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Maximally localized Gabor orthonormal bases on locally compact Abelian groups

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A Gabor orthonormal basis, on a locally compact Abelian (LCA) group A, is an orthonormal basis of $L^2(A)$ that consists of time-frequency shifts of some template $f \in L^2(A)$. It is well known that, on $\mathbb{R}^d$, the elements of such a basis cannot have a good time-frequency localization. The picture is drastically different on LCA groups containing a compact open subgroup, where one can easily construct examples of Gabor orthonormal bases with f maximally localized, in the sense that the ambiguity function of f (i.e., the correlation of f with its time-frequency shifts) has support of minimum measure, compatibly with the uncertainty principle. In this note we find *all* the Gabor orthonormal bases with this extremal property. To this end, we identify all the functions in $L^2(A)$ that are maximally localized in the time-frequency space in the above sense – an issue that is open even for finite Abelian groups. As a byproduct, on every LCA group containing a compact open subgroup we exhibit the complete family of optimizers for Lieb's uncertainty inequality, and we also show previously unknown optimizers on a general LCA group.

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1. Introduction

Let A be a locally compact Hausdorff Abelian (LCA, in short) group, and denote by $\widehat{A}$ its dual group. We will write $\langle x, \xi \rangle$, with $x \in A$ and $\xi \in \widehat{A}$, for the corresponding duality; hence $|\langle x, \xi \rangle| = 1$. For $x \in A$, $\xi \in \widehat{A}$ we define the translation and modulation operators T_x and M_ξ on $L^2(A)$, and the corresponding time-frequency shifts $\pi(x, \xi)$ as

$$T_x f(y) = f(y - x), \quad M_\xi f(y) = \langle y, \xi \rangle f(y), \quad \pi(x, \xi) f(y) = M_\xi T_x f(y), \tag{1.1}$$

where $y \in A$.

A Gabor orthonormal basis on A is an orthonormal basis of $L^2(A)$ of the type

$$\mathcal{G}(f, \Gamma) := \{\pi(z) f : \ z \in \Gamma\} \tag{1.2}$$

where $f \in L^2(A)$ and Γ is a (possibly uncountable) subset of $A \times \widehat{A}$. The function f is called *window*. In other terms, a Gabor orthonormal basis is an orthonormal basis in an orbit of the Schrödinger representation of the Heisenberg group associated to A.

There is a considerable amount of work on the construction of Gabor orthonormal bases on $\mathbb{R}$; see [2,7,12,22,41,42,47,46,53] and also [27] for far reaching generalizations on nilpotent Lie groups. In fact, already in 1929 J. von Neumann [59, footnote 10] considered the idea of using the functions $\pi(z)f$ above, when $A = \mathbb{R}$, f is a Gaussian function and z belongs to a suitable lattice of $\mathbb{R}^2$, to construct orthonormal bases of $L^2(\mathbb{R})$ (via the Gram-Schmidt orthogonalization procedure). The above mentioned papers focus mainly on windows that are characteristic functions of some compact subset, hence with a poor frequency localization. Indeed, the celebrated Balian-Low theorem states, roughly speaking, that there are no Gabor orthonormal bases on $\mathbb{R}^d$ generated by a window with a good time-frequency localization (see, e.g., the survey [4]), and similar obstructions also apply to LCA groups having no compact open subgroups [13,36]; see also [26] for the issue of the time-frequency localization of Riesz bases on $\mathbb{R}^d$.

The situation is drastically different for LCA groups containing a compact, open subgroup. Indeed, if $H \subset A$ is such a subgroup, the function $f = |H|^{-1/2}\chi_H$ (where $|H|$ stands for the measure of H) generates an orthonormal basis as in (1.2) if Γ is any set of representatives of the cosets of $H \times H^\perp$ in $A \times \widehat{A}$ (namely, Γ contains exactly one element of each coset) [23,28], and this window is, in a sense, maximally localized in the time-frequency space. To explain this latter point properly, we need some terminology.

For $f, g \in L^2(A)$, the short-time Fourier transform of f with window g is the complex-valued function on $A \times \widehat{A}$ given by

$$V_g f(x, \xi) = \langle f, \pi(x, \xi) g \rangle_{L^2(A)} \qquad (x, \xi) \in A \times \widehat{A}. \tag{1.3}$$

It is known that, if $\|g\|_{L^2(A)} = 1$, then $V_g : L^2(A) \to L^2(A \times \widehat{A})$ is an isometry, so that the quantity $|V_g f|^2$ can be regarded as a time-frequency energy density of f

(see [23,24,44]). More generally, the time-frequency localization of a function $f \in L^2(A)$ can be measured in terms of the L^p-norm of $V_f f$ – the so-called *ambiguity function* – and corresponding versions of the uncertainty principle, such as Lieb's uncertainty inequality, can be stated (see [23,43] and Section 7 below). We recall, in particular, the following elementary result (see [18,23,38], Theorem 2.2 below, and also [5,25] for other formulations of the uncertainty principle in terms of the short-time Fourier transform), which corresponds to a limiting case of an L^p-estimate as $p \to 0$ (cf. Remark 7.3 below):

Let $f \in L^2(A) \setminus \{0\}$. Then

$$|\{z \in A \times \widehat{A} : V_f f(z) \neq 0\}| \geq 1. \tag{1.4}$$

Here we used the notation $|S|$ for the measure of a set $S \subset A \times \widehat{A}$, where the Haar measures on A and $\widehat{A}$ are chosen so that the Plancherel formula holds true. The inequality (1.4) can be regarded as a time-frequency version of a lower bound for the product of the measures of the supports of f and $\widehat{f}$, first proved by Matolcsi and Szűcs in [48] (see also [6,11,57,58,61]) and usually referred to as the Donoho-Stark uncertainty principle.

The inequality (1.4) is sharp on every LCA group containing a compact, open subgroup H. Indeed, for the function $f = |H|^{-1/2}\chi_H$ considered above, we have $V_f f = \chi_{H \times H^\perp}$ (cf. [23] and Proposition 4.4 below) and therefore f is maximally localized in the time-frequency space, in the sense that the inequality (1.4) is saturated ($|H \times H^\perp| = 1$ by the Plancherel formula). This is a desirable property that guarantees that the "analysis operator" associated with the corresponding basis $\mathcal{G}(f, \Gamma)$ (cf. (1.2)), that is $L^2(A) \ni h \mapsto V_f h(z)$, with $z \in \Gamma$, is able to resolve the "blobs" of energy of h in the time-frequency space, with the highest possible resolution – at least in the measure theoretic sense.

This discussion raises the problem of identifying *all* the Gabor orthonormal bases generated by a function f maximally localized in the above sense, namely such that the set where $V_f f \neq 0$ has measure 1 (this implies that the same extremal property holds for every element of the basis). This issue is also motivated, on one hand, by the recent advances [13] on the Balian-Low theorem on general (second countable) LCA groups having *no* compact open subgroups and, on the other hand, by the recent, considerable interest in optimizers of uncertainty inequalities on Euclidean spaces and Riemannian manifolds [1,10,21,35,37,39,40,45,50–52,54]. In a sense, the results in this note can be regarded as complementary to the no-go results in [13].

The main problem therefore lies in the identification of all the functions $f \in L^2(A)$ for which equality occurs in (1.4), which is an open issue even for finite Abelian groups. This should not come as a surprise, in light of other extremal problems, e.g., for the Young and Hausdorff-Young inequalities, which are notoriously difficult even on LCA groups containing a compact, open subgroup (see [20, Theorem 3] and [30, Section 43]). Indeed, for the finite cyclic group $\mathbb{Z}_N$ all the optimizers for the inequality (1.4) were obtained only recently, in [49]; see also [18] for a particular case.

Our first result (Theorem 4.5) provides the complete answer to this problem and can be summarized as follows:

Equality occurs in (1.4) *if and only if* $f = cT_{x_0}h$, *for some* $c \in \mathbb{C} \setminus \{0\}$, $x_0 \in A$ *and some subcharacter of second degree* h *of* A. *In this case,* $\|f\|^{-2}_{L^2(A)} V_f f$ *is a subcharacter of second degree of* $A \times \widehat{A}$, *and its support is a maximal compact open isotropic subgroup of* $A \times \widehat{A}$.

