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Zero-dimensional schemes: curves, singularities and tensors

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Summary

In this thesis we deal with some problems regarding zero-dimensional schemes related to different areas of Algebraic Geometry. In particular, we draw inspiration from the investigation of the Jacobian scheme of a plane algebraic curve to define a new class of schemes, which we call *symmetric schemes*. We study the algebraic and geometric properties of these schemes and we use them to define some new varieties parameterising symmetric and partially symmetric tensors. Thanks to these new varieties we recover some already known results about tensors and we are able to prove some generalisations of them. Finally, we study the Hilbert functions of zero-dimensional reduced schemes lying on Veronese surfaces and we use our characterisation to classify the complete intersection zero-dimensional schemes lying on Veronese surfaces and rational normal curves.

Lasciamo il paladin ch'errando vada:
ben di parlar di lui tornerá tempo.
Quanto, Signore, ad Angelica accada
dopo ch'uscí di man del pazzo a tempo;
e come a ritornare in sua contrada
trovasse e buon navilio e miglior tempo,
e de l'India a Medor desse lo scettro,
forse altri canterá con miglior plettro

L. Ariosto, Orlando Furioso (XXX, 16)

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Introduction

In the wide panorama of Algebraic Geometry, a key role is played by Scheme Theory and a very noteworthy class of schemes is the one of zero-dimensional schemes. Indeed, apart from their intrinsic interest, zero-dimensional schemes deserve a particular and careful consideration in light of their several interactions with other fields of Algebraic Geometry. For instance:

i) many problems concerning secant varieties of projective varieties can be translated, via Apolarity Theory, in problems concerning zero-dimensional schemes; see [44] and [66] for more details about this topic;

ii) some intensely studied topics regarding plane algebraic curves, such as freeness and computation of Tjurina and Milnor numbers, are strictly related to the analysis of Jacobian schemes, which are zero-dimensional schemes encoding all the information about the singularities of the curve; see [82] and [104] for more details on free curves and [8] for more details on computations of Tjurina and Milnor numbers;

iii) zero-dimensional schemes are the constitutive elements of Hilbert schemes of points, a widely studied branch of Algebraic Geometry; see [72] and [80] for more details on Hilbert schemes.

Zero-dimensional schemes allow establishing deeper connections between these three topics; see [29], [30], [31], [32] and [80] for connections between i) and iii), and see [37] and [55] for connections between i) and ii). Also see [48] for a collection of topics about zero-dimensional schemes.

Beyond Algebraic Geometry, some other research fields where zero-dimensional schemes find applications are:

- Commutative Algebra, where they can be used, for instance, for a geometrical approach to Artin algebras and Gorenstein rings; see [81] for a recent state of the art;

- Code Theory for error-correcting codes associated to 0-dimensional schemes; see [15] for some application.

A particularly interesting class of zero-dimensional schemes is represented by fat points, which have long been, and still are, at the core of many Algebraic Geometry problems. Indeed, they represent a powerful tool for the study of many problems, such as the computation of the defectivity of some secant varieties and the study of singular points of projective varieties. These two aspects are among those that will be addressed in this thesis. Nowadays, our knowledge of fat points is certainly very rich, but nonetheless, there are still important open problems associated with them, such as the Gimigliano-Harbourne-Hirschowitz-Segre conjecture and, more in general, the complete classification of fat point schemes with bad postulation. See [66] for an exhaustive state of the art on fat points and see [2], [4], [5], [6], [7], [14], [16], [20], [21] for some applications of fat points to the study of secant varieties.

The main purpose of this thesis is to generalise fat points by introducing a new class of zero-dimensional schemes. In the literature, there already are some examples of such generalisations that broaden, for instance, the definition from $\mathbb{P}^n$ to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. An important example can be found in [43], where fat points in multiprojective spaces are used to study a classical problem concerning the dimension of certain secant varieties of Segre varieties. However, to our knowledge, there are no generalisations that, given the ambient space, specifically pertain to the geometry of the scheme itself. This is the type of generalisation that we are seeking in the thesis. More specifically we state the following question:

Question 1. *How can one define a new class of schemes in $\mathbb{P}^n$ that generalises the class of fat points?*

There are several motivations behind this question, some more related to the topics of this thesis.

The first one sprouts from studying the Jacobian scheme of a plane algebraic curve, which, in recent years, has proven to be of great interest. The two aspects most carefully considered in connection with the Jacobian scheme are the study of free divisors and the computation of Tjurina number for isolated singularities; see [53] for more details on the former, and see [8] for more details on the latter. However, despite the extensive research on these topics, we are not aware of any analysis on the geometric structure of the Jacobian scheme. We address this gap and, carrying on this analysis for ordinary singularities, we note that fat points play an important role, but are not sophisticated enough to provide a satisfactory geometric analysis. For this reason, it is necessary to extend the definition of fat point and consider a broader class of schemes. This concept will be clarified below.

