes of thinking in linear algebra that are connected to the type of representations used. The synthetic-geometric mode of thinking (SG) is used when a concept is thought about in terms of its geometric properties. In the analytic-arithmetic (AA) way of thinking the focus is given to the numerical components of the elements of vector spaces and to algebraic ways to compute them. The last one is the analytic-structural (AS) way of thinking, that considers structural entities and uses a formal language.

Let us now show an example considering a specific linear algebra topic: eigenvectors and eigenvalues. Examining the relationship between the three modes of thinking regarding eigenvectors and eigenvalues can be particularly interesting. Indeed, literature on the teaching and learning of this topic confirms that students' difficulty in understanding these notions is mainly due to the fact that they are usually presented with a strictly analytic-arithmetic procedure to compute them, hindering the understanding of their structural nature. Most of the research on the topic agrees that providing visual geometric representations of them might help students in their understanding, but how they can support what, according to Sierpinska's framework, is called the analytic-structural mode of thinking, is still to be deepened. The case study presented in this paper indeed deals with these concepts, hence it might be helpful to provide examples of what it means to think of these within the three modes introduced by Sierpinska.

The AS way of thinking of an eigenvector, is thinking of it as a vector, of whatever vector space, whose image to a given transformation T is itself multiplied by a scalar. Let us consider the formal standard definition of eigenvalue and eigenvector: "Suppose $T \in L(V)$[5]. A number λ is called an eigenvalue of T if there exists $v \in$V such that $v \neq 0$ and T v =λ v. [...] $v \in$V is called an eigenvector of T corresponding to λ " (Axler, 2015). There is no reference to arithmetic or geometric properties of vectors. Eigenvectors and eigenvalues are defined using general properties of vectors, vector spaces and linear transformations considered as structural entities. Thinking about an eigenvector in the SG mode would mean thinking about a vector [in R^2or R^3] whose image, with respect to a given transformation, lies on the same line [as the vector itself]. Likewise, it can be thought of as a vector which gets stretched or shrunk by a linear transformation. A number of authors (Caglayan, 2015; Gol Tabaghi & Sinclair, 2013) have investigated the effects of using Dynamic Geometry Software to enhance students' understanding of eigenvectors, proving that indeed such tools foster students'

[5] The symbol $L(V)$ is used in the book to represent an endomorphism from the vector space V to itself. An endomorphism is a linear transformation that maps a vector space onto itself.

development of the SG mode of thinking regarding the topic. Computing the eigenvalues of a transformation through the classic algorithm that takes the matrix associated to the transformation, subtracts to the diagonal, calculates the determinant and finds its roots that are the eigenvalues is a typical way of using the AA mode of thinking when dealing with eigentheory. Obviously, the algorithm is determined through a structural interpretation of the concepts. $Tv = \lambda v$ implies that $Tv - \lambda v = 0$ and this implies that $(T - \lambda I)v = 0$. This homogeneous linear equation has non trivial solutions if and only if the determinant of the matrix $(T - \lambda I)$ is null. Hence, the algorithm. The problem is that the reasoning behind it usually escapes students, who learn the procedure by rote without truly understanding its foundations (Thomas & Stewart, 2011). Various studies (Gol Tabaghi & Sinclair, 2013; Thomas & Stewart, 2011) show that students mostly develop the AA mode of thinking with respect to eigentheory, and such mode, in isolation, is likely to lead to the development of procedural knowledge to the detriment of a deep comprehension of the topic.

According to Sierpinska (2000), each mode of thinking in linear algebra uses a specific system of representations. The SG mode uses the language of geometric figures and transformations and their graphical representations: planes, lines, intersections, etc. In the AA mode of thinking usually n-tuples or matrices of numbers are produced and manipulated. Due to the non-ambiguity in the use of these different kinds of representation, when observing students' activity, it seems straightforward to know whether they are within the SG or the AA mode of thinking. The same cannot be said regarding the third mode. "Analytic-structural thinking goes beyond this type of analysis and synthesizes the algebraic elements of the analytic representations into structural wholes" (Sierpinska, 2000, p. 235). The structural nature of elements within this mode of thinking makes it difficult to detect a type of representation which is peculiar to it.

Most of the researches using Sierpinska's framework argue that students struggle in using the AS way of thinking (Caglayan, 2015; Sierpinska, 2000). In acknowledgment of the novelty and increased difficulty associated with this mode of thinking for students, the idea underpinning this work is that it is less likely that it explicitly appears in students' productions also because of its peculiar structural nature that carries with it the difficulty in being unambiguously represented. It seems worthwhile to deepen the issue of how to recognize in students' activity or discourses the activation of the AS way of thinking and how to describe the way in which it emerges. The assumption guiding this paper is that the emergence of the AS mode can be identified and described by studying, observing their behavior in guided interviews, the dynamic evolution of the different semiotic resources activated by students.

This paper presents a case study showing an example of how the analysis via the conceptual blending theory, can provide interesting insights about the process of construction of the AS way of thinking in linear algebra.

1.2 Theoretical framework

1.2.1 The semiotic bundle

A large branch of research in mathematics education investigates how the study of the semiotic resources used by teachers and students can deepen the understanding of processes that are part of the teaching and learning of mathematics (for an overview on the topic, see Presmeg et al., 2016). The importance of a semiotic approach in mathematics education research, stands in the fact that mathematics is an intrinsic symbolic activity (Radford et al., 2008): we need signs to do mathematics. Renowned is Duval's statement "there is no noetic without semiotic" (Duval, 1995). While tradition in the study of semiotics within mathematics education has focused mainly on oral and written kinds of representations (speech, written languages, the algebraic register, etc.), starting from the early years of 2000, new attention has been paid to considering a wider set of semiotic resources (see for example Radford, 2002). Among these theories, the semiotic bundle theory (Arzarello, 2006) frames mathematical learning processes according to a multimodal paradigm (Radford et al., 2017), considering students and teachers' use of a variety of semiotic resources, adding to the aforementioned written and oral resources, also extra-linguistic modes of expression (gestures, glances, actions, etc.) and instruments (from the pencil to the most sophisticated ICT devices). The semiotic bundle theory adopts the Peircean conception of sign (or semiotic resource), understood as anything that "stands to somebody for something in some respect or capacity" (Peirce 1931/1958, vol. 2, paragraph 228). It has been developed by Arzarello to analyze the variety of semiotic resources and their relationships and evolution in students' and teachers' productions and interactions. The semiotic bundle is composed of a collection of semiotic sets and a set of relationships between these sets. It is important to highlight that "semiotic set" has a different meaning with respect to the notion of "semiotic system". As explained in Arzarello (2006) a semiotic system, according to the definition given by Ernest (2006), consists of a set of signs, a set of rules of sign production and a set of relationships between the sign. Nevertheless in this vision of semiotics (Duval, 1995; Ernest, 2006), only compositional systems, as formal languages, are taken into account. The notion of semiotic set introduced by Arzarello encompasses also open sets of signs, such as informal drawings, sketches and gestures. Thus, a semiotic bundle is a system of signs (that can be written words or diagrams, uttered words, but also gestures and actions) which is produced by one or more interacting subjects and that evolves in time. Not only the collection of sets that are present in the bundle may evolve, but also the relationships existing between them can vary as the subject or the interacting subjects produce them. These relationships are analyzed through two types of lenses within this framework: a synchronic analysis allows to study the relationships among signs that are produced simultaneously, while the diachronic analysis studies the relationships among the signs in the semiotic sets activated by the subject (or subjects) in successive moments, thus their evolution. This last kind of analysis has been used by other theories, but the element of novelty brought by the theory of the semiotic bundle is the chance to use it to observe phenomena considering semiotic sets instead of the more restricted semiotic systems. This means, for example, that the dynamical relationship between gestures and speech can be analyzed, and such an analysis can add important information about the subject's ideas and thinking. In this case, gestures and language are a semiotic bundle, made of two deeply intertwined semiotic sets, of which only the second represents also a semiotic system.

To align with the multimodal view of sign set in the semiotic bundle theoretical framework, it is important that there should be no ambiguity when talking about "mode of thinking" in this article. Intending to overcome the mind/body, thought/action dualities, it is important to highlight that in this work any sign, in its broadest conception as understood in the semiotic bundle theory, thus also any action, is intended not only as evidence of a way of thinking but also as an integral part of it. Therefore, the theoretical frameworks outlined is enriched introducing the idea of semiotic control, which may be useful in the description and analysis of the case study that will be presented. Arzarello and Sabena (2011), in the context of argumentation and proof activities, introduce the notion of semiotic control to refer to when students' or teachers' "decisions concern mainly the selection and implementation of semiotic resources, namely when the decisions concern activities featured by the treatment of signs" (p. 191). In this paper the terms geometric control and arithmetic control will be used to refer to those situations where the student in the case study employs or manipulates signs that can be easily ascribed to respectively the SG mode or thinking, or the AA mode of thinking.

1.2.2 Conceptual blending

According to the theory of conceptual blending, developed by Fauconnier and Turner (2002), meaning construction involves the selective blending of elements from two or more different mental spaces, to create a new blended space. Mental spaces are defined as "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40). Following the two authors' iconicity, talking about mental spaces and blends between these, it is useful to use diagrams, where mental spaces are represented by circles, elements by points or icons in these circles and lines are used to represent connections between elements. In the network representing the conceptual blending, it is possible to define two (or more) input mental spaces. Then, counterparts in the input spaces are connected by a partial cross-space mapping, while a generic mental space maps onto each of the input spaces and contains what is present in both input spaces. Structure from the input mental space is projected to the blended space. This projection is selective, as it does not involve all the elements and relations in the input spaces. In the blend, an emergent structure that is not copied directly from one or both the input spaces, arises. Within this framework, the terms "emergence" or "emerging" are employed to explain inferences that arise from integrating two or more conceptual inputs, which may not be readily apparent when analyzing the inputs in isolation.

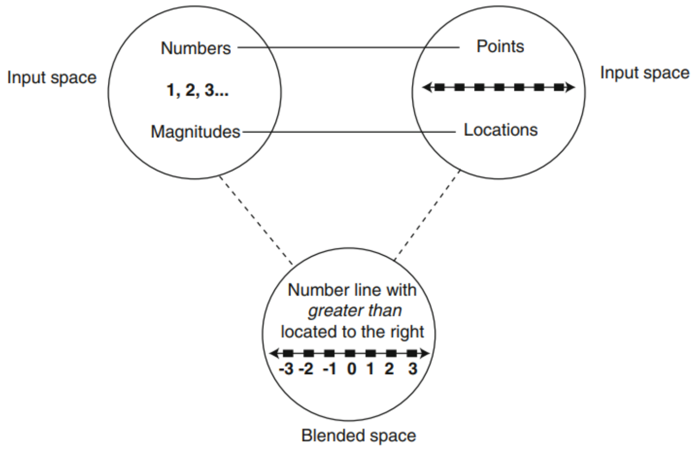


Figure 1.1. A conceptual blend that gives rise to the notion of a number line. Diagram found in Yoon et al (2011) and adapted from Edwards (2009, p. 129)

The diagram in Figure 1.1 is an example, retrieved from Edwards (2009) and reused with this graphical adaptation by Yoon and colleagues (2011), of a conceptual blend where the notion of "number line" emerges as a blend between the two distinct mental spaces: knowledge of number and our imagery and knowledge of the geometric entity of a "line" (p. 128).

In the field of mathematical education, the theory of conceptual blending has been used to study how the mathematical gesture space is endowed with meanings (Yoon et al., 2011; Edwards, 2009). Arzarello et al. (2015), analyzing three different case studies, have shown how the metaphoric[6] dimension of gestures (McNeill, 1992)

[6] In his study, McNeill (1992) categorized gestures based on their iconic, deictic, or metaphoric dimensions. The iconic dimension pertains to how a gesture visually resembles the entity it represents; the deictic dimension involves pointing gestures;

can emerge alongside the evolution from a grounded to a more conceptual blending of iconic gestures and speech. Zandieh et al. (Zandieh et al., 2014) have used the theory to illustrate the evolution of university students' proof writing processes, and more recently Apkarian et al. (2019) have used it to investigate mathematics education master students' reasoning in relation to properties of the Sierpinski triangle. These last two works, show how the conceptual blending theory can be particularly relevant for analyzing situations in which students must combine different "inputs" in order to build conceptions of advanced mathematical notions.

1.3 Research question and methodology

How can the emergence of the AS way of thinking be described as a blending between the SG and the AA way of thinking? In order to answer to this question a case study will be presented and analyzed, serving as a model for the description of the emergence of the AS way of thinking as a blend between the SG and the AA ways of thinking.

The selected case study is an interview conducted with Anna (a pseudonym), a second year mechanical-engineering student in a public university in Italy. She completed the linear algebra and geometry course approximately one year before being interviewed. The program of the course covered algebraic structures, matrices, vector spaces, linear transformations, linear systems, eigentheory, Euclidean vector spaces, affine spaces and bilinear and quadratic forms. Course attendance was optional, but Anna attended classes regularly. In the meantime, she had followed some engineering courses in which linear algebra notions were used, such as "Rational mechanics" and "Mechanical behavior of materials". The here reported dialogue was recorded during a series of semi-structured interviews conducted by the author (hereafter referred to as "INT") with engineering students who had completed the linear algebra course approximately one year before. The interviews aimed at observing what

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the metaphoric dimension entails the way gestures convey abstract concepts rather than concrete ones.

representations students more naturally use for the purpose of reconceptualizing concepts that they had encountered and learned a long time before and that perhaps might have forgotten by the time of the interview. In this article, only one of the conducted interviews is reported, as a case study. This has been completely video-recorded in order to perform a multimodal semiotic analysis of the discourse. This means that it was possible to analyze the different semiotic resources activated by the student (speech, drawings, gestures,…) and specifically their evolution and interactions, with the aim of detecting specific points in her activity where it is possible to identify elements of the blending resulting in the emergence of the AS way of thinking.

1.3.1 Data analysis

The entire interview was videotaped for a total of 35 minutes. The entire video was analyzed, but here we report the analysis of only the part of the interview related to eigenvalues, for a total of 12 minutes of video. In accordance with the semiotic bundle theory, I conducted both synchronic and diachronic analyses. Initially, I segmented the video by identifying focal moments in Anna's discourse. This facilitated the synchronic analysis, wherein I examined the different semiotic resources she utilized and their interrelationships within each segment. Specifically, I categorized each realized sign (spoken, written, gesture, etc.) into one of the three modes of thinking, where sufficiently clear. Subsequently, I conducted the diachronic analysis, identifying the evolution of the various signs. For instance, I identified instances of repeated use of a similar sign (e.g., the same gesture) throughout the interview, but apparently manifesting different modes of thinking.

Finally, utilizing diagrams analogue to the one depicted in Figure 1, I examined how elements pertinent to AA and SG ways of thinking, revealed by the synchronic and the diachronic analysis, are interconnected and how they are mapped into a blended space, highlighting the properties that emerged as a result of this blending. A second researcher, who was also involved in the research project as my doctoral dissertation supervisor, independently conducted the same analysis. We then compared our findings where disagreements arose and reached a consensus.

1.4 Description of the interview

Before the interview started, Anna did not know what the interview would have been about, so she was not expecting questions concerning linear algebra. INT started then with some warming up questions to help her re-getting some familiarity with basic concepts such as linear dependence and linear transformation. After a while INT asked her if she could remember what an eigenvalue and an eigenvector are. Anna admits that she does not remember it. She thinks for a few seconds and then states:

> 1. mmh wait… I remember these lambda…[she thinks without talking for some seconds]
> 2. ...it was like a vector that, starting from a linear transformation let you obtain a multiple of the vector that you had transformed [she performs the gesture in Fig. 1.2], let's say, something like this, and, the parameter that defined multiplicity was the eigenvalue, something like this.

When she talks about "linear transformation", "multiple", "vector", etc., it is not possible to discern only from her choice of terms if, using Sierpinska's terminology, she is thinking in a SG, AA or AS way. Analyzing the whole bundle of signs that she activates in her discourse can help in this direction. While pronouncing this sentence, indeed, she performs a gesture (Fig. 1.2).

Figure 1.2. Anna introduces with a gesture a geometric representation of a vector which gets overturned

This gesture, repeated different times during the utterance of the sentence, helps recognize that she is probably thinking in terms of geometrical vectors. It seems that she is gesturing a vector with her thumb, and then rotates it, as if the vector gets overturned, but remaining on the same direction. Of course, this remains a conjecture and it is not possible to really know what Anna is specifically referring to. However, the analysis of all the signs present in the discourse, besides the uttered words, can add information regarding the "interpretant"[7] Anna is probably implicitly referring to.

Seeing that Anna is struggling with remembering these concepts, INT suggested she starts from an example, trying to support in this way her process of remembering. INT proposed a geometrical example (that is, an example that considers the vector space of two-dimensional vectors within the Euclidean plane, with the canonical orthonormal basis). Namely, she asked Anna to think about a symmetry with respect to the y-axis in the Euclidean plane and to find if there are eigenvectors and eigenvalues. The choice to use a geometrical example was led by the assumption that a more familiar context could help Anna in recognizing the key notion that, probably with little awareness, she had just mentioned, such as "multiple" or "parameter". She sketches the situation on the blackboard (Figure 1.3), introducing in this way the graphic register. This enlarges the bundle, so far made of speech and gestures, making

[7] In Peirce's triadic conception of signs, the Interpretant is the effect that a sign vehicle (Representamen) determines upon a person (Peirce, 1992).

the ideas, that she had hinted before with the support of a metaphoric/iconic gesture, more tangible. Then, she starts thinking out loud.

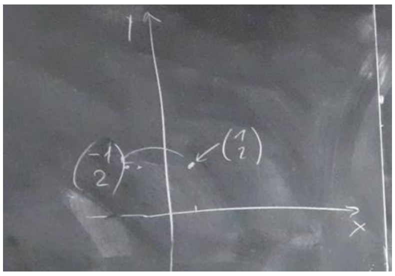

Figure 1.3. Anna's sketch of the situation suggested by the interviewer. Both geometric and arithmetic components appear on the sketch

> 3. If you have a vector made of, a vector in a 2D space, what you have to do is: you change the *x* sign, but you don't change the *y* sign, isn't it?
> 4. But in case…if you wanted potentially to have a similar transformation, but in which you also want eigenvalues and eigenvectors to appear you need that, I mean for the definition of eigenvector itself, you need that that starting vector is completely transformed, I mean it is completely multiplied for a parameter, which is the eigenvalue isn't it?

As shown in Fig. 1.3 she takes the vectors (1,2) and its symmetric (-1,2) as an example. At the beginning of her reasoning aloud, she starts referring to the algebraic notation to refer to these vectors. She focuses on the fact that if you have a vector and look for its symmetric with respect to the *y* axis, "you change the *x* sign, but you don't change the *y* sign". Thus later, when she wonders how an eigenvector should be and states that "the starting vector is completely transformed, [...] completely multiplied for a parameter", it is likely that she is thinking in analytic-arithmetic terms. Indeed, for a vector to be an eigenvector, its image must be its opposite, meaning that both components must have the same absolute value but opposite sign with respect to the original components. Nevertheless, observing the gestures performed by Anna while uttering the previous sentences, something interesting can be noticed.

Figure 1.4. Another key Anna's gesture

With her thumb and index finger she seems to be representing a horizontal segment. The thumb then rotates around the index finger (Figure 1.4). This happens while the words "you change the x sign" are being pronounced. It is highly likely that she is representing the geometrical x-coordinate of the vector (so probably the horizontal segment joining the tip of the vector with the y axis) and showing how it keeps the same length but with opposite sign, when the vector is transformed into its symmetric. So far, she is simply joining, using two different semiotic registers, the geometric representation of a vector with its numerical components. Her discourse gains complexity when she starts trying to understand if such a transformation can have eigenvectors and what characteristics they should have. When saying "completely transformed" she repeats the same gesture, with a slight difference. At the beginning she seems to be representing a vector that, considering the finger's inclination, is probably the vector in the drawn example. So, it appears that now she is not referring to the x component, but to the whole vector that needs to be "completely transformed". A possible interpretation is that in this moment, when she rotates one finger around the other, she is probably representing the vector, which is an alleged eigenvector, whose image appears to be in the same line. This gesture could indeed be a repetition of one of the gestures that she had performed earlier when trying to reconstruct the definition of eigenvalue/vector at first. In the end, it seems that when she talks about the vector that must be "completely multiplied by a parameter", it is not manifestly clear if she is thinking about an algebraic multiplication, namely in this case $k * (1,2)$, or a geometric multiplication, namely the vector stretching, reducing or overturning. According to our theoretical framework, at this stage of her reconstruction, she is simultaneously thinking in the SG and in the AA mode.

This simultaneous arithmetic and geometric control remains evident in the following part of the interview. Anna aims at finding some possible vectors that have the just stated property.

5.Eh, in my opinion there are only on the *x* axis, because if you want to have an eigenvalue you need at least, to have such a thing, you should multiply, I don't know, both by -1.
6. But if you, here, from this starting vector you multiply by -1, you don't end up here (Fig. 1.5a e 1.5b), you end up here.
7. The only way you have to obtain such a transformation and have an eigenvalue too, is moving on the *x* axis, because the *y* becomes 0, you can multiply it by -1, but it stays 0 anyway.
8. Then there is the eigenvalue which is -1, which multiplies the *x* thing by whatever you want and then you move on the other side yes, but at the same time you stay on the same line (Fig. 1.5c), and so you actually have a transformation that brings a vector to the other side of the x, which is a transformation with eigenvalues and eigenvectors too.

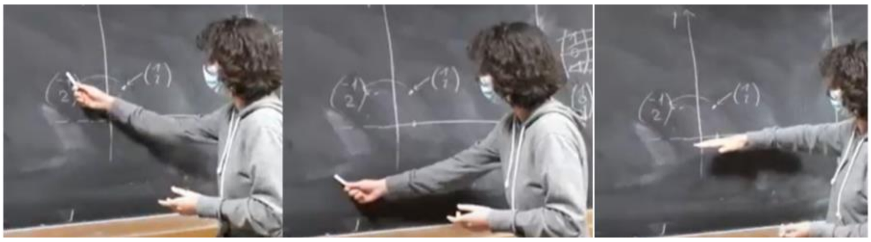

Figure 1.5 a-b-c. Anna's gesturing while pronouncing utterances 6 to 8

Here the simultaneous geometric and arithmetic control is evident both in the uttered words, that refer both to the geometric and algebraic realm, and in her gestures. She keeps replicating gestures that probably represent geometric properties and transformations, but also pointing at the algebraic components of the vectors and replicating gestures that likely represent geometrically these components. Here, the different registers in the semiotic bundle made by Anna's speech, diagrams and gestures, are coordinated.

The significant outcome of this sequence of reasoning becomes apparent when finally INT ask her if she can generalize the obtained results, that is finding a definition that could work within all possible vector spaces. At first she seems troubled by this question, because she seems to continue to have no memory of the formal definition. She then begins to says:

> 9. An eigenvector for a specific linear transformation T is a vector, let's say u (she writes it down as shown in Figure 1.6), belonging to the vector space V, such that u is mapped (she pauses to think for a few seconds) to λu (she looks at what she wrote for a few seconds). Well, I would tell you like this, honestly.

She is able to formalize the general definition of eigenvalues and eigenvectors. It is possible to state that she highly likely reached a formal structural consciousness of what these concepts are, thanks to the observation of her process of reconstructing them. The written formula $u \in V\ t.c.\ \ u \rightarrow \lambda u$ would not say anything about how the student thinks about its meaning, if observed alone. Instead, the process shown, with a combined and simultaneous use of different registers and different ways of thinking, allows us to interpretate that, Anna is now able to define eigenvalues and eigenvectors in their generality. There was a further evidence of this when, later, INT asked her to find the eigenvectors and eigenvalues of a transformation between polynomial vector spaces, to which Anna provided the correct answer. She could not use a geometric interpretation neither an algebraic procedure, and what allowed her to answer correctly cannot be but the reached structural notion of eigenvectors and eigenvalues.

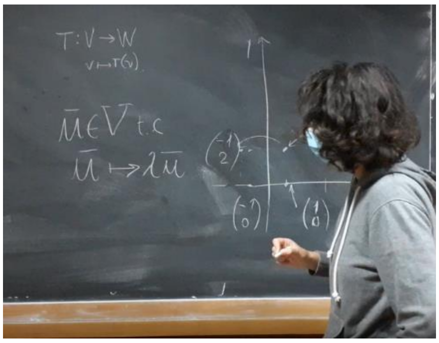

Figure 1.6. Anna's final elaboration of the definition of eigenvectors and eigenvalues

1.4.1 Analysis of the interview with the conceptual blending framework

Retaking the notions described in the theoretical framework section, it could be said that one input space is the mental space where the analytical-arithmetic way of thinking is used; this will be called the AA input space. The second input space would be that in which vectors and their transformations are thought in terms of synthetic-geometric properties and will be referred to as SG input space. The AS way of thinking emerges then in the blended space. The scheme in Figure 1.7 resumes the general situation.

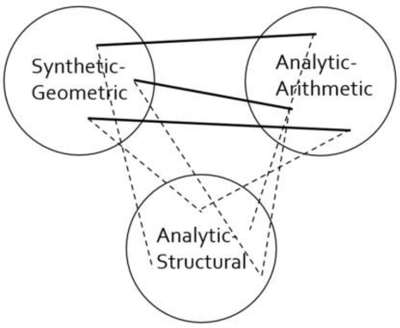

Figure 1.7. A scheme of the blending between modes of thinking

Considering the specific case studied, the conceptual blending theory has been used to describe the emergence of a structural view of eigenvectors and eigenvalues. In the AA input space are the arithmetic representations for vectors, multiple of vectors, linear combinations. For example, it is possible to attribute to this mental space the inscriptions (1,2), (-1,2), etc. in the blackboard representing analytically vectors (Fig. 1.3), and the utterances such as “you should multiply times -1”, referring to the idea of multiple of a vector. Also, deictic gestures as pointing to components of the vectors stay in this input space. Representations, gestures and utterances referred to the geometrical representations of vectors and transformation in the two-dimensional space are part of the SG input space. These are the drawings representing vectors, gestures representing properties of the vectors or of the transformation – as alignment or rotation – and uttered words as “completely transformed” or “moving on the *x*-axis”.

At the end Anna is able to make these elements of the input spaces map into the AS blended space. There, for example, the arithmetic/algebraic idea of multiple of a vector as multiplication of all its components for the multiplying scalar and the geometric idea that a vector and its multiple stay on the same line, converge into the blended abstract notion of multiplication of a scalar and a generic vector (visible in Fig. 1.8b). In line with Fauconnier and Turner’s framework, this blended notion is much richer than the simple addition of the two arithmetic and geometric interpretations. The emergent structure is abstract and general in the sense that it is valid for whatever vector space.

In the same way, from the AA idea of an eigenvector as a vector whose image has the coordinates that are multiples of the initial coordinates, and the SG idea of an eigenvector as a vector whose image stays on its same direction, the abstract general idea of eigenvector as a generic vector whose image is given by itself multiplied for a scalar, emerges (Fig. 1.9). The diagrams in Figure 1.8 and Figure 1.9 schematize the generic space, the projection of elements of the input spaces onto the blended space and the emergence of new elements in this last one.

In line with the work of Arzarello at al. (2015), the described blending process is supported by sequences of catchments. A catchment is defined as a “thematic discourse unit realized in an observable thread of recurring gestural imagery” (McNeill, 2005, p. 18), and Arzarello and colleagues have shown how “the evolution from the prominence of an iconic to a metaphoric dimension of a gesture, part of a sequence of catchments, can indicate a process of construction of a mathematical concept" (p. 35). In this case study a similar phenomenon can be noticed in Anna's utterances [3] and [4]. Anna repeats the same gesture (that in Figure 1.4) in a series of catchments. The words pronounced with the repeated gesture, though, change, passing from a reference to the algebraic representation of the vector – "you change the x sign, but you don't change the y sign" - to its geometric interpretation - "that starting vector is completely transformed". The recurrence of the same gesture seems to create the connection between the two corresponding elements in the input spaces, allowing then their blending. At the end of the series of catchments, the same gesture is indeed used one last time while pronouncing [4] "it is completely multiplied for a parameter, which is the eigenvalue, isn't it?", where an AS view seems to emerge.

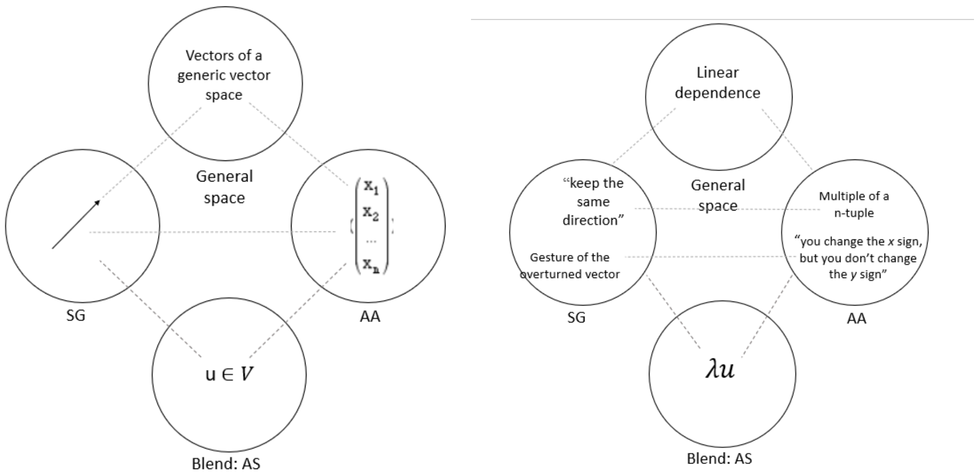

Figure 1.8 (a and b). A scheme of the projection of some elements of eigenvectors and eigenvalues in the SG and AA input space onto the AS blended space

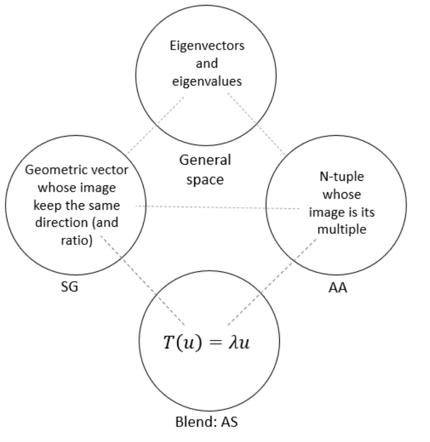

Figure 1.9. A scheme of the projection of eigenvectors and eigenvalues in the SG and AA input space onto the AS blended space

1.5 Discussion

This paper presented a case study showing one student's process of blending algebraic and geometric notions, and the way the formal idea of eigenvectors and eigenvalues emerged via this blend. It has been possible to capture this phenomenon thanks to the multimodal diachronic analysis of the semiotic resources activated by the student. Prior works using Sierpinska's framework, only described passages between one mode to the other, highlighting how students are more used to the AA way of thinking and/or how letting students explore elements in the SG way of thinking can help them better visualizing the notions. Nevertheless, most of these works only mentioned that the AS way of thinking is the most difficult to achieve by students, without really exploring how it could be reached starting from the two more familiar and manageable ways of thinking and related representations. This paper does not intend for the blending process to be perceived as a simple occurrence that naturally unfolds in students. Instead, it emphasizes the importance of creating teaching environments conducive to fostering such a process. This aspect will be further addressed in the conclusion. The primary objective of this paper is to illustrate how, starting from an understanding of the SG and AA types, which are more readily accessible to students as evidenced in the literature, AS thinking can emerge under appropriate conditions. This is crucial, as while it is documented in the literature that the use of geometric representations can enhance students' comprehension of linear algebra concepts, it is also acknowledged that such representations can be misleading. As Sierpinska points out: “It is not enough to just make the structural content more concrete through working in low dimensions and using visualizations. In fact, visualizations themselves are problematic; they may lead to irrelevant interpretations which make the understanding more, not less difficult” (Sierpinska, 2000, p. 244). The key point lies in the selective projection inherent in the blending process, which highlights the properties of example vector spaces (such as the Euclidean plane) shared by all vector spaces. This allows the structural aspects of eigenvectors and eigenvalues, and not only the sum of the shared features of the input spaces, to emerge in the blend.

Sierpinska was not the only researcher providing a framework that distinguishes working with different representations and conceptions of linear algebra notions. In her doctoral thesis, Stewart (2008) presented an adaptation of Tall’s three worlds to the study of linear algebra teaching and learning. The framework has been used in different works analyzing specific notions of linear algebra, included also eigentheory (Stewart, 2008; Thomas & Stewart, 2011). Following Tall, according to Stewart and Thomas, the learning of linear algebra concepts can take place in three worlds: the embodied, the symbolic and the formal world. Their works mostly focus on the effect of leveraging embodied thinking in learning linear algebra, while the problem of evaluating how students reach formal thinking is not addressed. Stewart herself, providing suggestions for future research, indicates this issue as worthy of study. Nevertheless, Tall himself (Tall, 2008) suggests how blending between embodiment, symbolism and formalism can allow transitions in thinking. Conceptual blending is not used in this work either to describe how formal thinking can emerge as a blend between elements in the embodied and symbolic world, but such a modelization is easily feasible considered that Tall has already set the ground for valuing blending between the three worlds.

It is important to mention that in many researches regarding the teaching and learning of linear algebra, using either Sierpinska’s framework or Tall’s three worlds, emphasis is given to how fostering the SG way of thinking (within the former framework) or working in the embodied world (within the latter framework), can enhance the understanding of linear algebra notions. Nevertheless, they do not specify how this can help students building a formal/structural conception of the same notions. The model proposed here confirmed that geometric representations can be particularly powerful in helping students building abstract/formal thinking, when they are recognized and used as heuristic for the general case and not just as features of specific cases. This happens via the blending between mathematically relevant elements of the SG input space and its corresponding elements in the AA input space.

1.6 Conclusions and implications for future research

In conclusion, there are some aspects that need to be accounted for. Firstly, what has been observed and analyzed is one student's reconstruction of some linear algebra concepts. Indeed, as specified in the methodology section, the student had already followed the linear algebra course the year before and thus already encountered these notions. Although it is clear that at the beginning of the interview she has no recollection of eigentheory - a process she achieves later – this is not her first encounter with eigenvectors and eigenvalues. However, it is also evident from the transcript that she does not retain from the previous year the memory of either the definition of eigenvectors and eigenvalues or their structural meaning. What she does recall, prompted by the example proposed by INT, is the geometric interpretation. Subsequently, she can easily reconstruct an analytic-arithmetic description and notice properties within it. The ensuing process is a genuine thinking process (always understood, of course, from a multimodal perspective as action) that can be described as a blending between the elements of the input spaces SG and AA remembered and reconstructed. From this blending, Anna is able to recognize the emerging structure of the mathematical concepts at stake. Evidence of this is the fact that when she writes the generic definition, she clearly derives it from the earlier considerations rather than as a resurfaced memory. In this sense, the entire process can be conceptualized as a form of knowing, aligning with Sinclair's (2023) proposition of 'Knowing as remembering,' which rejects the separation and sequencing of the learning experience. Hence, I believe that the blending process described can similarly occur during initial encounters with the mathematical content involved. Future research will undoubtedly explore this direction further.