A subcharacter of second degree of A is a continuous function $h : A \to \mathbb{C}$ such that, for some compact open subgroup $H \subset A$, the restriction of h to H is a character of second degree of H in the sense of Weil [60] (see Section 2.3 below) and $h(x) = 0$ for $x \in A \setminus H$. The term "isotropic" refers to the standard symplectic structure of $A \times \widehat{A}$ (see Section 3). Hence, maximal compact open isotropic subgroups of $A \times \widehat{A}$ are the minimum uncertainty phase-space cells – the so-called "quantum blobs" – and play the role of the symplectic images of the unit ball (or box) in $\mathbb{R}^d \times \mathbb{R}^d$ [9,15]. As a further motivation, notice the formal analogy with the extremal problem for the Hausdorff-Young inequality on a LCA group A, that is $\|\widehat{f}\|_{L^{p'}(\widehat{A})} \leq \|f\|_{L^p(A)}$, $1 < p < 2$, where the optimizers are the constant multiples of translates of subcharacters [30, Theorem 43.13].

The study of the cases of equality in (1.4) on $\mathbb{Z}_N$ [49] relies on the explicit description of the subgroups of order N of $\mathbb{Z}_N \times \mathbb{Z}_N$ and the equally explicit construction of "finite chirps", which is available in that case (see [8,16]). In the present generality we have to follow a more conceptual pattern. To give a flavor of the argument let us briefly outline how the above mentioned subcharacter of second degree arises. By using the covariance property for the short-time Fourier transform we will show that, if the set $G = \{z \in A \times \widehat{A} : V_f f(z) \neq 0\}$ has measure 1, then G is a maximal compact open *isotropic* subgroup of $A \times \widehat{A}$. The projection onto the first factor allows us to regard G as an extension of a compact open subgroup H of A by $H^\perp \subset \widehat{A}$. This induces, in a natural way, a continuous *symmetric* homomorphism $H \to \widehat{H}$, to which we can associate the desired character h of second degree of H – here we use an extension theorem for characters of second degree from [31]. The analogous problem for the short-time Fourier transform $V_g f$ is addressed in Theorem 5.2.

Let us now come back to the above problem of maximally localized Gabor orthonormal bases. We anticipate here the main result (Corollary 6.2), which provides the desired, full characterization.

Let $f \in L^2(A)$, $\|f\|_{L^2(A)} = 1$ *and* $G := \{z \in A \times \widehat{A} : V_f f(z) \neq 0\}$. *Let* $\Gamma \subset A \times \widehat{A}$. *Then* $\mathcal{G}(f, \Gamma)$ *(cf.* (1.2)*) is an orthonormal basis of* $L^2(A)$ *and* $|G| = 1$ *if and only if* $f = cT_{x_0}h$ *for some* $c \in \mathbb{C}$, $x_0 \in A$ *and some subcharacter* h *of second degree of* A, *and* Γ *is a set of representatives of the cosets of* G *in* $A \times \widehat{A}$.

The subsets of $A \times \widehat{A}$ defined by $V_f(\pi(z)f) \neq 0$, with $z \in \Gamma$, are precisely the cosets of the maximal compact open isotropic subgroup G, and define therefore a tiling of $A \times \widehat{A}$. In fact, it turns out that for every tiling of this type there is a corresponding Gabor orthonormal basis (see Remark 6.3). However the main point of the above result is clearly represented by the necessary condition. We refer the reader to [32,62] and the references therein for other constructions of (not necessarily maximally localized) Gabor orthonormal bases in the setting of finite Abelian groups and to [13] for general LCA groups.

In Section 7 we will show that, on any LCA group containing a compact open subgroup, every extremal function for the uncertainty inequality (1.4) is also an extremal function for Lieb's uncertainty inequality [23,43] and vice versa. This will also provide, as a byproduct, previously unknown optimizers for Lieb's inequality on every LCA group, because such a group is topologically isomorphic to $\mathbb{R}^d \times A_0$ for some integer $d \geq 0$ and some LCA group A_0 containing a compact open subgroup. It would certainly be interesting to identify *all* the optimizers for Lieb's uncertainty inequality – as well as for related uncertainty inequalities – on a general LCA group, but that would lead us too far, so that we decided to postpone this issue to a future work. We also refer the reader to [21,40] and the references therein for recent advances on Lieb's inequality on Euclidean spaces and Riemannian manifolds.

2. Notation and preliminary results

2.1. Notation

We use the notation introduced at the beginning of the introduction. Hence A denotes a locally compact Hausdorff Abelian (in short, LCA) group, and $\widehat{A}$ its dual group, namely the group of continuous homomorphisms $\xi : A \to \mathbb{T}$ (the multiplicative group of complex numbers of modulus 1). When endowed with the topology of the uniform convergence on the compact subsets, $\widehat{A}$ becomes a LCA group ([29, Theorems 23.13 and 23.15]). We denote by $\langle x, \xi\rangle$ the value of $\xi \in \widehat{A}$ at $x \in A$. The pairing $A \times \widehat{A} \to \mathbb{T}$ given by $(x, \xi) \to \langle x, \xi\rangle$ is therefore well defined. We will write the group laws in A and $\widehat{A}$ additively, hence $\langle x + y, \xi\rangle = \langle x, \xi\rangle\langle y, \xi\rangle$ and $\langle x, \xi + \eta\rangle = \langle x, \xi\rangle\langle x, \eta\rangle$.

For a subgroup $H \subset A$ we denote by $H^\perp = \{\xi \in \widehat{A} : \ \langle x, \xi\rangle = 1 \text{ for all } x \in H\}$ the annihilator of H in A (cf. [29, Definition 23.23]). It is clearly a closed subgroup of $\widehat{A}$. Also, if $H \subset A$ is a subgroup, we denote by A/H the corresponding quotient group, whose elements are the cosets $x + H = \{x + y : \ y \in H\}$, $x \in A$, of H in A (these cosets define a partition, sometimes called "tiling", of A). In the following, by "a set of representatives" of the cosets of H in A, we will mean a set that contains exactly one element of each coset.

If $H \subset A$ is a subgroup, A/H is a discrete space if and only if H is open in A ([29, Theorem 5.21]). If H is closed, A/H has a natural structure of LCA group ([29, Theorems 5.21 and 5.22]), $\widehat{A/H} \simeq H^\perp$ ([29, Theorem 23.25]) and $\widehat{A}/H^\perp \simeq \widehat{H}$ [29, Theorem 24.11] in the sense of topological isomorphisms.

The groups $A, \widehat{A}$ are equipped with Haar measures related so that the Plancherel formula holds true. The Haar measure on $A \times \widehat{A}$ is given by the Radon product measure. The inner product in $L^2(A)$ is denoted by $\langle \cdot, \cdot\rangle_{L^2(A)}$. We denote by $|S|$ the measure of a subset S (of A or $\widehat{A}$, or $A \times \widehat{A}$), and by χ_S its characteristic function.

We refer to (1.1) for the definition of the translation operators T_x, $x \in A$, the modulation operators M_ξ, $\xi \in \widehat{A}$, and the phase-space shifts $\pi(x,\xi) = M_\xi T_x$. They are unitary operators on $L^2(A)$. The short-time Fourier transform $V_g f$, for $f, g \in L^2(A)$, was defined in (1.3); more explicitly

$$\begin{aligned} V_g f(x,\xi) &= \langle f, \pi(x,\xi) g\rangle_{L^2(A)} \\ &= \int_A \overline{\langle y,\xi\rangle} f(y) \overline{g(y-x)}\, dy \qquad x \in A,\ \xi \in \widehat{A}. \end{aligned} \tag{2.1}$$

2.2. Preliminaries from time-frequency analysis

We recall some basic results about time-frequency analysis on LCA groups. We refer the reader to [17,23] for details (see also [24] for the analogous results in $\mathbb{R}^d$).

The following result generalizes a well known property of the short-time Fourier transform in $\mathbb{R}^d$ [24].