Another motivation comes from the application of algebraic geometry to the study of tensors. Indeed, zero-dimensional schemes have proven to be very useful tools for studying many problems in this context. Some important examples in this regard can be found in [14], [20], [21], [22], [29], [30], [32], [43], [45], [66], where zero-dimensional schemes are used to study and generalise secant varieties of some classical projective varieties, such as Segre, Veronese and Segre-Veronese varieties. In particular, new classes of zero-dimensional schemes can give new information on the geometry of tensors. We will show how to get this information via our new schemes.

This thesis is divided into five chapters. Chapter 1 is totally devoted to present the mathematical entities which are the objects of our research and to introduce the tools we will use to describe them. We start by recalling some general definitions and properties of Hilbert functions and fat points, and by giving a quick overview on Apolarity Theory and Inverse Systems. After that, we introduce Segre, Veronese and Segre-Veronese varieties in the setting of Waring-like problems, stressing the interchangeability of the algebraic, geometrical and tensorial interpretations of these varieties. We also present the machinery of secant varieties, showing how zero-dimensional schemes can be used to study the defectivity of secant varieties. In particular, we briefly describe how the postulation of zero-dimensional schemes can be studied via the Horace method and the differential Horace method. Finally, we give some definitions about singularities of plane algebraic curves and we recall the Jacobian and Milnor schemes related to a plane algebraic curve.

In Chapter 2 we devote our attention to a special type of zero-dimensional schemes: the Jacobian scheme of a plane algebraic curve. To this purpose we start by proving an algebraic version for plane curves of the famous Mather-Yau theorem, stated in [89], which allows us to simplify the next results. After that, we focus on the Jacobian schemes at ordinary singularities, and this study suggests us the introduction of a new class of schemes answering Question 1: symmetric schemes. In Chapter 2 we give the definition only for the projective plane. We also provide some examples of ordinary singularities whose Tjurina number is strictly less than the Milnor number, so partially recovering, with more algebraic tools, some results of [27] and [87].

In Chapter 3 we give the definition of symmetric scheme for any $\mathbb{P}^n$ and we point out how symmetric schemes are a generalisation of fat points. We also introduce the definition of *superfat points* and we study the geometry of these new schemes. Since it is quite difficult to manage symmetric schemes in $\mathbb{P}^n$, after some general results, we narrow down to symmetric schemes of $\mathbb{P}^2$. After showing some of their properties, we use them to define some new varieties parameterising symmetric and partially symmetric tensors. We study the defectivity of these varieties and the shape of the tensor parameterised by them.

In Chapter 4 we prove the good postulation of generic unions of 2-squares in $\mathbb{P}^2$. To do that, we use the Horace method ad we provide two different proofs. The two proofs only differ in proving the good postulation with respect to curves of even degrees, which are the hardest ones: in the first proof we use the differential Horace method, while in the second one we avoid using the differential Horace method and we solve the problem giving an argument based on a particular property of 2-squares.

Finally, in Chapter 5 we deal with the classification of reduced zero-dimensional schemes lying on Veronese varieties. We show how this problem can be considered as a generalisation of the Cramer-Euler problem and we completely solve it for the case of Veronese surfaces. The main tool we use to give our classification is an accurate study of the possible Hilbert functions of reduced points on Veronese surfaces. We conclude the chapter with a conjecture on complete intersections lying on Veronese varieties, inspired by the case of Veronese surfaces and by other experimental evidences.

From Jacobian schemes to symmetric schemes

The reasons that led us to pose Question 1 arose from studying a particular type of zero-dimensional schemes: the Jacobian scheme of a plane algebraic curve. However, before bringing up the "more sophisticated" Jacobian schemes, let us explore the origin of the idea of using zero-dimensional schemes, in particular fat points, to study the singularities of plane algebraic curves.

The first well-known remark is that if $\mathcal{C} : F = 0$ is a reduced curve of $\mathbb{P}^2$ passing through a point P, then saying that $\mathcal{C}$ has a singular point of multiplicity m at P means that $\mathcal{C}$ contains the fat point mP but not the fat point $(m+1)P$. This is a very rough information, because it does not allow to distinguish different analytical classes of singularities having the same multiplicity. Nevertheless, there are other 0-dimensional schemes contained in $\mathcal{C}$ which could characterise the singularity more carefully. For example, if P is an A_n singularity, then P is a nodal-type singularity if and only if for any $\ell \geq 1$ there is a curvilinear scheme supported at P of length ℓ contained in $\mathcal{C}$, while P is a cuspidal singularity A_{2r} if and only if for any $\ell \leq 2r+1$ there is a curvilinear scheme supported at P of length ℓ contained in $\mathcal{C}$, and no curvilinear scheme supported at P of length $> 2r+1$ is contained in $\mathcal{C}$ (see [68], Theorem 2.3). So, one possible approach to study a singularity is to understand which kind of "maximal" zero-dimensional schemes supported at P are contained in $\mathcal{C}$ but, since the curve $\mathcal{C}$ is 1-dimensional, it might contain curvilinear schemes supported at P of arbitrary lengths. We can undertake another way by using $\mathbb{X}(\mathcal{C})$, the Jacobian scheme of $\mathcal{C}$, which is defined as the subscheme of $\mathbb{P}^2$ associated to the Jacobian ideal

$$\mathbb{J}(\mathcal{C}) := \left(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) \subseteq \mathbb{C}[x_0, x_1, x_2].$$

Indeed, the Jacobian scheme is the zero-dimensional scheme encoding all the information, up to analytical equivalence, of all the singularities of $\mathcal{C}$.