It is interesting to notice how the support of the suggested geometric visual example helped Anna starting the recollection process, where by only trying to remember the definition verbally she was stuck. In their work on gestures as simulated actions, according to which gestures emerge from perceptual and motor simulations underlying embodied language and mental imagery, Hostetter and Alibali (2008) highlight the mnemonic benefits of imagery. Referring to Paivio's dual coding theory (Paivio, 1991), they state that "successful verbal recall is strongly associated with how easy a concept is to visualize" (Hoestetter & Alibali, 2008, p. 499). The construct of gestures as simulated actions, and the dual coding theory, which explains memory as being the product of both symbolic and non-symbolic (e.g., imageable) representations, could be used as additional framework to study Anna's recollection of an abstract notion by means of symbolic and visual representations and the role of her gestures in this process. This could be also an interesting future line of research to deepen this study.

Another aspect to consider is that this paper has presented a mean for analyzing the emergence of the AS way of thinking, without providing guidance on how this process can be facilitated. The presented research is part of the pilot study for a wider doctoral project on the teaching and learning of linear algebra, whose following steps consist in trying to answer to this here unanswered didactical issue. In Piroi (2023), the first results of this second part are presented. In the cited work it is analyzed how suitable activities in small groups can foster the understanding of eigenvectors and eigenvalues. One element enhancing this phenomenon is the fact that in small groups usually each student prefers to resort to a specific register. Thus, in one group working on a task, it can happen that one student prefers working with SG notations, while the others are more confident with the AA way of thinking. In order to align their discourses, they need to make the elements of the two ways of thinking communicate, that is assembling elements from two input spaces that are likely to be progressively projected into a shared blended AS space. The conceptual blending described for a single student case study, can then work, and even be enhanced with group activity, and this can already be an element to be considered in the design of activities that aim at fostering this blending process. Although conceptual blending is rarely characterized in the literature as a group phenomenon, Megowan and Zandieh (2005) have shown how students' interactions during collaborative learning activities foster the assemblage of different elements into a blended shared space. It would be undoubtedly interesting to investigate whether and under which didactical conditions the AS way of thinking can emerge as a shared blended space in collaborative learning. Future works concerning the following steps of the overall research project, will tackle this issue and present more in details implications for activity design derived from the results of the here presented analysis.

CHAPTER 2 - OBJECTIFICATION PROCESSES IN ENGINEERING FRESHMEN WHILE JOINTLY LEARNING EIGENTHEORY

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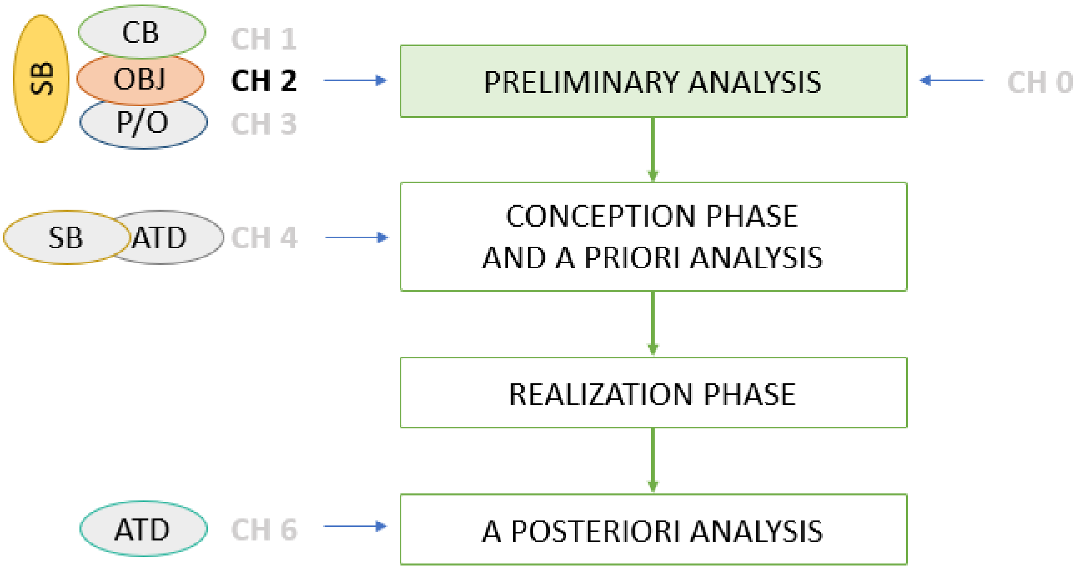
THE CHAPTER IN THE RESEARCH PROJECT



Abstract

In this paper we present the first results of an ongoing PhD study which investigates eigentheory teaching and learning processes. Drawing on a sociocultural theory, namely the theory of objectification, we study students' collective meaning-making processes. A specifically designed activity, aimed at supporting these objectification processes, is described. University engineering freshmen, working in small groups, are prompted to jointly reconceptualize eigentheory notions and rules and to solve some problems. Then a few excerpts of one small group's work are presented and analysed with a focus on students' use of different semiotic resources, their mutual relationship and evolution.

Keywords: Teaching and learning of specific topics in university mathematics, teaching and learning of linear and abstract algebra, eigentheory, objectification, embodiment.

2.1 Introduction

Linear algebra is widely recognised to be a major obstacle for university freshmen. A growing body of literature has investigated the sources of these difficulties and the way students comprehend linear algebra concepts. Nevertheless, only a small number of studies has focused on eigentheory teaching and learning processes, despite its importance in different applications in STEM subjects. This paper describes the first results of an ongoing PhD project, concerning the didactics of this specific topic.

As described by Stewart and Thomas (2006), when eigenvector and eigenvalue concepts are introduced to students, the focus is turned too soon to the manipulation of algebraic representations. In a standard instructional sequence, the formula to compute eigenvalues, i.e. $det(A - \lambda I)x = 0$, follows their formal definition almost without delay. Immediately after, the algorithm to compute the eigenvectors associated to each eigenvalue is given. We agree that in this way students are provided with a trusty procedure and do not feel the need to elaborate further these concepts' definitions. As a result, "the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust" (p.185). Most of the few studies concerning this topic, agree on the fact that consequently students prefer to rely on the standard algebraic procedure rather than draw on conceptual understanding to solve exercises and problems (Bouhjar et al., 2018; Salgado & Trigueros, 2015). Nevertheless, some of these researches bring evidence on how students' understanding of eigentheory could be enhanced by the use of dynamic-geometry software (Gol Tabaghi & Sinclair, 2013), inquiry-oriented instruction (Bouhjar et al., 2018; Wawro et al., 2019) or modelling activities (Salgado & Trigueros, 2015). However, the comprehension of how students develop and coordinate the interpretations needed for a deep conceptual understanding of eigentheory is not so clear and deserves further investigation (Bouhjar et al., 2018).

This research tries to fill this gap, analyzing how students collectively reinterpret an introductory standard frontal lecture on eigentheory, in order to construct a robust meaning for the presented concepts. We build on a sociocultural theory on mathematics teaching and learning, namely the Theory of Objectification (Radford, 2021). Hence, we are particularly interested in collective forms of knowledge production, with a focus on their multimodal features (Arzarello, 2006; Radford, 2014).

2.2 Theoretical framework

Radford (2010, 2021), defines the process of objectification as "the process through which cultural knowledge (Objekt) is progressively transformed into an object of consciousness" (Radford, 2021, p. 99). Students must engage in suitable activities in order to be able to transform cultural knowledge into *knowing* (p. 49). Through this activity, the student has the chance to encounter and to attend mathematics as a cultural-historical system of thinking. This encounter does not happen all of a sudden but must be considered as a process; a process which is highly determined by the student's effort to attend the object of knowledge. The word *Activity* in the theory of objectification "refers to a dynamic system where individuals interact collectively in a strong social sense" (p. 29). To distinguish this specific formulation from activity as merely meaning "doing something", the notion of *joint labour* has been introduced (Radford, 2021). In joint labour, the acts of teaching and learning are not distinguished from each other any longer. In particular, students do not passively receive the knowledge in an "alienated" form of learning, but actively take part, through collective work, to the production of cultural social knowledge. Joint labour not only includes language as a means for collective activity, rather encompasses the agency of body, matter, movement, rhythm, passion and sensations. Indeed, in order to become objects of consciousness, concepts must be actualised through material, sensuous activities (Radford, 2014). During the objectification process, students and teachers resort to multiple semiotic resources: written symbols, uttered and written words, diagrams, gestures, etc. These, together with object and tools, are intentionally used in social meaning-making activities in order to carry out actions aimed at fulfilling the goal of such activity: in Radford's theory (2001) they are called *semiotic means of objectification*. Since we are interested in analysing how these different signs jointly contribute to the process of knowledge objectification, they must be looked at in an integrated and systemic way, with attention to relationships and dynamics between them (Radford & Sabena, 2013). For this reason, methodologically speaking, it is important to analyse *semiotic nodes*, namely those segments of students' activities, in which different kind of semiotic resources intertwine and play a key role. In this investigation we emphasise the importance of the genesis of new signs, their evolution and the evolution of their mutual relationships in the process of objectification. Hence, we adopt the notion of *semiotic bundle* (Arzarello, 2006), in order to perform an analysis of students' sign production and their evolution in time. It allows one to have a more precise view on the way objectification is occurring. A semiotic bundle has been defined as:

a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p.100).

In this study, specifically, we will use two theoretical constructs originating in the field of gestures study, namely those of *growth point* and *catchment* (McNeill, 2005*)*, to show how the evolution of the relationship between gestures and other semiotic resources can provide information about students' cognitive processes. A growth point is a cognitive mechanism that integrates linguistic and imagistic components (McNeill, 2005) and in a discourse is identified as "the starting point for the emergence of noteworthy information prior to its full articulation" (Arzarello et al., 2015, p. 22). The information condensed in a growth point could be progressively unpacked through a catchment, defined as an observable sequence of recurring gestural imagery (McNeill, 2015). Arzarello and colleagues (2015) have shown how catchments are produced by students in meaning-making processes of a new mathematical concept (Arzarello et al., 2015).

2.3 Research aim and methodology

The investigation here presented has been conducted in an Italian public university in the fall term of 2021. The aim of the study was to analyse if/how students can objectify the concepts of eigenvector and eigenvalue, while engaged on joint labour in a specifically designed activity. Data were collected in three different linear algebra and geometry courses offered to first-year engineering students; in the Italian curriculum this is the unique linear algebra course offered to students in their first year of engineering studies, and covers standard vector-space theory (approximately: vector spaces, matrix algebra, linear systems, eigentheory, euclidean spaces). In total 64 students attended the activity and they worked divided in small groups of three, or in a few cases four students each. Sheets of paper used were collected for all the groups, while eight of them were video-taped during the whole activity. This last kind of data was necessary to collect, considering the theoretical framework that we have outlined. Indeed, from a methodological point of view, "the identification of the semiotic nodes and the semiotic means of objectification mobilised by the teacher and the students provides a kind of window to the investigation of objectification processes" (Radford, 2021, p. 106). We made sure that the recordings would capture not only the whole discussions, but also gestures and gazes produced by the students.

2.3.1 Activity design

As previously emphasised, activity is a key component of the objectification process. Even more, it is a key component of the investigation of this process, meaning that the design of an appropriate activity not only can support the process of meaning-making, but can also provide the observer with important information about how this process occurs and develops (Radford & Sabena, 2015). Another key component of the theory of objectification is classroom interaction, and this is why we shaped our activity as a small-groups work.

Because of institutional constraints - among others, the deeply-rooted habit in Italian engineering first year courses of performing traditional blackboard frontal lectures and the extremely high number of attending students (around 200) per course - we had to accommodate the planning of our activity to the standard schedule of the linear algebra courses involved in the research and were not able to plan the activity as a first introduction to the topic. Consequently, we decided to perform a pilot study after the teachers would have conducted their frontal lecture of introduction to eigentheory. Because of this, we designed the first part of the activity as a collective review of the lecture to be performed during a two-hours tutoring class, which occurred a few days after the teacher's introductory lectures on the topic. The activity was guided and attended by the course tutor and/or the researcher author of this paper. We prepared guidelines that could direct the small groups in the meaning-making process. These guidelines comprised very open questions such as "How would you explain the concept of eigenvector to someone who has never heard of that before?". Students were not specifically asked to answer the question in a written or oral form, but could freely benefit from trying to answer to these questions in order to jointly make sense of eigenvalue and eigenvector concepts. They were free to use any tool and encouraged to use other resources that they had encountered, besides the book or notes taken during the lessons. In fact, the teachers of all the three courses had shown or suggested use of a GeoGebra applet to explore eigenvectors in two-dimensional space and to watch some videos about this topic retrieved from the web.

For the second part of the activity, we prepared a set of five problems. In this paper we focus on students' engagement in the first part of the activity, while students' solution strategies to the problems are left for future works. For this reason we will not further elaborate here on the design of the problems.

2.3.2 Research questions

Considering the outlined theoretical framework, we can phrase our research questions as follows:

1. Can our designed activity trigger and support first year university students' objectification process of eigenvector and eigenvalue concepts, and if so, how?

2. What information can the analysis of the evolution in time of the semiotic means of objectification mobilized by students give about these objectification processes?

For space reasons, we will limit to the description and analysis of one small-group's work, which we consider as illustrative of a trajectory for the objectification process towards eigenvectors and eigenvalues: we refer to it as Group 1. We will present three particularly significant extracts from their first part of activity and describe key semiotic nodes in their objectification process.

2.4 Results

2.4.1 Tackling obstacles with the definition of eigenvalue and the formula $Ax = \lambda x$

The three students start from the guiding question "What are eigenvalues and eigenvectors and how would you explain these concepts to someone who is following a linear algebra course but still has not encountered this topic?"

They decide to write the answers on a sheet of paper and one student, that we will call A, takes on the task of writing. They glance at their lecture notes and start focusing on the term "eigenvalue". At the beginning, they seem to focus on writing a correct definition of the term, without really trying to make sense of the concept or to look for specific and possibly clear examples.

> A: So I would say, starting from eigenvalues, that eigenvalues are values that can represent a linear transformation with a number.
>
> B: Yes
>
> A: Via a value …
>
> B: Yes, at the end, if you think about it, if I'm not wrong, it is like multiplying the matrix of the associated function …

Student B, immediately starts focusing on procedures to find eigenvalues and A stops him and goes back to trying to find a definition. They keep looking for a reasonable definition until B's intervention leads them to facing another conflict:

B: because λ can be a 2x1 matrix

A: [thinks about it some seconds] No, λ is just a number

B: eh!

A: λI is the matrix

B: yes, ok, but you can think about λ also as a matrix, can't you?

The two students discuss about this conflict, each persuaded by his own idea. After a while, B understands that he is not able to make A understand his point with verbal language only. He starts writing formulas on his tablet. This is a first significant semiotic node to be analyzed in the group's activity. He insists on the fact that when finding the image of a vector, $f(v)$, a matrix that he calls M must be multiplied by that vector. He links then this idea to the formula used by the teacher and the textbook to define eigenvalues, namely $f(v) = \lambda v$. He correctly deduces the equality $Mv = \lambda v$, but interprets it as if λ must be a matrix as well, for the equality to stand. Stewart and Thomas (2006) have indeed described how the use of this formula can be a source of difficulty for students:

One serious problem with $Ax = \lambda x$ for students is that the two sides of the equation are quite different processes, but they have to be encapsulated to give the same mathematical object. In the first case the left hand side is the process of multiplying (on the left) a vector by a matrix; the right hand side is the process of multiplying a vector by a scalar. Yet in each case the final object is a vector that has to be interpreted as the product of the eigenvalue and its eigenvector. (p. 186)

B's explanation of why he thinks that λ could be a matrix, shows how he has encountered this misconception. A seems to understand the reason of B's error, and tries to solve the conflict by rewriting the equality as $Mv = \lambda Iv$, so to make clear that λ is a scalar, while λI is a matrix. He keeps using this formulation from that moment on. We cannot say from the analysis of the rest of this segment of activity if B has understood his error; surely, as argued in Stewart and Thomas' work, reasons behind and ways to avoid this misconception need to be better studied.

As we have shown in this subsection, students struggle in finding a suitable verbal definition of eigenvalue. In our opinion, their difficulty might be due to the fact that, ontologically speaking, it is challenging to think of an eigenvalue before even considering the existence of a linear transformation and of eigenvectors. In the following subsection, we will see how the comprehension of what an eigenvalue is can be supported by a geometric context. In fact, in it, we can define a linear transformation, and what happens to different vectors under its effect becomes more tangible.

2.4.2 Picturing a geometric example and gesturing as a meaning-making tool

An important shift in the advance of the activity, occurs when B suggests to use an example. In particular he suggests to consider an example offered by the teacher during the lecture. He refers to the teacher using a GeoGebra applet to explore and show the students a possible representation of eigenvectors in the two-dimensional space. Student C, who had not particularly got involved in the first part of the discussion, suddenly appears interested. He tries to recall the way eigenvectors could be identified in the applet, by gesturing with his two index fingers: first he moves them towards each other and then overlaps them (Fig. 2.1). These gestures allow a shift in the focus of the discussion: it moves from trying to define eigenvalues, to attempt to understand what eigenvectors are. After different efforts to verbally describing the situation, finally A states:

> A: […] It is possible to find an eigenvalue associated with an eigenvector when the image of the linear application coincides…[B and C look baffled]
>
> B: How to say it? Can we say "overlapping"?
>
> A: When the eigenvector and its image overlap.

The three of them seem happy with this definition, but C, again with the help of gestures to make himself understood, shows that the words “coincides” and “overlapping” are not satisfactory because

C: With this definition it means that the vectors reach the same point (Fig. 2.1)

A then refines his definition with:

A: When they have the same direction.
C: Same direction and same sense.

At this point B steps in and, he too gesturing (Fig. 2.2a and 2.2b), shows that actually the eigenvector and its image can have opposite senses. The so refined definition satisfies the whole group.

Figure 2.1

Figure 2.2 (a and b)

2.4.3 Objectifying eigenspaces

One last episode deserves being mentioned. Later in the discussion, the doubt about the number of eigenvalues that can exist for a same direction, triggers the need to bring eigenspaces into play. Talking about eigenvectors laying on the same line, B asks:

B: There are different values for λ, aren't there?

A: No

C: Why not?

A: No, because if a linear application let's say multiplies an eigenvector times 3, if you multiply the eigenvector times 3, its image is times 3, then times 9 with respect to the first one.

Providing this answer, A performs a gesture (Fig. 2.3) that is the first one of a series of repeated and very similar gestures that will have a key role in the development of the discussion. Apparently he starts gesturing – he almost hadn't done that yet during the activity – in order to align with his group mates' discourses. Obviously this is just our interpretation. In order to convince B and C that all vectors lining on an eigenvector's direction are associated to the same eigenvalue, he starts with this embodied idea of stretching different vectors in the span of (1,1) by the same factor 3:

A: If you, the vector (1,1)..(3,3), the vector (3,3) goes into (9,9). On the other way round if you take (-1,-1) (Fig. 2.4a) it goes into (-3,-3) (Fig. 2.4b)

The interesting part of this excerpt is the way the semiotic bundle evolves: from a gesture used to convey an embodied conceptualization of this property, A progressively moves to the use of written diagrams and then to symbolic formulas. Firstly, he converts the idea shown with his gestures into a diagram and then from this he shows to B and C how this idea can be formalized with symbols (Fig. 2.5) and to provide an almost correct proof of the fact that each vector laying on the same direction of an eigenvector is an eigenvector as well, associated to the same eigenvalue.

Figure 2.3

Figure 2.4 (a and b)

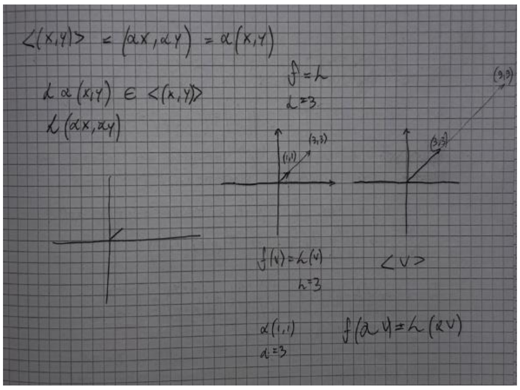


Figure 2.5

We can identify A's recurring gesture as a catchment. He keeps replicating it, or a slightly modified versions of it, throughout his whole process of development of thought: from the example grounded in embodied reality, to the more formal formulation. The repeated gesture appears as the element of cohesion between these different levels of conceptualization, and that allows the other members of the group to follow and comprehend this development.

Moreover, the catchment generates from that first gesture (Fig. 2.3), accompanied by language. The idea guiding the described process seems to arise from this language-gesture integration that we have indeed identified as a growth point.

2.5 Conclusions

From the presented results we can outline some, however partial, conclusions. We can assert that the designed activity was suitable to make students engage in an objectification process. Firstly, students' management of time is a relevant indicator. As already stated, the whole activity lasted two hours. We had not recommended a partition of the whole available time, but were expecting students not to engage in the first part for longer than 25 minutes and that they would have hurried to start solving the problems. Unexpectedly, all the small groups engaged for at least 40 minutes in the first part of the task, before moving to the second one. We interpret this fact as an indication of the fact that students felt the need to really grasp the meaning of the concepts at stake. As we could notice from the recordings, students never settled for just repeating the definitions seen during the lecture. Rather, they tried with conviction to build strong meanings for those concepts and to pinpoint connections with other linear algebra concepts. As well, they tried to ensure that all the members of their group grasped the same meaning. Secondly, the analysis of students' means of objectification and their evolution and mutual relationships actually allowed us to study their collective meaning-making process. Thanks to the use of the semiotic bundle as an analytical tool, we could detect semiotic nodes in which the emerging and evolving relationships between signs help accomplish the objectification process (Radford & Sabena , 2015). It is particularly interesting to notice how students preferably appealed to different semiotic registers. Student A from the beginning privileged the use of oral or written verbal discourse, and this, despite his evident confidence on the topics, represented an obstacle for the objectification of eigenvalues. B apparently was more confident with symbolic manipulation and resorted different times to this kind of representation in order to connect to A's discourse. The role of C was relevant, even if from the beginning he seems to be the least confident on the subject. In the first part of the activity, he struggles in following the conversation and easily gets distracted.

When they switch to a geometric example, he is able to actively engage in the dialogue using gestures, with which he is able to convey the intended meanings. In this case, it is clear how gestures, as also highlighted by previous researches (e.g., Arzarello et al., 2015), are not only a means for communication, but can be productive resources that help constitute thought. They are indeed key actors in the objectification process. Even more, it is the combination of these different semiotic resources in the bundle and conflicts arising between them, that allowed objectification to occur. “In fact, the activity through which knowledge is actualized is an activity of conflicting significations” (Radford & Sabena, 2015, p. 164). The intertwining of means of objectification activated by different students was possible only thanks to their joint labour. One last remarkable aspect is the fact that the observed group, despite required to deal with eigenvalues and eigenvectors, autonomously felt the need to deeply investigate the concept of eigenspace, in order to really understand them. This is a quite informative result, considering also the fact that research concerning eigenspaces teaching and learning is really limited (cf. Wawro et al., 2019), and will be more deeply described in future works.

To conclude, it is important to remark the fact that in this activity the course’s teacher was almost absent. The issue of considering teachers’ lectures and students’ reflections in two separated moments poses a relevant question which requires further research also because of still scarce consideration in the literature. How can the teacher’s role be integrated with a students’ joint activity as that described? In future stages of our research, we are planning to move the focus to this aspect, whose investigation might provide further insights and perspectives to the same process of objectification.

CHAPTER 3 – STUDENTS' DIFFICULTIES WITH EIGENVALUES AND EIGENVECTORS. AN EXPLORATORY STUDY

This is the authors' original manuscript of an article accepted for publication by the *Italian Journal of Pure and Applied Mathematics*.

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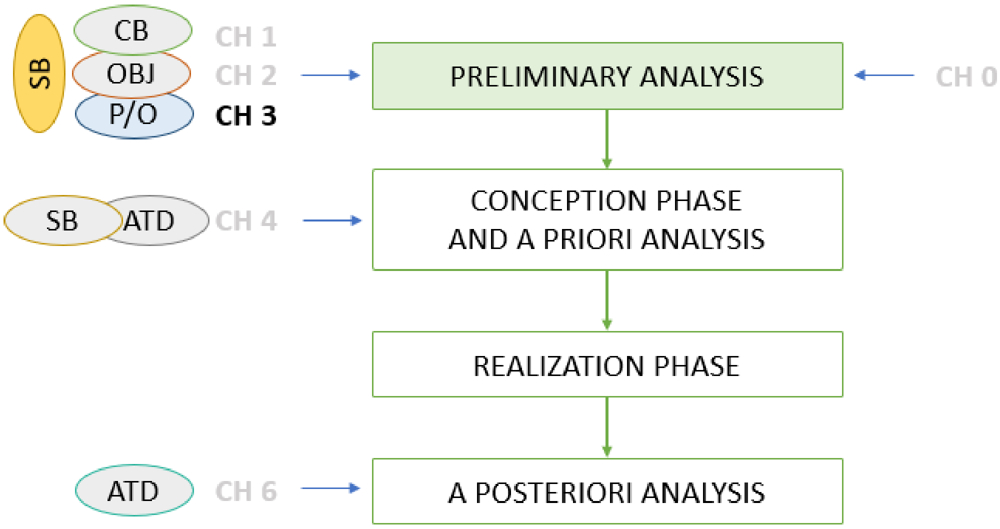


Abstract

In this paper we study the problem of students' difficulties in interpreting the definition of eigenvalues and eigenvectors, extending the research of other scholars. We analyse the activity of small groups of students, during an optional specifically designed extra class, while trying to make sense of the definitions seen in class. Using Sfard's notions of process/object, we scrutinise students' speech and written productions to detect those aspects in the formal definition of eigenvectors and eigenvalues that might hinder their understanding. We consider also how commognitive conflicts present in students' discourse with their classmates may encourage difficulties in grasping the meaning of these mathematical concepts.

Keywords: Eigentheory, process/object duality, commognitive conflict, undergraduate mathematics education

3.1 Introduction

In recent years, research in mathematics education has been taking an interest in the teaching of linear algebra, recognizing its importance not only in the purely mathematical sphere, but also as a basic subject for the study of STEM subjects, given the many applications this topic has in disciplines such as computer science or engineering. Early studies in the area have focused on the general difficulties university students usually experience in dealing with the "formalism" of linear algebra (Dorier, 2000). Some studies then investigated the teaching and learning processes of basic concepts of the discipline, such as linear dependence, determinant, span, bases, etc. Recently, several researchers have delved into investigating processes relative to more advanced topics, for which a coordination of basic concepts such as those mentioned above is necessary. One such example is the theory of eigenvalues which, in our opinion, deserves special attention for several reasons, as pointed out by other researchers who have already investigated educational aspects related to them (i.e. Thomas & Stewart, 2011; Wawro et al., 2019). Firstly, in accordance with what already mentioned regarding linear algebra in general, applications of eigentheory are widespread in STEM subjects, for example in the study of dynamical systems, or of rotating bodies or for modelling quantum mechanical systems (Serbin et al., 2020). Moreover, the understanding of eigenvalues and eigenvectors implies a good understanding and coordination of other linear algebra concepts such as linear dependence, linear transformation, determinant, kernel, etc. Hence investigating its learning processes can let difficulties related to the aforementioned notions emerge; lastly, according to both literature and our experience, the topic seems to represent a major obstacle in the study of a subject which is in itself a hindrance for students. Although the literature on eigentheory is still scarce, it is evident how interest in the topic is growing in the research community. Most of the work on the topic is concentrated in the last fifteen years or so and concerns different institutional settings. Some investigate students' difficulties in dealing with such concepts, while others describe the implementation of innovative instructional pathways and their impact on different aspects of student learning - such as a better coordination of algebraic and visual/geometric aspects (Beltran-Meneu et al., 2016; Caglayan, 2015; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015). Some other researchers (Plaxco et al., 2018; Zandieh et al., 2017) have investigated how properly designed task sequences can assist students in understanding eigenvalues and eigenvectors.

In their article "Process-object difficulties in linear algebra: eigenvalues and eigenvectors", Stewart and Thomas (2006) showed some difficulties related to the definitions of eigenvector and eigenvalue, faced by students when learning linear algebra. Two main problems related to learning these notions are highlighted by these authors: (i) in a linear algebra course, students are usually given the concept definition in words and then an easy formula to algebraically compute them is soon provided. "In this way the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust" (Stewart & Thomas, 2006; p.186); (ii) another serious problem with the formula $f(v) = \lambda v$ for students is that the two sides of the equation are quite different processes, since on the left it is a matrix which is multiplied by a vector, and this is equated on the right side to a multiplication of a scalar by the same vector.

Both these difficulties are confirmed by a more recent study conducted by the first author of this paper (Piroi, 2023a). Nevertheless, as highlighted also by Stewart and Thomas themselves, the role and genesis of these obstacles, particularly regarding the reading of the formula $f(v) = \lambda v$, requires further investigation. In this paper we will give a possible explanation for the difficulty in dealing with this formula, as emerged from the pilot study of the first author's PhD project, concerning eigentheory teaching and learning processes. This hypothesis is related to the way the formula $f(v) = \lambda v$ is read by students and to the way it, together with the traditional sequence for introducing eigenvalues and eigenvectors, supports what Sfard calls a "pseudostructural" conception of them, to the detriment of their conceptualization as objects.

3.2 THEORETICAL FRAMEWORK

3.2.1 The process/object duality

In different works, Sfard (Linchevski & Sfard, 1991; Sfard, 1992) has drawn attention to the operational/structural dual nature of mathematical conceptions. This means that in all fields of mathematics, the same mathematical concepts, or to be more precise, their representations, may at times be interpreted as processes, and other times as objects. Although these two aspects seem to be conflicting, Sfard emphasizes the importance of considering them as complementary, meaning that processes and objects must be seen as two different sides of the same coin, and not as two completely distinct aspects of mathematical activities. Sfard has brought many examples to show how the learning process may follow a pattern which is similar to the way mathematical concepts have developed historically: a process (primary process) slowly becomes an entity in its own right through a process of reification whose outcome is a mathematical object. This new object-like entity serves then as an input for new processes (secondary processes) at a higher level. Let us show an example from the realm of linear algebra to make things clearer. If we consider the notion of determinant, this achieves, both in its historical development and in the learning process, an object-status just after having been considered and manipulated operationally. It is first introduced as the result of a computation, to then become, hopefully also to the students' eyes, a mathematical object that gives important information about a matrix or the linear transformation the matrix represents. The reified object "determinant" then becomes a useful tool to perform new procedures at a higher level, such as computing the inverse of a matrix.

As already mentioned, the dual nature of mathematics can be observed in the different representations used. Although "such property as structurality lies in the eyes of the beholder rather than in the symbols themselves" (Sfard, 1991, p. 5), some of these are more likely to be interpreted as structural, while others as operational. Sfard brings the example of different representations of a function (see Figure 3.1) stating that the computer program encourages an operational view while the graph supports a structural interpretation.

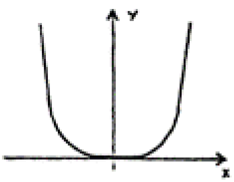
Graph	Algebraic expression	Computer program
	$y = 3x^4$	10 INPUT X 20 Y = 1 30 FOR I = 1 TO 4 40 Y = Y • X 50 NEXT I 60 Y = 3 • Y

Figure 3.1 Different representations of a function, from (Sfard, 1991, p. 6)

The algebraic expression can be easily interpreted in both ways and this may represent a benefit for the learning process, or an obstacle, as in the case we will present. It is important to notice, as Sfard points out, that this double interpretability of the algebraic representation also reflects the different ways the "=" sign can be read:

"it can be regarded as a symbol of identity, or as a 'command' for executing the operations appearing at its right side" (p. 6).

The operational/structural duality of the equality sign as well as its misuse by primary and lower secondary school pupils has been widely discussed (Behr et al., 1980; Godfrey & Thomas, 2008; Powell, 2012 ; Sfard, 1991). However more recent studies (see for example Fyfe et al., 2020) have shown that an exclusively operational interpretation of the equal sign can persist in college students. Moreover, Das et al. (2020) show how university students can struggle with the bidirectional flexibility in reading an equality, confirming a permanence well beyond elementary school of the difficulty with the structural interpretation of the equals sign.

3.2.2 Commognitive conflicts

It is crucial to remember that, according to Sfard's theory of commognition, thinking, and thus thinking mathematically, is a form of communication. Therefore, it is important to consider all the elements related to the development of mathematical discourse to comprehend phenomena such as the persistence of the operational view of mathematical concepts and the challenges associated with transitioning to a structural interpretation of them. In this context, one significant phenomenon to consider is what Sfard refers to as *commognitive conflict* (Sfard, 2008). When a new term is introduced in the discourse, or a new meaning of that term is presented, our impression of understanding it can be misleading. There is no guarantee that other participants in the discourse are using it in the same way, and this potential ambiguity exposes the interlocutors to the risks associated with the use of non-operationalized terms. Specifically, when one encounters a new discourse governed by metarules different from those with respect to which he or she has acted up to that point, a commognitive conflict may arise. This situation involves different participants in the discourse following different metarules. In such cases, the two discourses are compatible but incommensurable, meaning they do not share the same criteria for approving or disapproving the current narrative. According to Sfard, the resolution of these conflicts generates learning. However, this typically occurs when an 'oldcomer' in the mathematical discourse participates in the discussion and can provide meaning to the newcomer's discourse using shared metarules. Unfortunately, this does not always happen, and if a commognitive conflict remains unresolved, it can lead to a failure to achieve a structural interpretation of the mathematical object in question or to the development of a pseudostructural conception of it. Pseudostructural conceptions (Radford et al., 2017) are the conceptions which develop when the student, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result, remains unaware of the relations between the secondary and primary processes. Consequently, they remain unaware of the relationships between secondary and primary processes.

3.3 Research questions and methodology

In autumn 2021, as part of the pilot study for the first author's PhD thesis on eigentheory teaching and learning processes, 67 first-year engineering students participated in an experimental activity. This activity took place during tutoring or extra hours a few days after the introduction of eigenvalue and eigenvector notions in a standard lecture by the linear algebra course's teacher.

During this two-hour activity, students were organized into groups of three. The first part involved a collective re-reading of the lecture notes and other study resources to reorganize newly encountered notions and deepen their understanding. To encourage a true comprehension of eigenvectors and eigenvalues, rather than mere repetition of the teacher's or textbook definitions, a guiding question was posed: "How would you explain what eigenvectors and eigenvalues are to someone who only knows basic notions of linear algebra such as linearity, dependence, basis, etc.?" Following this, four problems were presented. Written protocols from all groups were collected, and eight groups were audio and video recorded throughout the activity. A multimodal semiotic analysis (Radford et al., 2017) was conducted to detect moments where the simultaneous use of different semiotic resources (e.g., utterances, inscriptions, gestures) provided interesting insights into students' understanding of the concepts.

For a more detailed description of the methodology used and some further results, refer to Piroi (2023b). In this discussion, our focus is on presenting two specific segments of activity from two small groups, highlighting difficulties in reading the formula "$f(v) = \lambda v$" and various interpretations of the equal sign within it.