Proposition 2.1. *Let $f, g \in L^2(A)$. Then $V_g f$ is a continuous function on $A \times \widehat{A}$, which vanishes at infinity.*

Proof. Since the unitary representations $A \ni x \mapsto T_x$ and $\widehat{A} \ni \xi \mapsto M_\xi$ are strongly continuous on $L^2(A)$, $V_g f$ is a continuous function on $A \times \widehat{A}$. Moreover, by the definition of $V_g f$ in (2.1) we have $|V_g f(x,\xi)| \le |f| * |\tilde{g}|(x)$ and similarly, $|V_g f(x,\xi)| \le |\widehat{f}| * |\tilde{\widehat{g}}|(\xi)$, with $\tilde{g}(y) := g(-y)$. On the other hand, functions in $L^2 * L^2$ tend to zero at infinity. $\square$

As a consequence, we also see that the set $\{z \in A \times \widehat{A} : \ V_g f(z) \neq 0\}$ is σ-compact.

We will need a few elementary formulas concerning time-frequency shifts and the short-time Fourier transform.

First, for $x, y \in A$, $\xi, \eta \in \widehat{A}$ we have the commutation relations

$$\pi(x,\xi)\pi(y,\eta) = \langle y,\xi\rangle \overline{\langle x,\eta\rangle} \pi(y,\eta)\pi(x,\xi). \tag{2.2}$$

As a consequence, the following covariance-type properties hold true, for $x, y \in A$, $\xi, \eta \in \widehat{A}$:

$$V_g(\pi(x,\xi)f)(y,\eta) = \langle x,\xi\rangle \overline{\langle x,\eta\rangle} V_g f(y-x, \eta-\xi) \tag{2.3}$$

and

$$V_{\pi(x,\xi)g}(\pi(x,\xi)f)(y,\eta) = \langle y,\xi\rangle \overline{\langle x,\eta\rangle} V_g f(y,\eta). \tag{2.4}$$

An application of the Cauchy-Schwarz inequality gives at once the following pointwise estimate:

$$|V_g f(x,\xi)| \le \|f\|_{L^2(A)}\|g\|_{L^2(A)} \qquad x\in A,\ \xi\in\widehat{A}. \tag{2.5}$$

We finally recall the Parseval equality

$$\|V_g f\|_{L^2(A\times\widehat{A})} = \|f\|_{L^2(A)}\|g\|_{L^2(A)}. \tag{2.6}$$

The following uncertainty inequality was first proved in [38] in the case of finite Abelian groups; see also [18,23,49].

Theorem 2.2. *Let $f,g\in L^2(A)\setminus\{0\}$ and $S=\{z\in A\times\widehat{A} : V_g f(z)\neq 0\}$. We have $|S|\ge 1$. If $|S|=1$ then $|V_g f| = c\chi_S$, with $c=\|f\|_{L^2(A)}\|g\|_{L^2(A)}$, and therefore S is compact and open.*

Proof. We can suppose $\|f\|_{L^2(A)} = \|g\|_{L^2(A)} = 1$. By (2.6) and (2.5), we have

$$1 = \int_S |V_g f(x,\xi)|^2 dx\, d\xi \le \int_S dx\, d\xi = |S|.$$

If equality occurs in the above inequality, then $|V_g f(z)| = 1$ for almost every $z\in S$, and therefore for every $z\in S$, since $V_g f$ is continuous (Proposition 2.1), and S is open. Hence $|V_g f| = \chi_S$, which implies that S is also closed. Moreover, since $V_g f$ tends to zero at infinity, S is contained in a compact subset, and therefore is compact. □

We will also need the following uniqueness result for the ambiguity function $V_f f$.

Proposition 2.3. *Let $f,g\in L^2(A)$. Then $V_f f = V_g g$ on $A\times\widehat{A}$ if and only if $f=cg$ for some $c\in\mathbb{C}$, $|c|=1$.*

Proof. For the sake of completeness we provide a proof similar (but not exactly equal) to that given in [24, Section 4.2] for $A=\mathbb{R}^d$.

For every $x\in A$, we can regard $V_f f(x,\cdot)$ as the Fourier transform of the L^1 function $f\overline{T_x f}$. Hence, if $V_f f = V_g g$, by the Fourier uniqueness theorem, for every $x\in A$ we have

$$f(y)\overline{f(y-x)} = g(y)\overline{g(y-x)}$$

for almost every $y\in A$. By the Fubini theorem (both sides vanish on the complement of a σ-compact set in $A\times A$; cf. [19, page 44]), the above equality holds for almost every $(x,y)\in A\times A$. Multiplying by $f(y-x)$ and integrating with respect to x yields $\|f\|^2_{L^2(A)} f = \langle f,g\rangle_{L^2(A)} g$, which gives the desired conclusion.

The converse implication is obvious. □

2.3. Characters of second degree

We recall from [60] (see also [55]) the notion of character of second degree.

Definition 2.4. Given a continuous symmetric homomorphism $\phi : A \to \widehat{A}$ (hence $\langle x, \phi(y)\rangle = \langle y, \phi(x)\rangle$ for $x, y \in A$) a character of second degree of A, associated to ϕ, is a continuous function $f : A \to \mathbb{T}$ such that

$$f(x+y) = f(x)f(y)\langle x, \phi(y)\rangle \qquad x, y \in A.$$

The following result provides the existence and uniqueness, up to multiplication by characters, of characters of second degree associated to a given homomorphism ϕ as above.

Theorem 2.5. *Given a continuous symmetric homomorphism $\phi : A \to \widehat{A}$ there exists a character of second degree associated to ϕ. Two characters of second degree associated to the same ϕ differs by the multiplication by a character.*

The uniqueness is an immediate consequence of the definition. The existence was first proved in [31, Lemma 6] (see also [55, page 37] for an easier proof due to M. Burger, and [3, Theorem 2.3] for generalizations). Explicit constructions are available in particular cases; for example, if multiplication by 2 is an automorphism of G then one can take $f(x) = \langle x, \phi(2^{-1}x)\rangle$ [60, page 146]. We refer the reader to [55, Section 7.7] for an explicit construction when G a finite dimensional vector space over a local field, [16] for $\mathbb{Z}_N$, and [34] for finite Abelian groups.

3. Some symplectic analysis on $A \times \widehat{A}$

In this section we prove some auxiliary results concerning subgroups of $A \times \widehat{A}$, in connection with the standard symplectic structure of $A \times \widehat{A}$, that is the bicharacter $\sigma : (A \times \widehat{A}) \times (A \times \widehat{A}) \to \mathbb{T}$ given by (cf. (2.2))

$$\sigma((x, \xi), (y, \eta)) = \langle y, \xi\rangle \overline{\langle x, \eta\rangle} \qquad (x, \xi),\ (y, \eta) \in A \times \widehat{A}. \tag{3.1}$$

The following description of the compact open subgroups of $A \times \widehat{A}$ will be crucial in the following.

Proposition 3.1. *Let $H \subset A$, $K \subset \widehat{A}$ be compact open subgroups, and let $\phi : H \to \widehat{A}/K$ be a continuous homomorphism. Then the set*

$$G = \{(x, \xi) \in A \times \widehat{A} :\ x \in H,\ \xi \in \phi(x)\} \tag{3.2}$$

is a compact open subgroup of $A \times \widehat{A}$, and $|G| = |H||K|$.

Every compact open subgroup of $A \times \widehat{A}$ *arises in this way for a unique triple* (H, K, ϕ) *as above.*

Proof. It is easy to see that the set G in (3.2) is indeed a subgroup of $A \times \widehat{A}$. Since H and K are compact and ϕ is continuous it follows from some general (not completely trivial) results from the theory of topological groups (cf. [29, Note 5.24]), that G is compact. However, since H and K are open, we can apply a more direct argument, that also has the advantage to give some more information, namely that G is in fact a finite union of pairwise disjoint compact open subsets of $A \times \widehat{A}$ of product type. Precisely, since K is open, $\widehat{A}/K$ is discrete, and $\phi(H) \subset \widehat{A}/K$ is compact, therefore finite. On the other hand we have

$$G = \cup_{B \in \phi(H)} \phi^{-1}(\{B\}) \times B.$$

The sets $B \subset \widehat{A}$ above are compact and open, because they are cosets of K in $\widehat{A}$. The subsets $\phi^{-1}(\{B\}) \subset H$ are open and closed and therefore compact, because H is compact. This shows that G is compact and open.

For every $x \in H$, $\phi(x)$ is a coset of K in $\widehat{A}$, and therefore has the same measure as K. Hence

$$|G| = \int\limits_H \int\limits_{\phi(x)} d\xi \, dx = |H||K|.$$

Now, given any compact open subgroup $G \subset A \times \widehat{A}$, let $\pi_1 : G \to A$ be the projection onto the first factor, and set $H = \pi_1(G)$ and $K = G \cap \widehat{A} = \operatorname{Ker} \pi_1$ (where $\widehat{A}$ is regarded as a subgroup of $A \times \widehat{A}$). It is clear that K is a compact and open subgroup of $\widehat{A}$ and H is a compact open subgroup of A, because the projection $A \times \widehat{A} \to A$ is an open map. Hence we have the short exact sequence

$$0 \to K \to G \to H \to 0.$$

The map $G \to H$ is open, and therefore the algebraic isomorphism $\beta : G/K \to H$ is in fact a topological isomorphism ([29, Theorem 5.27]) (this also follows from the fact that β is a continuous bijection, G/K is compact and H is Hausdorff). Now set $\phi = \pi_2' \circ \beta^{-1}$, where $\pi_2' : G/K \to \widehat{A}/K$ is the natural epimorphism induced by the projection $\pi_2 : G \to \widehat{A}$ onto the quotient spaces. Then $\phi : H \to \widehat{A}/K$ is a continuous homomorphism, and clearly (3.2) holds for the triple (H, K, ϕ).

Finally, it is clear from (3.2) that the triple (H, K, ϕ) is uniquely determined by G. □

Remark 3.2. It is easy to see that G and $H \times K$ in Proposition 3.1 are homeomorphic. Indeed, with the notation of the above proof, choosing a representative ξ out of any $[\xi] \in \phi(H)$ ($\phi(H)$ is a finite set), yields a continuous section $\alpha : \phi(H) \to \widehat{A}$ and therefore a lifting $\tilde{\phi} := \alpha \circ \phi : H \to \widehat{A}$ of ϕ. We now extend $\tilde{\phi}$ to A by setting $\tilde{\phi}(x) = 0$ for

$x \in A \setminus H$. Since H is both open and closed, $\tilde{\phi} : A \to \widehat{A}$ is continuous, and the map $A \times \widehat{A} \to A \times \widehat{A}$ given by $(x, \xi) \mapsto (x, \tilde{\phi}(x) + \xi)$ is a homeomorphism (although not a group homomorphism) which maps $H \times K$ onto G.

We now single out a class of subgroups and provide a convenient characterization in the spirit of Proposition 3.1.

Definition 3.3. A subgroup $G \subset A \times \widehat{A}$ is called isotropic if $\sigma(z, w) = 1$ (cf. (3.1)) for every $z, w \in G$.

Consider a triple (H, K, ϕ) as in Proposition 3.1 and assume, in addition, that $K \subset H^\perp$. Then a natural epimorphism $\widehat{A}/K \to \widehat{A}/H^\perp \simeq \widehat{H}$ is induced. Hence, for $x \in H$, we can regard $\phi(x) \in \widehat{A}/K$ as a character of H, whose value at $y \in H$ will be denoted by $\langle y, \phi(x)\rangle$. In concrete terms,

$$\langle y, \phi(x)\rangle := \langle y, \xi\rangle \quad \text{for any} \quad \xi \in \phi(x).$$

Definition 3.4. If $K \subset H^\perp$, a continuous homomorphism $\phi : H \to \widehat{A}/K$ is called symmetric if

$$\langle x, \phi(y)\rangle = \langle y, \phi(x)\rangle \qquad x, y \in H.$$

We recall, for future reference, that if $H \subset A$ is a compact open subgroup, then $H^\perp$ is a compact open subgroup of $\widehat{A}$ (see e.g., [23, Lemma 6.2.3 (b)]).

Proposition 3.5. *Let $G \subset A \times \widehat{A}$ be a compact open subgroup of $A \times \widehat{A}$ and let (H, K, ϕ) be the corresponding triple (cf. Proposition 3.1). Then G is isotropic if and only if $K \subset H^\perp$ and $\phi : H \to \widehat{A}/K$ is symmetric. Moreover $|G| \leq 1$.*

Proof. Let G be isotropic. Let $x \in H$, $\xi \in \phi(x)$ and $\eta \in K$, so that $z = (x, \xi) \in G$ and $w = (x, \xi + \eta) \in G$. We have

$$1 = \sigma(z, w) = \langle x, \xi\rangle \overline{\langle x, \xi + \eta\rangle} = \overline{\langle x, \eta\rangle},$$

and therefore $K \subset H^\perp$. Now, if $x, y \in H$ and $\xi \in \phi(x)$, $\eta \in \phi(y)$, so that $z = (x, \xi) \in G$ and $w = (y, \eta) \in G$, we have $1 = \sigma(z, w) = \langle y, \xi\rangle \overline{\langle x, \eta\rangle}$, which means that ϕ is symmetric.

Vice versa, it is clear from the above computation that if $K \subset H^\perp$ and ϕ is symmetric then G is isotropic.

Finally, by Proposition 3.1, $|G| = |H||K| \leq |H||H^\perp| = 1$, where the last equality follows from the Plancherel formula (see [56, Formula (4.4.6)]). □

We finally characterize the compact open isotropic subgroups of maximum measure.

Proposition 3.6. *Let $G \subset A \times \widehat{A}$ be a compact open isotropic subgroup and let (H, K, ϕ) be the corresponding triple (cf. Propositions 3.1 and 3.5). The following statements are equivalent:*

(a) $|G| = 1$.
(b) $K = H^\perp$.
(c) *G is a maximal compact open isotropic subgroup, i.e., if $G' \subset A \times \widehat{A}$ is a compact open isotropic subgroup with $G \subset G'$ then $G' = G$.*

Proof. (a) $\Rightarrow$ (b). We know from Proposition 3.5 that $K \subset H^\perp$. If this inclusion were strict then $|K| < |H^\perp|$, because $|H^\perp| < \infty$ and $|H^\perp \setminus K| > 0$, being $H^\perp \setminus K$ open. Hence, since $0 < |H| < \infty$, by Proposition 3.1 $|G| = |H||K| < |H||H^\perp| = 1$, which is a contradiction.

(b) $\Rightarrow$ (c). Let $G' \subset A \times \widehat{A}$ be a compact open isotropic subgroup with $G \subset G'$. Let (H', K', ϕ') be the corresponding triple, as in Propositions 3.1 and 3.5. Since $G \subset G'$ we have $H \subset H'$ and $K \subset K'$. On the other hand, by Proposition 3.5 and the assumption, we have $H' \subset K'^\perp \subset K^\perp = H$. Hence $H = H'$ and $K = K'$. Since, for $x \in H$, $\phi(x)$ and $\phi'(x)$ are two cosets of K in $\widehat{A}$ and $\phi(x) \subset \phi'(x)$ we have $\phi(x) = \phi'(x)$ and therefore $G' = G$.