Using the described theoretical framework, we address the following research questions: (Q1) In what ways is a commognitive conflict likely to arise in a discourse on eigenvectors, when newcomers participate in that specific discourse? (Q2) What aspects in the formulation of the equation "$f(v) = \lambda v$" may hinder the development of a structural view of eigenvectors and eigenvalues?

To answer these questions, we present excerpts from discussions between students in two of the observed groups, showcasing their utterances, gestures and drawings produced and used in the discourse. Our analysis seeks elements suggesting the presence of a commognitive conflict, where participants use the same terms or visual mediators but according to different discursive rules. We aim to detect if and how these conflicts produce or reinforce a pseudostructural conception of eigenvectors in, at least, one participant in the discourse. Being our focus on how students interpret the commonly provided definition of eigenvectors and eigenvalues, $f(v) = \lambda v$, we do so examining the segment of the video recording where students respond to the leading question mentioned earlier.

We start presenting and describing the chosen extracts and then we analyze them in order to show the elements that help us answer our research questions.

3.4 Results

3.4.1 Description of discourse in Group 1

From now on, we will call the three students in the considered group, Antonio, Bruno and Claudio. Right after having read the assignment, the students agree to start with reading the examples given by the teacher in the lecture:

1. Bruno: We must look at the examples
2. Claudio: Yes, exactly, the definition here is useless.

Their choice of trying to understand the examples, instead of interpreting the definition, seems effective. Indeed, it seems that the chosen example, eigenvectors and eigenvalues of a geometrical transformation in R^2, helps them understand what these actually are.

3. Bruno: Then, basically, when is it that they form. . . when they overlap?
4. Claudio: When the vector remains itself, so when nothing changes.
5. Bruno: When it is rescaled times λ.
6. Claudio: Mmh. . .
7. Bruno: Yes, because f of v is equal to λ times v (he points to the formula "*f (v) = λv*" written on his lecture notes) so this means that the image of the vector is the vector rescaled times λ.
8. Claudio: Yes!

9. Bruno: So it means that it remains on the same line basically, when the two end on the same line (while pronouncing this sentence he performs the gesture in Figure 3.2), so what they do is to increase or decrease.

They also re-read another example, an algebraic one, and also in this case it seems that they have grasped the idea of what it means for a vector to be an eigenvector. Nevertheless, a first doubt emerges, expressed by Claudio:

10. Claudio: So, does the eigenvector always coincide with the linear vector?
11. Bruno: Yes, sure, the one that you must have, isn't it? It is the vector of which you take the image.
12. Claudio: I don't know. . .

Figure 3.2 The gesture produced by Bruno (line 9)

This exchange of lines is very brief, and the two students do not dig deeper in this miscomprehension, even if it is clear that Claudio is not satisfied with Bruno's answer. Nevertheless, by analyzing the whole activity, this perplexity emerges at different times and seems to be related to the fact that in the formula "$f(v) = \lambda v$", v appears both as the argument of the function and as the vector which is multiplied by λ. After having, almost, convinced themselves of this characterization of eigenvectors and eigenvalues, Bruno tries to write down a definition and he shares it with his group classmates:

13. Bruno: Is it good? If I say that the eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue.
14. Antonio: Can you repeat?
15. Bruno: The eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue.

The three of them agree on this definition. By reading this, it might seem that they have indeed understood what eigenvectors and eigenvalues are. This is probably right for Bruno, and this can be inferred by the way he solves the exercises in the second part of the activity. Nevertheless, the following part of their discussion makes clear that Antonio has misinterpreted the definition given by Bruno.

16. Antonio: But, I don't understand, this e1 (they are now looking at an example where e1 is an eigenvector for the given function f) would be v which finds v? Bruno and Claudio ask him to explain it with clearer words.

17. Antonio: I mean, f of v is λ times v? So in this case. . .
18. Claudio: It is a vector rescaled by its image.
19. Bruno: Because v is 3 times e1.
20. Claudio: Basically, the image of e1 is e1 rescaled by the eigenvalue.
21. Antonio: Ok!

Are Bruno, Claudio and Antonio's discourses really aligned? When they say "f of v is times v" or when they read "$f(v) = \lambda v$", are they really interpreting it in the same way?

Let us move a little forward, when the three students are solving the problems. The first one consists of finding the eigenvectors and eigenvalues related to the symmetry with respect to the line $y = x$, in R^2. In this problem-solving phase, students' conceptions of the meaning of "=" appear more evidently. For example, after having realized that all vectors lying on the line $y = x$ are eigenvectors, they cannot find the related eigenvalue. The researcher leading the activity intervenes to help them, by asking which is the image of a generic vector (3, 3). Claudio answers "It is λ times (3, 3)". He seems convinced of this statement because in the following lines he adds

"It is (3, 3) rescaled for whatever eigenvalue", and then again restates "[the image of (3, 3) is] (3, 3) times λ". A few lines later, Antonio asks:

> 22. Antonio: if we had the vector (3, 3), and the vector (6, 6) [as its image], is it wrong to say that it is the vector times λ that is 2?

3.4.2 Interpretation of Antonio and Claudio's statements

Antonio and Claudio's confusion in solving this problem seems to be related to the way they interpret "$f(v) = \lambda v$", as well as the literal definition given by Bruno at the beginning of the activity, namely "The eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue". It seems that they are interpreting "=" in the formula as a defining equal sign. This means that they do not recognize the equality in the formula as the fact that the vector obtained computing v's image is equal to the vector obtained multiplying v by λ. They seem to be reading the formula as "the image of v is given by λ times v". Bruno's definition unconsciously helps Antonio and Claudio build this interpretation of the formula. When he says "is defined" he is improperly using this verb and it seems that he uses it to intend "is equal to". Nevertheless, the use of the verb "is defined" likely reinforces Antonio and Claudio's interpretation of "=" as a defining equal sign. This way of reading the equality is evident in the fact that, when I ask what is the image of (3, 3), instead of calculating its actual image and checking if it actually is equal to a multiple of (3, 3) itself, Claudio immediately answers "λ times (3, 3)", as if to calculate a vector's image one had to multiply it times some eigenvalue. Also Antonio seems to think that one is allowed to arbitrarily consider a vector's multiple as its image, without actually applying the transformation considered. This would also shed light on the perplexity shown before by these two students. When Claudio asks "does the eigenvector always coincides with the linear vector?", he seems to be confused by the fact that v appears in both the left and the right side of the formula as if he did not know which one to take to compute the eigenvalue. The same doubt seems to emerge for Antonio when he states that the eigenvector "would be v which finds v".

3.4.3 Description of discourse in Group 2

The second observed group is also formed by three students, Davide, Emma, and Flavio. The discussion predominantly involves Davide and Emma, with Flavio primarily listening and contributing minimally. Once they initiate the task, Davide and Emma deliberate on how to approach answering the guiding question. Emma suggests starting "by saying what it [an eigenvalue] is", while Davide insists on the importance of starting with an example to make it clear what it is. He draws a cartesian plane and provides an example that he recalls the lecturer presenting in class:

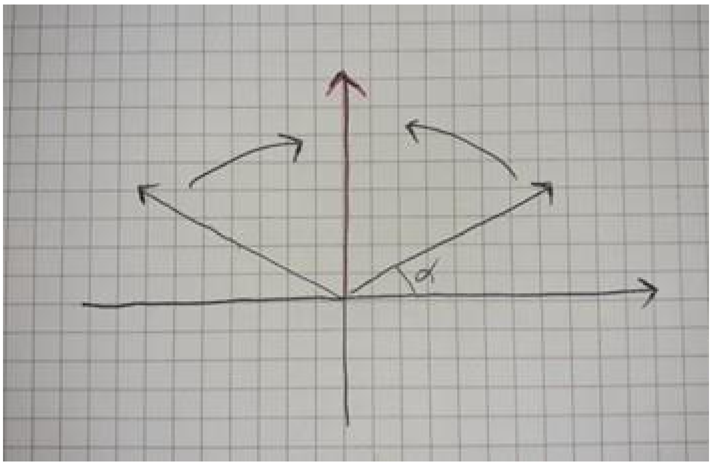


Figure 3.3 Davide's drawing

1. Davide : There was a rotation. That if this goes this way (drawing a random vector in the first quadrant of the cartesian plane, and showing with an arrows that it rotates anticlockwise, as visible in Figure 3.3), the other one goes there (drawing a second vector on the second quadrant and show with an arrow that it rotates clockwise – Figure 3.3). And so, obviously, these two don't have the same direction, but for some specific coordinates, they do have the same direction (drawing a vector lying on the y-axis). For equal to specific values, these two overlap or we could say that one v is equal to v (and writes 1v = v).

Davide talks about a "rotation" but what he is really talking about and representing is in reality a symmetry. The thing that probably confuses him and makes him talk about "rotation" is that he immediately represents a vector and its image, speaking of them as two different vectors. After that, he "rotates" the first vector and consequently the second. What he is doing with this apparent rotation (and probably doing in imitation of what he has seen the teacher do in class) is exploring the various positions that the starting vector and consequently its image can take according to the transformation involved, that is, its symmetry with respect to the y-axis. He then points out that "for specific values of α", by which he means the vector of the same (arbitrarily chosen) length lying on the y-axis, the two vectors (vector and its image) overlap.

At this point, Davide attempts to provide additional examples, but they appear to confuse the other two members of the small group. The first author, who is present during the activity, tries to redirect the focus to the initial example given. She emphasizes that they are exploring a specific transformation; one that, when given a vector, yields its image symmetrical with respect to the y-axis. She then proceeds to inquire about the eigenvalues and eigenvectors of this particular transformation.

2. Flavio: So, I would say that the eigenvalues are 1, because it brings the vector into the multiple of itself and so it is a multiple of 1.
3. Davide: But how can you explain that?
4. Emma: Well. . . it is all those times in which the vectors overlap, they coincide.
5. Davide: But it is not that they always coincide. You must explain the generic concept of eigenvalue.
6. Emma: But you made that example!
7. Davide: Yes but I would make the example saying, in all these cases, when the direction is the same, if we make the ratio between the lengths of the two vectors, the result is 1. [. . .]
8. Davide: In this case it is 1.
9. Emma: And this is the scalar that multiplied to the vector. . .
10. Davide: But if this example was like this. . .

Davide replicates the previous illustration, this time doubling the length of the image vector. He endeavours to illustrate to his classmates that with the vectors configured in this manner, as they both converge on the y-axis (reiterating the concept from earlier that the vectors rotate towards the *y*-axis), they share the same direction. However, the crucial distinction lies in the ratio between their lengths, which is now 2 instead of 1. After reflecting on this example, the group tries to synthesize their thoughts.

11. Davide: I would say so: for all those case where a vector is image of itself, this function..
12. Emma: But this is a specific case.
13. Davide: Yes, but I started with the example of this function and then I extend it to all the functions.
14. Emma: And how can you extend it?
15. Davide: It is a concept that I am trying to say. For a generic function. . .
16. Emma: So, any function that has this ratio, I mean, this symmetry. . .
17. Davide: Exactly! But not for a symmetry. I am
18. Emma: In general
19. Davide: For any function where the image of a vector is equal to a multiple of the vector itself, that is a number that is multiplied for the vector itself that in this case (referring to the example in Figure 3.3) is 1 and in this other -1 (referring to the same example but to the vectors lying on the *x*-axis), then we will call that value "eigenvalue".
20. Emma: And the vectors "eigenvectors".
21. Davide: And all the vectors that respect this criterion "eigenvectors".
22. Emma: So, eigenvalue and eigenvectors are. . . the eigenvalue, in this case -1, always works for those functions that go into the same direction.
23. Davide: Yes, if you talk about the plane.

3.4.4 Interpretation of Davide and Emma's interaction

The analysis of this excerpt, while rooted in results, may not be entirely straightforward, and the forthcoming interpretation is acknowledged to be somewhat subjective. Nevertheless, it appears that a commognitive conflict is arising here as well. Upon examining the entire sequence, it becomes apparent that Davide has grasped the concepts of eigenvectors and eigenvalues. However, it seems that the discourse between Davide and Emma is incommensurable. In other words, due to the different representations employed and the terminology used to describe them, it appears that at certain points in their conversation, the two students are using the same terms to refer to different concepts.

Davide's drawing (Figure 3.3) represents a crucial element in the discussion. In his representation, the distinction that one of the two vectors is independent, while the other is dependent— with its image dictated by the prescribed transformation (symmetry with respect to the *y*-axis)— is not immediately evident. Specifically, his discourse involves rotation, and he mentions that, for certain values of λ, the two vectors overlap. The commognitive conflict here seems to emerge in two ways. Firstly, the fact that Davide uses the term 'overlap' initially makes Emma think that in order to be eigenvectors, the vectors must overlap. Davide recognizes the presence of this misconception by Emma and attempts to resolve it by providing an example in which the two vectors have twice the length of one another. However, in this case, the dependence between the vector and its image is even more tenuous. In fact, it appears that in this instance, Davide himself is constructing the function arbitrarily, ensuring that one of the two vectors is twice the length of the other. In our opinion, this may further reinforce the notion that eigenvalues exist for functions that are defined by taking a multiple of the initial vector as its image. Furthermore, the fact that it is not evident that the dependence between the two vectors is dictated by the transformation in question, seems to make Emma think that what for Davide are different vector-image configurations for the same transformation, are instead different transformations. In fact, later in the conversation, she explicitly states: "any function that has this ratio . . . " and "the eigenvalue [. . .] always works for those functions that go into the same direction". It seems, therefore, that for Emma, there are functions for which all vectors have the property of having their image aligned, and in those cases, "the eigenvalue works"—that is, probably, it exists. However, she does not grasp the fact that for every transformation, there can be some vectors with this property. The way in which she and Davide use terms like "exists" or "any" in the dialogue certainly supports this commognitive conflict.

3.5 Conclusions

In the two excerpts, recurring elements are noticeable. Despite their different manifestations, it appears that in both instances, a participant in the discourse fails to grasp a fundamental aspect of the definition of eigenvalues and eigenvectors: a precise mapping is provided, and for that specific mapping, vectors may exist whose images are multiples of themselves. Instead, it seems that these students (Antonio in the first example and seemingly Emma in the second) are under the impression that the function itself is defined in such a way that each vector has, as its image, a multiple of itself. This misconception is underpinned by a commognitive conflict, wherein the same sentences or representations are employed and interpreted differently by various participants in the discourse based on distinct metarules - in the sense of Sfard (2008, p. 204). In the first example, the conflict arises from the use of the term "is defined" while in the second example, it appears to stem from how the example application is represented and described. Both conflicts are closely tied to the classical formulation of the definition of eigenvalues and eigenvectors. For instance, the definition provided by the teacher of one of the involved courses, which can be taken as a model for the classically given definition, is: "Let $f: V \to V$ be an endomorphism on the vector space V . We say that a vector $v \in V, v \neq 0,$ is an eigenvector of f if there exists a value $\lambda \in$ K (the field of scalars), called its eigenvalue, such that $f(v) = \lambda v$ stands". In this formulation, two elements are likely to cause confusion. Firstly, the expression $f(v) = \lambda v$ might be misinterpreted as implying that $f(v)$ is defined as $f(v) = \lambda v$, leading to a potential reinforcement of an interpretation as that by Antonio. Secondly, the term 'exists' appears to be associated with λ. The way it is formulated assumes, almost taken for granted, the existence of vectors with such a property for that particular application. Consequently, the emphasis is placed on the value λ rather than on what happens to v, considering the specific application. This tendency is observed in both groups, mirroring Antonio's and Emma's interpretations of their peers' examples. In these cases, there seems to be a tendency to overlook the crucial point that a vector's image is determined by the given application: only instances where the image is a multiple of the initial vector qualify this as an eigenvector. This interpretation also stems from a challenge in understanding the inherent logical dependencies in the classical definition through the formulation $f(v) = \lambda v$ (or equivalently $Av = \lambda v$). A correct reading of it would be that there are some linear applications f , for which $\exists v\ s.t. f(v) = \lambda v$ for some $\lambda \in K$[8]. Contrary to the intended interpretation, students like Antonio or Emma tend to interpret it as: there exist linear applications f such that $\forall v, \exists \lambda: [(v, \lambda v) \in f \leftrightarrow f(v) = \lambda v]$. In our experience, we have observed this interpretation in many first-year STEM students taking linear algebra courses. What seems to happen in the two groups is that somehow the definition they have already seen in class guides them in their reasoning. However, the meaning of the definition had remained obscure to them. In their reasoning, they start from an example, which would have the great potential of being able to start them from an operational interpretation of eigenvalues and eigenvectors. However, they have the bias of having already seen the structural definition, which they somehow want to make sense of at the same time as exploring the example. The fact that in both groups, there are those who have actually understood what eigenvalues and eigenvectors are but use representations and terms that are interpreted by their peers according to different metarules, generates commognitive conflicts. As a result, a pseudostructural conception of the concepts involved seems to be reinforced.

These results confirm students' difficulties in dealing with the formula $f(v) = \lambda v$ identified also by Stewart and Thomas (2006). Moreover we have tried to delve into the reasons behind such difficulties, showing how also the way students engage in discourses about these concepts might reveal and reinforce some misconceptions. It seems that, in their attempts at giving an operational meaning to eigenvalues and eigenvectors, through the analysis of examples, students are strongly biased by the fact that they have already encountered their definition. In this way, the passage between process and object is not smooth and results in reaching a pseudostructural conception of eigenvalues and eigenvectors. One potential didactical implication is the importance

[8] Equivalently, in prenex normal form (Shoenfield, 1967), the formula would be: $\exists f \exists v \exists \lambda\ (f(v) = \lambda v)$.

of starting with an operational interpretation of mathematical concepts, as suggested by Sfard, even at the advanced level of university instruction. Specifically, for eigentheory, instead of immediately presenting the definition of eigenvectors and eigenvalues, it might be beneficial to begin with the exploration of examples. Introduce a linear transformation (in the form of a geometrically known transformation or as a matrix) and ask students to determine if there are vectors whose image is a multiple of themselves. Only then could these vectors be designated as eigenvectors, with the ratio between the image and the vector representing the corresponding eigenvalue. This approach can enable students to initiate their understanding from exploring a process before transforming it into a mathematical object. Furthermore, it might reveal to students the inherent logical dependence in the formal definition through concrete examples.

Following this pilot study, we have indeed formulated the subsequent phase of the research project, taking into account these considerations. Specifically, emphasis has been placed on the utilization of dynamic digital representations of linear applications, such as videos or an explorable GeoGebra applet. This approach aims to support the initial process-like conceptualization of eigenvectors, preventing misconceptions likely to arise in a static drawing, as exemplified by Davide in the second example.

CHAPTER 4 - INTRODUCING EIGENSPACES: SEMIOTIC ANALYSIS AND DIDACTIC ENGINEERING

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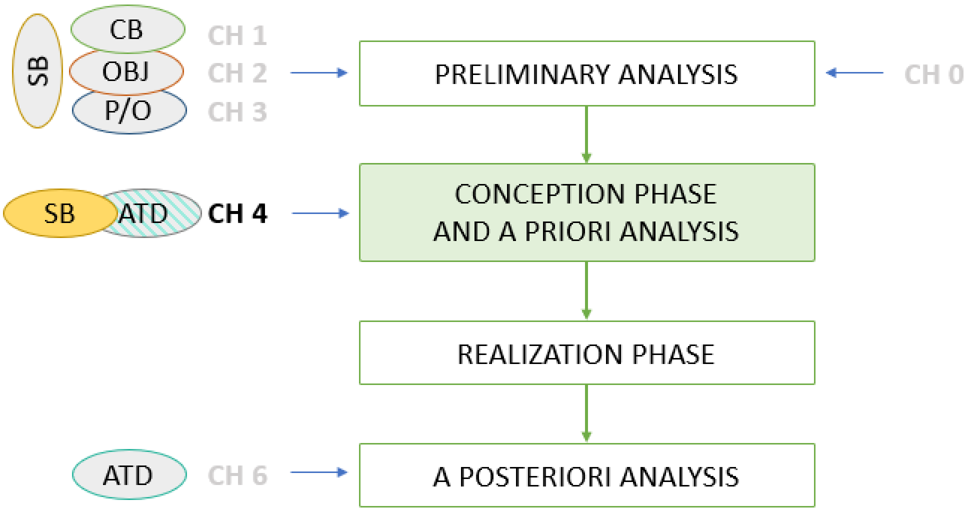
THE CHAPTER IN THE RESEARCH PROJECT



Abstract

This paper reports part of the author's PhD research, that focuses on eigentheory teaching and learning processes. We describe how the results of a pilot study, based on multimodal semiotic analysis, together with a broader epistemological, didactical and ecological analysis of the mathematical knowledge at stake and the learning context have informed the conception and a priori analysis of activities and methodologies designed within the didactic engineering methodology. These activities have been designed by the author and the teacher of a linear algebra course, with a high number of class attendees, offered in a mechanical engineering degree program in a public university in Italy. We report the multimodal analysis of a fragment of students' activity during the course implementation, to validate the hypotheses guiding their design.

Key-words: Eigentheory, didactic engineering, university mathematics education, semiotic bundle

4.1 Introduction

In the last decades, a growing body of literature has been investigating the obstacles encountered by students when learning linear algebra (Stewart et al., 2019). Among the topics generally covered in a linear algebra course, that of eigentheory has been studied in a number of researches, given its wide applicability in STEM disciplines and the particular difficulties students usually encounter in studying this topic. Thomas and Stewart (2011) investigated eigenvalues and eigenvectors concept development, and provided interesting results regarding the way students understand the equation $Ax = \lambda x$. They showed how students seem to struggle in coordinating the two different processes encapsulated in the two sides of the equation: on the left side there is a matrix multiplied by a vector, while on the right side a multiplication of a scalar by the same vector. Another problem highlighted by the same authors is that the dominant way eigenvectors and eigenvalues are taught is through their algebraic definition and providing an algorithm to compute them. Consequently, the focus is turned too soon to the manipulation of algebraic representations, and in this way "the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust" (p. 186). As detailed in a broad overview provided by Wawro et al. (2019) on the educational research that has focused on eigentheory, other researchers have confirmed Thomas and Stewart's results and detected different reasons that add complexity to the understanding of the topic. Among these, there is the difficulty in coordinating solutions to $Ax = \lambda x$, solutions to the homogeneous systems of equations resulting from $(A - \lambda I)x = 0$, and the null space of the matrix $A - \lambda I$. In general, research on this topic agrees on the fact that integrating a geometrical interpretation of eigenvectors would enhance the understanding of eigentheory. Different researchers, summarized in (Wawro et al., 2019), have shown that the use of dynamic geometric software can help in this direction. Nevertheless, this seems to be rarely done and the algebraic approach continues prevailing.

One of the notions of eigentheory whose understanding has been rarely investigated (Wawro et al., 2019), is that of eigenspace, that is the set of all the eigenvectors associated to an eigenvalue of a linear transformation, together with the null vector. The fact that all the linear combinations of independent eigenvectors related to the same eigenvalue are eigenvectors of the same eigenvalue too, is not trivial at all for students (Salgado & Trigueros, 2015; Wawro et al., 2019). For this reason, there is a high risk that when the definition of eigenspace is given, students assume it, without really grasping the power of the fact that all the eigenvectors associated to an eigenvalue form indeed a vector subspace.

The author's PhD research project faces these issues, trying to investigate teaching and learning processes of eigentheory, in a linear algebra course offered in the first year of a degree in engineering. A pilot study was first realized, followed by the implementation of a teaching-learning sequence designed according to the didactic engineering methodology (Artigue, 2015).

4.2 Didactic Engineering: main features

Didactic engineering (DE) has been acknowledged as a research methodology since the early 1980s (Artigue, 2015). As such, it is structured into different phases: preliminary analyses, conception and a priori analysis, realization, observation and data collection, a posteriori analysis and validation. Preliminary analyses lay the foundation for the whole design and take into consideration different aspects that together contribute to the choices shaping the activity conception. The principal dimensions studied at this preliminary stage are the epistemological, the institutional, and the didactical ones. The aim of the institutional analysis is to identify the specificities of the Institution in which the DE takes place, highlighting the conditions and constraint to its implementation, while that of the didactical analysis is to investigate what information previous research has provided about the teaching and learning of the concept at stake. This last dimension is substantially cognitive, and in order to study it, different theoretical frameworks can be integrated into the methodology of DE.

The preliminary analysis guides the phase of conception and a priori analysis, where research hypotheses are made explicit and employed in the design of the activities to implement. Choices made in the conception phase can be made at the macro-level of the design, involving the global project, or at a micro-level, concerning the design of specific activities or situations. The a priori analysis clarifies how the choices made in the conception relate to the preliminary analysis, taking into account its different dimensions.

In the realization phase, the activity or activities are implemented, under the researchers' observations. The type of data collected strongly depends on the theoretical framework accompanying the DE. It is to be expected that the project will not follow exactly the path imagined during the conception phase, so the researcher should carry out an in-vivo analysis during the course of the activities, in order to determine if some teaching variables or choices need to be changed during the course of the project and to make the necessary changes as soon as possible.

A key aspect of this design research methodology is that its validation is internal and realized through a comparison between the a priori and a posteriori analyses. This comparison allows to put to the test the research hypotheses. Again, based on the theory behind the DE, the specific research questions posed and the type of data collected, a posteriori analysis can be conducted using different methodological tools. For example, in the here presented DE, a semiotic lens, namely, the semiotic bundle theory, was used both to enrich the preliminary analysis and conception phase and to realize the a posteriori analysis.

4.3 The semiotic bundle

Contributing to a broad strand of research in mathematics education concerned with the semiotic aspects of mathematical practice, Arzarello introduced the notion of semiotic bundle (SB):

"a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question". (Arzarello et al., 2009, p. 100)

Not only can the collection of signs that are present in the bundle evolve, but also the relationships existing between them can vary as the subject or the interacting subjects produce them. These relationships are analyzed through two types of lenses within this framework: synchronic analysis allows to study the relationships among signs produced simultaneously, while diachronic analysis studies the relationships among semiotic sets activated by the subject (or subjects) in successive moments, thus their evolution. This last kind of analysis has been used also by other theories, but the element of novelty brought by the theory of the semiotic bundle is the chance to use it to observe phenomena considering all the signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact, etc. This means, for example, that the dynamical relationship between gestures and speech can be analyzed, and such an analysis can give important information about the subject's ideas and thinking. In this case, gestures and language are a semiotic bundle, made of two deeply intertwined semiotic sets, of which only the second represents also a semiotic system.

4.4 Context, preliminary analyses and research question

The DE presented in this report has been designed for a Linear Algebra and Geometry course offered at the first year of the mechanical engineering degree program at a public Italian university, and taught by Teacher "A". Topics of this course span from basic vector spaces theory (approximately: vector spaces, matrix algebra, linear systems, eigentheory) to Euclidean spaces. A pilot study had been conducted in the Fall term of 2021 with students of the same course taught by Teacher A, and of other two linear algebra courses for different engineering degree courses, taught by different teachers. In the pilot the teachers were not involved. In the week following the classes on eigentheory, taught by the teachers without any researcher's intervention, some students voluntarily attended an extra two-hours tutoring class lead by this paper's author. During the optional activity, students had to work in groups of three or four and re-elaborate the notions of eigentheory encountered in the recently attended lecture, guided by guidelines provided by the researcher. These comprised very open questions such as “How would you explain the concept of eigenvector to someone who has never heard of that before?”. Students were allowed to answer in the way they preferred, orally, in written form, with diagrams, etc., and were free to use whatever tools were available to them and encouraged to use other resources that they had encountered, in addition to the book or notes taken during the lessons. For the analysis, all the written protocols of the students, and audio and video recordings of eight of the small groups, were collected. This type of data allowed a multimodal analysis, according to the construct of the semiotic bundle (Arzarello, 2009). After the implementation and analysis of the results of the pilot study, teacher A was involved in the redesign of the module of her course regarding eigentheory for the following year. The DE here described regards this redesign of the course and its implementation in the Fall term of 2022. The results of the pilot study, together with the observation of the whole module on eigentheory as taught by teacher A before our intervention (all classes were video recorded by teacher A), contributed to the preliminary analysis for the DE.

Before our intervention, the module on eigentheory of the course followed the sequence traditionally used in Italian linear algebra courses: the definition of eigenvectors and eigenvalues, as those vectors v and scalars λ for which $Av = \lambda v$ stands, was given; some mainly algebraic examples were shown by the teacher (this is the case for teacher A, who also used to present a few geometrical examples, but it rarely happens in other courses); the procedure for computing eigenvalues and eigenvectors of a given matrix was shown and algebraically justified; the definition of eigenspace was presented; the theorems regarding diagonalization were introduced.

Our analysis of the most commonly used Italian books and of notes of some other linear algebra courses taught in the same university where we conducted our research, ensures us that this teaching sequence is dominant in Italian universities. Descriptions of curricula and courses found in the literature seem to confirm that the same path is dominant in other countries too. The raison d'être of eigentheory is only hinted at the beginning of the course, when the teacher explains that eigenvalues and related notions are useful for diagonalizing matrices, but this rationale is soon lost by the students, also because in the following introduction of definitions and theorems, it is no longer taken into consideration. Furthermore, this is only the mathematical raison d'être of the knowledge at hand, whereas the reason why it is taught in an engineering course is not addressed at all. This problem affects the entire linear algebra course and is shared by mathematical courses for engineering and other STEM degree programs in Italian universities. In such a course structure, all the difficulties highlighted in the existing literature are likely to arise. In addition, the geometric interpretation, which, as confirmed by the literature, would help overcome these difficulties, although presented by the teacher with some examples, is completely absent in students' practices, which consist solely of algebraic computations to find eigenvalues, eigenvectors and eigenspaces.

4.4.1 Analysis of part of the pilot study

The analysis with the SB lens of students' activities in the pilot study allowed us to expand the epistemological and didactical analysis. Due to space constraints, we will present here only some results concerning students' conceptualization of eigenspaces. In the small-group out-of-lecture activity, most of the groups struggled in understanding what an eigenspace is, by trying to interpret its definition. Almost all the students, after reading the main eigentheory related definitions, began to review their class notes, including examples of transformations and related "eigen-objects" provided by the teacher. They struggled to understand why those indicated by the teacher were the transformation's eigenvalues, vectors and spaces, likely because they had not a clear idea of what these were. Nevertheless, a very interesting aspect emerged from two of the groups whose activity we have recorded. Students in these groups, before even reading on their notes the definition of subspace, began to wonder how many eigenvalues could exist for the same direction. The need to answer this question arose from the desire of better grasping what eigenvalues and eigenvectors are. In one particular group we observed how the attempt to tackle this issue, brought to an autonomous conceptualization of what eigenspaces are. When these students later encountered the definition in their lecture notes, they could make sense of it, due to their prior spontaneous inquiry. It is particularly interesting the way the evolution of the semiotic resources used by students played an important role in this process of discovery. The student who initiated the inquiry in the group starts considering the type of representation he is more confident with, namely the geometric one, that he represents in an embodied way, using gestures (in Fig. 4.1a and 4.1b the student shows to his peers that an eigenvector is a vector whose image gets "stretched" but remains in the same direction). Then, to align with the discourse of his fellow group members, he gradually shifts to using written diagrams (Fig. 4.1c) and then to symbolic formulas to ultimately arrive to an almost correct abstract formulation of the fact that any multiple of an eigenvector is also an eigenvector related to the same eigenvalue. This excerpt of students' activity is described and analyzed in greater detail in Piroi (2023). Students in this example, however, do not investigate the case where there are independent eigenvectors related to the same eigenvalue, most likely limited by the choice of a two-dimensional geometric example, and this limits their deep understanding of what an eigenspace is. The crucial point of this example, however, is that students naturally inquire about how many different eigenvalues might correspond to a given direction and, subsequently, what all the eigenvectors related to a given eigenvalue are. This exploration leads them to understand, with the stated limitation regarding dimension, that all eigenvectors related to a same eigenvalue form a vector subspace. This confirms the few results present in the literature about students' difficulties in understanding eigenspaces, but further suggests that letting students actively explore examples of geometrical transformations, they are likely to mobilize more visual and dynamic semiotic resources, such as gestures, and thus to visualize the eigenspaces formed by all the eigenvectors related to an eigenvalue. This might help them make sense of the definition of eigenspace, when they encounter it.

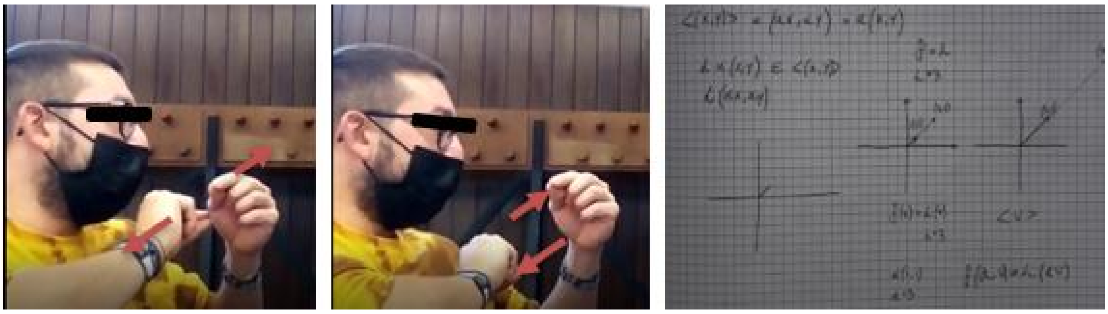


Figure 4.1 (a,b,c): Evolution of the signs produced by the group of students

Thus, the pilot study not only allowed us to perform a fine-grained analysis of students' difficulties in understanding eigenvalues and eigenvectors. It also revealed that the proposed activity actually helped students make sense of these concepts. It is when the students start collectively reinterpreting the examples given by the teacher, mobilizing different semiotic resources, such as written and oral language, drawings and gestures, that they begin to create and share an effective conception of eigenvectors, etc. The work in small groups allowed the students to mobilize different semiotic resources for the sake of communication, but those resources became fundamental tools for comprehension itself. This aspect is evident in the presented extract. As a matter of fact, educational research has widely demonstrated that collective active learning is beneficial to students' learning also in Engineering courses (Prince, 2004). Nevertheless, the considered learning context has a number of constraints that highly hinder the implementation of collective activities. Firstly, the course was attended by more than 200 students and the physical disposition of the desks in the classrooms, lined up in long rows, does not allow working easily in small groups, making it also very difficult for the teacher to walk around the desks to observe the students' interactions. Moreover, the thick curriculum and on the other side the little time allocated to the course, urge the teacher to follow the standard teaching sequence.

Taking into account the epistemological, didactical and ecological dimensions of the mathematical knowledge to be taught and of the specific learning context, we formulate the following research question: What kind of mathematical-didactic proposal, compatible with the learning context described with constraints such as the very high number of students attending the classes, can be implemented to foster the understanding of eigentheory? To answer this question, we will describe some choices made in the redesign of the course and will analyze part of the data collected during the implementation using the SB lens. The results will answer the research question and validate the research hypotheses generated during the conception and a priori analysis phase.

4.4.2 Design, implementation and analysis

Considering the various aspects that emerged in the preliminary analysis, we redesigned the eigentheory module, devoting a significant portion of the lecture to students' collective exploration of eigentheory concepts. We designed a series of tasks to be progressively proposed to students, that could support their understanding of eigenvectors, etc., avoiding the difficulties highlighted by the literature and emerged in our pilot study. We addressed the institutional constraints that hindered the implementation of collective activities by introducing a technological tool, namely the online notice board 'Padlet'. Through this platform, the teacher could assign problems to be solved on the spot by pairs or small groups of students during the lectures. Then, at the end of the time allocated for solving the problem, the groups could anonymously post a picture of their solution on the shared padlet. The teacher would project the padlet so that everyone could see the answers posted by others. She would also be able to check, as solutions were posted, if there were any particularly interesting solutions to present to the class later, as a bridge between the groups activity and what she would explain immediately afterwards. For the a posteriori analysis, we not only had all the pictures posted by the students in the padlet at our disposal, but we also audio and video recorded three randomly chosen small groups during all the activities about eigentheory conducted in the classes. This type of data was necessary to analyze students' activity according to the multimodal paradigm (Radford et al., 2017) at the core of the SB framework.

As for eigenspaces, we did not present the definition; instead, we let students explore geometrical examples where the question of "how many eigenvectors related to the same eigenvalue can exist" is likely to arise. In the second part of the first class on eigentheory we proposed a series of known bi and tri-dimensional geometrical transformations (rotations, reflections, projections) and asked the groups of students to find their eigenvectors and eigenvalues. We chose the examples so that eigenspaces of dimension one (lines) and two (planes) would appear. The hypothesis behind the choice of this task was that students are expected to notice that all vectors on a line or on a plane - depending on the example - are eigenvectors with respect to the same eigenvalue, and thus to visualize the set of eigenvectors related to a same eigenvalue as a vector subspace. In the following activity, students had to answer the question, expressed in more general and formal terms: "If I have two eigenvectors that are independent of each other but related to the same eigenvalue, are all their linear combinations also eigenvectors?". By answering to this question, students were expected to transpose in algebraic form the geometrical ideas encountered in the previous task and to discover the generalizability of the statement for all vector spaces. We will present here a fragment of a couple of students' activities related to the task "consider the reflection with respect to a plane passing through the origin, and find, if any, its eigenvectors and eigenvalues".

At the beginning, they start drawing the transformation on their notebooks. They reason a bit on the drawing but struggle to visualize the three-dimensional transformation and thus which are the vectors that keep the same direction when transformed. Then, they start representing the situation with their hands, gesturing in the air (Fig. 4.2a). Doing so, they immediately realize that all the vectors staying on the student on the right's hand (representing the plane of reflection), remain fixed by the transformation, that is they stay on the same direction and keep the same magnitude, meaning that they are all eigenvectors related to the eigenvalue 1. They easily visualize that this property is shared by all the vectors staying on the plane. Later they start looking for other different eigenvectors. They state that all the vectors lying in the plane perpendicular to the reflection plane (represented in Fig. 4.2b by the student's left hand) are eigenvectors related to -1. They do not seem very sure about it, and explore various similar configurations by moving their hands. At one point, one of the students (Fig. 4.2 c and d) begins representing the vectors with his right index finger and, by moving it in different directions, realizes that all and only the vectors lying on the line passing through the origin and perpendicular to the reflection plane (represented by his left hand) are eigenvectors related to -1.

Figure 4.2 (a,b,c,d): key gestures performed by the students while solving the task

At the end of this excerpt, when the two students have to summarize what they have found, they realize that in the two explored cases (the previous task required them to find the "eigen-objects" of the reflection in the plane with respect of a line), the set of eigenvectors related to an eigenvalue always formed a line or a plane. For space reasons, we cannot continue the analysis, but in the following group activity they are able to generalize this finding, proving algebraically that the linear combinations of eigenvectors with the same eigenvalue are always eigenvectors of the same value.

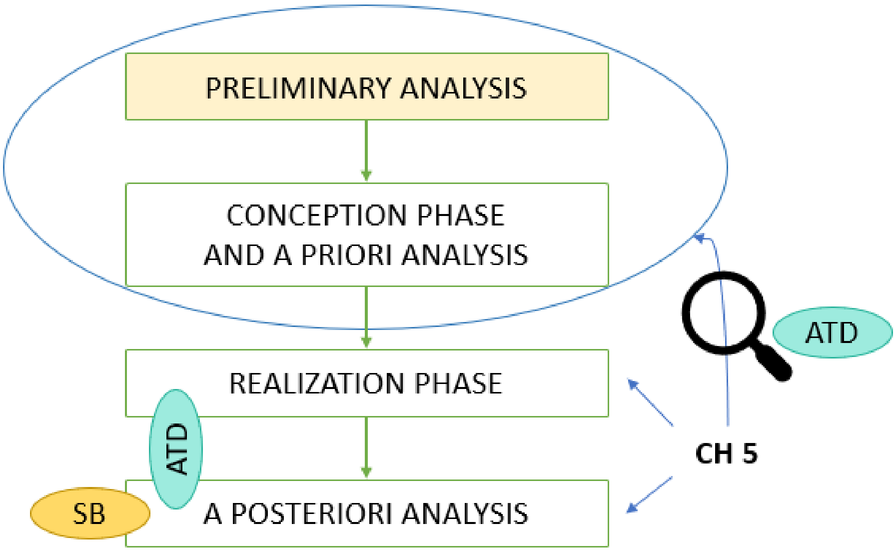
4.5 Conclusions

The analysis of the video recordings allowed us to observe how students' mobilization of multimodal, dynamical, resources, such as their gestures, enabled them to visualize that the set of eigenvectors related to the same eigenvalue form a vector subspace. In the following activity, which unfortunately we do not have the space to analyze here, they were able to switch to a different semiotic register, the algebraic one, to generalize this statement and form a correct conceptualization of eigenspace. The choice of letting students explore geometrical transformations where they could visualize eigenspaces of dimension one or three helped overcome the difficulties highlighted by Wawro and colleagues (2019) and observed also in our pilot study. The SB lens was used to analyze all the excerpts of activity from the three video-recorded groups, in order to validate the hypotheses guiding the redesign of the course and of the tasks assigned to the students as small groups activities in the classes on eigentheory. Moreover, additional data were collected, including questionnaires, in order to validate (or not) other research hypotheses regarding the students' attitudes toward the new didactical organization of the course module. The analysis of these aspects will be reported elsewhere.

CHAPTER 5 - DESIGNING AND ANALYSING A TEACHING PROPOSAL ABOUT LINEAR ALGEBRA THROUGH THE DIALOGUE OF TWO THEORIES

This is the authors' original manuscript of an article published in *International Journal of Mathematical Education in Science and Technology*.

THE CHAPTER IN THE RESEARCH PROJECT



Abstract

In this paper we propose a new strategy of dialogue between theories, considering a semiotic cognitive analysis and a praxeological analysis. We examine the implementation of a teaching and learning sequence for introducing eigentheory in a linear algebra course taught during the first year of an engineering degree programme. Initially, the semiotic cognitive analysis based on the APC approach is used in the preliminary, *a priori* and *in vivo* phases of the didactic engineering process to propose a teaching strategy that incorporates a variety of semiotic resources. In a second moment, the praxeological analysis intervenes to reformulate the preliminary analysis and to progress on the consideration of a reference epistemological model about eigentheory in a first course of mathematics for engineers. This model is then used in the *a priori* and *a posteriori* analysis to integrate the semiotic and epistemological dimensions in the design and implementation of the teaching proposal. The results show the productivity of this new networking strategy and the complementarity of connecting the semiotic with the epistemological analysis.

Keywords: eigentheory; engineering; semiotic; APC approach; Anthropological Theory of the Didactic; reference epistemological model.

5.1 Introduction

In the past decades, a growing body of literature has been focusing on issues about teaching and learning linear algebra (Andrews-Larson et al., 2018; Stewart et al., 2019). Among the topics generally taught in linear algebra courses, that of eigentheory has garnered particular interest (Altieri & Schirmer, 2019; Beltrán-Meneu et al., 2016; Bouhjar et al., 2018; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Stewart et al., 2019). The reasons for its interest are multiple. First, eigentheory is a complex set of notions that highly rely on many basic linear algebra concepts (linear dependence, basis, kernel, etc.), thus investigating its learning process can inform also the understanding of these other notions. Second, the possible application of eigenvectors and eigenvalues is an important topic of STEM subjects. Thus, research on students' understanding of this topic can impact not only the field of mathematics education but also the teaching of a variety of other scientific disciplines (Wawro et al., 2019).

Students' difficulties with the study of eigentheory have been analysed from different perspectives. Drawing on the theoretical framework of Tall's three worlds of mathematics and APOS theory (Dubinsky & McDonald, 2001), Thomas and Stewart (2011) investigated eigentheory concept development in the symbolic, embodied and formal worlds. They provided interesting results regarding the way students understand the equation $Ax = \lambda x$. In particular, students seem to have difficulties in coordinating the two different processes encapsulated in the two sides of the equation. On the left side, there is a matrix multiplied by a vector, which is equalised on the right side to a multiplication of a scalar by the same vector. The complexities inherent in understanding this equation have been further investigated by Bouhjar et al. (2018).

Stewart and Thomas (2006, 2011) highlight another problem related to the definition and understanding of eigenvalues and eigenvectors, faced by students when learning linear algebra. As the authors highlight, the dominant way students are introduced to these concepts is through their algebraic definition, to then be introduced to an algorithm to compute them. Consequently, the focus is turned too soon to the manipulation of algebraic representations. In a standard instructional sequence, the formula to compute eigenvalues, i.e. $det(A - \lambda I)x = 0$, follows their formal definition almost without delay. Immediately after, the algorithm to compute the eigenvectors associated to each eigenvalue is given. In this way "the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust" (Thomas & Stewart, 2011, p. 186). Complementarily, other researchers have brought evidence of how the exploration of eigentheory through dynamic geometric software (Caglayan, 2015; Gol Tabaghi & Sinclair, 2013), inquiry-oriented instruction (Bouhjar et al., 2018; Wawro et al., 2019) or modelling activities (Salgado & Trigueros, 2015) can enhance its understanding.

Complementarily to these results, investigations in the framework of the anthropological theory of the didactic problematise the teaching and learning of mathematics at the university level. Many undergraduate science-oriented courses continue to be significantly influenced by the dominant epistemological model of '*applicationism*' (as characterised in Barquero et al., 2013). This entails the standardisation of first-year undergraduate courses, and a notable reduction of a modelling perspective regarding mathematics, as an application of previously taught mathematical concepts. Such standardisation results from the shared vision of the use of mathematics in other disciplines as an application of formerly constructed knowledge (Hochmuth et al., 2021). This has been also attributed to the fact that scholarly mathematics and engineering mathematics can be considered as two different scholarly institutions, where mathematical knowledge coexists in two different ways (Castela & Romo, 2011). Yet, those responsible for teaching the foundational mathematics courses for engineering students are usually trained within the first of the two institutions and have little to no knowledge of the *raison d'être* in the scholarly engineering knowledge for the mathematical contents to be taught. Thus, the reasons that historically led to the development of eigentheory, related to the mechanics of the body, remain hidden in mathematics courses, even for those students (e.g., mechanical engineering students) who will use those concepts mainly in that specific realm. For them, a geometrical representation of eigenvectors would be more meaningful than the algebraic one usually adopted in standardised linear algebra courses.

In addition, the dominant epistemological model of *applicationism* is accentuated by the dominance of *monumentalism* (Chevallard, 2015) in university institutions, where contents are presented as already made works to be 'visited' by students, and which are rarely questioned or problematised. Regarding eigentheory, this results in presenting the definition of eigenvalues and eigenvectors and the algebraic algorithm to compute them without even problematising the need for studying those specific vectors. The significance of examining vectors that remain coordinate-independent for a given linear transformation is rarely emphasised. Even adhering to the dominant algebraic matrix-representation of linear transformations, the need to find a basis for which the matrix of the linear transformation is diagonal (if possible) is not used as a prompt to look for eigenvalues.

The study presented in this paper focuses on the teaching and learning of linear algebra, specifically on eigentheory at the undergraduate level. The primary objective of the research project is to investigate the semiotic resources activated during the implementation of a teaching proposal about eigentheory and its analysis. To achieve this, two theoretical approaches have been employed, thus initiating a dialogue between them. First, the APC-space theory (Arzarello, 2006) was initially considered for the design of the teaching proposal. This approach is used for the analysis of the cognitive processes associated with the construction of the core concepts here considered: eigenvectors and eigenvalues. Second, the anthropological theory of the didactic (ATD) (Chevallard, 2015 and 2019) helps expand the scope of the analyses and integrates the institutional approach to the teaching and learning processes around eigentheory, as well as a macro-analysis of the mathematical knowledge to be taught and learnt in the proposed teaching institution.

This paper presents the first step of the dialogue between these two theories. The first one provides tools for the design and implementation of a teaching proposal for eigenvalues and vectors, while the second one contributes to the *a posteriori* epistemological analysis of the knowledge at stake. With this purpose, the following section introduces the main tools considered from both theories and the main aims of the research. It is followed by a discussion on the collaboration strategy adopted to facilitate the dialogue between both approaches and their progressive interaction along the specific phases of the didactic engineering process (Artigue, 2014; Barquero & Bosch, 2015). We assume that this coordinated approach might provide a more comprehensive understanding of the educational phenomenon under consideration.

5.2 Theoretical framework

The APC-space theory (Arzarello, 2006) frames mathematical processes according to the multimodal paradigm, considering that mathematical practice employs a variety of semiotic resources. In addition to the written and oral resources, it incorporates extra-linguistic modes of expression (gestures, glances, actions, etc.) and instruments (from the pencil to the most sophisticated ICT devices). Within this theoretical approach, Arzarello has developed a semiotic tool suitable to analyse this variety of semiotic resources and their relationships and evolutions in the students' and teachers' productions and interactions, namely the "semiotic bundle". A semiotic bundle is a system of signs which is produced by one or more interacting subjects and that evolves in time. Not only the collection of sets of signs that are present in the bundle may evolve, but also the relationships existing between them can vary as the subject or the interacting subjects produce them. These relationships are analyzed through two types of lenses within this framework: the *synchronic* analysis allows to study the relationships among signs that are produced simultaneously, while the *diachronic* analysis studies the relationships among semiotic sets activated by the subject (or subjects) in successive moments, thus their evolution. This last kind of analysis has been used by other theories, but the element of novelty brought by the theory of the semiotic bundle is the chance to use it to observe phenomena taking into account semiotic sets (including all types of signs, as even gestures or actions) instead of the more restricted semiotic systems (composed uniquely of linguistic and written signs). This means, for example, that the dynamical relationship between gestures and speech can be analysed, and such an analysis can give important information about the subject's ideas and thinking. In this case, gestures and language are interpreted as a semiotic bundle, made of two deeply intertwined semiotic sets, of which only the second represents also a semiotic system.

The s*emiotic resources*, as considered by the APC-space, are defined in the ATD in terms of ostensive objects—or simply *ostensives*—to refer to all objects that can be shown or perceived in any way (including material objects, sounds, gestures, discourses, written symbols, among others) (Bosch & Chevallard, 1999). The word *ostensive* has been preferred to semiotic not to limit the usage of ostensives as sign vehicles of other non-ostensive objects (such as concepts, ideas, etc.) but to constantly retain their role as tools of mathematical practices. In many cases, mathematical activities mobilise a variety of ostensives simply as tools to *do something*, not always—or not only—to *mean something*. To take a simple example, $\sqrt{x}$ and $x^{0.5}$ are two different written ostensives that represent the same function (non-ostensive). We attribute to them a semiotic value (or *valence*, as in Chemistry) according to their capacity to refer to this non-ostensive object. However, when we use them to calculate a derivative, they appear as different tools having different *instrumental valence*. In the first case, one has to know that $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$. In the second case, one can just "bring down the exponent value and subtract one from the exponent" to get $(x^{0.5})' = 0.5x^{-0.5}$, a written action that cannot be performed with the first tool. Of course, the ostensives obtained do not have the same *semiotic valence* either: in the first case, it is more visible than in the second one that the derivative is not defined for $x = 0$.

As in other approaches, different ostensive registers are considered, depending on the type of materiality of the ostensive objects and the more or less determined syntax, semantic and pragmatic rules governing them: gestures, written symbolism, informal graphisms, oral and written verbal discourses, etc. The specificity of the ATD approach is the way the ostensive dimension of mathematics activity relates to the epistemological dimensions and the reference epistemological model(s) provided to analyse mathematical knowledge and mathematical activities in terms of *praxeologies* (Arzarello et al., 2008).

A *praxeology* is the union of two blocks, a *praxis* and a *logos*. The praxis block corresponds to the know-how or the practical work, encompassing two entities: *types of tasks* to be addressed and *techniques* to carry them out. In linear algebra, determining whether a matrix is diagonalisable, proving that the eigenvalue of an isometry is 1 or -1, calculating the eigenspace of a linear application or finding the matrix expression of a space transformation are examples of *types of tasks*. The associated *techniques* can sometimes be rather algorithmic, but not necessarily. The *logos block* of a praxeology, corresponding to the know-why or justificative work about the praxis, also includes two entities: the *technology* (or *logos* about the *techne*), that is, a discourse elaborated to describe, explain and justify the techniques; and a *theory* containing notions, principles, properties (among other elements) supporting the technology. Regarding praxeologies and ostensives, two main principles are assumed. First, the fact that *techniques*, *technologies* and *theories* are made of ostensive and non-ostensive objects: notions, concepts, ideas, properties, etc. are non-ostensive objects supported by ostensives; the techniques are also mainly made of ostensives, the use of which is guided by non-ostensives. The *instrumental valence* of ostensives is closely related to their facility to develop the praxis block, while their *semiotic valence* can be more linked to the logos. However, there is a dialectic between the semiotic and instrumental valences of ostensives; their use helps create new meanings that, in turn, might suggest new ways of doing.

An initial dialogue between APC-space theory and ATD (Arzarello et al., 2008) has shown how the two approaches can complementarily frame the semiotic (or ostensive) dimension of mathematical activity through their specific approaches to teaching and learning phenomena. The first approach prioritises a comprehensive semiotic analysis using semiotic bundles, that is, to be integrated into a more specific epistemological analysis of mathematical activity. On the other hand, ATD presents a general model of mathematical knowledge and practice using the notion of *praxeologies*, which requires a more detailed analysis of the role of ostensive objects in the evolution of mathematical activities within the classroom.

In this paper, our objective is to advance the ongoing dialogue by delving deeper into the first direction: enhancing the analysis conducted with the semiotic bundle through praxeological analysis. This dialogue will be supported by the collaboration along the process of designing, implementing and analysing the experience of a teaching proposal about eigentheory for undergraduate students. The collaboration strategy, facilitating the dialogue between both approaches, consists of progressively interacting within certain steps of the *didactic engineering process* (Artigue, 2014; Barquero & Bosch, 2015) in the selected case study of a linear algebra course. In the following, we introduce the context of the case study selection and the strategy for establishing the research collaboration between both approaches.

5.3 Context of the research and essential results of the pilot study

The research project has undergone a pilot phase and subsequent implementation within the setting of a linear algebra course offered in the first year of a mechanical engineering degree programme at the University of Bologna in Italy. The course is attended each year by more than 200 students and its syllabus,[9] with 6 credits ECTS, is highly dense, covering a wide range of topics (Linear systems. Matrices. Vector spaces. Linear transformations and matrices. Inner product spaces. Projections. Eigenvalues and eigenvectors; similarity of matrices and diagonalisable matrices. Symmetric matrices and spectral decomposition. Quadratic forms and matrices).

A pilot study —partially described in Piroi (2023a)— has been conducted in the fall term of 2021 and has been used to formulate hypotheses on the way students understand eigentheory. It consisted of a collective review of eigentheory notions carried out during optional two-hours tutoring classes after the topic had been presented in the previous weeks by the teacher with traditional lectures. Students attending the activity worked in small groups of 3-4 students. The activity aimed to

[9] The syllabus of the "Linear Algebra" course can be retrieved from: https://www.unibo.it/en/teaching/course-unit-catalogue/course-unit/2022/387288

collectively re-elaborate the notions of eigentheory encountered in the recently attended lecture to better grasp their meaning. We provided students with guidelines that could direct them in this process, such as asking them to try to answer the question "How would you explain the concept of eigenvector to someone who has never heard of it before?". Students could answer in the way they preferred: orally, in written form, with diagrams, etc. Moreover, they were free to use any tool and encouraged to use other resources that they had at their disposal, besides the book or notes taken during the lectures. Written protocols of all the students were collected, as well as audio and video recordings of eight of the working groups. This type of data allowed us to observe the different semiotic resources activated by the students within a multimodal perspective (Radford et al., 2017), the relationship between these resources, and their evolution in time. The theoretical construct of the semiotic bundle was used as the semiotic lens through which to observe the nature and the dynamics of the ostensives in the APC-space (Arzarello et al., 2008) shared by the students in each small group.

This paper does not delve into the analysis of the pilot study, as it has already been covered in a previous work presented by the first author (Piroi, 2023a). In that work, the evolution of the semiotic bundle generated and progressively enriched by one of the observed working groups is described and analysed. Nevertheless, we include here some of the results that have highlighted some interesting aspects regarding the way students approach eigentheory, and consequently guided the design of the following phase of the research project. First, it was clear that what students retained from the standard lecture was almost only the algebraic formulation of eigenvectors and eigenvectors. That is, those vectors v and values λ for which, given a linear transformation A, the equation $Av = \lambda v$ stands. This formulation carries some difficulties and misconceptions that have already been highlighted by Thomas and Stewart (2011). The static nature of this formulation, due to its algebraic representation, obscures the rather dynamicity of the concepts at stake. This dynamicity could be, on the other side, grasped by the students if visual ostensives, such as drawings, are activated, and even more, if dynamic visual semiotic resources, such as gestures or dynamic geometry software, are used.

The extracts analysed in the aforementioned paper reveal that two students in this group initially attempted to write a definition of eigenvalue, struggling with the meaning of the formula $Av = \lambda v$. The third student even skipped this part and focused directly on the algorithm provided for computing eigenvalues. Recognising their difficulties in understanding what eigenvalues are by only analysing their formal definition, one of them suggested consulting different resources. He recalled a GeoGebra applet shown by the course teacher representing eigenvectors of a symmetry with respect to a line in the two-dimensional space. He recreated the scenario for his group, using gestures to represent vector relationships: he moved synchronously his index fingers one toward the other as if they were representing a vector and its image, the first one being an eigenvector when it overlaps with its image (Fig. 5.1).

Figure 5.1. A student gesturing the overlapping of an eigenvector with its image

The dynamicity offered by the GeoGebra applet, as noted by the first student, and by his gesture, effectively conveyed the idea of what an eigenvector, and consequently its eigenvalue, are. In another noteworthy extract from the same paper, students were observed utilising various ostensives to explore the number of possible eigenvectors related to the same eigenvalue. This exploration led them to autonomously conceptualise the idea of eigenspace, that is the set of eigenvectors associated with the same eigenvalue. The process began with a gestural exploration of vectors aligned on the same line. The idea generated by the gestures was transferred onto paper through a drawn diagram. Ultimately, the properties illustrated by the diagram were described algebraically using a generalised formula stating that every multiple of an eigenvector is likewise an eigenvector with the same eigenvalue. Despite demonstrating the potential of integrating different registers in working with eigenspaces, the extract also reveals a common misconception among students. Starting with the recognition of eigenvectors through overlap, students tend to identify an eigenspace with a line. This leads to a failure in conceiving that eigenspaces can be represented by a plane or even the entire space in the three-dimensional context, and that, more generally, eigenspaces of dimension other than one can exist. The results obtained are in line with previous literature on the topic (e.g., Wawro et al., 2019; Gol Tabaghi & Sinclair, 2013). They show that introducing and articulating a geometrical approach alongside the algebraic one in teaching and learning eigentheory might facilitate the emergence of a variety of semiotic resources beyond the algebraic written and oral registers. This approach permits a departure from the rigidity of solely relying on commonly employed algebraic techniques, providing a deeper understanding of eigenvectors, eigenvalues, and eigenspaces.

After analysing the pilot study, the first author, together with the lecturer responsible for the linear algebra course, worked on the redesign of the module dedicated to eigentheory for the following academic year. The module included an adaptation of the pilot study activities considering the numerous institutional constraints present in that specific context. In the following sections, we will provide a detailed presentation of the redesigned module.

5.4 Research questions and methodology

The purpose of this paper is to present the design and implementation of the eigentheory part of the course, but more importantly to carry out an analysis in terms of praxeologies to better highlight the kind of relationship students and the teacher can establish with the mathematical knowledge involved, in the specific context. To do this, it is necessary to reinterpret the results of the pilot study obtained through the semiotic framework in terms of praxeologies, and to make explicit the epistemological model that supported the design. Furthermore, this work has a broader objective of advancing the dialogue between the APC-space theory and ATD. Therefore, we aim to highlight here the networking strategy used, to coordinate two theories with a unit of analysis distinct from each other but sharing a common interest in examining the relationship between ostensive objects appearing in mathematical activity and their corresponding non-ostensive objects. The research questions can be formulated in the following terms:

RQ1: What kind of instructional proposal regarding eigentheory can be implemented to foster the emergence and use of different kinds of ostensives rather than the solely algebraic ones, compatible with a traditional first-year university lecture-based linear algebra course?

RQ2: How to interpret in terms of the ATD an instructional proposal about linear algebra designed and implemented in the APC-space? What strengths and limitations does the ATD analysis reveal?

RQ3: What networking strategy can support this kind of dialogue between ATD and APC-space?

In this paper, we use the phases of *didactic engineering* as a research methodology (Artigue, 2014; Barquero & Bosch, 2015) and as a main strategy for the dialogue between the two theoretical frameworks: the APC-space and ATD (RQ3). Following the four different phases or steps of didactic engineering, the *preliminary analysis* focuses on identifying the didactic phenomena related to the teaching of eigentheory. Moreover, it includes the analysis of conditions under which eigentheory is taught and learned in first-year courses in linear algebra at the university, as well as the formulation of research hypotheses. In our case, the preliminary analysis consisted of a literature review on the topic, to identify common didactic phenomena related to eigentheory. Additionally, it included a naturalistic observation of the classes where the topic of eigentheory was introduced in the same course, as planned and taught by the lecturer in the year preceding the implementation. The preliminary analysis was further enriched by the realisation, through the semiotic bundle lens, of the pilot study briefly described earlier.

The subsequent phase consists of the conception and *a priori* analysis of a teaching proposal able to address and contribute to the study of the didactic phenomena under consideration. In our context, the preliminary analysis, which incorporates insights from the pilot study, directed the design of the activities to be implemented in the eigentheory-related classes and the overall redesign of the course. In (Piroi, 2023b), the connections between the results obtained through the semiotic bundle lens in the pilot study and the decisions guiding the redesign are explicated.

This phase was followed by the implementation, for a total of four sessions of the instructional proposal and the *in vivo* analysis. The direct observation of the teaching was the main way for the researcher and lecturer to comment on the carrying out of the proposal. With this respect, RQ1 about the kind of instructional proposal, compatible with a traditional first-year university course on linear algebra, is addressed through the *a priori* and *in vivo* analysis.

The *a posteriori* analysis was based on the analysis of the audio and video recordings of all the lectures and three selected working groups during all the small groups' activities. The recordings are used to observe all the semiotic registers present in the students' activities. This study enables a fine-grained multimodal analysis of the students' activity in the small groups, as well as the emergence and evolution of praxeologies in both the working groups and the whole class. Complementarily, this stage also included the praxeological analysis of the implemented mathematical activities, which aids in addressing RQ2. The interdependence of the use of theories across the didactic engineering phases corresponds to our networking strategy. We use these different phases to structure the next sections.

5.5 Preliminary analysis reinterpreted within the ATD

The pilot study can be reinterpreted in terms of the ATD to highlight the praxeologies emerging in the small group activity, which typically do not appear, nor are used in a traditional sequence for learning eigentheory. For instance, the initial example, where a student gestures the vector and its image overlapping, can be interpreted in praxeological terms as follows. An implicit task is posed—“what are the eigenvectors and eigenvalues of a symmetry in a plane with respect to a line?”. The technique employed to address this task involves using one's index fingers to represent a vector and its image in relation to the given linear transformation. Various configurations are explored, and when the vector and its image (ostensively represented by the two fingers) coincide, the vector is identified as an eigenvector. The ratio between the image and the vector determines the eigenvalue. The technological justification guiding this technique lies in the geometric principle that "an eigenvector is a vector whose image maintains the same direction". This justification is supported by the theories of linear algebra and vector spaces, specifically geometrical vector spaces. Alternatively, other techniques could have been employed (as observed in different groups), such as drawing the same situation and simulating the movement with subsequent drawings. Regardless, such a task requires the activation of non-algebraic ostensives, that let features of the related non-ostensives emerge. In the second example presented in the description of the pilot study, the implicit task, in this case echoing a question posed by the students in the group themselves, is: “find how many eigenvectors exist related to the same eigenvalue”. The technique used by the observed students involves different ostensives: gestures, drawings and algebraic formal language. The ATD perspective not only enables to reinterpret students’ activity in terms of praxeologies and analyse the emergence of new ostensives as “instruments” in the techniques used, acquiring an epistemic value in their relationship with their non-ostensive counterpart. The framework is also suitable for enriching the preliminary analysis with an epistemological analysis of the considered mathematical knowledge, and an ecological analysis of the institution at stake. The latter involves analysing the conditions enabling and the constraints hindering the implementation of specific activities, in this case in the university context and specifically in the context of mathematics service courses in scientific degree programmes. For example, the aspects mentioned in the introduction section regarding the dominant pedagogical model in university teaching and the dominant epistemological model specific to eigentheory teaching become fundamental components of the preliminary analysis.

Explicitly questioning the dominant epistemological model is crucial in this phase, as is elucidating the reference epistemological model guiding the design. Reference epistemological models (REM, Bosch & Gascón, 2006) are widely used tools in ATD for analysing praxeological organisations. These models enable researchers to articulate their own vision about the mathematical knowledge under construction and, in particular, to emancipate from prevailing epistemological models in the institution considered (Gascón, 2014). Recently, question and answer maps (Q-A maps) have been employed to materialize a REM (Barquero, 2009; Florensa et al., 2020). Q-A maps are tree-like graphs presenting a series of questions and answers generated from an initial question Q_0. According to Florensa and colleagues (2020), Q-A maps are a valuable tool to "(1) describe the knowledge involved during a study process overcoming the limitations of the previous conception of knowledge, (2) contrast the new study process with the previous one especially in terms of responsibilities assumed and richness of the media and the milieu and (3) make explicit the raison d'être of the knowledge to be taught" (p. 245).

Thus, even if not explicitly conceived, the preliminary analysis guiding the redesign of the module on eigentheory can be described in terms of a REM in the form of a Q-A map. Indeed, although this is rarely made explicit or emphasised in linear algebra courses, the study of eigentheory, in a dominantly algebraic, matrix-oriented course as the one considered, essentially serves to answer a question, which we will consider as our Q_0: "Can I find a diagonal matrix similar to the one given? How? Is it always possible?". All eigentheory can be developed starting from attempting to answer these questions, following different possible paths where new questions emerge and pieces of information appear as answers to these questions.

In Figure 5.2, we have represented such a reference Q-A map, highlighting the wealth of semiotic resources that might appear to answer each of the questions that emerged. For the sake of clarity, the map includes simplified labels for each question and answer. In Table 1, the questions and answers present in the map are written extensively to help the reader interpret the map as an actual sequence of subsequent questions and answers stemming from Q_0.

Starting from Q_0 three branches develop and soon interact, with answers leading to questions present in a different branch. The central one and the one on the right respectively generate from questions (Q_A and Q_G) referring to the algebraic and the geometrical interpretation of diagonal matrices and hence of eigenvectors and eigenvalues. All three branches contribute to the emergence of the technique to compute eigenvalues and eigenvectors (indicated with + and *). Nevertheless, in each branch, many different questions leading to different answers and praxeologies emerge.

This REM (briefly described here) first serves to contrast and propose possible study paths, concerning the existing or dominant ones, in the institution considered. Indeed, as observed in the course path of the same course in previous years, or in the textbooks and notes mostly used in engineering courses in that university, the only praxeology used by students, and the relative semiotic counterpart, is the algebraic one (marked with a green ellipse in the Q-A map). Moreover, usually, it is mainly its *praxis* block that is used, while its theoretical justification is quickly neglected because it does not intervene in the techniques activated to solve the tasks. Consequently, the only semiotic resources introduced and used by students to perform the strictly algebraic technique of computing a determinant and finding its roots are written algebraic symbols.

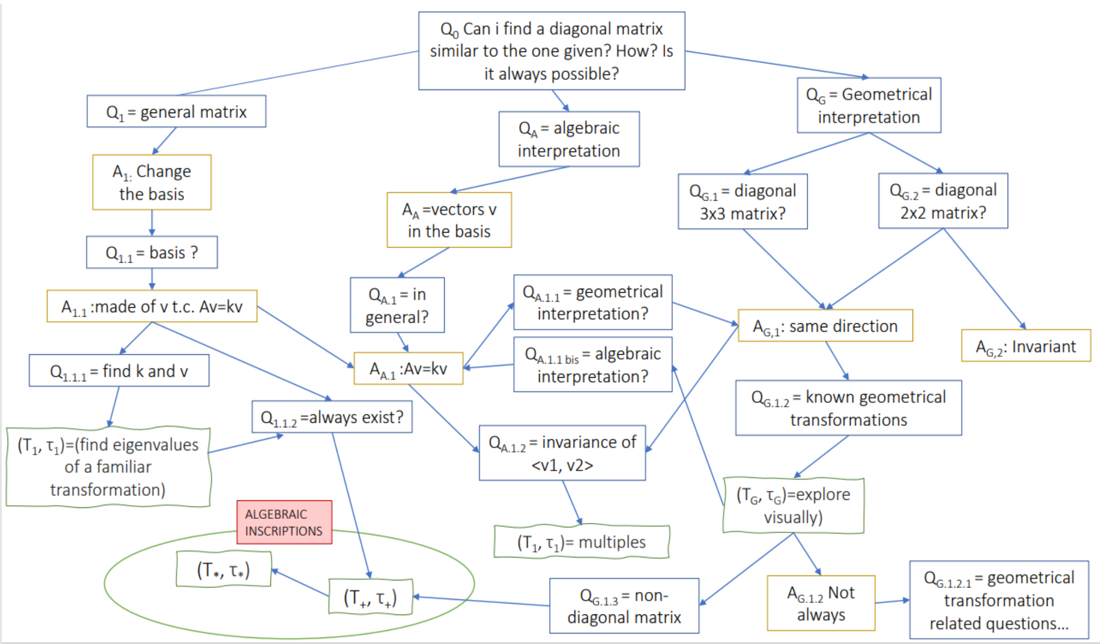


Figure 5.2. Reference epistemological model for the study of eigentheory, described as a Questions-and-Answers map

Table 5.1. Description of the questions and answers in the Q-A map of Figure 5.2

Q_1	Given a non-diagonal matrix, what can be done?
Q_A	What does it mean algebraically to have $A = D$?
Q_G	What does it mean geometrically to have $A = D$?
A_1	Change the basis of the matrix.
$Q_{1,1}$	How must the basis be so that the matrix is diagonal?
$A_{1,1}$	It must be made of vectors v so that $Av = kv$.
$Q_{1,1,1}$	How do I find k and v (if they exist)?
$Q_{1,1,2}$	Do they always exist?
A_A	Every vector v_i in the basis must have its image that is made by all 0, but in position i.