(c) $\Rightarrow$ (a). Suppose, by contradiction, that $|G| = |H||K| < 1$. Then the inclusion $K \subset H^\perp$ is strict. Consider the subgroup $G' \subset A \times \widehat{A}$ associated to the triple $(H, H^\perp, \phi')$, with $\phi' = \alpha \circ \phi : H \to \widehat{A}/H^\perp$, where $\alpha : \widehat{A}/K \to \widehat{A}/H^\perp$ is the natural epimorphism. Since ϕ is symmetric, the same holds for ϕ'. Hence, by Proposition 3.5, G' is a compact open isotropic subgroup of $A \times \widehat{A}$ and $G \subset G'$ with strict inclusion, because $G \cap \widehat{A} = K \subset H^\perp = G' \cap \widehat{A}$ strictly. This is a contradiction. $\square$

Corollary 3.7. *Every compact open isotropic subgroup of $A \times \widehat{A}$ is contained in a maximal compact open isotropic subgroup.*

Proof. Let $G \subset A \times \widehat{A}$ be a compact open isotropic subgroup and (H, K, ϕ) be its associated triple (Propositions 3.1 and 3.5), hence $K \subset H^\perp$ and ϕ is symmetric. Then the compact open isotropic subgroup associated to the triple $(H, H^\perp, \phi')$, with $\phi' : H \to \widehat{H}$ as in the proof of "(c) $\Rightarrow$ (a)" in Proposition 3.6, is maximal by Proposition 3.6 and contains G. $\square$

Remark 3.8. Notice that the result of the above corollary is no longer valid if we drop the adjective "isotropic". For example, consider the p-adic field $\mathbb{Q}_p$. Its topological dual $\widehat{\mathbb{Q}}_p$ can be identified with $\mathbb{Q}_p$. The balls $B_j := \{x \in \mathbb{Q}_p : \ |x|_p \leq p^j\}$, $j \in \mathbb{Z}$, are compact open subgroups of $\mathbb{Q}_p$. Hence the sets $B_j \times B_j$, are compact open subgroups of $\mathbb{Q}_p \times \widehat{\mathbb{Q}}_p$. However, if $K \subset \mathbb{Q}_p \times \widehat{\mathbb{Q}}_p$ is a compact subset, then $K \subset B_j \times B_j$ for some j, and $K \subset B_{j+1} \times B_{j+1}$ strictly. We refer to [14] for a quick review of the p-adic number system from the perspective of time-frequency analysis.

We conclude this section with a result which will be useful below.

Proposition 3.9. *Let $G \subset A \times \widehat{A}$ be a maximal compact open isotropic subgroup, hence $|G| = 1$ (cf. Proposition 3.6). Let $(H, H^\perp, \phi)$ be the corresponding triple (cf. Propositions 3.1, 3.5 and 3.6). If f is a character of second degree of H associated to ϕ (cf. Definition 2.4), then the function $G \to \mathbb{T}$ given by*

$$(x, \xi) \mapsto \overline{f(x)} \qquad (x, \xi) \in G$$

is a character of second degree of G associated to the continuous symmetric homomorphism $\phi' : G \to \widehat{G}$ given by

$$\langle (x, \xi), \phi'(y, \eta) \rangle = \overline{\langle x, \eta \rangle} \qquad (x, \xi), (y, \eta) \in G. \tag{3.3}$$

Proof. For $(x, \xi), (y, \eta) \in G$ we have

$$\overline{f(x+y)} = \overline{f(x)}\,\overline{f(y)}\,\overline{\langle x, \phi(y) \rangle} = \overline{f(x)}\,\overline{f(y)}\,\overline{\langle x, \eta \rangle}$$

because $\eta \in \phi(y)$ (where $\phi(y) \in \widehat{H} \simeq \widehat{A}/H^\perp$ is now regarded as a coset of $H^\perp$ in $\widehat{A}$). $\square$

4. Optimizers for the ambiguity function

From (2.5) we see that, for $f \in L^2(A)$ and $z \in A \times \widehat{A}$,

$$|V_f f(z)| \leq V_f f(0) = \|f\|^2_{L^2(A)}.$$

The following result provides some properties of the set where $|V_f f|$ attains its maximum value.

Proposition 4.1. *Let $f \in L^2(A) \setminus \{0\}$, and*

$$G = \{z \in A \times \widehat{A} : \ |V_f f(z)| = V_f f(0)\}.$$

Then G is a compact isotropic subgroup of $A \times \widehat{A}$ and the restriction of the function $\|f\|^{-2}_{L^2(A)} V_f f$ to G is a character of second degree associated to the continuous symmetric homomorphism $\phi' : G \to \widehat{G}$ given in (3.3).

Indeed, for $(x, \xi) \in G$, $(y, \eta) \in A \times \widehat{A}$ we have

$$V_f f(y + x, \eta + \xi) = \|f\|^{-2}_{L^2(A)} V_f f(x, \xi) V_f f(y, \eta) \overline{\langle x, \eta \rangle}. \tag{4.1}$$

In particular, $|V_f f|$ is constant on every coset of G in $A \times \widehat{A}$.

Proof. Without loss of generality we can suppose $\|f\|_{L^2(A)} = 1$, hence $V_f f(0) = \|f\|^2_{L^2(A)} = 1$.

Since $V_f f$ is a continuous function which vanishes at infinity (Proposition 2.1), G is compact.

For $z \in G$ we have

$$|\langle f, \pi(z)f\rangle_{L^2(A)}| = 1,$$

and therefore

$$\pi(z)f = c(z)f$$

for some $c(z) \in \mathbb{C}$, $|c(z)| = 1$. As a consequence,

$$V_f(\pi(z)f)f(w) = c(z)V_f f(w)$$

for $w \in A \times \widehat{A}$. Setting $z = (x, \xi)$ and $w = (y, \eta)$, by (2.3) we have

$$\langle x, \xi\rangle \overline{\langle x, \eta\rangle} V_f f(y - x, \eta - \xi) = c(x, \xi)V_f f(y, \eta).$$

Setting $y = 0, \eta = 0$, we obtain

$$\langle x, \xi\rangle V_f f(-x, -\xi) = c(x, \xi),$$

which gives

$$V_f f(y - x, \eta - \xi) = V_f f(-x, -\xi)V_f f(y, \eta)\langle x, \eta\rangle \tag{4.2}$$

for $(x, \xi) \in G$, $(y, \eta) \in A \times \widehat{A}$.

On the other hand, it is clear from the very definition (2.1) of $V_f f$, that $|V_f f(-w)| = |V_f f(w)|$ for $w \in A \times \widehat{A}$, and therefore if $(x, \xi) \in G$ then $(-x, -\xi) \in G$ as well.

Hence we obtain

$$V_f f(y + x, \eta + \xi) = V_f f(x, \xi)V_f f(y, \eta)\overline{\langle x, \eta\rangle} \tag{4.3}$$

for $(x, \xi) \in G$, $(y, \eta) \in A \times \widehat{A}$, which proves (4.1).

By (4.3), if $(x, \xi), (y, \eta) \in G$ then $(x + y, \xi + \eta) \in G$, hence G is a subgroup of $A \times \widehat{A}$ ($(0, 0) \in G$, of course). Finally, exchanging the roles of (x, ξ) and (y, η) in (4.3) yields that G is isotropic. □

Remark 4.2. For $A = \mathbb{R}^d$ we recapture the well known radar correlation estimate [24, Lemma 4.2.1]: if $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ and $z \in \mathbb{R}^{2d}$, $z \neq 0$, then $|V_f f(z)| < V_f f(0)$.

We now introduce the notion of *subcharacter of second degree*. This terminology – in fact non-standard – is inspired by the definition of subcharacter [30, Definition 43.3], that is a character of a compact open subgroup $H \subset A$, extended by 0 on $A \setminus H$.

Definition 4.3. Let $H \subset A$ be a compact open subgroup and let $\phi : H \to \widehat{H}$ be a continuous symmetric homomorphism. A function $h : A \to \mathbb{C}$ is called subcharacter of second degree associated to (H, ϕ) if its restriction $h|_H$ is a character of second degree of H associated to ϕ and $h(x) = 0$ for $x \in A \setminus H$.

The following result provides the ambiguity function of a subcharacter of second degree.

Proposition 4.4. *Let $h : A \to \mathbb{C}$ be a subcharacter of second degree associated to (H, ϕ) (cf. Definition 4.3); hence $H \subset A$ is a compact open subgroup and $\phi : H \to \widehat{H}$ is a continuous symmetric homomorphism. Then*

$$V_h h(x, \xi) = |H| \overline{h(-x)} \chi_G(x, \xi) \qquad (x, \xi) \in A \times \widehat{A}, \tag{4.4}$$

where G is the maximal compact open isotropic subgroup of $A \times \widehat{A}$ corresponding to the triple $(H, H^\perp, \phi)$ (cf. Propositions 3.1, 3.5 and 3.6).