$Q_{A,1}$	How do I write it in general?
$A_{A,1}$	$Av = kv$
$Q_{A,1,1}$	What does it mean geometrically?
$Q_{A,1,2}$	Why is <v_1, v_2> invariant if <v_1> and <v_2> are?
$Q_{G,1}$	What is the geometrical interpretation of a diagonal 3x3 matrix?
$Q_{G,2}$	What is the geometrical interpretation of a diagonal 2x2 matrix?
A_G	The vectors of the basis have the same direction as their images
$Q_{G,1,2}$	What happens with simple geometrical transformations (symmetry, rotation, projection)?
$Q_{G,1,3}$	Given a non-diagonal matrix, are there invariants? How many?
$A_{G,1,2}$	Not always.
$Q_{G,1,2,1}$	What non-diagonal matrices do not have invariants, apart from rotations?

5.6 Design phase and the *a priori* analysis of the instructional proposal

As mentioned in previous sections, following the pilot study, we designed and implemented a teaching and learning sequence, to elicit different ostensives in the study of eigentheory. The sequence was also intended to integrate small group activities that could foster students' active use of different ostensives during the lessons. The conditions of the linear algebra course, as well as that of the institution where the sequence was implemented, presented a considerable number of constraints hindering the implementation of students' inquiry activities during the classes. Firstly, as mentioned before, the course was attended by more than 200 students. Moreover, the physical disposition of the desks in the classrooms lined up in long rows hindered easy collaboration in small groups and made it challenging for the teacher to walk around the desks to observe students' interactions.

Despite these challenges, the teacher of the course and the first author, who collaborated on the design of the teaching and learning sequence, identified some conditions favouring a blended format that integrated a more transmissive lecture style with active collective explorations. One of these conditions was the possibility of using the online notice board "padlet" as a technological facilitator for sharing students' ideas and for conducting a collective discussion oriented by these ideas. With the support of this tool, the teacher could assign, at certain moments in the class, short tasks for the students to approach in pairs or small groups. These tasks have been designed based on the results of the pilot study, as described in (Piroi, 2023b). When the time at their disposal for addressing the task was finished, each group could post anonymously a picture of their hand-written answer on the shared padlet. The possibility of publishing a picture allowed the students to use and share different types of representations, but also to share only a partial idea without the need to complete the task. The teacher then projected the padlet (Fig. 5.3) for everyone to see the posted answers. While the pictures were being posted, she could identify particularly interesting solutions to present to the class later, as a bridge between the groups' activity and the subsequent explanation she provided to the entire class.

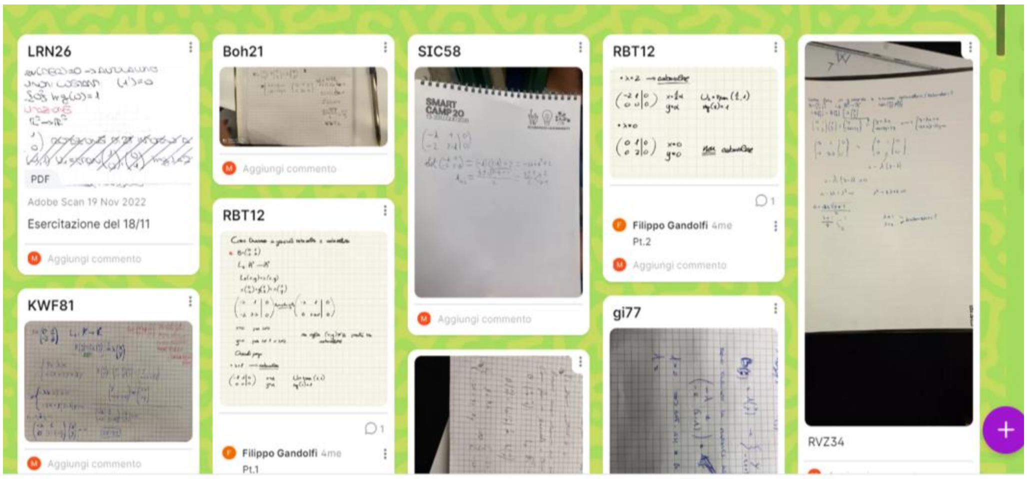


Figure 5.3. An example of pictures with students' productions posted in the Padlet

The development of tasks and lessons was carefully planned to facilitate a comprehensive exploration of the potential praxeologies involved. To provide a general overview (the details will be presented in the next section), the sequence began by introducing the concepts of eigenvalue and eigenvector through a geometrical task, utilising a GeoGebra application (see Fig. 5.5). This task was then complemented by an algebraic counterpart more aligned with the traditional course content. Then, students were asked to work on other geometrical tasks requiring visual geometrical techniques. The following task required students to try to formalise the fact that all the linear combinations of eigenvectors related to the same eigenvalue are eigenvectors as well. The final task required students to develop a general procedure for finding eigenvalues and eigenvectors for a given matrix. The approach involved introducing new geometrical tasks and emphasising less common algebraic explorations, collectively leading to a justified understanding of the technique for computing eigenvalues and eigenvectors. What has been briefly described here is the sequence as it has been conceived and then implemented. In the next section, we will delve into the actual teaching and learning sequence, describing more finely the tasks and questions posed to students, elucidating the rationale behind their conception, as well as students' answers

5.7 Implementation, *in vivo* analysis and *a posteriori* analysis

For the sake of text economy, in this section, we present together the description of the implementation, the *in vivo* analysis conducted during the implementation, and the praxeological analysis carried out *a posteriori* for each step of the designed sequence. The *in vivo* analysis was realised through a naturalistic observation by the first researcher, present during the implementation, and through the real-time analysis of the pictures posted on the padlet during the classes. These pictures served also as data for the *a posteriori* analysis. However, they only provide a record of students' final productions, lacking insight into the processes leading to the final solution, including all the semiotic resources activated in this process. To address this limitation, we audio and video recorded three of the small groups during all the small groups' activity in the four lessons concerning eigentheory. This type of data allows us to observe all the ostensives present in the students' activity, including utterances and gestures, as well as their relationships and evolution. It enables us to perform a multimodal semiotic analysis. The semiotic bundle construct, through a fine-grained analysis, facilitates the study of events occurring within a few minutes or even seconds. Relevant fragments of students' activity can be selected to analyse the semiotic resources mobilized and how their use, evolution, and connection to other resources contributed to the development of knowledge about eigentheory. Nevertheless, as explained earlier, this paper focuses also the *a posteriori* analysis within the praxeological framework.

Students' first encounter with eigentheory actually happened before the first class on the topic. Indeed, another novel didactic choice involving the whole course was the consistent use of dynamical geometric representations of linear algebra notions. The teacher decided to include YouTube videos among the course resources, and specifically videos retrieved from the YouTube channel 3Blue1Brown. The channel's playlist "The essence of linear algebra"[1] presents notions of linear algebra (linear dependence, determinant, eigenvectors, etc.) with a highly visual and dynamic approach. For instance, eigenvectors are visually introduced as those vectors (represented as geometric vectors in the plane, and later in the 3D space) whose image lays in the same direction (Fig. 5.4), differently from other vectors' images that get "knocked out from their span".

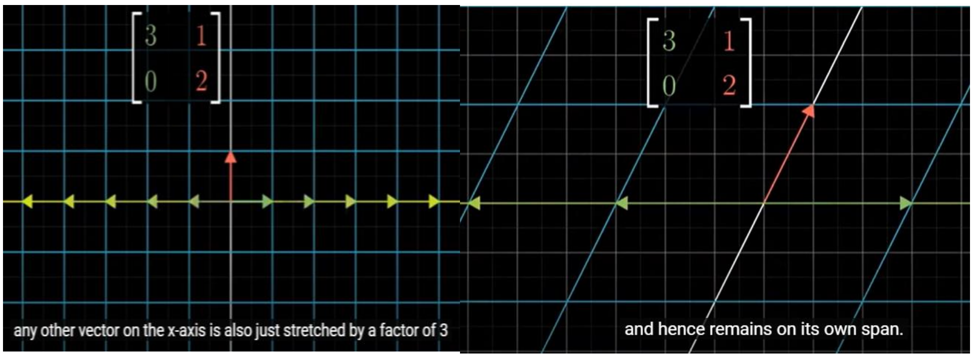


Figure 5.4. Two subsequent fragments of 3B1B's video on eigenvectors and eigenvalues

Selected videos of this playlist were uploaded on a social learning platform called "Perusal" (Miller et al., 2018). Students were asked to watch each video before the class in which the topic covered in the video would have been addressed, comment on it (the platform allows to comment on the video and others' comments) if desired, and answer a question posed by the teacher, that served as a warm-up for the class and as a first step in formalising the notions "viewed" in the video. Thus, the students' first encounter with eigenvectors was in watching the corresponding video, before the first lecture of the module. As an adjunct to the video, the teacher provided a GeoGebra applet that showed a draggable vector on the plane and its relative image. Students were then asked the following question: "Find the matrix B canonically associated with the represented transformation and any, if existing, eigenvalues and eigenvectors of the transformation".

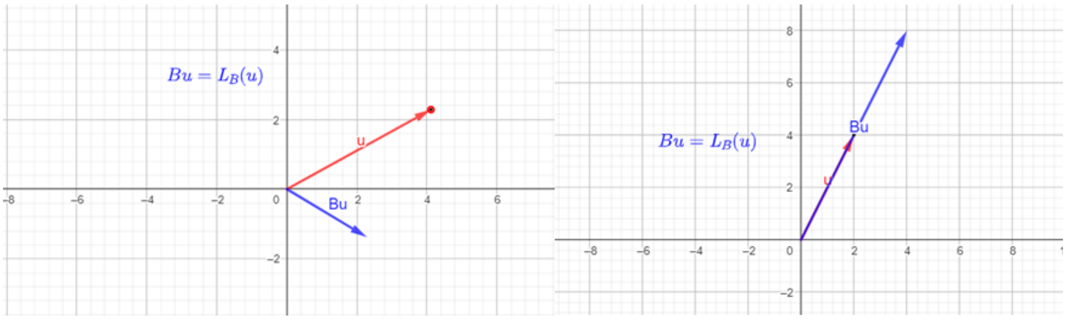


Figure 5.5. The GeoGebra applet explored by students

Students could write their answers as comments on the Perusal platform. The technological elements they had encountered so far were only what they had seen in the YouTube video: the vectors of a given transformation with the special property of staying on their span once the transformation is applied, are the eigenvectors of that transformation; the eigenvalue is the factor by which the eigenvector is stretched or squashed during the transformation. Thus, the technique that they used to find the eigenvectors and eigenvalue of the transformation in the GeoGebra applet was dragging the *u* vector, observing its image and recognising those situations where the two overlap. In those cases, *u* would be an eigenvector and the ratio between the length of *Bu* and *u*, is its eigenvalue. This (summarised in Table 5.2) is the first praxeology on eigentheory encountered by students. To recognise it later, we code it as P_1.

Table 5.2. Description of praxeology P1

Type of task	Find eigenvalues and eigenvectors (if existing) of a linear transformation represented geometrically in GeoGebra.
Technique	Drag the vector and find those configurations where its image overlaps with it. Those are eigenvectors, and the ratio between the image and the vector is the eigenvalue.
Technology	Geometrical definition of eigenvectors and eigenvalues seen in YouTube video: eigenvectors are those vectors whose image keeps the same direction; their eigenvalue is the ratio between the length of the image and the length of the vector.
Theory	Theory of vector spaces and linear transformations (linear algebra).

The first lesson on eigentheory started with an introduction on the need for introducing eigentheory, that is, in the teacher's words: "Is there a matrix among those associated with the various bases that I can choose for the domain and codomain, that is simpler, easier to calculate than the others?". The question was left written on the blackboard as a reminder that the concepts and theorems that would be addressed in that and subsequent lessons would be used to answer it. As the lesson started, a small group activity was proposed. The teacher asked the students to compare their solutions to the GeoGebra homework task and find a way to algebraically check that those found were indeed eigenvectors and eigenvalues. Students were implicitly asked to "translate" in algebraic terms the ideas of overlapping and stretching factors. They had at their disposal the canonical matrix representing the transformation that they had to find as homework. What almost all groups did was multiply the found matrix for the vectors (each at a time) that they had recognized as eigenvectors and check algebraically if the resulting image was a multiple of the vector itself. This praxeology (P2, described in Table 5.3. From now on we will omit the line "Theory" in the description of the different praxeologies, being it always the same) involved the coordination of two different semiotic registers, and the interpretation of the geometrical "overlapping" or "alignment" with the algebraic multiplication of the vector for a scalar. The second part of the task proposed to students was to write the matrix of the transformation with respect to the basis formed by its eigenvectors. Students could discuss in pairs or groups of three for around ten minutes and then post a picture of what they had written, in the padlet.

Table 5.3. Description of praxeology P2

Type of tasks	Checking algebraically if a vector in R^2 given by its coordinates is indeed an eigenvector for a given linear transformation.
Technique	Multiply the matrix of the linear transformation for the considered vector, and check if the resulting image is a multiple of that vector.
Technology	The fact that two vectors have the same direction is translated in algebraic terms, as one vector is equal to the other multiplied for a scalar value.
Theory	Theory of linear transformations and matrix operations.

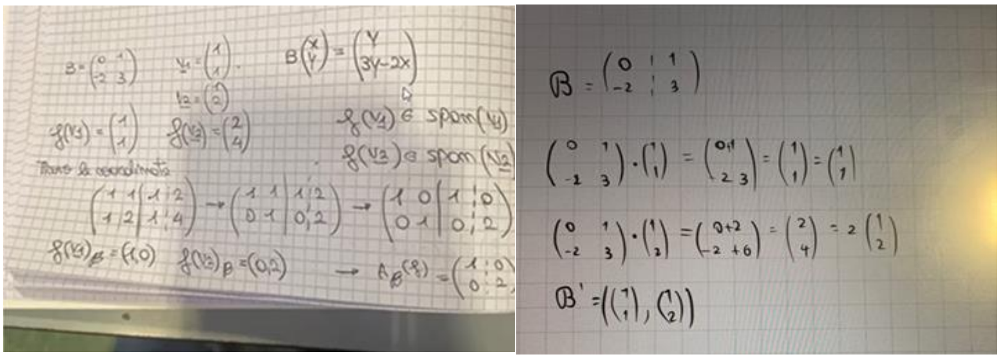


Figure 5.6. Examples of students' shared pictures of their solution to the first task

As the images were posted (Fig. 5.6), the teacher would look at them to get a general idea of the processes the students had carried out. Once the time at students' disposal was finished, the teacher started commenting with the whole class some selected posts that she projected. This task was chosen to introduce eigenvectors in a dynamic way, to prevent the arising of difficulties related to the formula (as highlighted by Thomas & Stewart, 2011). Not only the dynamic geometrical representations help in this sense, but also making students engage with an algebraic task for which the quest for eigenvectors results in a dynamic process rooted in the geometrical interpretation of eigenvectors and not in the mere application of an algorithm (as it is with the classical technique which is usually the only one students use) contributed.

The teacher then showed some examples of known geometrical transformations, like the rotation of a given angle, in the space around the origin, or in the space around an axis. She showed how it is possible to find all the eigenvectors and eigenvalues (if existing) of such transformations just by representing or imagining the movement of the vectors in the transformations and thinking in geometrical terms. The technique used then was only representing, with drawings or gestures, the transformation and trying to visualize what vectors have their image staying in the same direction, without any arithmetic computation. In the next step, the teacher proposed the students to do the same, always working in pairs or small groups, with different geometrical transformations: the symmetry with respect to a line in the plane, or with respect to a plane in the space, and the projection on a line in the plane. The choice of the second and third examples to be explored by the students was guided by the semiotic analysis performed in the pilot. This, indeed, showed how students struggled in making sense and imagining an eigenspace of dimension greater than one, and in making sense of what a null eigenvalue could represent.

Having students explore these examples, in the form of tasks, allowed them to use techniques that are not classical. Indeed, as part of the *in vivo* analysis, during this class moment, we noticed how almost all the small groups of students were gesturing or using objects at hand to represent the given transformations and solve the task. The video recording of the three groups was particularly helpful in accessing students' techniques. For instance, in one group, while addressing the task (corresponding to praxeology P3) “find the eigenvectors and related eigenvalues of a symmetry with respect to a plane in the space”, students used their hands to represent the configuration (as depicted in Figure 5.7). Both students employed their left hand to represent the symmetry plane, and their right index finger to represent vectors. With the same finger they represented the image of that vector, and checked if the two had the same direction. Observing that all vectors lying “on the left hand” maintained their direction, orientation, and length when transformed, helped them recognize that the entire plane constitutes an eigenspace for the eigenvalue 1. With the same pattern as before, the teacher collected all the answers in the Padlet, selected some interesting ones, and shared them with the whole class (Table 5.4).

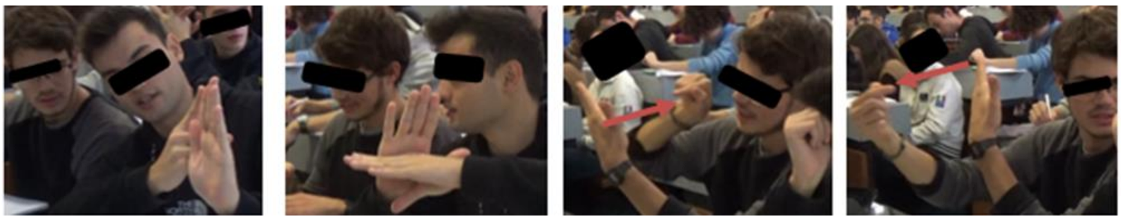


Figure 5.7. Students using gestures to address the task in P3

Table 5.4. Description of praxeology P3

Type of tasks	Finding eigenvectors and eigenvalues of a given geometrical linear transformation (such as a rotation or symmetry in the Euclidean plane or space).
Technique	Use drawings or gestures to explore different configurations to find those vectors that have the same direction of their image. Those are eigenvectors, and the ratio between the image and the vector is the eigenvalue.
Technology	Geometrical definition of eigenvectors and eigenvalues: eigenvectors are those vectors whose image keeps the same direction; their eigenvalue is the ratio between the length of the image and the length of the vector.
Theory	Theory of linear transformations and geometrical interpretation of eigentheory as the one underlying 3B1B's videos.

The first lecture ended with an interesting fact. At the end of the class, a student approached the teacher and asked a question, probably prompted by the activities proposed in class: "If I have two eigenvectors that are independent of each other but related to the same eigenvalue, are all their linear combinations also eigenvectors?". Since the question appeared as a genuine doubt by one student, we decided to pose the same question as a task (generating Praxeology P4) to solve at the beginning of the second class. Specifically, the teacher asked the students to relate the question to a specific case: let v_1 and v_2 be independent eigenvectors with the eigenvalue 3 and prove that all their combinations are eigenvectors with eigenvalue 3. The teacher then used the productions posted in the Padlet to generalise the property (Table 5.5).

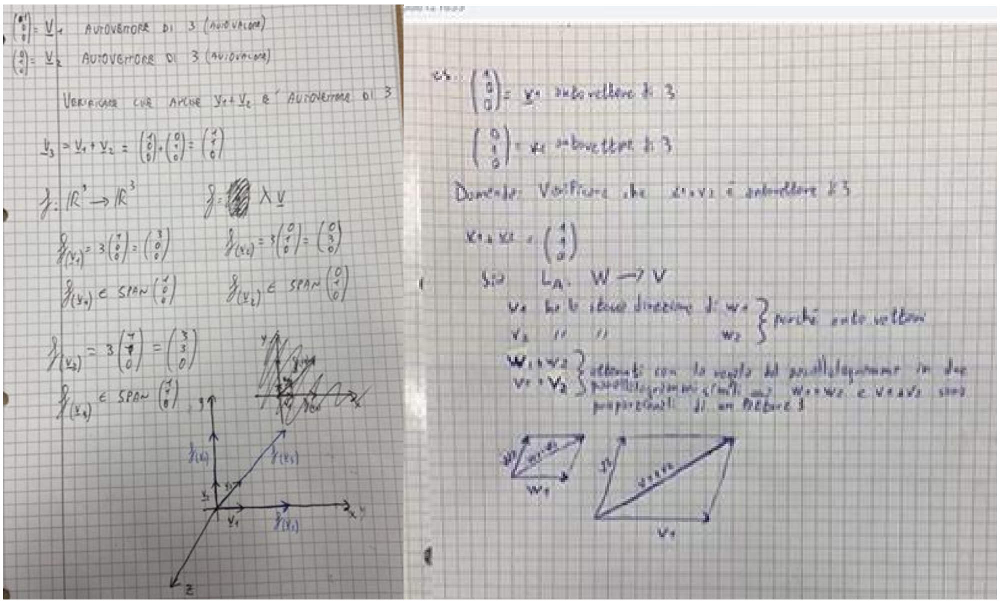


Figure 5.8. Example of students' shared pictures of their solution to the third task

Table 5.5. Description of praxeology P4

Type of tasks	Showing that, given two eigenvectors that are independent of each other but related to the same eigenvalue, all their linear combinations are also eigenvectors related to the same eigenvalue.
Techniques	1. Verify algebraically the relation $T(av_1 + bv_2) = 3(av_1 + bv_2)$ holds if v_1 and v_2 are eigenvectors related to the eigenvalue 3. 2. Verify geometrically that if v_1 and v_2 have their respective images in their same direction but with a triple length, the vector obtained as a sum of multiples of v_1 and v_2 also has its image in its same direction and with three times its length.
Technology	Algebraic/Geometrical definition of eigenvectors and eigenvalues and of linear transformation.
Theory	Algebraic *vs* geometrical definition and interpretation of eigentheory.

As Figure 5.8 shows, students resorted to different semiotic resources to solve the task. The teacher could then give voice to students' different techniques in the discussion phase following the small group activity, liaising arithmetic/algebraic and geometric ostensives.

The last part of the second class was dedicated to searching for a general technique for finding a transformation's eigenvectors and eigenvalues given its matrix. This was presented as a task and students were asked to work in small groups to find such a general technique, considering the case of the first matrix explored in the module, that is the 2x2 matrix associated with the transformation represented in the GeoGebra applet. We were not expecting them to find a correct technique, but at least to get some ideas that could be used to introduce the classical technique in a meaningful way for them, that is, justified by a well-established logos block, and not presented just as a mere procedure. After allowing the students to work for a dozen minutes, the teacher picked a couple of the posted pictures (Fig. 5.9).

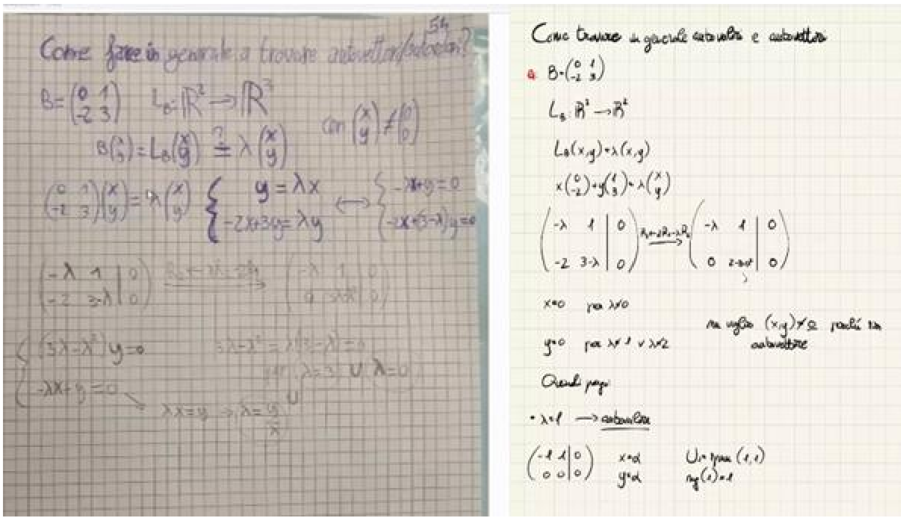


Figure 5.9. Example of students' shared pictures of their solution to the fourth task

In both works, students have set up the system, written the associated matrix, and carried out the row reduction. They then set the last pivot equal to zero, since this is a homogeneous system with at least one non-trivial solution—that means that the matrix's rank cannot be the maximum—and computed the possible values for λ

accordingly. The teacher commented and elaborated these posts with the following synthetically described steps:

1. The procedure is correct, and the idea of setting the last pivot equal to zero so that the system satisfies the condition of having one non-trivial solution is good. Nevertheless, the procedure can be long, and computational errors can easily occur, mainly with bigger matrixes. Moreover, in the Gaussian reduction, students should be careful when multiplying by λ, and do not forget to consider the case when λ is equal to zero.

2. The teacher poses the question to the whole class: "Is there another way to set the condition that the rank is not maximum (in this case equal to 2)?" Some students answer: "By setting the determinant equal to zero". The teacher shows how this method is easier and does not need a prior row reduction.

3. The teacher shows how the equation "determinant = 0" gives as root values for lambda that are the same found at the beginning of the first lesson.

4. She poses the question: "How do I write the matrix used for this system, starting from the initial one?". Students answer: "By using the initial one and subtracting lambda only in the diagonal". One student better reformulates "We take the matrix B and subtract another matrix that has all zeros, but lambda in the principal diagonal".

5. The teacher goes back over all the steps and shows how in general, for any matrix, you can take the matrix given by subtraction to the initial matrix of the identity matrix multiplied by lambda, calculate its determinant, set it equal to zero, and calculate its roots. Those are the eigenvalues of the transformation. She writes on the blackboard the general proposition.

We will not describe here, due to lack of space, the phase of exploration of how to find eigenvectors and the following lessons regarding diagonalisation strategies.

5.8 Further results obtained through the praxeological analysis

The analysis conducted helps identify the possibility to provide other praxeologies that facilitate the emergence of a variety of different semiotic resources, following the path we have designed and implemented. Moreover, the REM presented in Figure 5.2 (complemented by Table 5.1) can be used to confront the implemented praxeologies, emerging in the selected course, with the ones that could exist in a wider study process. The institutional constraints described in the "conception phase" section hindered the realisation of a genuine inquiry process, as advocated by the complete development of the Q-A map. Nevertheless, the implemented sequence allowed students to encounter—even if not systematically address—some of the sub-questions present in the Q-A map, as well as different praxeologies the techniques and technologies of which make use of many more semiotic resources than the sole algebraic register: dynamic animations, drawings, gestures, natural language. The appearance of these ostensive objects contributes to the emergence of their non-ostensive counterparts, letting students encounter more flexible praxeological elements about the notions of eigenvectors and eigenvalues.

For example, even if it is quite common that the teacher presents some geometrical examples, these often do not become part of students' praxeological equipment, since they do not appear as tasks to solve. In our instructional path, students immediately encountered a geometrical representation of eigenvectors (in the video) in both the praxis and the logos blocks. Later, they also had to activate geometrical techniques, mobilising new semiotic resources as drawings and gestures, to actually solve the types of tasks proposed (*praxis*) and answer more theoretical questions (*logos*). Surely, this phenomenon was enhanced by the distribution of responsibilities among the teacher and students, provided by the general design of the classes, interspersed with moments of small groups' activity.

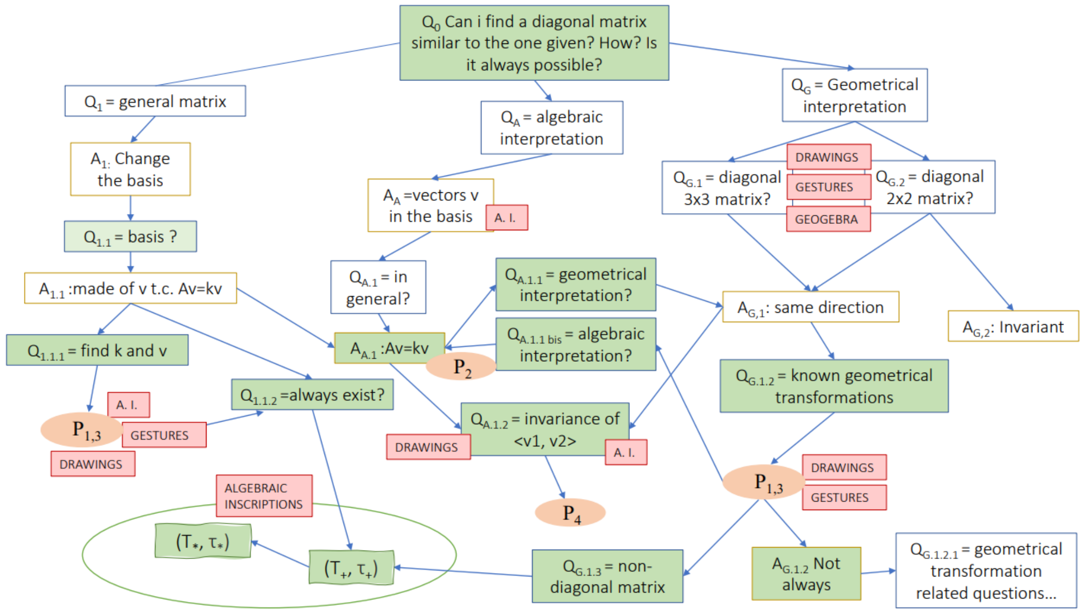


Figure 5.10. Reference epistemological model (as presented in Figure 5.2) for introducing eigentheory confronted with the actual implementation of the sequence

The Q-A map shows different interesting facts. First, how the phenomenon of "algebraic reduction" corresponds to a praxeological reduction in the map (green ellipse). The instructional proposal nourished by the APC approach aimed at developing students' semiotic resources beyond written symbolism to incorporate gestures and figures. The strategy led to the incorporation of new questions and answers (green parts of the map in Figure 5.10) and consequently new potential praxeologies (in orange circles). Moreover, this praxeological—and mathematical—incrementation was not purely formal ("teach more ostensives") but went with a development of a new type of questioning: the ostensives appear in praxeologies that are needed to answer new (theoretical and practical) questions about linear mapping classification. Secondly, when the relationship between the geometrical and the algebraic praxeologies is systematically addressed, new questions arise (white parts of the map) leading to new praxeologies and, consequently, new developments of the corresponding semiotic resources. We are then faced with the critical problem of the curriculum: what mathematics must be taught to engineering students concerning linear algebra? What praxeologies—with what ostensive and non-ostensive elements—about linear maps, classification and matrix diagonalisation are the most suitable to them? What questions might motivate the construction and development of these praxeologies?

The third big issue is the problem of the institutional conditions that seem necessary to better sustain the study of these algebraic and geometrical praxeologies. Among them, we can find the pedagogical organisation of the teaching and learning processes (course length, class schedule, size of groups, classroom structure, online resources, pedagogical tradition, etc.). However, we do not have to forget other conditions of a more epistemological nature about the conceptions of "mathematics for engineers" that prevail in university settings, such as the phenomenon of *applicationism*, and the pedagogical conditions related to the dominant paradigm of *monumentalism*.

5.9 Conclusions

We have presented the development and implementation of a teaching and learning sequence to introduce eigentheory in a linear algebra course taught during the first year of an engineering degree programme. We retraced the initial phases of the didactic engineering process, initially designed using the semiotic bundle lens, and reanalysed under the lens of the ATD through praxeological analysis. The ATD perspective has enabled us to identify some strengths and limitations of the teaching proposal, addressing RQ2. First, the *a priori* epistemological analysis has shed light on the epistemic significance of the types of tasks presented to the students in the sequence. This was facilitated by the preliminary analysis of the tasks themselves, alongside the utilisation of a reference epistemological model in the form of a Q-A map.

Furthermore, the ATD approach also considers the conditions and constraints set up for the implemented sequence, relating the epistemological analysis to some limitations due to the school organisation of the linear course, the instructional devices and pedagogical decisions traditionally prevalent at the university level (groups with many students, 2-hour lectures per week with only one lecturer, problem sessions closely link to the lectures' content, etc.). It shows how the instructional proposal still responds to the prevailing model of "monumentalism", assuming the importance of the praxeological organisations to be "visited" and neglecting the questions that could give them a raison d'être.'

In addition, the analysis of the implemented sequence in terms of praxeologies sheds light on what types of tasks can prompt students to use techniques beyond mere algebraic computation, introducing various ostensive objects. The diverse use of ostensives significantly strengthens their epistemic values by connecting them to their related non-ostensive objects. This connection, however, necessarily passes through their utilization as instruments in the applied techniques. The incorporation of a reference epistemological model in the form of a Q-A map facilitates a comparison of the dominant epistemological model with the knowledge organization on eigentheory resulting from the designed and implemented sequence.

The emergence of ostensives as gestures, informal drawings or speech supports the appearance of more flexible activations of eigenvectors and eigenvalues. Nevertheless, what students will use in their university practice will be almost only the algebraic technique to compute them. We believe though that, differently from a standard teaching sequence for eigentheory, in our teaching sequence, all the ostensive objects and the different praxeologies encountered by students, such as the geometrical praxeologies to find eigenvectors of a known geometrical transformation, can end up integrating the *logos* part of the standard algebraic technique. Thus, on the one side, the so-called phenomenon of "chirographic reduction" (Arzarello et al., 2008) becomes apparent: material objects, gestures, informal graphs, or oral expressions acting as real tools in the mathematical activity end up disappearing from the formal presentation of the results, leaving only the types of signs that can be "chirographied", transferred to a sheet of paper. On the other hand, even if many ostensive objects disappear from the final technique used, they represent a consistent part of the justification that has brought to the introduction of that technique and can remain as part of the *logos* of the praxeology. This example shows how praxeologies are “sensitive” to the ostensives they are composed of. Even if the final praxeology to compute eigenvectors and eigenvalues is the same in whatever standard course in linear algebra, the logos part, as well as the evolution of such praxeology, can be significantly different since its evolution and development has been affected by the availability of different ostensives encountered in the study path travelled.

Adopting the perspective of the semiotic tools activated by students (and teachers) when learning, teaching, or just doing mathematics leads to the consideration of new types of praxeologies that have been discarded, rejected, or just “lost on the way” when linear algebra has been incorporating more and more algebraic tools. Helping students think with their gestures and drawings, to complement and give more flexibility to the linear algebra strategies (both theoretically and practically) not only requires introducing these ostensives but especially posing new questions and elaborating new techniques and theories—new praxeologies—to address them. We postulate the need to address the semiotic dimension of mathematical activities from a more mathematical perspective, that is, by restituting the praxeologies that support this dimension. When doing this, we can observe a development of not only the semiotic or ostensive tools used, but also of the type of mathematical questioning that is developed.

Finally, the praxeological approach of the ostensive dimension helps address the variety of semiotic registers that intervene in the mathematical activity from a certain cultural neutrality, without giving a greater epistemic or cognitive value to some registers in front of others. In the initial stage of this research, students’ activity comprising gestures and drawing arrows or verbally describing different spatial configurations appeared as a richer cognitive activity than the merely written calculations with vectors and matrices. However, when deepening into the ingredients of the mathematical activities that motivate and sustain the use of gestures and drawings, we find different praxeologies—some more algebraic, other more geometrical—that provide different ways to approach linear problems, with different formulations of the types of tasks, different techniques to carry them out and different *logos* to describe and justify these *praxes*.

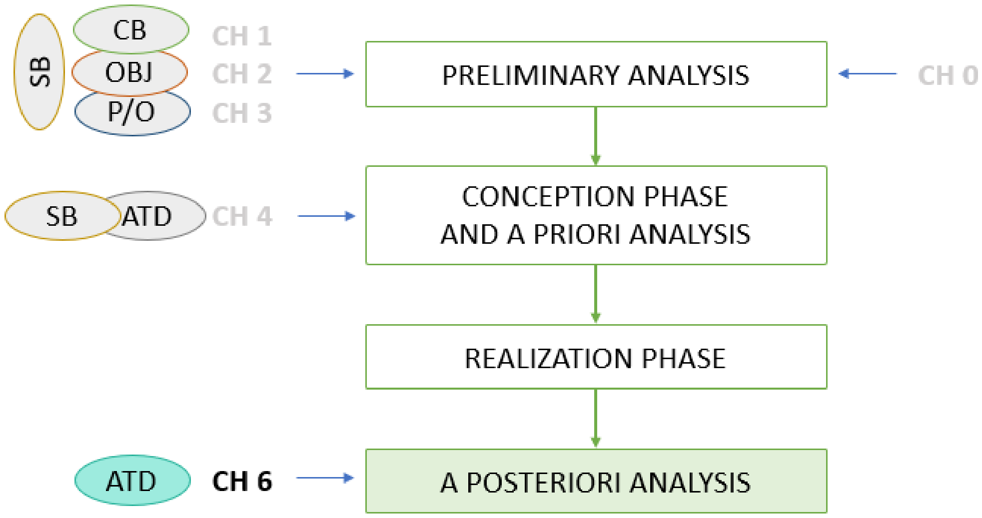
With this work, we aimed to extend the dialogue initiated by Arzarello et al. (2008) between the APC-space theory and the ATD. The interaction between these two theories have been indicated (Presmeg et al., 2016) as a paradigmatic example of the networking strategy of ‘coordinating theories’ given their coherence but with complementary analytical lenses. However, their coordination and complementarity have not been challenged. After the initial attempt by Arzarello et al., (2008), the two theories had never been used together until now to study a concrete example of the implementation of a teaching sequence. In this paper, we have presented and used a new strategy to develop a dialogue between theories guided by the didactic engineering process. In its different phases, the APC-space and the ATD have established different kinds of collaboration. In the initial steps of the pilot experience, the APC-space was the central theoretical framework acting in the design, where the ATD has externally contributed to the a posteriori analysis of this initial experience. This has allowed us to highlight new aspects of the didactic phenomena addressed, as well as to propose new developments of the a priori analysis for a second-round implementation. Furthermore, in the redesign of the activities and their *a priori* analysis, the ATD has contributed with the praxeological analysis to integrate the previous activities in a more global epistemological reference model making visible some strengths and limitations, which the redesign has been based on.

The use of the didactic engineering tool in the dialogue between theories presents several potentials. It provides more control over the way two approaches can interact, in the identification of didactic phenomena during the preliminary analysis, in the way to address them through empirical investigations, and in the interpretation of the results obtained during and after instructional implementations. It also clarifies the roles of the theories in the concrete way of approaching didactic phenomena, the unit of analysis delimited, and the assumptions made, pointing towards the construction of new collaboration tools. We hope that this type of strategy will prove valuable for future research.

CHAPTER 6 - UNIVERSITY STUDENTS' ATTITUDE TOWARDS A NON-STANDARD INSTRUCTIONAL PROPOSAL FOR INTRODUCING EIGENTHEORY

This is the author's original manuscript of an article accepted for publication by *LUMAT: International Journal on Math, Science and Technology Education.*

THE CHAPTER IN THE RESEARCH PROJECT



Abstract

In this paper, I present the responses provided by first-year engineering university students to a questionnaire investigating their appreciation of a novel teaching sequence used for introducing eigentheory in a linear algebra course. I analyse these responses using the notion of the scale of levels of didactic codeterminacy, as conceived and employed in the Anthropological Theory of the Didactic to study the ecology of mathematical practices in a specific institution. Specifically, I show how the recurring themes in students' answers are likely rooted in higher levels of the scale, such as that of the school and pedagogy. This implies that it is necessary to act at these levels as well in order to change university students' attitudes towards non-standard didactical approaches.

Keywords: Eigentheory, secondary-tertiary transition, ATD, levels of codeterminacy

6.1 Introduction

Linear algebra is widely recognized as a significant obstacle for first-year university students (Stewart et al., 2019). While research has delved into specific aspects of the subject to understand this phenomenon, it is also valuable to contextualize it within the broader issue of Secondary-Tertiary Transition (STT). The transition from secondary to university education has attracted the interest of many researchers in recent years. According to Gueudet (2008), numerous studies have focused on the cognitive and epistemological aspects of this transition, leaving room for further exploration of its sociocultural and affective dimensions. A recent literature review on STT by Di Martino et al. (2023) reveals a growing interest in sociocultural and cognitive aspects within the mathematics education research community. However, the topic still deserves attention and further study, considering its multifaceted nature. For instance, as highlighted by Gueudet (2008), the STT can be approached from an institutional perspective, given that transition between two institutional cultures occurs when entering the university. A key aspect of this transition is the difference in didactical contracts between secondary school and university (Brousseau, 1997). In encountering the new didactical contract, students develop new expectations regarding teachers' approaches. The didactical contract can be examined at various levels (Gueudet & Pepin, 2018), including the specific subject or mathematical content level, as well as at a general institutional level where the rules of the contract apply to all subjects taught in the institution. In university courses, this type of contract appears particularly influential, potentially hindering the adoption of non-standard pedagogical practices such as student-centered instruction.

This paper presents results regarding students' attitudes towards a non-classical teaching and learning sequence designed to introduce eigentheory. The sequence was implemented in a linear algebra course for first-year engineering students at a public Italian university. The focus is on studying the extent and manner in which students' attitudes are influenced by institutional aspects.

Recognizing the complexity of the construct of attitude in mathematics education (Di Martino & Zan, 2015), the term "attitude" in this paper refers generally to students' disposition toward a proposed teaching approach. However, the primary focus is not on analyzing this "attitude" as an individual aspect of the student, but as a phenomenon reflecting a trend at the institutional level. Therefore, as a framework to guide the analysis of the results, I will use notions pertaining to the Anthropological Theory of the Didactics (ATD), acknowledging the contribution that it has made to the study of transitions, particularly in Undergraduate mathematics education (Artigue, 2022)

6.1.1 The ATD perspective: the hierarchy of levels of didactic codetermination

The ATD perspective shifts the lens of analysis from individual students and teachers to the institutions shaping their relationship with mathematical knowledge. It does so by studying the genesis and diffusion of institutional *praxeologies*. A praxeology is defined as a quadruplet $[t, T, \theta, \Theta]$, comprising a praxis block made of the different types of tasks t and their associated techniques T, as well as a theoretical block with the technological discourse θ used to produce and justify these techniques, along with the overarching theory providing such technological discourse (Chevallard, 2019).

A particularly valuable theoretical construct within ATD for approaching institutional transitions is the *hierarchical scale of levels of didactic codetermination*, which helps us to better understand "the complex system of conditions and constraints that condition the ecology of mathematical and didactical praxeologies" (Artigue, 2022, p. 273). This hierarchy scale spans from the most generic level of humankind and societies down to the lower levels of disciplines, up to specific subjects and the relative questions. Precisely, it consists of the following sequence (Bosch et al., 2020, p. 39):

Humankind ⇆ Civilisation ⇆ Society ⇆ School ⇆ Pedagogy ⇆ Didactic System ⇆ Discipline(s) ⇆ Domain ⇆ Sector ⇆ Theme ⇆ Subject

Figure 6.1. Scale of levels of didactic codetermination (Bosch et al., 2020)

This scale highlights the reciprocal influence between mathematical contents and the way of organizing their teaching, evident in the structure of a didactic system or, at a higher level, of a society. For this reason, it serves as a valuable tool for studying the ecology of mathematical practices that currently exist or could potentially exist within a teaching institution. Furthermore, it aids in identifying the appropriate level of the scale at which intervention is necessary to enhance the conditions for a specific mathematical or didactical praxeology to flourish within that institution.

6.2 Design of the sequence and ecology of the context

The teaching and learning sequence has been designed and implemented as part of my PhD research project concerning the didactics of eigentheory. The sequence was collaboratively designed by the author and the course teacher following the methodology of didactic engineering (Artigue, 2015). According to this methodology, a preliminary epistemological, didactical and institutional analysis should guide the design and a priori analysis of the didactic intervention. Barquero and Bosch (2015) highlighted the importance of considering the ecological dimension of the institution where the teaching and learning process happens, that is studying the conditions and constraints present in it.

Our proposal involved the implementation of small group activities during class hours. Thanks to the use of the online notice board "Padlet", the teacher could assign, at certain times in the lectures, problems to be solved on the spot by pairs or small groups of students. These problems were not exercises requiring the mere application of a procedure taught earlier by the teacher, but rather problems aimed at exploring new concepts. For instance, it was required to find eigenvalues and eigenvectors of known linear transformations (such as geometric transformations in the plane or space), even before the algebraic algorithm for computing eigenvalues and eigenvectors of a matrix was introduced. The generalized procedure for calculating eigenvalues and eigenvectors was not directly provided by the teacher; instead, students were encouraged to hypothesize during group work. Then, at the end of the time at their disposal to solve the problem, the groups could anonymously post a picture of their solution on the shared Padlet. The teacher would project the Padlet so that everyone could see the answers posted by others, and simultaneously she was able to check, as solutions were posted, if there were any particularly interesting solutions to present to the class later, as a bridge between the groups activity and what she would explain immediately afterwards. A more detailed description of the entire teaching sequence can be found in Piroi et al. (submitted). It is important to mention that this teaching-learning modality was exclusively used in the eigentheory module, one of the final topics covered in the course. Previous topics were taught using the standard frontal lecturing approach.

The newly designed teaching sequence had to face different institutional constraints hindering its implementation. First of all, it had to deal with physical limitations present in the typical university classroom in the case study setting: the aforementioned linear algebra course is in fact generally attended by more than 200 students, and takes place in a classroom in which it is impossible to change the arrangement of desks to facilitate group work. Additionally, the dense nature of the course curriculum often pressured teachers to adhere to traditional methods due to concerns about time constraints. In the a priori analysis, we had to consider also the variable of university students' attitude towards this type of teaching and learning modality, and in particular towards active and cooperative learning.

6.2.1 A priori analysis regarding students' attitude and research questions

In the conventional model of university teaching, face-to-face lectures by the professor dominate, reflecting what Chevallard (2019) refers to as the paradigm of "monumentalism". In this paradigm, a mathematical work is "visited" by students in a class under the supervision of the teacher as if it was a monument. This dominant view, on the one hand, pushes teachers to play the role expected from them, i.e., that of the lecturer, and on the other hand, creates expectations in students about what university lectures are like and, consequently, that their task consists of taking notes, studying, and being able to solve the exercises proposed in the exam (Gueudet, 2008). This strong didactic contract undoubtedly challenges the implementation of alternative teaching approaches that involve direct student engagement and cooperation during lectures. For this reason, we hypothesized that, although students might appreciate the new learning mode, at least for the dynamism it would give to lectures by making them less boring for them, they would have struggled to fully grasp its pedagogical value. Moreover, the disruption of the typical university didactical contract could potentially disorient them.

To validate this hypothesis, this paper addresses the following research questions:

(Q1) "What is the attitude of first-year engineering students toward a nonstandard educational proposal that involves their active participation and cooperative work?". In answering this question, we aim to address the broader question:

(Q2) "To what extent can students' expectations of mathematics teaching in undergraduate STEM courses be a constraint to the effective implementation of innovative teaching proposals at this school level?".

6.3 Metodology

At the end of the module on eigentheory, we administered to the students an anonymous online questionnaire investigating their appreciation of the new teaching and learning proposal and their perceived knowledge attainment regarding the topic at stake. Specifically, the questionnaire consisted of 5 questions:

A. Compared to other lectures in Linear Algebra and Geometry, in these last 4 lectures related to the "Eigenvalues and Eigenvectors" module, how much more involved did you feel?

B. Did you feel that the activities proposed in the last 4 lectures (reasoning or solving exercises in pairs and or small groups) increased your attention during the lectures?

C. Do you feel that the addition of the pair or small group work during these lessons enhanced your learning of the concepts covered in these lessons?

D. Justify your previous answer if you like by specifying which concepts you feel you understood well and which remain unclear.

E. Would you like this mode (interspersing the frontal lecture with moments of group activities, shared via the Padlet) to be used in the remaining lessons of the course as well?

F. Justify your previous answer.

Questions A, B, C and E were closed-ended and students could choose a value between 1 and 4 where 1 stood for "not at all" and 4 for "absolutely yes". Questions D and F were open-ended and required students to justify their responses to C and E, respectively. To answer the outlined research questions, I focused on students' responses to these two open-ended questions, since "It is the open-ended responses that might contain the 'germs' of information that otherwise might not be caught in the questionnaire" (Cohen et al. 2007, p. 330). 92 students in total answered the questionnaire.

Since I did not have predefined theoretical categories to categorize the students' responses, I conducted a qualitative content analysis through inductive coding. Following Mayring's (2000) approach to inductive qualitative content analysis, I initially read all the answers to questions D and F (related to the numerical values given in response to C and E, respectively) to gain a general understanding of the students' answers. I then identified macro categories based on emerging themes and whether or not students appreciated specific aspects, such as 'appreciate group work,' 'do not appreciate group work,' 'appreciate active participation in class,' etc. Next, I identified recurring themes as subcategories of the previous ones, assigning specific codes to each occurrence. Since question F was often linked to D, I coded all responses to question D. I omitted responses to question F from students who had given very similar responses traceable to the same categories as response D, and coded only responses to question F in which new themes emerged.

6.4 Results

A brief overview of the responses provided for close-ended questions C and E, as depicted in Figure 6.2, validates the a priori hypothesis that the majority of students favored the innovative teaching approach. However, it is through the analysis of open-ended responses that we can thoroughly explore the research problem and address the research question. In this discussion, my focus will be on aspects that are particularly pertinent to contributing to the resolution of Q2.

One recurring theme I observed in the responses of students who enjoyed doing problems during the activity is the concept of "applying theory to exercises". I assigned this theme the code "*apply*", which I identified 14 times in the responses. Responses such as "[....] because I can better understand how to apply theory to exercises than instead of just following the professor's examples in class" fall into this category.

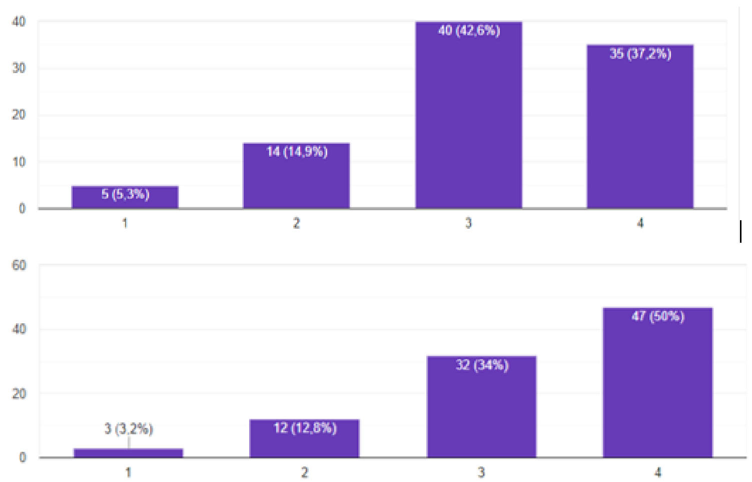


Figure 6.2. Answers, in order, to questions C and E

A closely related category is "*solidifying explained arguments*", also identified in numerous responses. For instance, one student states: "After seeing theory with the professor, doing exercises in the classroom seems to me to be a great way to test what has just been explained"; another expresses: "I think doing exercises in the classroom on topics that have just been explained helps to fix new concepts". The other 13 responses assigned this code share similar sentiments.

It is particularly striking that a large number of students interpreted the proposed small-group activities as the application of notions explained during the lecture. The prevalent perception is that they were required to "apply" these notions to the exercises. However, as briefly explained in the section on the context of the research, the activities proposed during the lectures were true insights into the theory, and sometimes even moments of constructing notions and properties related to eigentheory.

It appears that students are so accustomed to the idea that, in mathematics teaching, especially at the university level, typically theory is taught by the teacher and then they are required to put it into practice by doing exercises, that even when the teaching proposal is different and does not follow this pattern, they do not realize it and interpret it according to the dominant model. This phenomenon, reflected in students' attitudes, is likely generated at a higher level of the codetermination level scale, specifically at the school level, with likely influences arising from even higher levels.

The same can be said for another recurring theme, coded (and found 41 times) as "*exam*". An exemplary instance of this category is the statement: " I think this mode is essential in preparation for the written exam." In general, this code was assigned whenever a student's focus on the exam was evident, that is when the activities proposed were appreciated because of their positive impact on the preparation for the final exam of the course. Once again, the attitude of students, whose perceived task in the institution seems to be not so much to learn but rather to succeed in passing the final exam, reflects a teaching contract strongly rooted in the university institution.

This is reflected not only in the attitudes of the students, but also in that of the teacher. Indeed, although the teacher of the studied case willingly lent herself to the implementation of this new didactic sequence, she maintained the course assessment method unchanged. The dominant epistemological model is so entrenched at higher levels of the codetermination scale that a change at the didactic level, although it may occur thanks to the commitment of the teacher, struggles to carry with it a broader pedagogical change that also includes a revision of the mode of assessment. Not surprisingly, this code co-occurs several times with the two described above.

The motivations behind the appreciation of group work also partially reflect this strong influence of the dominant pedagogical model in students' perceptions. For example, in the macro category "appreciate group work", the three subcategories described above recur several times, confirming the results also found by Bächtold et al. (2023), regarding the fact that group work is appreciated by students mainly because it provides an opportunity to discuss and adapt methods for completing tasks and thus better prepare for the final exam.

On the other hand, fortunately, there is no shortage of responses in which students' appreciation of the proposed activities related to a better understanding of the topic itself seems to emerge. This category, identified 23 times, in 14 of its occurrences is found in responses belonging to the macro category "appreciate group work". This is certainly an interesting aspect to explore in future work, including comparing it with the results found by Bächtold et al. (2023), regarding university students' attitudes towards cooperative learning.

6.5 Discussion

I have presented part of the obtained results concerning the attitudes of first-year engineering students towards the described novel educational pathway, which represent an answer to Q1. In order to address Q2, I focused on those aspects that highlight how these attitudes may be a constraint to the hoped-for realization of the designed course. Indeed, the fact that students have deeply rooted expectations about how university teaching works and the belief that what is required from them is only being able to solve the exercises that will be proposed on the final exam, hinders the rich construction of meaning that was intended to be achieved by the design of the specific teaching sequence. It is particularly useful to observe the level of codetermination on which these expectations are based, that is usually at a much higher level than that of the discipline. In the presented example, although the teacher's effort to make a change in her course was generally appreciated by the students, the reasons for this appreciation diverge from the intentions behind the educational project. This makes it clear that specific actions need to be taken at the highest levels of pedagogy and school to create the right conditions for the effective implementation of such an educational path and remove the constraints to it.

In this paper, my primary focus was on presenting students' reactions to the proposed teaching sequence and analyzing them by attempting to identify the level of codetermination to which they are grounded. Additionally, I explored how these reactions affect lower levels, acting as constraints for a deeper understanding of eigentheory concepts. The conducted analysis specifically centers on institutional aspects at the university level. The results do not address precisely the issue of Transition, but together with a parallel institutional analysis at the secondary level, they might contribute to the research strand about affective aspects in the SST.

CHAPTER 7 - DISCUSSION AND CONCLUSIONS

The academy is not paradise.
But learning is a place where paradise can be created.
The classroom, with all its limitations,
remains a location of possibility.

bell hooks

In Chapter 0, I presented the main research questions guiding the overall research project, that are:

In what ways can the use of geometric representations of eigenvectors and eigenvalues support students' structural understanding of these and related concepts? **(GQ1)**

and:

What teaching scenario can support a meaningful integration of geometry in the teaching of eigentheory in a linear algebra course for engineering?

What conditions can support and what constraints can hinder such implementation? **(GQ2)**

In this section, I will attempt to address these overarching questions. To begin, I will provide a summary of the key findings derived from each chapter.

7.1 Summary of results

7.1.1 Answering the specific research sub-questions

In Table 7.1 I present the answers to the research sub-questions posed in each chapter.

Table 7.1. Answers to the specific research sub-questions

Research sub-questions	Answers and key findings
1.1 How can the emergence of the Analytic-Structural (AS) way of thinking be described as a blending between the Synthetic-Geometric (SG) and the Analytic-Arithmetic (AA) way of thinking?	By employing multimodal diachronic analysis of the semiotic resources utilized by students, we can observe the emergence of the Analytic-Structural mode of thinking. The crucial aspect lies in showing how specific elements from different representations (in the SG and AA modes) are projected into the blended space (AS), as they are shared across all vector spaces. This facilitates the emergence of structural aspects of eigenvectors and eigenvalues, going beyond merely summing up shared features of input spaces. Emphasis is placed on the importance of creating teaching environments conducive to fostering such a process.
2.1 Can our designed activity trigger and support first year university students' objectification process of eigenvector and eigenvalue concepts, and if so, how?	The designed activity proved suitable to support students' objectification of eigenvectors and eigenvalues. This is suggested by the fact that students engaged for a long time in the part of the task requiring them to collectively reconstruct the notions of eigenvector and eigenvalue. Also, students

2.2 What information can the analysis of the evolution in time of the semiotic means of objectification mobilized by students give about these objectification processes?	never settled for just repeating the definitions seen during the lecture. Rather, they tried with conviction to build strong meanings for those concepts and to pinpoint connections with other linear algebra concepts. As well, they tried to ensure that all the members of their group grasped the same meaning. It is particularly interesting to notice how different students preferably appealed to different semiotic registers. The semiotic resources used, especially gestures, clearly appear not only as means of communication, but also as productive resources that contribute to the constitution of thought and are key actors in the process of objectification. Even more, it is the combination of these different semiotic resources in the bundle and conflicts arising between them, that allowed objectification (Radford, 2021) to occur and the intertwining of means of objectification activated by different students was possible only thanks to their joint labour. The analysis of students' means of objectification and their evolution and mutual relationships actually allowed us to study their collective meaning-making process. Thanks to the use of the semiotic bundle as an analytical tool, we could detect semiotic nodes in which the emerging and evolving relationships between signs help accomplish the objectification process. Another remarkable aspect is the fact that the observed group, despite required to deal with eigenvalues and eigenvectors, autonomously felt the need to deeply investigate the concept

	of eigenspace, in order to really understand them.
3.1 In what ways is a commognitive conflict likely to arise in a discourse on eigenvectors, when newcomers participate in that specific discourse? 3.2 What aspects in the formulation of the equation "*f(v)* = *λv*" may hinder the development of a structural view of eigenvectors and eigenvalues?	In both the presented examples, it appears that participants in the discourse struggle to grasp a fundamental aspect of eigenvalues and eigenvectors: the definition of eigenvectors and eigenvalues refers to a precise linear map *f(v)*, that can or cannot have vectors v such that *f(v)* = *λv*. This misunderstanding stems from a commognitive conflict, wherein the same representations are interpreted differently based on distinct metarules. The classical formulation of the definition contributes to this confusion, particularly due to the interpretation of *f(v)* = *λv* as implying that *f(v)* is defined as *λv*. Students' attempts at giving operational meaning to these concepts are hindered by their preconceived notions shaped by the structural definition encountered in class. Consequently, there's a reinforcement of a pseudostructural conception of eigenvalues and eigenvectors. A didactical implication is proposed: starting with an operational interpretation before presenting the formal definition, allowing students to initiate their understanding from exploring examples. This approach aims to smooth the transition between process and object and reveal the inherent logical dependence in the formal definition.
4.1 What kind of mathematical-didactic proposal, compatible with the learning context described with constraints such as the very high number of students attending the classes, can	The designed and later implemented didactical proposal is described. In particular, micro level choices as the tasks to be posed to students in the sequence are analyzed a priori, showing how the fine-grained semiotic

be implemented to foster the understanding of eigentheory?	analysis of the pilot study informed the design. In particular, the a posteriori analysis of an excerpt of activity in the implementation of the teaching sequence, allowed us to observe how students' mobilization of multimodal, dynamical, resources, such as their gestures, enabled them to visualize that the set of eigenvectors related to the same eigenvalue form a vector subspace.
5.1 What kind of instructional proposal regarding eigentheory can be implemented to foster the emergence and use of different kinds of ostensives rather than the solely algebraic ones, compatible with a traditional first-year university lecture-based linear algebra course? 5.2 How to interpret in terms of the ATD an instructional proposal about linear algebra designed and implemented in the APC-space? What strengths and limitations does the ATD analysis reveal? 5.3 What networking strategy can support this kind of dialogue between ATD and APC-space?	Sub-question 5.1 retakes 4.1, deepening it. A novel contribution of this Chapter is the construction of a Reference Epistemological Model for the study of eigentheory in the form of a Questions-and-Answers map. It allows to contrast the actual implementation with it, highlighting the conditions and constraints actually influencing the implementation. The study uncovers a variety of ostensive objects within students' praxeologies when they engage with the teaching sequence, facilitating more adaptable approaches to eigenvectors and eigenvalues. The use of gestures, informal drawings, and speech fosters more flexible approaches to eigenvectors and eigenvalues. Our teaching sequence integrates various ostensive objects and praxeologies, such as geometric methods for finding eigenvectors, will converge in the logos part accompanying the standard algebraic technique. While the phenomenon of "chirographic reduction" may lead these alternative semiotic resources to disappear in the technical block of eigentheory-related praxeologies, they remain an integral part of

	the logos block. This underscores how praxeologies are influenced by the ostensive objects they incorporate. The use of APC-space theory (Semiotic Bundle), through fine-grained analysis, in the preliminary analysis and in the conception phase and a priori analysis guided the design of the teaching sequence. ATD allowed one to consider a broader unit of analysis in the analysis a posteriori of the implementation, using the notion of praxeology and analysing the ecology of the Institution. This use of different lenses at different stages of the Didactical Engineering process is a possible strategy for coordinating the two theoretical frameworks.
6.1 What is the attitude of first-year engineering students toward a nonstandard educational proposal that involves their active participation and cooperative work? 6.2 To what extent can students' expectations of mathematics teaching in undergraduate STEM courses be a constraint to the effective implementation of innovative teaching proposals at this school level?	The deeply rooted expectation among students regarding the traditional model of university teaching, primarily focused on learning how to solve exercises in view of the final exams, impedes the intended rich construction of meaning envisioned by the specific teaching sequence. This highlights the significance of examining the level of codetermination underlying these expectations, often surpassing that of the discipline itself. While the teacher's initiative to innovate her course was generally well-received by students, their reasons for appreciation diverged from the educational project's intentions. This underscores the necessity of implementing targeted actions at higher levels of pedagogy and within educational institutions to establish conducive conditions for effective implementation of such

	educational approaches and to alleviate associated constraints.

7.1.2 Answering the general research questions GQ1 and GQ2

In what ways can the use of geometric representations of eigenvectors and eigenvalues support students' structural understanding of these and related concepts? **(GQ1)**

Chapters 1, 2 and 3 contribute to answer to this research question. In particular, the main elements arising from the results of the specific analyses are:

- Using geometric representations of eigenvectors, eigenvalues and eigenspaces is likely to support students' understanding of the concepts. In particular, the mnemonic benefit of imagery and embodied language (Hoestetter & Alibali, 2008) can aid not only in understanding but also in recalling these concepts at later stages.
- It is fundamental that geometric examples are associated with different representations so that elements in common within the different representations - or modes of thinking in Sierpinska's terminology (Sierpinska, 2000) – are selected and projected into the blended structural space.
- Small-group work can facilitate this process. Our results indicate that different students may prefer various languages and modes of thinking when representing eigenvectors and eigenvalues. This can result in a blending, albeit unconscious, that can support an abstract understanding of the concepts by recognizing those elements that are structural and do not depend on the type of vector space in question.
- In particular, the use of dynamic geometric representations can support a procedural definition of eigenvectors that starts with a given linear map and, with respect to it, looks for those vectors that maintain their direction when transformed. This could lead to a correct object-like conceptualization of

eigenvectors and eigenvalues, avoiding their pseudo-structural interpretation. This can reduce difficulties that are likely to arise from the standard formal definition of eigenvectors and eigenvalues.

Chapters 4, 5 and 6 continue this investigation, but contextualizing it in the institutional reality of university mathematical courses for STEM degree programs. They contribute indeed to answer to GQ2:

What teaching scenario can support a meaningful integration of geometry in the teaching of eigentheory in a linear algebra course for engineering? What conditions can support and what constraints can hinder such implementation? **(GQ2)**

The teaching scenario proposed in this project, as well as the ecological analysis of the context are extensively described in chapter 5 (and partly also in chapters 4 and 6). We focus here on the main aspects emerging from the analysis:

- The construction of a Reference Epistemological Model (here in the form of a Questions-and-Answers map) and its comparison with the actual implementation of the teaching sequence can be particularly helpful in recognizing the conditions and constraints influencing the implementation.
- The Q-A map allows also to shows the variety of ostensives used in different praxeologies, thereby highlighting those that are actually likely to appear in the teacher's and students' praxeologies within the ecology of the considered teaching and learning context.
- Students generally appreciated the innovative teaching mode proposed, as visible in the chart in Figure 7.1, representing the answers to the closed-ended questions posed in the questionnaire (The questions are retrievable in Section 6.3). Nevertheless, students' inertia towards an innovative approach such as that proposed in this project are likely to be strongly influenced by elements that relate to high levels of the scale of didactic codeterminacy, such as those of the school and pedagogy, making it difficult for the individual teacher to act

to remove these barriers. It is important in the conception phase and in the a-priori analysis to consider this aspect.

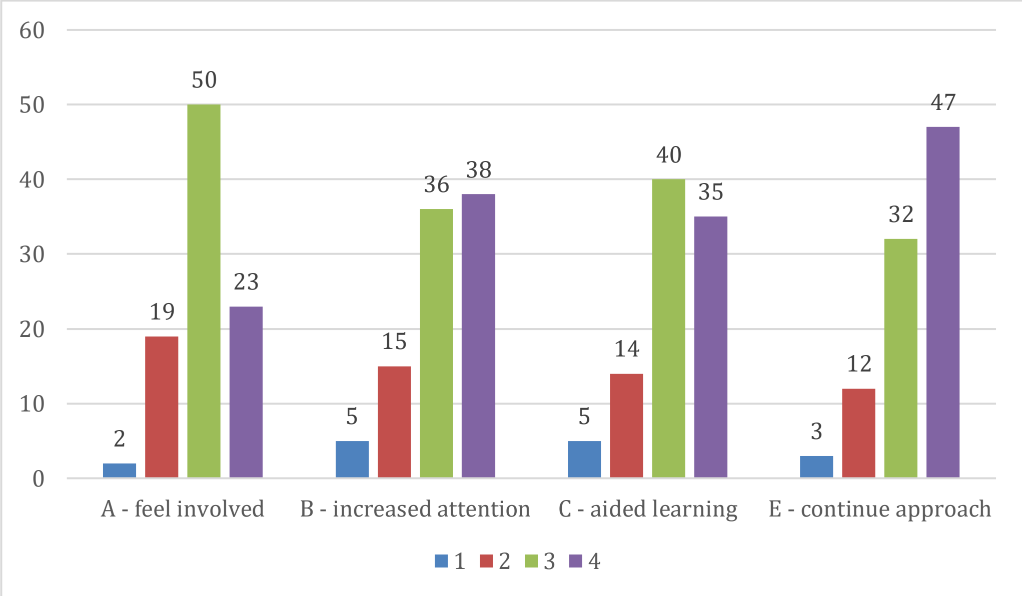


Figure 7.1. Answers given by students to closed-ended questions in the questionnaire

7.2 Discussion of the results

I will now present some reflections on the obtained results, focusing on the potential impact of this research project both on a didactical level and on a theoretical and methodological level.

7.2.1 Didactical implications of this research

The initial part of this thesis (Chapters 1, 2, and 3) primarily delved into a detailed analysis of students' understanding of eigentheory and the associated challenges. While these findings hold significance for didactic purposes, they may not directly inform the teaching of eigentheory in university mathematics courses, particularly those catering to STEM degree programs with large class sizes and traditional lecture formats. Recognizing this, the project incorporated a design research component aimed at providing practical solutions to common didactic issues. Chapters 4 and 5, in particular, explore the constraints encountered in implementing teaching activities as intended, alongside conditions that facilitated their realization. The integration of the technological tool Padlet emerged as a key mediator between students' active group work and classroom discussions led by the course instructor.

Additionally, in Piroi and Cattabriga (to appear), the authors (the second author is the teacher of the course involved in the research) have presented a description of the teaching sequence focusing on its didactical aspects and extendibility to different linear algebra contents and different STEM courses. In particular, they present how the specific features of Padlet (including its ease of use) supported the integration of small group activities during the lessons related to the teaching sequence on eigentheory. I have not included this article in this dissertation because, also due to the characteristics of the journal in which it was published[10], it is more didactic and less research-oriented.

It is noteworthy that the teaching method proposed in this project is not only applicable to other linear algebra topics but also easily extends to other STEM subjects. Indeed, this approach facilitates the integration of inquiry activities into university classes, which often adhere to a transmissive pedagogical model due to various institutional constraints. Moreover, it allows for the sharing of not only written natural language text but also various semiotic resources, such as mathematical symbols and drawings, which are essential for scientific subjects.

7.2.2 Changing the unit of analysis: from the single student to the whole course. And beyond?

As stated by Barquero et al. (Barquero et al., 2019), when a teaching problem becomes the subject of research, "the way in which it is interpreted, the type of entities that are considered, and the empirical domain that is considered as the minimal unit of analysis can vary significantly depending on the research framework chosen" (p. 315).

[10] The Journal in question is PRIMUS (Problems, Resources, and Issues in Mathematics Undergraduate Studies), whose focus is on pedagogical initiatives in college-level mathematics, targeted for a readership comprised primarily of practitioners.

This research project originated from a pedagogical problem, namely how to integrate geometric representations of eigenvectors and eigenvalues into a linear algebra course for engineers. This problem then became the subject of the research. In this process, the research problem was approached from different angles. These different points of view are reflected in the unit of analysis chosen and consequently in the theoretical framework chosen for the analysis.

Indeed, the research began (Chapter 1) with the analysis of the semiotic resources used in the discourse of a single student, Anna, as she attempted to recall concepts from the linear algebra course she had attended the previous year. The unit of analysis was Anna's actions and the signs she produced and how they evolved over time during the interview she was involved in.

In Chapters 2 and 3, the unit of analysis was extended to groups of three students. I analyzed the students' actions, productions and communications during a group activity aimed at reconstructing and giving meaning to the concepts of eigenvectors and eigenvalues, a few days after these concepts had been introduced in class by the teacher. Again, the analysis was fine-grained and limited to a very small number of students, but this time it included their interaction.