Moreover the function $|H|^{-1} V_h h$ is a subcharacter of second degree of $A \times \widehat{A}$ associated to the pair (G, ϕ'), where $\phi' : G \to \widehat{G}$ is the continuous symmetric homomorphism in (3.3).

Proof. Since $h|_H$ is a character of second degree of H associated to ϕ, for $x, y \in H$ we have

$$h(y - x) = h(y) h(-x) \overline{\langle x, \phi(y) \rangle}. \tag{4.5}$$

We now compute $V_h h$. Observe that, if $x \in A \setminus H$, $\xi \in \widehat{A}$,

$$V_h h(x, \xi) = \langle h, M_\xi T_x h \rangle_{L^2(A)} = 0,$$

because $H \cap (x + H) = \emptyset$ in that case.

On the other hand, if $x \in H$, $\xi \in \widehat{A}$, by (4.5),

$$\begin{aligned} V_h h(x, \xi) &= \int_H \overline{\langle y, \xi \rangle} h(y) \overline{h(y - x)} \, dx \\ &= \overline{h(-x)} \int_H \overline{\langle y, \xi \rangle} |h(y)|^2 \langle y, \phi(x) \rangle \, dy \\ &= \overline{h(-x)} \int_H \langle y, \eta - \xi \rangle \, dy, \end{aligned}$$

where η is any element of the coset $\phi(x) \subset \widehat{A}$ (recall $\phi : H \to \widehat{H} \simeq \widehat{A}/H^\perp$). Now, in the last integral the measure dy can be regarded as a Haar measure of the compact *open* subgroup H. Therefore, the integral does not vanish if and only if ξ and η induce the same character of H ([29, Lemma 23.19]), namely if $\eta - \xi \in H^\perp$, that is $\xi \in \phi(x)$. This proves (4.4).

From (4.4) we see that

$$V_h h(x, \xi) = |H|\,\overline{h(-x)} \qquad (x, \xi) \in G.$$

It is easy to see that the function $h(-x)$, $x \in H$, is still a character of second degree of H associated to the same homomorphism ϕ. Hence, it follows from Proposition 3.9 that the function $|H|^{-1}V_h h(x, \xi)$, restricted to G (that is $\overline{h(-x)}$) is a character of second degree associated to the homomorphism ϕ' in (3.3). $\square$

We can now state the main result of this section.

Theorem 4.5. *Let $f \in L^2(A)$ and let $G = \{z \in A \times \widehat{A} : \ V_f f(z) \neq 0\}$. The following statements are equivalent:*

(a) $|G| = 1$.
(b) *G is a maximal compact open isotropic subgroup of $A \times \widehat{A}$.*
(c) *There exist $c \in \mathbb{C} \setminus \{0\}$, $x_0 \in A$ and a subcharacter h of second degree of A such that $f = cT_{x_0}h$.*

If any of the above condition is satisfied, the function $\|f\|^{-2}_{L^2(A)} V_f f$ is a subcharacter of second degree of $A \times \widehat{A}$ associated to (G, ϕ'), where $\phi' : G \to \widehat{G}$ is given in (3.3).

Proof. We can assume, without loss of generality, that $\|f\|_{L^2(A)} = 1$.

(a) $\Rightarrow$ (b) By Theorem 2.2 we have $|V_f f| = \chi_G$, and G is a compact open subset of $A \times \widehat{A}$. By Proposition 4.1, G is an isotropic subgroup of $A \times \widehat{A}$. Since $|G| = 1$, it is maximal by Proposition 3.6.

(b) $\Rightarrow$ (c) By Proposition 3.6 we have that $|G| = 1$ and therefore $|V_f f| = \chi_G$ by Theorem 2.2. In fact, by Proposition 4.1, the restriction of $V_f f$ to G is a character of second degree of $A \times \widehat{A}$ associated to the homomorphism $\phi' : G \to \widehat{G}$ in (3.3).

Let now $(H, H^\perp, \phi)$ be the triple associated to G (cf. Propositions 3.1, 3.5 and 3.6), and let h be a subcharacter of second degree of A associated to (H, ϕ) (cf. Definition 4.3), which exists by Theorem 2.5. We know from Proposition 4.4 that the function $|H|^{-1}V_h h$, restricted to G, is a character of second degree associated to the same homomorphism ϕ' as above. Hence by Theorem 2.5 there exists a character g of G such that

$$V_f f(x, \xi) = g(x, \xi)|H|^{-1}V_h h(x, \xi) \qquad (x, \xi) \in G.$$

The character g extends to a character of $A \times \widehat{A}$ ([29, Corollary 24.12]) and therefore there exist $y \in A$, $\eta \in \widehat{A}$ such that

$$g(x,\xi) = \langle x, \eta\rangle \overline{\langle y, \xi\rangle}.$$

We deduce that

$$V_f f(x,\xi) = |H|^{-1}\langle x,\eta\rangle\overline{\langle y,\xi\rangle} V_h h(x,\xi) \qquad (x,\xi) \in G.$$

In fact, this formula holds for every $(x,\xi) \in A\times\widehat{A}$ because both sides vanish on $A\times\widehat{A}\setminus G$, by (4.4) and the fact that $|V_f f| = \chi_G$. By comparison with (2.4) we deduce that

$$V_f f = |H|^{-1} V_{\pi(y,\eta)h}(\pi(y,\eta)h).$$

By Proposition 2.3 we obtain

$$f = c|H|^{-1/2}\pi(y,\eta)h$$

for some $c \in \mathbb{C}$, $|c| = 1$.

Setting $h' := M_\eta h$, we have $f = c'|H|^{-1/2}T_y h'$, $|c'| = 1$, and h' is a subcharacter of second degree associated to (H,ϕ), which gives the desired conclusion.

(c) ⇒ (a) This is clear by Propositions 4.4 and 3.6.

The last part of the statement is also clear by Proposition 4.1. □

5. Optimizers for the short-time Fourier transform

In this section we identify the functions $f, g \in L^2(A)$ such that

$$|\{z \in A\times\widehat{A} :\ V_g f(z) \neq 0\}| = 1.$$

The following result will reduce the problem to the case $f = g$, that we addressed in the previous section.

Proposition 5.1. *Let $f, g \in L^2(A)$, with $\|f\|_{L^2(A)} = \|g\|_{L^2(A)} = 1$. Let $S = \{z \in A\times\widehat{A} : |V_g f(z)| = 1\}$ and $G = \{z \in A\times\widehat{A} :\ |V_g g(z)| = 1\}$. Let $z_0 \in S$. Then $f = c\pi(z_0)g$ for some $c \in \mathbb{C}$, $|c| = 1$, and $S = z_0 + G$.*

Proof. Since $|\langle f, \pi(z_0)g\rangle_{L^2(A)}| = |V_g f(z_0)| = 1$, we have $f = c\pi(z_0)g$ for some $c \in \mathbb{C}$, $|c| = 1$. Hence, if $z \in A$,

$$|V_g f(z)| = |\langle \pi(z_0)g, \pi(z)g\rangle_{L^2(A)}| = |V_g g(z - z_0)|,$$

which implies $S = z_0 + G$. □

We therefore obtain the following characterization.

Theorem 5.2. *Let $f, g \in L^2(A)$ and let $S = \{z \in A \times \widehat{A} : \ V_g f(z) \neq 0\}$. The following statements are equivalent:*

(a) $|S| = 1$.
(b) *S is a coset in $A \times \widehat{A}$ of a maximal compact open isotropic subgroup.*
(c) *There exist $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $z_1, z_2 \in A \times \widehat{A}$ and a subcharacter h of second degree of A such that $f = c_1 \pi(z_1) h$ and $g = c_2 \pi(z_2) h$.*

Proof. The result follows easily from Theorem 2.2, Proposition 5.1 and Theorem 4.5. □

6. Maximally localized Gabor orthonormal bases

We recall that a Gabor orthonormal basis of $L^2(A)$ is an orthonormal basis of the form $\mathcal{G}(f, \Gamma)$ (cf. (1.2)), where $f \in L^2(A)$ and Γ is a (possibly uncountable) subset of $A \times \widehat{A}$.