Then, when ATD was included in the overarching theoretical framework, with the aim of also considering the institutional aspects of teaching and learning eigentheory, the unit of analysis was extended to the whole class. In fact, the focus of analysis became the praxeologies related to eigentheory that were encountered by all students in the course. On the "time level", the analysis was shifted back again. In this analysis we were interested in the students' first encounter with the concepts in question.

In fact, the lens of analysis was not limited to the specific linear algebra course. By adopting the ATD constructs as ecological analysis and the scale of levels of didactic codeterminacy, the analysis has involved, even if not so explicitly, the whole degree program setting and curriculum, the Institution of the university and the pedagogical paradigm prevailing in it.

Figure 7.2 schematize the different temporal and unit-of-analysis-related levels addressed in the different phases.

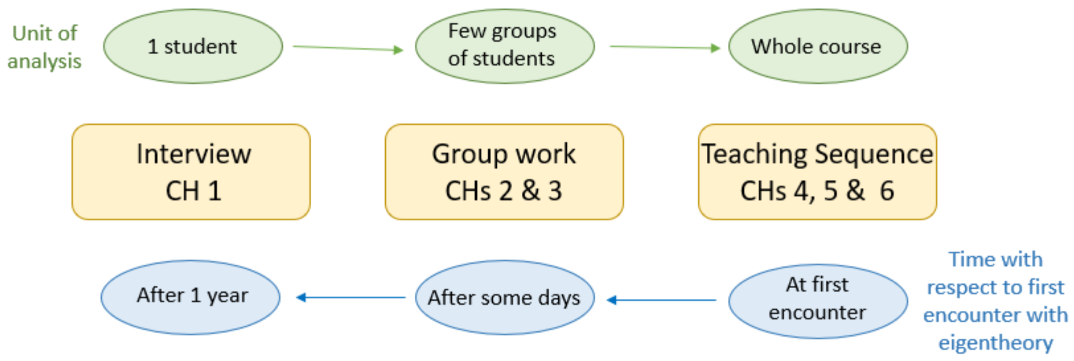


Figure 7.2. Sequence of chapters with respect to unit of analysis and time

The way in which the different units of analysis are considered at different stages of Didactic Engineering makes them far from incommensurable. In fact, as shown in Chapter 5, the coordination between ATD and SB theory (with their respective different units of analysis) is made possible by situating the analyses with the different lenses at different stages of DE. In particular, the analyses carried out with a limited unit of analysis (individual students or small groups of students) are part of the preliminary analysis (as shown in Figure 7.3) and help to guide the design phase and the a priori analysis, as shown in Chapter 4. This type of analysis further informs the didactic analysis, extending the results already available in the literature, but also making it possible to deepen the didactic aspects of the specific research question at hand.

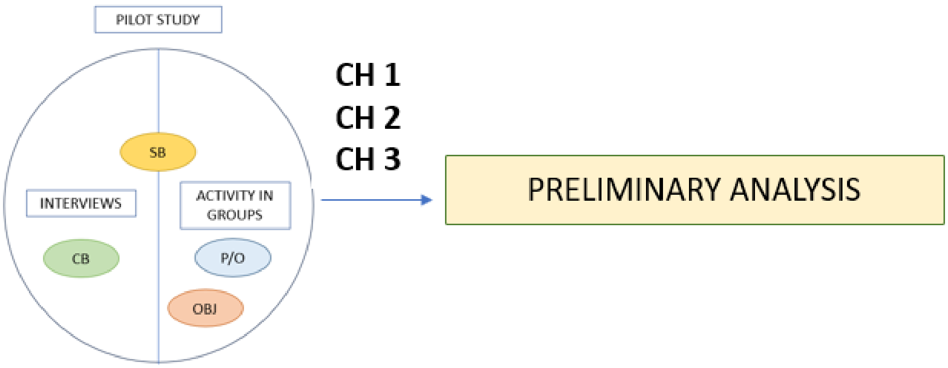


Figure 7.3. Scheme representing the pilot study as part of the preliminary analysis

This aspect has both didactic and methodological/theoretical implications. On the one hand, it shows how fine-grained analysis can inform the design of tasks and didactic sequences (as shown in Chapter 4). On the other hand, it suggests a fruitful strategy for coordinating theoretical lenses that differ in the unit of analysis (as explained in Chapter 5).

7.2.3 Transforming research limits into research results

The pilot study was conducted within a "fabricated" context, particularly evident in the small-group activity described in its second part. While yielding interesting results, it was acknowledged that replicating this activity exactly in a real university class context would be challenging. Initially viewed as a research limitation, these constraints took on the status of research findings when approached from a new institutional perspective. Analyzing these obstacles through an ecological approach allowed them to be recognized as constraints to implementing the intended activities. In addition to promoting the need to individualize favorable conditions, such as the use of the technological tool Padlet, the recognition of current and unchangeable constraints also informed the adaptation of the activities proposed in the pilot study to the real classroom context. Furthermore, these constraints were carefully described and situated within the scale of levels of didactic codeterminacy.

In my opinion, this as an important theoretical and methodological outcome of this research project. Coordinating theoretical frameworks that focus on different units of analysis can have also this positive effect. One fine-grained lens of analysis (as the Semiotic Bundle, together with the Conceptual Blending Theory, the Theory of Objectification and Sfard's theory on process/object duality in this case) used on a limited unit of analysis can provide interesting didactical and epistemological results that can inform and guide the design of a teaching sequence to be implemented in the real institutional context. While different limitations may appear and force to reshape the implementation accordingly, identifying these limits as constraints, to be associated with the conditions actually supporting the implementation, can enrich the research outcomes. Indeed, by showing how they impede the realization of the intended teaching sequence, possibly designed on the basis of a carefully crafted reference epistemological model, they will become visible and can be clearly positioned in the scale of levels of didactic codeterminacy. In this way usually unquestioned elements, as the organization of schools and curricula, or the prevailing pedagogical *paradigm of visiting works*[11](Chevallard, 2015), will be highlighted prompting their critical questioning and analysis.

7.2.4 Research limits that are inherently limits

Besides the previous considerations, there are some limitations in this research project that must be considered as such.

Firstly, both the results of the pilot study and of the implementation and analysis of the teaching sequence, rely on single case studies, which inherently make them context-dependent. This implies that results may vary when considering linear algebra courses offered in different degree programs or countries.

Additionally, we have considered the course taught by a single teacher (even if in different years). The obtained outcomes, both in the pilot and in the second phase, are then strongly dependent on her teaching style. It would have been surely interesting

[11] Or "Paradigm of monumentalism". See Section 6.2.1.

to analyze the classes on eigentheory she had taught the years preceding the research experiment, focusing on the semiotic resources she had used in the teaching process. This is surely an area where this research is lacking.

Another limitation consists in the fact that a lot of emphasis has been put on the preliminary and a priori analysis of the implemented teaching sequence. On the other side, the a posteriori analysis has focused mainly on praxeologies and students' attitude, while I have neglected to a large extent the a posteriori analysis of the semiotic resources activated by students during the small-group activities in the teaching sequence. According to the Didactical Engineering methodology (Artigue, 2014), a crucial feature of this design research methodology is that its validation is internal and achieved through a comparison between the a priori and a posteriori analyses. The limited scope of the a posteriori analysis in this case significantly diminishes the validation capacity of the project, particularly concerning semiotic aspects.

The fact that the a-priori analysis with the ATD has been conducted a-posteriori clearly represents also a critical issue in this study. As explained in Chapter 0, the idea of using ATD appeared in a second stage of the project. For this reason, making the preliminary and a-priori analyses really a-priori was impossible.

However, conducting these analyses using the praxeological model proved to be particularly helpful, even though it was done at an unsuitable stage of the project. Despite this, we decided to proceed with it and to adhere to the actual sequence of analysis in presenting the results.

7.3 Future directions

This research project, multifaceted in its nature, opens the road to many different directions of research.

Firstly, it would be surely interesting to repropose the same teaching sequence in a different course, taught by a different teacher, to deepen the issue of its replicability.

Extending the research to other topics, rather than just eigentheory, is also a research path to follow. Initially, it was our intention to do this in the year after implementation. Unfortunately, the course teacher was assigned to another course, so this could not be done, as she felt the need to get used to the new course with the new syllabus (a linear algebra course for the physics degree) before starting a new teaching experiment. However, as she appreciated the way the teaching sequence was carried out during the course involved in this research, she confirmed that she would use the same mode (using Padlet for supporting active participation and small-groups work), although less structurally, the following year with the new course.

Furthermore, delving into the transition from high school to university could serve as an intriguing follow-up to this thesis. Exploring how certain concepts typically covered in a linear algebra course could be introduced in a manner that supports this transition could be particularly insightful. For instance, regarding eigentheory, this could be done when introducing geometric transformations in the plane or space, especially when addressing fixed points, lines, or planes.

As previously discussed, the design of the instructional sequence was significantly constrained by institutional limitations beyond our control, operating at a higher level of didactic codeterminacy. Explicit constraints included the large class size, inadequate classroom settings for group activities, and a densely packed curriculum, all of which hindered the implementation of a genuine inquiry-based activity. In response, we developed a "guided path" toward exploring eigentheory, stemming from a question posed to students: "Can I find a diagonal matrix similar to a given matrix? How? Is it always possible?" However, this question was limited by a broader constraint not readily apparent, namely its disconnection from the field of interest of the mechanical engineering students involved.

This disconnect reflects a norm within institutional contexts heavily influenced by the phenomenon of *applicationism* (Barquero et al., 2013), characterized by the standardization of first year undergraduate courses and the associated lack of a modelling perspective about mathematics. Researchers involved in the study of mathematical practices in other university disciplines have recognized and confirmed that students, and usually also instructors, often lack detailed knowledge of discipline-related mathematical practices and their rationales (Hochmuth & Peters, 2022; Winslow et al., 2018). From an epistemological standpoint, the failure to utilize mechanics as a context for introducing eigenvectors in a mechanical engineering course represents a missed opportunity. Considering the historical development of eigentheory, rooted in mechanics, starting from a mechanical problem to introduce eigentheory would seem more natural and meaningful for students in this specific course. Even within the realm of "non-applied" mathematical theory, rephrasing the initial question in terms of finding vectors independent of chosen coordinate systems would be more relevant and meaningful than focusing solely on algebraic representations of similar matrices. While we acknowledge that the organization of the school and degree program lies beyond our scope, it would be worthwhile to explore designing a pathway, possibly in collaboration with the mechanics course instructor, that begins with mechanical properties where the need for considering eigenvectors arises.

As elaborated in sections 8.2.2 and 8.2.3, this thesis offers not only empirical findings but also a methodological and theoretical contribution. Specifically, it proposes a strategy for coordinating – in the sense of Networking of Theory (Prediger et al., 2008) - a theoretical framework that emphasizes cognitive aspects at a micro-level of analysis with another framework focused on institutional aspects at a broader unit of analysis. Beyond the scope of eigentheory and linear algebra, I foresee another avenue for research emerging from this work: the application and refinement of this coordinated lens in diverse domains or across various educational institutions at different school levels.

7.4 Conclusions

In recent decades, there has been a growing consensus among researchers and educators that teaching also at university level should extend beyond simply transmitting knowledge, and instead, equip students with the tools to investigate real-world phenomena and utilize mathematics as a fundamental modeling tool (i.e. Jaworski et al., 2021). This underscores the need to shift away from prevailing pedagogical models that primarily emphasize the transmission of established knowledge. Instead, new models for teaching and learning mathematics are envisioned, where responsibilities are redistributed among teachers and students (Barquero et al., 2019), and learning is driven by action and exploration. Yves Chevallard (2015) refers to this transition as moving from the predominant "visiting works" paradigm to the more dynamic "questioning the world" paradigm.

However, implementing such a shift poses challenges due to institutional constraints specific of the university level, such as large class sizes and rigid scheduling, which hinder the implementation of active inquiry activities. Therefore, there is a need for innovative teaching practices that bridge the gap between pure transmission of knowledge and inquiry.

The teaching sequence on eigentheory presented in this work represents a step in this direction, creating conditions conducive to integrating moments of active student participation in traditionally lecture-based classes. By drawing on epistemological and fine-grained didactical analyses focusing on the role of semiotic resources in students' understanding of eigenvectors and eigenvalues, this study not only delves deeper into the roots of students' difficulties with eigenvectors and eigenvalues, but also offers a practical solution in the form of a proposal for a teaching sequence on the topic.

CHAPTER 8 (EXTRA) - COLLECTIVE DOCUMENTATIONAL GENESIS FOR TEACHING LINEAR ALGEBRA: TRANSFORMING ONLINE ANIMATED VIDEOS INTO CURRICULUM RESOURCES

This is the authors' original manuscript of an article submitted to a journal.

THE CHAPTER IN THE RESEARCH PROJECT

As stated in Chapter 0, this paper has been included as an Extra Chapter to inform the reader about the design considerations pertaining to the incorporation of YouTube videos as official resources for the Linear Algebra course involved in the research project reported in this thesis.

Abstract

In this paper, we present the design and implementation of a teaching sequence for a linear algebra course within the Mechanical Engineering degree program at the University of Bologna (Italy). This sequence integrates standard lecturing with a digital environment, providing students with the opportunity to access YouTube videos and interact with each other regarding these videos. The sequence aims to structure YouTube videos for dynamic visualisation of linear algebra topics, commonly utilised by students for autonomous study, as course resources, via an online social annotation platform. We describe the different phases of the sequence's design through the lens of the Documentational Approach to Didactics and analyse the choices behind the ongoing changes made by the instructors, regarding student and instructor interactions with the platform in terms of instrumentation and instrumentalization.

Keywords: Documentational genesis, Linear Algebra, Digital Resources, Dynamic Visualization, Asynchronous Discussion

8.1 Introduction and related literature review

The past decade has witnessed a proliferation in the availability of freely accessible online resources not expressly designed for educational use but widely utilised by students in their studies (Kempen & Liebendörfer, 2021; Poshka, 2020). This holds especially true for university students, who have greater independence in selecting the resources they use for studying. In order to study the way university students use and sequence different resources in the study of mathematics, Pepin and Kock (2021), following Pepin and Gueudet (2018), distinguish between (a) *curriculum resources*, that are "all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom" (p. 172); (b) *social resources*, such as the teacher, classmates, tutors, etc.; and (c) *cognitive resources*, that might be, for example, conceptual frames used in professional development courses. Pepin and Kock (2021) add the category of *general resources* that are those resources students choose beyond the ones proposed by the university (p. 308). This last kind of resource seems to gain importance for students at the university level, compared to high school (Pepin & Kock, 2019).

Examining the role of these general resources is increasingly imperative, particularly in the digital age, characterised by the continuous expansion of free access to diverse digital tools and repositories. It is essential not only to explore students' utilisation of this material and its potential impact on learning, but also to consider how educators can effectively incorporate online content or tools originally intended for non-educational purposes into their teaching methodologies, to transform them into curriculum resources.

This necessity has been validated also through our firsthand experience, as evidenced by the findings of a pilot study conducted in preparation of the broader research project within which this study is situated. In such context, we conducted interviews with second-year engineering students who had completed the linear algebra course the previous year. Among various inquiries, we asked about the resources they utilised and preferred for studying and comprehending linear algebra during their first year of university. In response to this question, one student stated: “Every time the teacher suggested videos by this English guy, I was amazed! [...] These videos make it clear! I remember one particular video where he explained the concept of span. I had already discovered the meaning of span on my own before watching the video, but now I can't recall the exact definition. However, I do remember that as he explained it, he made it visible: "Look, a span is this". He illustrated this vector, showing the set of all possible linear combinations... whatever they may be, and he made you see this vector gradually expanding in three dimensions. Sketching it on a blackboard would never make it as clear.” The videos she refers to are videos from the YouTube Channel “3Blue1Brown” (2015), created by the mathematician Grant Sanderson, and specifically pertaining to a playlist called “The Essence of Linear Algebra”. These videos are designed to make visible the geometric intuitions that underlie numerous topics covered in a typical linear algebra course. They utilise a range of innovative visual animations to achieve this goal. The same channel was mentioned by other students in the same series of interviews, and not only as an additional resource; as one other student said: “It is thanks to it that I understood the linear algebra concepts that I still remember”.

YouTube videos are a widely independently used resource by university students (Eichler et al., 2022; Kempen & Liebendörfer, 2021; Kolbe & Wessel, 2023). Various recent studies have assessed and confirmed the usefulness of YouTube videos in students' study (see for example: Lange, 2019; Ploetzner et al., 2021). Gyeltshen and Dorji (2023), for example, specifically explored the integration of YouTube videos in mathematics instruction for sixth-grade students, demonstrating their effectiveness in enhancing student performance. Comparing the outcomes of an experimental group, which utilised educational videos, with those of a control group, they revealed that the experimental group exhibited superior performance in the annual examination. Notably, the use of videos was associated with increased motivation among participants in this group. Undoubtedly, further investigation into the effectiveness of various types of mathematical videos available on YouTube is warranted. While existing efforts in this realm predominantly scrutinise explanatory videos (Dorji, 2023) - those that emulate a lecture on a blackboard -, to our knowledge, no research has been conducted on videos presenting mathematical content in a different style. Specifically, there is a dearth of studies on videos that prioritise dynamic visualisation of concepts, akin to those found on the 3Blue1Brown (3B1B) channel.

In this work, our aim is not to study the effectiveness of videos in students' learning, but rather to address the previously identified research gap by investigating how to incorporate and integrate these general resources into teaching practices. This aligns with a broader research necessity, which involves investigating how university instructors structure the resources they utilise in their teaching. This becomes particularly pertinent due to the extensive amount of material mathematics educators are required to cover in early university-level courses (Gueudet et al., 2014). This aspect prompted the redesign of a linear algebra course - the same course taken the previous year by the interviewed students in the pilot study - by integrating videos from the 3Blue1Brown channel as 'official' course resources, and thus transforming them into curriculum resources.

8.1.1 Teaching and learning linear algebra

The issue of how to integrate videos for the dynamic visualisation of concepts into the teaching of linear algebra is particularly interesting. Indeed, various research studies in this field have highlighted the positive impact of dynamic representations on students' learning outcomes in linear algebra. The question of whether and to what extent the use of geometric and visual representations can enhance students' understanding of linear algebra notions has been widely discussed in this branch of research (Geudet-Chartier, 2003). Integrating geometry in the teaching of linear algebra cannot be taken up without a deep reflection on how to do it. Gueudet-Chartier (2003) and Harel (2019) report that researchers that have studied the use of geometry for teaching linear algebra have proved both its usefulness in helping students and the risk that it generates specific difficulties. As Gueudet-Chartier (2003) states, "the concreteness that seems to lack in linear algebra could be more efficiently provided by the use of drawings" (p. 500). At the same time, it often happens that, without proper guidance by the teacher, if a result is given and associated with a geometric example, students might not be able to apply the same result in other contexts such as the vector space of polynomials or of functions.

The emergence of advanced digital tools has prompted a fresh call for integrating technology, particularly dynamic geometry software, into the teaching of linear algebra. For example, Gol Tabaghi and Sinclair (2013) demonstrated the beneficial effects of using Sketchpad to explore the concepts of eigenvalues and eigenvectors. Romero Félix (2016), as well as Turgut and colleagues (2022), have found that emphasising visualisation of linear transformations through the use of GeoGebra can enhance students' understanding of them. In particular, Turgut and colleagues (2022) claim that dynamic geometry software allows for the design of digital contexts where students can manipulate objects in R^2 and R^3, and concurrently visualise the results of these manipulations both algebraically and geometrically. This is fundamental in avoiding the risks associated with using only geometric representations (Gueudet-Chartier, 2003). Indeed, it is commonly acknowledged by scholars in this field (see, for example, Sierpinska, 2000; Dogan, 2018) that establishing and strengthening relations between different representations helps students deal with the level of abstraction that often causes difficulties in learning linear algebra.

The didactical value of letting students engage with dynamic visualisations of linear algebra concepts is confirmed by Dogan (2018). He found that it is mainly the aspect of dynamism in visualisations that enhances students' understanding of linear algebra concepts. In his study, he found that students who were shown dynamic visualisations as part of instruction tended to integrate geometry into their abstract reasoning, whereas students shown static images tended to keep relying on purely algebraic and computational reasoning. Despite a growing number of studies focusing on the positive effects of integrating interactive dynamic visualisations, such as those provided by dynamic geometry software, the use of "passive" dynamic animations of linear algebra concepts, such as those present in the 3B1B channel's videos, has never been investigated. Such an investigation seems particularly interesting since it allows us to address the need for research on how to integrate online general resources into class practice, as a preliminary step in investigating the effectiveness of providing students with dynamic geometric representations of linear algebra concepts.

8.2 Theoretical background: the Documentational Approach to the Didactics

In the previous section we have presented a distinction between different kinds of resources used for teaching and learning mathematics. The role of resources in mathematics education research has received increasing interest over the years, given their growing availability, especially of digital ones. The interaction of teachers and students with a vastness of resources requires their selection and organisation, and this process can have an important effect on learning. For this reason different theoretical approaches have been developed in order to study this phenomenon. In particular, the Documentational Approach to Didactics (DAD) has been introduced by Gueudet and Trouche (2009) and later developed further in joint work with Pepin (Gueudet et al., 2012), in order to study teachers' work with resources including selecting, modifying and creating new resources, in-class and out-of-class (Trouche et al., 2020). Following Adler (2000, p. 207), they believe that "it is possible to think of a resource as the verb re-source, to source again or differently", thus it implies the idea of sourcing something anew or in a different manner. We decided to focus on resources since a resource is never isolated, it belongs to a set of resources (Gueudet & Trouche, 2009). This highlights the variety of artefacts the teacher has at disposal when teaching and the related choices s/he makes.

When a teacher adopts a set of resources to address a teaching situation, s/he creates a *document*, which can be regarded as a hybrid entity composed of the set of resources and a *scheme of usage* of them. According to Vergnaud (2009), whose theory on conceptual fields also helped lay the theoretical foundation for DAD, a scheme is a way somebody organises his/her activity for facing not only a situation (e.g. 'teaching how to compute the area of a trapezium in a given grade 9 class'), but a class of situations. This is critical because different teachers, even when using the same resource, may adopt different usage patterns and thus generate different documents. The process through which the document is developed, including the teacher learning involved, is called *documentational genesis.*

Considering a set of resources, or a document, requires taking into account three intertwined components (Gueudet & Trouche, 2009): (1) the material component, such as paper, computer, USB key, etc; (2) the mathematical content component, that is notions involved, mathematical tasks and techniques; and (3) the didactical component, that includes organisational elements, ranging from mapping over the whole year to planning a single one-hour session.

It is useful to introduce other terms from this theoretical framework: DAD calls "*resource system*" the set made of all the resources used by the teacher, and "*document system*" the system of documents developed by him/her.

DAD has been inspired by different theoretical stances on mathematics education and, regarding specifically the field of technology use, the instrumental approach (Rabardel & Bourmaud, 2003) has had significant influence on it. As a consequence, DAD takes into consideration the dialectical nature of teachers' interaction with resources, combining the processes of *instrumentation* and *instrumentalization* (see Figure 8.1). Indeed, not only the teacher's knowledge guides the selection, use and modification of resources (in what is called the *instrumentalization* process), but the affordances inherent in the resource itself have an impact on teachers' practices (i.e. the *instrumentation* process). These processes do not occur in a single step, but rather are time-dilated, encompassing in most cases *design*, *re-design*, and *design-in-use* practices, the last one occurring when teachers change a document in progress and according to their instructional needs or to unforeseen students' usage.

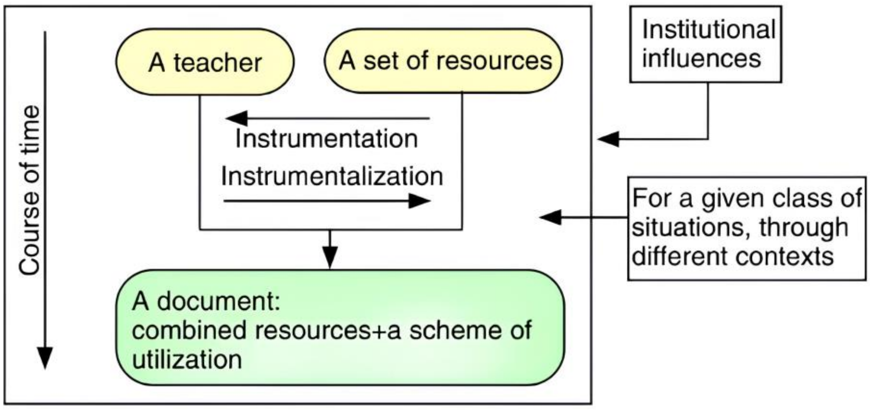


Figure 8.1. Schematic representation of a documentational genesis (Gueudet & Trouche, 2009, p. 206)

8.3 Research questions and research methodology

Based on the framework of the documentational approach to the didactics, we focus on the teacher's documentation work, that is looking for resources, selecting and designing tasks, planning their delivery and the associated time management (Gueudet & Trouche, 2009) in the initial design and in the design-in-use of a document system. In this paper, we would like to address the two following research questions: (RQ1) How can online animated videos aimed at the dynamic visualisation of linear algebra concepts, usually independently consulted by university students as general resources, be organised in a document system in order to make them become curriculum resources? (RQ2) How can the teacher develop his/her schemes of usage of online animated videos aimed at the dynamic visualisation of linear algebra concepts in the design-in-use in order to support students' engagement for the learning of the related mathematical concepts?

To investigate the ways in which general resources can be structured in order to become curriculum resources, we present and discuss a possible design to include online animated videos on linear algebra in a document system with respect to the documentational approach to didactics (Gueudet & Trouche, 2009), that is suitable for the descriptive analysis of the design of teaching devices (Gueudet et al., 2014).

To address the development of the teacher's schemes of usage in the design-in-use, we investigate his/her documentational genesis and the related processes of instrumentalization and instrumentation, with a specific focus on the ongoing changes made within the experiment to the digital environment as the didactical component of the document system.

During the meetings in the design phase, we documented all decisions made and the rationale behind them. We continued this practice in the implementation phase, which we refer to as the design-in-use phase. For the analysis, we scrutinised these notes, identifying various elements of documentational genesis, including the evolution of usage schemes over time, particularly in relation to the processes of instrumentation and instrumentalization. To address RQ1, we will outline the design of the document in question, elucidating the relationships established among the various material components of the resources and the hypothesised schemes of usage intended to transform them into a document. Subsequently, to address RQ2, we will delineate the processes of instrumentation and instrumentalization that guide the design-in-use. Specifically, we will analyse how the actual interaction between students and the set of resources has informed these processes.

8.4 The research context and the design of the experiment

The experiment presented in this paper was designed and implemented during the linear algebra course at the degree in Mechanical Engineering of the University of Bologna (Italy). This is a compulsory course scheduled for students enrolled in the first year. The students enrolled each year are about 220 (UNIBO, 2022), therefore, the students expected to attend the lessons and participate in the experiment were more than 240, composed of all freshmen and some students from previous years in debt of passing the exam. The course, held by the first author in 2022, included three two-hours lessons per week, for 12 weeks in total.

The experiment was designed to integrate some online video resources, generally consulted by university students in their self-study for the course of linear algebra and validated by the teacher, into a document system, so as to make them become curriculum resources. In order to support students in approaching the proposed

YouTube videos as curriculum resources, we decided to share the videos through the online social annotation platform Perusall[12] (Miller et al., 2018). The platform Perusall was chosen not only because it was particularly suitable for sharing videos with students making explicit their schemes of usage, but also to stimulate an asynchronous discussion between participants on the topics tackled in the videos. In fact, participating in discussions has been demonstrated to enhance the learning experience and, by incorporating asynchronicity, students who require additional time can actively contribute to the discussion, thereby fostering their participation (Andresen, 2009). Perusall allows the teacher to create a course sharing resources of different kinds, including videos, with students and to request different kinds of interactions with the resources, like commenting, commenting on other students' comments, or solving some tasks. The users of the course can have either a *student* profile or an *instructor* one, with different permissions. The instructor can create courses, organise the resources and act as a moderator in discussions between students. Moreover, Perusall collects and shares with the instructors *analytics*, that is quantitative data on students' activity: *viewing time* (how much time the student had the video open), *active engagement time* (how much time the student was watching the video actively, i.e., with some sort of mouse movement or keypress at least once every 2 minutes), *annotations posted* (how many comments, questions or answers the student submitted), *likes given and received* on annotations (how many times the student upvoted[13] another student's question or comment and how many times the student's annotations were upvoted by another student). Perusall also includes a system of automatic assessment based on multiple factors: the instructor can choose whether to select one of the presets recommended by the platform or indicate the weight of each single factor personally.

[12] https://www.perusall.com/

[13] This can be done by clicking the button indicating they have the same question, or clicking the button indicating that a particular comment was helpful for their understanding.

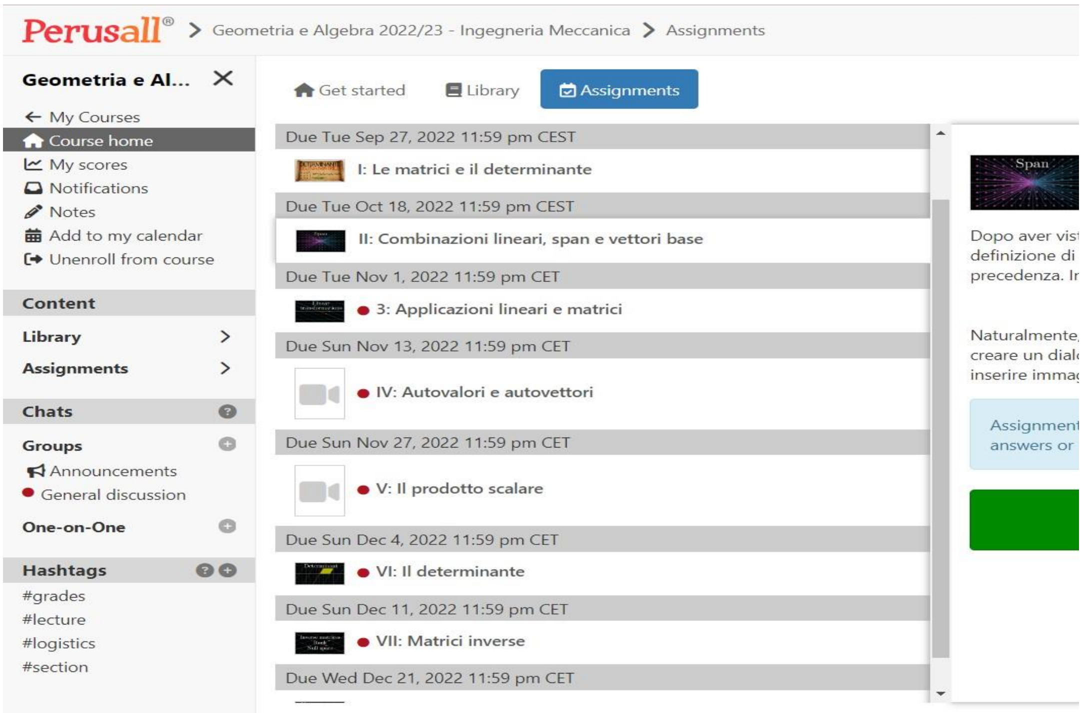


Figure 8.2. Screenshot of the student's view of the course on Perusall with, on the left, the list of the assignments

In our design, we created a course within Perusall (see Figure 8.2), with the four authors as instructors. The course consists of eight assignments as asynchronous activities each relating to a video that presents a specific topic of linear algebra: I. matrices and determinant, II. linear combinations span and basis vectors, III. linear transformation and matrices, IV. eigenvalues and eigenvectors, V. the dot product, VI. the determinant, VII. inverse matrices, VIII. other vector spaces. All the videos, with the exception of the first one, belong to the playlist "Essence of Linear Algebra" of the YouTube channel "3Blue1Brown" (2016). These videos aim at animating the geometric intuitions underlying many of the topics taught in a standard linear algebra course. In the videos the mathematical objects move on a completely black field, with a white grid, and a voice-over that tells the mathematical concepts the animation conveys (see Figure 8.3). Although there is no explicit epistemic control over the construction of such videos, we observe that the overarching principle of animation inherent in them resonates with research findings on the efficacy of dynamic visual representations for conveying concepts in linear algebra. Regarding linear transformations, for instance, Zandieh and colleagues (2012) propose representing them through the idea of morphing. “There must be a clear sense that the beginning entity did not simply move to the new location (ending entity), nor was it replaced by the new output (ending entity), but that there was actually a metamorphosis of the beginning entity into the ending entity” (p. 527). This adheres to what Eisenberg and Dreyfus (1994) defined as the *dynamic view* of a function transformation, that consists in viewing “it as a mapping which is moving every point in the plane to a new location” (p. 58). Oktac (2018) proposes a didactical principle for representing transformations, suggesting the use of a grid and illustrating the movement of vector tips as they 'morph' from the initial vector to the image. This animation design closely resembles the approach adopted by 3Blue1Brown in his videos when representing linear transformations in R^2.

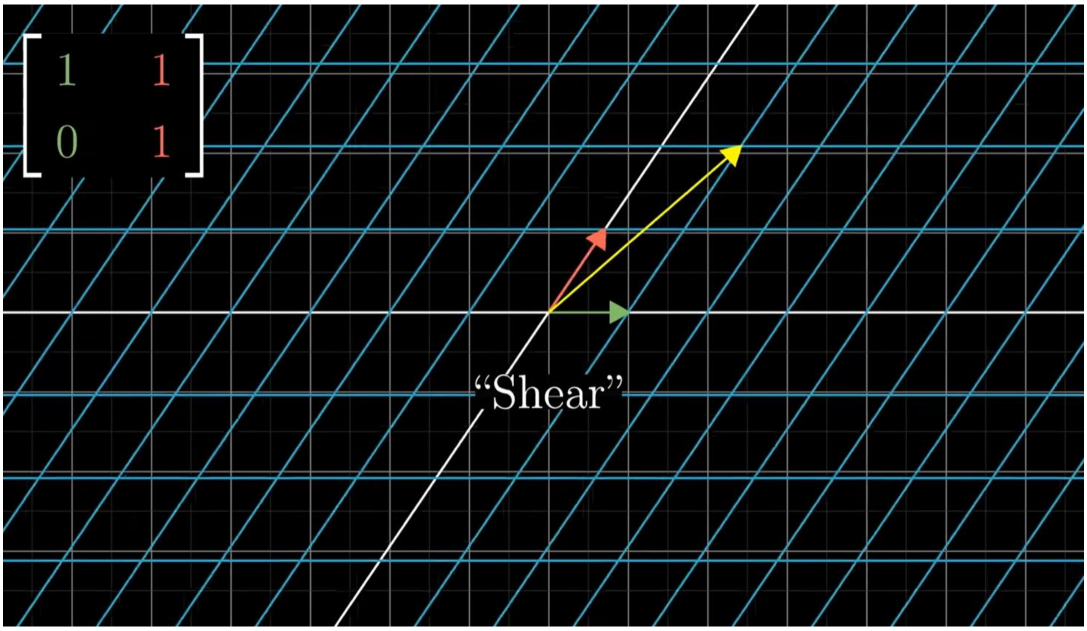


Figure 8.3. Screenshot from the video “linear transformations and matrices”, Chapter 2 of 3Blue1Brown (2016)

For the first assignment, we used an italian video of the YouTuber Elia Bombardelli (Bombardelli, 2017) for a twofold reason: the video on the determinant of 3Blue1Brown has linear transformations as prerequisite, while the determinant was the first topic of the course; moreover, the first assignment was designed to be a trial one in order to let the students make friends with the digital environment Perusall, so we decided to test them with a video in Italian that has a more traditional structure, in which the creator explains the topic by writing on a virtual blackboard without the use of any geometric animation.

We scheduled the first five assignments so that students were required to complete them before the lecture concerning the topics presented in the corresponding video[14]. The last three assignments were instead conceived as refresher activities: the topics associated with them were covered in the first half of the course and the assignments were given in the second half of the course to give students the opportunity to review the fundamental notions and concepts in view of the exam sessions, also in the light of new notions learned later on in the course.

We decided to present the asynchronous activities to students during the first lecture of the course, explaining to them how to register for the course on Perusall, and how to watch, comment, pose questions, answer or react to comments. In order to do so, we prepared and showed during the lecture a sample assignment consisting of the video *Vectors,* the first introductory video of the playlist "Essence of Linear Algebra" (3Blue1Brown, 2016).

In the *assignment description box* on Perusall, in which the instructor may add instructions for students regarding the assignment, it was initially decided not to leave any type of indication, waiting to know the spontaneous reactions and participation of the students in the first weeks of the experimentation. Furthermore, it was decided that the instructors would have participated as little as possible in the conversations, in order to be able to initially observe the dynamics that would have naturally created between the students. Once the behaviour of the students is known, the decisions regarding the descriptions within the assignments and how the instructors interact would be modified in the most appropriate ways.

Since, as recalled above, potentially more than two hundred students could participate in the asynchronous course, we decided to divide them randomly into groups of up to 25 members so that comments and questions posted by a student could

[14] To do this, the assignment deadlines were rescheduled during the course if the timing initially foreseen was not correct according to the evolution of the lessons.

be seen just by the members of its groups and by instructors. During the first week of the experimentation more or less 210 students registered for the course on Perusall: they were divided automatically by Perusall into 12 groups of 17 or 18 members each.

8.5 Design-in-use: instrumentation and instrumentalization

In this section, we present the process of documentational genesis followed for the design-in-use of the digital annotation platform Perusall for integrating the YouTube videos for the dynamic visualisation of linear algebra concepts as curriculum resources. We focus on the analysis of the instrumentation and instrumentalization processes accomplished within the design-in-use. We follow a chronological order in presenting the documentational genesis but focusing on different variables one at a time. In particular, we refer to the design-in-use related to:

- the groups' composition, that is modifying the groups' composition according to students' level of participation in the previous assignments considered with respect to the number of their written comments in Perusall;
- the assignment of tasks, that is adding tasks related to the videos that vary depending on the content of the assignment and on the level of participation of the students;
- the instructors' interventions, that is changing the way moderators interact with students on the platform.

We oriented ourselves towards actions that could contribute to fostering student engagement and exchange, considering it important to encourage dialogue among all students while taking into account that participation in activities proposed on Perusall was voluntary. Therefore, we aimed to promote discussion among students who were spontaneously interested in participating in the activity, without implementing actions to compel the rest of the students to participate (at least in this first experimentation).

In the next subsections we describe in detail these actions and analyse the reasons behind each of the related choices and we describe the effect that our actions had in terms of instrumentation/instrumentalization. We focus only on the first five assignments since, as mentioned before, the last three were designed for a different aim, that is they were conceived as refresher activities.

The initial design is guided by an instrumentalization process. In fact, the platform Perusall was chosen because it offers the possibility to interact easily with videos and allows students to engage in discussions. Moreover, the teacher's goal of fostering spontaneous interactions among students guides the decision to leave the assignment description box blank, while choosing to participate as a ghost in the wings (Mazzolini & Maddison, 2003), leads to a specific instructor-platform interaction, namely, reading students' contributions without answering them.

8.5.1 Groups’ composition

According to our initial design, Perusall, during the sign-in phase, randomly assigned students to twelve groups each consisting of 17 or 18 members. After the second assignment we decided to regroup students according to their engagement in the platform. The engagement on the platform was evaluated on the basis of the number of comments redacted by students and by the amount of active time spent on the assignments. The interaction of students on the platform informs instructors about the need for a new group management approach, that consists in the regrouping (instrumentation). As a consequence, the instructors decide how to reorganise the groups and intervene to actually restructure them on the platform (instrumentalization). Figure 8.4 shows the succession of these processes, where the blue arrow is the instrumentation process and the green one the instrumentalization. After the regrouping, the groups remained the same until the end of the experimentation.

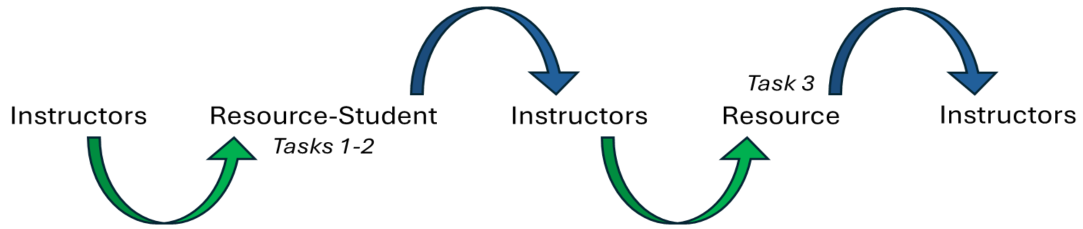


Figure 8.4. Instrumentation/instrumentalization process for groups' composition

The most active students, those who had added comments or reactions, as well as the users who had spent active time on both the first two assignments, in the regrouping were placed in groups 1 and 2. Students who had spent active minutes on only one of the first two assignments were divided into groups 3-9. Finally, the last three groups (10-12) consisted of the remaining students (low or no interaction with Perusall). This choice was made because in the initial weeks of monitoring, only a few students in each group were contributing to discussions. In some cases, certain users did not have any peers they could interact with, making it impossible to have a dialogue. For this reason, it was decided to modify the composition of the groups by bringing together students who had shown similar engagement and behaviour on the platform in the early weeks of the experiment. We also attempted to assign within the same group students who were originally together.

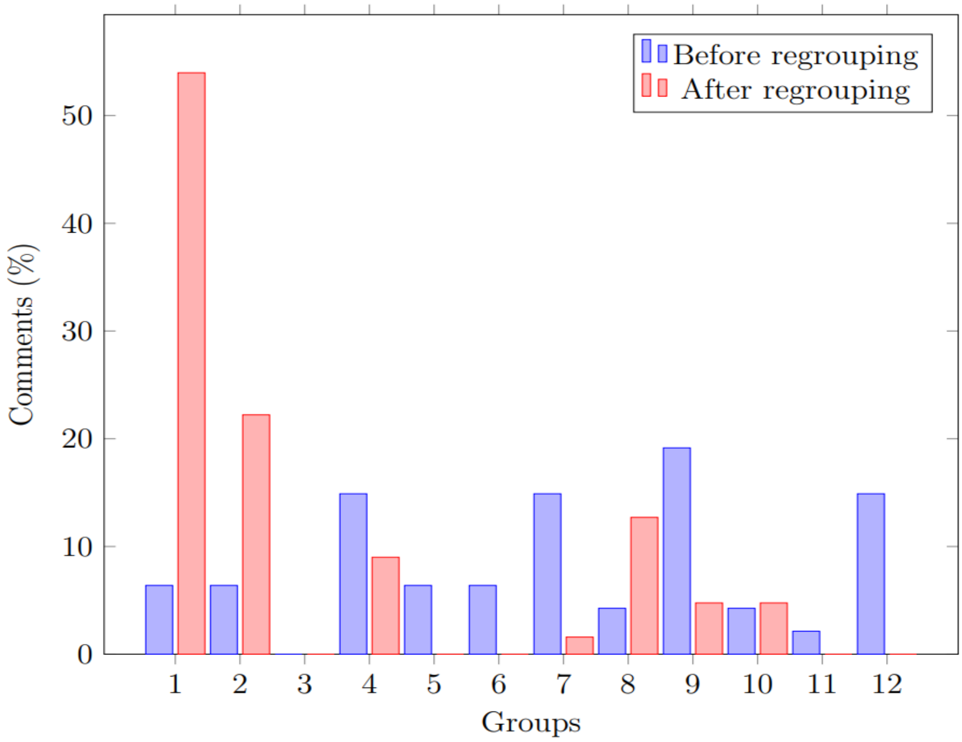


Figure 8.5. Distribution of comments among groups before and after regrouping students

Once the new groups were formed, a predictable change in the distribution of messages was observed. Prior to the change, the groups had posted a similar number of comments, but starting from the third assignment, groups 1 and 2 had a clear predominance over the others. Figure 8.5 shows the data regarding the groups to which students who made comments in the assignments belonged. Data are expressed as a percentage of the total contributions made in the reference periods. Clearly groups 3-12 drastically reduced their participation in discussions after the change. In particular, groups 5, 6, 11, and 12 completely ceased their activity on the platform.

8.5.2 Tasks added to the assignments

The first assignment, related to a video of the YouTuber Elia Bombardelli on matrices and determinants, was given to the students without any task nor specific indication on the way to contribute in the discussion except for the presentation of the activity done by the teacher during classes, that was to watch the video and feel free to comment or ask questions. This instrumentalization process (as discussed before, instructors leave blank the assignment description box) has an impact both on the platform itself (the box contains no writing) and on the relationship among students and the platform, since no specific information has been left to students concerning how to interact on Perusall.

Since students interacted less than expected in the first assignment, for the second one, concerning linear combinations, span and vector basis, instructors decided to accompany it by an initial message aimed at guiding the students (instrumentation). This decision had the effect, from the perspective of instrumentalization, of prompting instructors to write a message in the box (relationship instructors-platform) where students were asked to provide examples (relationship instructors-student interaction on the platform). The indication given to students in the description box on Perusall for the second assignment was the following: "After watching the entire video, try to answer the reflection proposed by 3Blue1Brown by comparing the basic definition provided in the last part of the video with the entire digression on span and linear combination done earlier. Furthermore, try to write an example for a basis of the spaces R^2, R^3, and $M_2(R)$".

Between the second and third assignment, it was decided to modify the composition of the groups (see Paragraph 5.1) and to provide differentiated activities for the different groups in the third assignment according to their level of participation in the previous assignments (instrumentation of the relationship student interaction on the platform-instructors). This decision entails, as in previous cases, a subsequent process of instrumentalization, as instructors write the tasks to be inserted into the platform (relationship instructors-platform) providing students with explicit questions to address during their interaction on the platform (relationship instructors-student interaction on the platform).

After watching the video on linear transformations, groups 1 and 2, consisting of the most active students, were asked to comment on the positions of two fictional students facing a linear transformation and the study of its image. The task was structured as a Who-Is-Right task (Tabach & Koichu, 2019) composed by the text of a problem and two given contradicting solutions. This kind of task was chosen for its potential in promoting looking back practices (Koichu et al., 2021), involving a first looking back practice when giving a self-solution to the question “Who is right?” and a second looking back practice when asking “Why?”, suitable for stimulating students’ interaction within the discussion on Perusall since a Who-Is-Right task requires students to engage with a series of why-questions coming from the need to compare the provided and the self-produced solutions. The text of the task was as follows:

In R^2, given the vectors v = (1; 1) and w = (3; 0), let v' and w' be the vectors obtained by applying the linear transformation associated with the matrix A = (-2 4; 1 -2). Is it correct to say that the vectors v' and w' are linearly independent?

LUC and MAR responded as follows:

LUC: Yes, because v and w are linearly independent, and applying the linear transformation does not alter this property.

MAR: No, because v' and w' are aligned.

What do you think? How can we determine who is correct? Discuss among yourselves.

For the remaining groups, instead, a multiple-choice quiz was chosen, focusing on some concepts presented in the aforementioned video. The quiz consisted of three questions:

1. Let O = (0, 0, 0) be a point in R^3, and let f be a linear transformation. What can we say about $f(O)$?

   a) It is not possible to determine the image of O without knowing f.

   b) $f(O) = O$

   c) $f(O)$ = (1, 2, 0)

   d) None of the above.

2. Let e_1, e_2 be the vectors of the canonical basis of R^2. Let f be a function such that $f(e_1)$=(4, 0) and $f(e_2)$=(1, 1). What can we say about the function f?

a) f is a linear transformation.

b) f is not a linear transformation.

c) $f(e_1 + e_2) = (5, 1)$

d) None of the above.

3. Let f be a linear transformation and let v be a vector in the domain. If we define $w=2v$, what can we say about $f(w)$?

a) $f(w) = v$

b) $f(4w) = 8f(v)$

c) $f(w) = 2v$

d) None of the above.

The activities were designed with the aim of being aligned with the characteristics of the students to whom they were proposed: the first two groups were primarily composed of those who had no issues in commenting and engaging with their peers, while members of groups 3-12 had shown a tendency to not contribute in written form, but rather to only view the presented content. In this way, the goal was to involve the students more actively in the modes that seemed most suitable to them. The contributions given by the students on Perusall as answers to the assigned tasks were different between groups 1-2 and groups 3-12. Concerning the latter, no comments were written in Perusall, even if the 30% of students in that groups answered the multiple-choice quiz. On the other hand, students from groups 1 and 2 discussed between each other, refining and elaborating on previous interventions and answering to other students' comments; these kind of comments could be an immediate consequence of the choice of posing a Who-Is-Right task, as students were required to justify their positions.

The interventions within the groups inform instructors on how to modify the design (instrumentation), as task differentiation proves ineffective for groups 3-12 (few students respond to the quiz and no student interacts on the platform) in terms of peer comparison and support, which is one of our educational objectives related to the choice of Perusall. So, starting from the fourth assignment, the activities accompanying the videos returned to being identical for every group, since we decided to adopt a different strategy, concerning the role of the instructors within the platform (see next subsection), to engage students with the assignments. Figure 8.6 shows in a schematic way the instrumentation (in blue) and instrumentalization (in green) processes presented above.

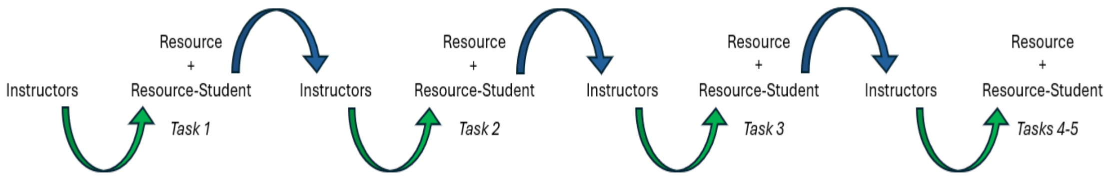


Figure 8.6. Instrumentation/instrumentalization process for the addition of tasks to the assignments

In the assignment related to the study of eigenvalues, it was decided to use the software GeoGebra in constructing the task associated with this topic. Unlike previous cases, only the first part of Sanderson's video on the topic was included in the assignment, not the entire video. This was because the american author devotes the second half to justifying the identity det(A-λI)=0, with a more operational approach that was intended to be addressed only later in the classroom. As Thomas and Stewart (2011) highlight, in dealing with eigentheory, moving straight to symbolic manipulation of algebraic representations and matrices can result in the fact that “the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust” (p. 280). For this reason we proposed to the students only the part of the video in which the “essence” of eigenvectors and eigenvalues was shown. Specifically, in the video, it is shown how each endomorphism tends to change the direction of vectors, while for each endomorphism there can be special vectors, whose image remains in the same direction: such vectors are called eigenvectors. We decided to assign an activity in which students could explore the directionality of vectors and their images with respect to a given linear transformation, thanks to the use of a dynamic geometry software, that resonates with the dynamic approach used in the video adding the interactive feature. There are a number of works in the literature that show how embodied views of eigenvectors can be effectively supported by the use of such software (Gol Tabaghi & Sinclair, 2013; Caglayan, 2015). Our proposed activity comprised a GeoGebra applet (Figure 8.7) in which it was possible to move a fixed vector *u* at the origin and dynamically visualise its corresponding image through the transformation matrix *B* = (0, 1; -2, 3). The matrix was not disclosed to the students, who had to deduce it by observing the image vectors. The text was as follows:

A linear transformation on the plane is represented. Try dragging the red vector "u" and observe the behaviour of its image "Bu". Then, try to answer these two questions, commenting at the end of the video and discussing among yourselves whether you agree or not:

1. Write the matrix B canonically associated with the represented transformation.
2. Does this transformation have eigenvalues and eigenvectors? If yes, what are they?

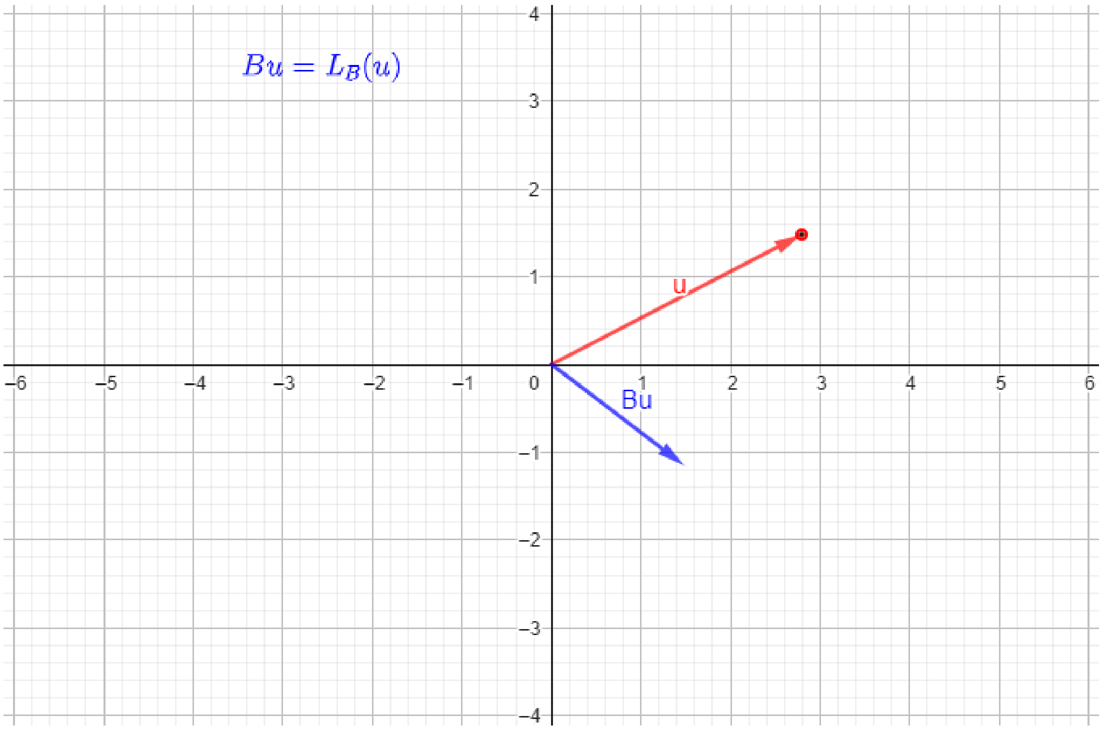


Figure 8.7. Screenshot of the GeoGebra applet proposed in the fourth assignment

8.5.3 Instructors' interventions

In literature, the role of the teacher is recognised as one of the most important components for a successful asynchronous discussion (see, for instance, Andresen, 2009). When facilitating asynchronous discussions in educational contexts, the teacher can interact with students in very different ways and the degree of visibility of the teacher depends also on the purpose of the discussion. According to Mazzolini and Maddison (2003), the role of the teacher in asynchronous discussions can vary from being *sage on the stage*, *guide on the side*, or *ghost in the wings*. Indeed, depending on the purpose of the discussion, the teacher may prefer to lead the discussion, answer most of the questions from the students and be amongst the most frequent contributor to the discussion (sage on the stage), or moderate the discussion maintaining a low profile and encourage students to initiate discussions and answer each other's questions (guide on the side) or be absent from the discussion (ghost in the wings).

During the implementation of the teaching sequence, instructors changed how they interacted and participated in discussions. Initially, they were silent and only contributed to the conversation when necessary. Later, they began to engage students more actively using facilitation techniques (Hew & Cheung, 2008), such as personally inviting students to contribute, tagging students, and asking for their opinions about the conversation they were involved in.

In the initial three assignments, the instructors simply observed the conversations initiated by students and intervened only in cases where the students encountered difficulties in finding solutions to problems or queries arising from watching the videos, following an approach in line with the role of ghost in the wings in order to favour student-student interaction. The instructors' ghost in the wings attitude is reflected in the complete lack of interventions by the instructors (instrumentalization). The low number of comments in the first assignments, compared to the number of students enrolled in Perusall, prompts the moderators (instrumentation) to encourage the discussion by referencing specific students to elicit their response to the points discussed thus far (Hew and Cheung, 2008). This was done by tagging them and asking them to respond and discuss their peers' ideas or solutions (instrumentalization), shifting to a role closer to the guide on the side. As a result of this adjustment, the fourth assignment witnessed an improvement with 25 comments coming from different groups, compared to the previous observations. The moderators randomly selected the students to tag among the ones who have poorly or never intervened within the discussion, asking them to express their opinions or further elaborate on the contributions written by other students. In some instances, this prompted students to post their first comment on the platform. However, the fifth assignment saw a decrease in the number of comments, with only 13 recorded, despite the moderators' continued personal invitations for student engagement. Consequently, it is challenging to determine whether the observed increase between the third and fourth assignments can be attributed solely to the moderators' revised approach or if other factors such as the topic covered or the accompanying task influenced the outcome.

Figure 8.8 shows the succession of the processes related to the documentational genesis in the design-in-use with respect to instructors' interventions, where the blue arrow is the instrumentation process and the green one the instrumentalization.

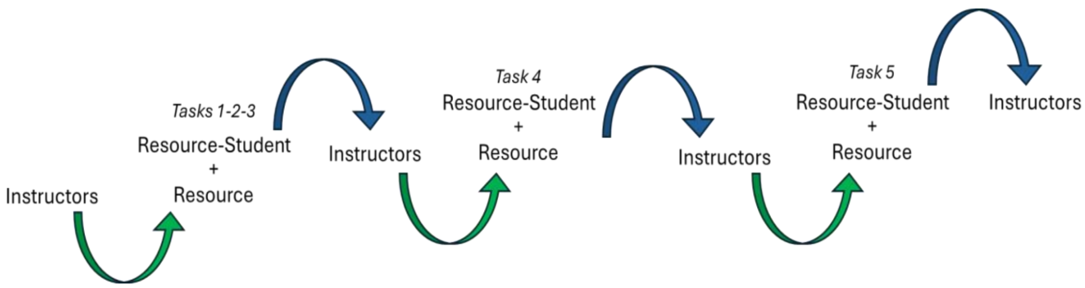


Figure 8.8. Instrumentation/Instrumentalization process of instructors' interventions

8.6 Discussion and conclusions

In this paper we have presented the design and the design-in-use constituting the genesis of a document for the teaching and learning of linear algebra and, specifically, for enhancing the dynamic visualisation of its core notions.

In paragraph 4, we tackled RQ1 regarding the design of a document to transform the video resources as curriculum ones, and we explored the three components of a document (material, mathematics, didactics). With respect to the three components of a document (Gueudet & Trouche, 2009), the document we designed consists of: (1) the material component, that is the social annotation platform Perusall and YouTube videos for the dynamic visualisation of some linear algebra concepts; (2) the mathematical content component, that is basic notions of linear algebra (linear combinations span, basis vectors, linear transformations, matrices, eigenvalues and eigenvectors, the dot product, the determinant, inverse matrices); and (3) the didactical component, that includes the design of the document, also in relation to the work carried out in the classroom. The final document, instead, has seen the addition of GeoGebra applets as material components, the tasks on the basic notions of linear algebra as mathematical content components, and the different actions implemented in the design-in-use as didactical components. In fact, apart from the initial design, we have also discussed the changes the instructors have adopted in the design-in-use in order to support students' engagement with the document. Specifically the teacher and the researchers involved in the design-in-use developed their schemes of usage of the Perusall platform integrated with the videos, with respect to the composition of the students groups, the formulation of the assignments with the addition of specific tasks related to each video, and the types of intervention adopted by the instructors.

In paragraph 5, we addressed RQ2 through the processes of instrumentation/instrumentalization (Gueudet & Trouche, 2009), examining both the teacher and the resource (Perusall), as well as the relationship that develops between students and the resource, which informs the teacher. This led us to view the processes of instrumentation and instrumentalization slightly differently compared to the original one (see Figure 8.1), in which the processes involve a set of resources and a teacher. In our case, the instrumentalization process extends from the instructors to the resource (Perusall), but also affects the relationship that forms between the platform and the students. This relationship is the one informing the teacher in the instrumentation process. A schematic representation of the subjects involved in the instrumentation/instrumentalization processes is reported in Figure 8.9. The "third" arrow, orange in the figure, in addition to the classical instrumentation and instrumentalization ones, should be further studied in future research.

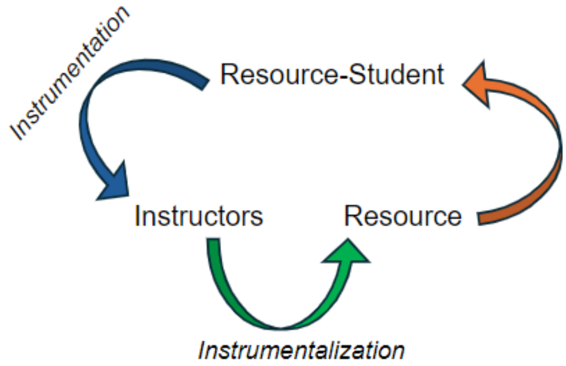


Figure 8.9 Instrumentation/instrumentalization process in our study

Concerning the instrumentation and instrumentalization processes presented and schematized in paragraph 5, it should be noted that the processes are sequentially intertwined and always ending with an instrumentation process that can inform instructors for future re-design.

It should be also noted that the term “resource” is used differently in paragraphs 4 and 5. In paragraph 4, we discuss videos as resources and their integration into a document to become curriculum resources. The decision to introduce these videos as curriculum resources is tied to the introduction and literature review, as we effectively articulate the potential of the specific videos we have chosen and why it makes sense to formalise them into a document. In paragraph 5, we examine how resources, such as Perusall, allow us to understand how the document we have created can evolve by informing the teacher's temporary schemes of usage, leading to a revision of the initial document as soon as there are interactions between the resource and students.

The employment of the platform Perusall in our study allowed students to engage also in asynchronous discussions with their group mates on the videos and tasks proposed within the course. As already mentioned, students' effective engagement plays a delicate role in the implementation of asynchronous discussions since usually a small number of students actively participate (Hew & Cheung, 2012). This is also the case of the experiment presented in the paper since, from the beginning of the course on Perusall, few students wrote comments and interacted compared to the high number of students enrolled in the platform. The regrouping of students and the facilitation techniques (Hew & Cheung, 2008) adopted by the instructors slightly influenced the participation of students, even if in some cases the regrouping activated new interactions between students and addressing specific students within the discussions led them to post their first comment. Although these positive cases, the data available do not allow us to determine whether and to what extent the design-in-use interventions influenced the unfolding of the asynchronous discussions.

The description of this design is specific to the teaching of linear algebra, but it includes features that make it replicable in other contexts. Certainly, the integration of Perusall as a material resource, which allows general resources such as videos to be officially incorporated, can be leveraged in various educational levels. It seems particularly beneficial, though, in university courses with a large number of students, where monitoring the extensive use of resources other than the curriculum would be challenging. While the selection of videos or general resources is at the instructor's discretion, utilising a platform like the one we employed, with design choices tailored to student engagement, could enhance the course's resource system. Although this data provides limited information, it enables immediate redesigns throughout the course. Gathering more comprehensive data, as we intend to do, can allow us to reconsider the design for subsequent implementation cycles.

One aspect that should be taken into account is that, in monitoring and managing the asynchronous discussion, the course teacher was helped by three other instructors. This poses an issue with respect to the replicability of the activity: indeed, even if in future experimentations and implementations it would be possible to tak advantage of the document we have created, the managing of the discussion on Perusall by a single instructor will take much more time. Anyway, for the teacher it could be a good investment for the reward of having important information on students' doubts and criticisms useful to plan more effective lectures. It could be also interesting to investigate the constraint on replicability using the lens of instrumental distance (Haspekian, 2014) between the sequence we designed and implemented and the “current university habits”.

Clearly there are other interesting aspects that emerged from the implementation of the teaching sequence that have not been considered in this article and deserve to be developed in future works, especially concerning the analysis of data on students. For example, the regrouping of the students according to their initial engagement in the document produced different results in a priori similar groups. In order to investigate this aspect we administered a questionnaire to the students and interviewed some of them. We plan to discuss in detail the results emerged by the questionnaire and interviews in a future work. Another aspect that needs to be investigated has to do with the kind of comments that the students posted on the platform: a preliminary analysis with the lens of Veerman and Veldhuis-Diermanse (2001) seems to reveal that associating tasks of different types to videos stimulates different kinds of comments (Cattabriga et al., 2023).

Such a project lays the groundwork not only for studying how the didactic transposition of resources not intentionally designed for educational purposes can take place, but also for understanding what role non-interactive dynamic visualisation (such as that available via video) can play in learning mathematics, an aspect still very little studied in this field of research. Future work may take advantage of a document consisting of videos, such as the one we have proposed, to investigate this specific aspect.

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APPENDICES

Appendix A – Activity proposed in pilot study

In the following, I present the handout used during the optional group activity proposed during the pilot study. First I present the original version in Italian and then the English translation.

Attività per progetto di ricerca

November 29, 2021

- Provate a schematizzare assieme i concetti dell'ultima lezione. Pensate di dover "stabilizzare" sui vostri appunti delle idee che l'insegnante ha mostrato durante la lezione e che rischiereste di perdere se non trovate il modo di fissare queste idee nei vostri quaderni.
- Cercate soprattutto di ricostruire il significato di autovalore e autovettore. Che cosa sono? Come si collegano le definizioni e gli esempi visti a lezione con il procedimento visto per trovare gli autovalori e i relativi autovettori? Discutetene tra di voi nel gruppo. Provate a schematizzare nel modo che ritenete più adatto queste idee, per far sì che da qui in poi non memorizziate solo il procedimento, ma riusciate a portare con voi anche "il perché" del procedimento.
- Ricordate che più tardi dovrete risolvere degli esercizi assieme, quindi trovate e accordatevi su delle parole, rappresentazioni, procedure che userete poi per discutere quando dovrete risolvere tali problemi. Possono essere anche parole e rappresentazioni di vostra invenzione se questo vi aiuta meglio a ricordare e comprendere i concetti e capirvi tra di voi!
- Ricordate che in questa fase non state preparando un compito sul quale sarete valutati (non ci sarà nessun giudizio su questo), ma il vostro lavoro è finalizzato ad aiutarvi a capire meglio i concetti e le procedure spiegate attraverso la vostra condivisione e discussione delle difficoltà nel gruppo. Noi da parte nostra, attraverso il vostro lavoro cercheremo di capire meglio le vostre difficoltà per preparare meglio i nostri interventi didattici. Cercate quindi di sfruttare quest'attività per chiarire i concetti visti e studiare o ripassare la lezione insieme a delle compagne/dei compagni di corso.

Un suggerimento delle cose importanti che dovreste considerare in questa schematizzazione è:

- Che cos'è un autovalore (non limitatevi a riscrivere la definizione. Provate a descrivere cosa secondo voi è un autovalore. Come lo descrivereste a qualcun altro che non ha ancora affrontato questo costrutto nel corso di algebra lineare)
- Che cos'è un autovettore ? (idem)
- Come si determinano gli autovalori e gli autovettori di un' applicazione? Perché funziona questo metodo? Non limitatevi a considerare applicazioni date tramite le rispettive matrici, ricordate che le applicazioni lineari possono essere descritte in tanti modi diversi.

Figure A.1. Handout of the activity - first part

1 Esercizi

1. Interpretate geometricamente la trasformazione lineare $T_A : R^2 \to R^2$ $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ e trovatene gli autovalori e autovettori.
2. Immaginate una trasformazione lineare $T : R^2 \to R^2$ con le seguenti proprietà:
   - Lungo la direzione della retta $y = -3x$, la trasformazione dilata tutti i vettori con fattore di dilatazione 2;
   - Lungo la direzione della retta $y = x$, la trasformazione mantiene tutti i punti fissi.

   Determinate cosa succede ai vettori $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ sotto questa trasformazione.
3. Dato l'endomorfismo $T : R^3 \to R^3$ definito da $T(x, y, z) = (2x + y + 2z, x, x)$, trovatene gli autovalori e relativi autovettori.
4. Dato l'endomorfismo $F : M_2(R) \to M_2(R)$ definito da $F(A) = \frac{1}{2}(A +^t A)$, trovatene gli autovalori e relativi autovettori.

Figure A.2. Handout of the activity - second part

- Try to sketch together the concepts from the last lesson. Think about "stabilising" on your notes ideas that the teacher showed during the lesson and that you would risk losing if you do not find a way to fix these ideas in your notebooks.
- Above all, try to reconstruct the meaning of eigenvalue and eigenvector. What are they? Discuss among yourselves in the group and try to schematise these ideas as you see fit. How do you think you should represent these concepts so that their meaning becomes apparent?
- Remember that you will have to solve exercises together later, so find and agree on words, representations, procedures that you will then use to discuss when you have to solve such problems. They can also be words and representations of your own invention if this helps you better remember and understand concepts and each other!
- Remember that at this stage you are not preparing an assignment on which you will be assessed (there will be no no judgement on this), but your work is aimed at helping you to better understand the concepts and procedures explained through your sharing and discussion of the difficulties in the group. We for our part, through your work will try to better understand your difficulties in order to better prepare our teaching interventions. So try to use this activity to clarify the concepts you have seen and study or review the lesson together with your classmates.

A suggestion of the important things you should consider in this outline is:

- What is an eigenvalue (don't just rewrite the definition. Try to describe what you think an eigenvalue is. How would you describe it to someone else who has not yet tackled this construct in linear algebra class)?
- What is an eigenvector? (idem)
- Which examples do you think would help you most to understand these concepts?

Figure A.3. Translation - first part

[15]

1 Exercises

1. Interpretate geometrically the linear transformation $T_A : R^2 \to R^2$ $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and find its eigenvalues and eigenvectors.

2. Imagine a linear transformation $T : R^2 \to R^2$ having these properties:
   - on the direction of the line $y = -3x$, the tranformation dilates all vectors by a factor of 2;
   - on the direction of the line $y = x$, the transformation keeps all the points fixed

   Determine what happens to the vectors $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ under this transformation.

3. Given the endomorphism $T : R^3 \to R^3$ defined by $T(x, y, z) = (2x + y + 2z, x, x)$, find its eigenvalues and related eigenvectors.

4. Given the endomorphism $F : M_2(R) \to M_2(R)$ defined by $F(A) = \frac{1}{2}(A +^t A)$, find its eigenvalues and related eigenvectors.

Figure A.4. Translation - second part

[15] Exercise 2 was taken from (Wawro et al., 2013).