The following result characterizes the Gabor orthonormal bases with f maximally localized, in the sense that the subset

$$G_f := \{z \in A \times \widehat{A} : \ V_f f(z) \neq 0\}$$

has measure 1. Observe that, by (2.4), $G_f = G_{\pi(w)f}$ for every $w \in A \times \widehat{A}$, so that all the elements of the basis are then maximally localized.

Theorem 6.1. *Let $f \in L^2(A)$, $\|f\|_{L^2(A)} = 1$, with $|G_f| = 1$; hence G_f is a maximal compact open isotropic subgroup of $A \times \widehat{A}$ (by Theorem 4.5). Let $\Gamma \subset A \times \widehat{A}$.*

$\mathcal{G}(f, \Gamma)$ is an orthonormal basis of $L^2(A)$ if and only if Γ is a set of representatives of the cosets of G_f in $A \times \widehat{A}$.

Proof. We know from Theorem 4.5 that $|V_f f| = \chi_{G_f}$.

Let Γ be a set of representatives of the cosets of G_f in $A \times \widehat{A}$. Since

$$|\langle \pi(z) f, \pi(w) f \rangle_{L^2(A)}| = |V_f f(w - z)|,$$

we see that $\pi(z) f$ and $\pi(w) f$ are orthogonal if $z, w \in \Gamma$, $z \neq w$, because Γ contains at most (in fact exactly) one element of each coset of G_f.

Let us verify that the set $\mathcal{G}(f, \Gamma)$ is also complete. We claim that

$$\operatorname{span}\big(\{\pi(z) f : \ z \in \Gamma\}\big) = \operatorname{span}\big(\{\pi(z) f : \ z \in A \times \widehat{A}\}\big).$$

To see this, observe that if $z \in A \times \widehat{A}$ there exists $w \in \Gamma$ such that $z - w \in G_f$, hence $|V_f f(z - w)| = 1$, which means that $\pi(z) f = c\pi(w) f$ for some $c \in \mathbb{C}$, $|c| = 1$, which yields the claim.

Now, the set $\{\pi(z)f : z \in A \times \widehat{A}\}$ is clearly complete, because if $g \in L^2(A)$ and $\langle g, \pi(z)f\rangle_{L^2(A)} = V_f g(z) = 0$ for every $z \in A \times \widehat{A}$ then $g = 0$, since the short-time Fourier transform is injective (cf. (2.6)).

Conversely, suppose that $\mathcal{G}(f, \Gamma)$ is an orthonormal basis. If $z, w \in \Gamma$, $z \neq w$, since $\pi(z)f$ and $\pi(w)f$ are orthogonal, we have $V_f f(z-w) = 0$, namely $z - w \notin G$, i.e., z and w belong to different cosets. Moreover, if Γ did not contain any element of some coset $z_0 + G$, then the function $\pi(z_0)f$ would be orthogonal to all the functions $\pi(z)f$, with $z \in \Gamma$, which is impossible since $\mathcal{G}(f, \Gamma)$ is a complete set by assumption. □

Combining Theorems 4.5 and 6.1 we deduce the desired characterization of the maximally localized Gabor orthonormal basis.

Corollary 6.2. *Let $f \in L^2(A)$, $\|f\|_{L^2(A)} = 1$, and $\Gamma \subset A \times \widehat{A}$.*

Then $\mathcal{G}(f, \Gamma)$ is an orthonormal basis of $L^2(A)$ and $|G_f| = 1$ if and only if $f = cT_{x_0}h$ for some $c \in \mathbb{C} \setminus \{0\}$, $x_0 \in A$ and some subcharacter h of second degree of A, and Γ is a set of representatives of the cosets of G_f in $A \times \widehat{A}$.

Remark 6.3. Observe that, in Corollary 6.2, the sets $\{V_f(\pi(z)f) \neq 0\} = z + G_f$, $z \in \Gamma$ (cf. (2.3)), define a tiling of $A \times \widehat{A}$ and $|z + G_f| = |G_f| = 1$. Vice versa, if $G \subset A \times \widehat{A}$ is a maximal compact open isotropic subgroup (hence $|G| = 1$), $(H, H^\perp, \phi)$ is the triple associated to G (cf. Proposition 3.6) and h is a subcharacter associated to (H, ϕ), the function $f = |H|^{-1/2}h$ generates a Gabor orthonormal basis corresponding to the tiling generated by G.

Example 6.4. Let $N \geq 1$ be an integer and let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}_N$ be the cyclic group of order N, equipped with the counting measure. We coherently choose the counting measure multiplied by N^{-1} as the Haar measure on the dual group.

On $\mathbb{Z}_N$ a subcharacter of second degree has the form $h = M_\xi h_{b,p}$, where $\xi \in \widehat{\mathbb{Z}}_N$, $b \geq 1$ is a divisor of N, $p \in \{0, \dots, b-1\}$, and,

$$h_{b,p}(x) = \begin{cases} \exp\left(\frac{\pi i p x^2 b(1+b)}{N^2}\right) & x \in a\mathbb{Z}_N \\ 0 & x \in \mathbb{Z}_N \setminus a\mathbb{Z}_N, \end{cases}$$

where $a = N/b$ (see [49, Remark 2.1] and [16, Section 3 (iii)]). We also have

$$G_h = \{(ma, nb + mp) : \ m = 0, \dots, b-1, \ n = 0, \dots, a-1\},$$

(see the proof of [49, Theorem 1.2]), which is indeed a subgroup of $\mathbb{Z}_N \times \widehat{\mathbb{Z}}_N$ of cardinality N, hence of measure 1 (incidentally, all the subgroups of cardinality N have this form), and Corollary 6.2 applies.

Of course, on $\mathbb{Z}_N$ there are Gabor orthonormal bases $\mathcal{G}(f, \Gamma)$ which are not maximally localized, e.g., we can take $f = 2^{-1/2}\chi_{\{0,1\}}$ and $\Gamma = 2\mathbb{Z}_N \times (N/2)\mathbb{Z}_N$, assuming $N \geq 4$ even. A straightforward computation shows that

$$G_f = (\{0, 1, N-1\} \times \mathbb{Z}_N) \setminus \{(0, N/2)\},$$

so that $|G_f| = 3 - 1/N > 1$.

We also obtain the following result for finite Abelian groups.

Corollary 6.5. *Let A be a finite Abelian group and $S \subset A \times \widehat{A}$. The following statements are equivalent, for the family of operators $\{\pi(z) : \ z \in S\}$:*

(a) *There exists a common eigenfunction.*
(b) *The operators $\pi(z)$, $z \in S$, commute.*
(c) *There is a Gabor orthonormal basis, which consists of common eigenfunctions, generated by a function $f \in L^2(A)$, with $|G_f| = 1$.*

Proof. (a)⇒(b) If $f \in L^2(A)$ is a common eigenfunction, with $\|f\|_{L^2(A)} = 1$, then $|V_f f(z)| = 1$ for $z \in S$, because the eigenvalues of $\pi(z)$ have modulus 1. Hence $S \subset G' := \{z \in A \times \widehat{A} : \ |V_f f(z)| = 1\}$, and G' is an *isotropic* subgroup of $A \times \widehat{A}$ by Proposition 4.1. Hence the operators $\pi(z)$, with $z \in S$, commute by (2.2).

(b)⇒(c) Since the operators $\pi(z)$, $z \in S$, commute, the subgroup generated by S is isotropic. It is moreover contained in some maximal isotropic subgroup G (whose existence is obvious, because A is finite; see also Corollary 3.7). Let $(H, H^\perp, \phi)$ be the triple associated to G (cf. Propositions 3.1, 3.5 and 3.6) and let h be a subcharacter of second degree associated to the pair (H, ϕ) (cf. Definition 4.3), which exists by Theorem 2.5 (see also [34]). By Proposition 4.4, for the function $f = |H|^{-1/2}h$ we have $\|f\|_{L^2(A)} = 1$ and $|V_f f| = \chi_G$, and by Theorem 6.1 f generates a Gabor orthonormal basis $\mathcal{G}(f, \Gamma)$, for a suitable subset $\Gamma \subset A \times \widehat{A}$. Since $S \subset G$, $|\langle f, \pi(z) f\rangle_{L^2(A)}| = |V_f f(z)| = 1$ for $z \in S$, so that f is a common eigenfunction of the operators $\pi(z)$, $z \in S$, and therefore, by (2.2), every function $\pi(w)f$, with $w \in A \times \widehat{A}$, is a common eigenfunction too.

(c)⇒(a) This is obvious. □

We point out that extensive numerical experiments on eigenfunctions of time-frequency shifts were done by H. Feichtinger (private communication), in connection with the work [33].

7. Lieb's uncertainty inequality

The following result was first proved in [43] for the group $A = \mathbb{R}$, and then extended to a general LCA group in [23], following essentially the same proof.

We recall that every locally compact Abelian group A is topologically isomorphic to $\mathbb{R}^d \times A_0$, for some integer $d \geq 0$ and some LCA group A_0 containing a compact open subgroup, and the dimension d is an invariant [29, Theorem 24.30].

Theorem 7.1 *(Lieb's uncertainty inequality). For $f, g \in L^2(A)$, we have*

$$\|V_g f\|_{L^p(A\times\widehat{A})} \le \left(\frac{2}{p}\right)^{d/p} \|f\|_{L^2(A)}\|g\|_{L^2(A)} \qquad 2 \le p < \infty \tag{7.1}$$

and

$$\|V_g f\|_{L^p(A\times\widehat{A})} \ge \left(\frac{2}{p}\right)^{d/p} \|f\|_{L^2(A)}\|g\|_{L^2(A)} \qquad 1 \le p \le 2. \tag{7.2}$$

Using only (2.5) one easily obtains similar estimates – in fact weaker, if $d > 1$ – with the constant $\left(\frac{2}{p}\right)^{d/p}$ replaced by 1 (see Theorem 7.2 below), namely

$$\|V_g f\|_{L^p(A\times\widehat{A})} \le \|f\|_{L^2(A)}\|g\|_{L^2(A)} \qquad 2 \le p < \infty \tag{7.3}$$

and

$$\|V_g f\|_{L^p(A\times\widehat{A})} \ge \|f\|_{L^2(A)}\|g\|_{L^2(A)} \qquad 0 < p \le 2, \tag{7.4}$$

where now the case $0 < p < 1$ is also included. These estimates are sharp if A contains a compact open subgroup (i.e., $d = 0$).

We are going to prove that the pairs of functions f, g for which equality is attained in (7.3) or (7.4) are precisely those for which the set where $V_g f \ne 0$ has measure 1, which have been characterized in Theorem 5.2.

Theorem 7.2. *Let A be any LCA group. Then (7.3) and (7.4) hold true.*

Equality holds in (7.3) for some $p \in (2,\infty)$ and $f, g \in L^2(A) \setminus \{0\}$ if and only if there exist $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $z_1, z_2 \in A \times \widehat{A}$ and a subcharacter h of second degree of A such that $f = c_1\pi(z_1)h$ and $g = c_2\pi(z_2)h$. In that case, equality occurs in (7.3) for every $p \in [2,\infty)$.

A similar uniqueness result holds true for the inequality (7.4), for $0 < p < 2$.

Proof. We can suppose $\|f\|_{L^2(A)} = \|g\|_{L^2(A)} = 1$.

Let $2 \le p < \infty$ and set $S = \{z \in A \times \widehat{A} :\ V_g f(z) \ne 0\}$. Using (2.5) and (2.6) we see that

$$\begin{aligned}\int_S |V_g f(x,\xi)|^p \,dx\,d\xi &= \int_S |V_g f(x,\xi)|^{p-2} |V_g f(x,\xi)|^2 \,dx\,d\xi \\ &\le \int_S |V_g f(x,\xi)|^2 \,dx\,d\xi = 1,\end{aligned}$$

which proves (7.3). If equality occurs in the above inequality and $2 < p < \infty$ then $|V_g f| = \chi_S$ and $|S| = 1$, which implies the desired conclusion for the functions f and g by Theorem 5.2.

The result for the inequality (7.4), hence $0 < p \le 2$, is analogous, using

$$1 = \int_S |V_g f(x,\xi)|^2\, dx\, d\xi = \int_S |V_g f(x,\xi)|^{2-p} |V_g f(x,\xi)|^p\, dx\, d\xi$$
$$\le \int_S |V_g f(x,\xi)|^p\, dx\, d\xi. \quad \square$$

Remark 7.3. If $f, g \in L^2(A)$, we have $V_g f \in L^\infty(A \times \widehat{A})$ by (2.5). Hence, by monotone convergence,

$$\lim_{p \to 0^+} \int_{A \times \widehat{A}} |V_g f(x,\xi)|^p\, dx d\xi = |\{z \in A \times \widehat{A} :\ V_g f(z) \neq 0\}|.$$

As a consequence, raising to the power p both sides of (7.4) and taking the limit as $p \to 0^+$, we obtain that, if f and g are non-zero,

$$|\{z \in A \times \widehat{A} :\ V_g f(z) \neq 0\}| \ge 1,$$

that is the inequality in Theorem 2.2.

It is easy to check that, on a general LCA group $A = \mathbb{R}^d \times A_0$, for $f_1, g_1 \in L^2(\mathbb{R}^d)$ and $f_2, g_2 \in L^2(A_0)$, we have

$$V_{g_1 \otimes g_2}(f_1 \otimes f_2) = V_{g_1} f_1 \otimes V_{g_2} f_2.$$

As a consequence, for fixed $p \in [1, \infty)$, if f_1, g_1 is a pair of optimizers for Lieb's L^p-inequality in $\mathbb{R}^d$ (Theorem 7.1) and similarly for f_2, g_2 on A_0, then $f_1 \otimes f_2, g_1 \otimes g_2$ is a pair of optimizers for the Lieb's L^p-inequality on A. We now show a family of such optimizers. To this end, we need some terminology, inspired by [43].

Definition 7.4. A function f on $\mathbb{R}^d$ is called a Gaussian if

$$f(x) = \exp(-\alpha x \cdot x + i\beta x \cdot x + \gamma \cdot x + \delta),$$

where α is a real symmetric positive definite $d \times d$ matrix, β is a real symmetric $d \times d$ matrix, $\gamma \in \mathbb{C}^d$ and $\delta \in \mathbb{C}$. Two functions f, g are called a *matched Gaussian pair* if f and g are both Gaussian with the same α's and β's but with possibly different γ's and δ's.

Similarly, a pair of functions f, g on a LCA group A is called a *matched pair of subcharacters of second degree* if $f = c_1 \pi(z_1) h$ and $g = c_2 \pi(z_2) h$ for some $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $z_1, z_2 \in A \times \widehat{A}$ and some subcharacter h of second degree of A.

It is easy to check that for a matched Gaussian pair f, g, equality occurs in (7.1) and (7.2) ($A = \mathbb{R}^d$). For $A = \mathbb{R}$ it was proved in [43] that these are in fact the only pairs of non-zero optimizers if $p \neq 2$.

The previous discussion therefore leads to the following result.

Proposition 7.5. *Let f_1, g_1 be a matched Gaussian pair on $\mathbb{R}^d$ and let f_2, g_2 be a matched pair of subcharacters of second degree on A_0. Then, for the functions $f := f_1 \otimes f_2$ and $g := g_1 \otimes g_2$ on $A = \mathbb{R}^d \times A_0$, equality occurs in* (7.1) *and* (7.2) *for every $p \in [1, \infty)$.*

The optimizers where f_1 and g_1 are time-frequency shifts of the Gaussian $\exp(-\pi|x|^2)$, and f_2 and g_2 are time-frequency shifts of the characteristic function of some compact open subgroup of A, were already known from [23].

We postpone to a future work the problem of identifying all the optimizers on a general LCA group – as already observed, the case $A = \mathbb{R}$ was addressed in [43], whereas the case $A = A_0$ is the content of Theorem 7.2 above.

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