An analogue of the Jacobian algebra $\mathbb{C}[x_0, x_1, x_2]/\mathbb{J}(\mathcal{C})$ can be defined, mutatis mutandis, also in the set of analytic geometry and it is a highly studied topic; see for instance [70] and [71]. As proved by a famous theorem of J. N. Mather and S. S.-T. Yau in [89], the Jacobian algebras of two analytical germs at O, both having an isolated singularity at O, are isomorphic as $\mathbb{C}$-algebras if and only if the two germs are analytically equivalent. Clearly, this result greatly simplifies the study of singularities up to analytical equivalence. However, it has the disadvantage, in case one wishes to work in an algebraic context, of requiring a transition from the algebraic setting to the analytic one. In order to avoid this transition, we prove, in Chapter 2, an analogue of Mather-Yau Theorem in a purely algebraic context.

Theorem 1. (Theorem 2.1.6) *Let $\mathcal{C} : f = 0$, $\mathcal{D} : g = 0$ be reduced algebraic curves in $\mathbb{A}^2$ with a singular point at O. Then the analytical germs of $\mathcal{C}$ and $\mathcal{D}$ at O are analytically equivalent if and only if their (algebraic) Jacobian schemes at O are isomorphic as schemes over $\mathbb{C}$.*

After proving this theorem, we focus on the geometry of Jacobian schemes at ordinary singularities and on their Tjurina and Milnor numbers. In doing so, we remark that these schemes possess a particular symmetry property: each line passing through their support intersects them with the same length. Let us be more precise.

Definition 1. (Definition 2.2.4) Let Y be a 0-dimensional scheme supported at one point $P \in \mathbb{P}^2$. We say that Y is *k-symmetric* if, for every line r passing through P, $\ell(Y \cap r) = k$. We say that Y is a *k-symmetric local complete intersection* (k-slci for short) if it is a local complete intersection of two curves $\mathcal{D}, \mathcal{E}$ with no tangent in common at P and such that $m_P(\mathcal{D}) = k, m_P(\mathcal{E}) = k$, this implying $\ell(Y) = k^2$.

Clearly, a k-slci is k-symmetric.

Theorem 2. (Theorem 2.2.7) *Let P be a multiple ordinary point of multiplicity m for a plane curve $\mathcal{C}$ in $\mathbb{P}^2$ and let Z_P be its Milnor scheme at P and X_P be its Jacobian scheme at P. Then:*

1. *the tangent cones of the derivative curves $\mathcal{C}_x, \mathcal{C}_y$ have no lines in common, hence $Z_P = (\mathcal{C}_x \cap \mathcal{C}_y)_P$ is a $(m-1)$-slci, so that $\mu = \ell(Z_P) = (m-1)^2$;*

2. *X_P is a $(m-1)$-symmetric scheme and $\tau = \ell(X_P) \leq (m-1)^2$;*

3. *in particular, if $\mathcal{C}$ is a union of m distinct lines through P, then $X_P = Z_P$, so that $\ell(X_P) = (m-1)^2$.*

It is precisely this theorem that has inspired Question 1. In fact, the only case in which the Jacobian scheme of an ordinary singularity is a fat point is the case of nodes, that is, double points with two distinct principal tangents. In all other cases, the obtained schemes are not fat points, but share with them the symmetry property stated in Definition 1. Note that one can wonder if all symmetric schemes arise as Jacobian schemes at ordinary singularities, however, there are many examples of symmetric schemes that cannot be obtained in this way; see Remark 2.2.6 for some examples.

We will shortly discuss how Theorem 2 not only inspired Question 1 but also a possible answer to it. Before that, however, we want to emphasise that Theorem 2 also suggests another question:

Question 2. *Do there exist ordinary singularities whose Tjurina number is strictly less than the Milnor number?*

Questions of this kind date back to Zariski and appear quite often in Algebraic and Analytic Geometry; see for instance [8] and [107]. In fact, Question 2 already has a complete answer, which can be recovered using some results of [27] and [87]. In Chapter 2, we state the result in the form of following theorem.

Theorem 3. (Theorem 2.2.10) *Let $\mathcal{C}$ be a plane algebraic curve and assume that $P \in \operatorname{Sing} \mathcal{C}$ is a multiple ordinary point of multiplicity $m \geq 2$. Then*

$$\left\lfloor \frac{3m^2 - 2m - 4}{4} \right\rfloor \leq \tau_P(\mathcal{C}) \leq (m-1)^2.$$

Moreover, the bounds are sharp and all the values of $\tau_P(\mathcal{C})$ occur.

Despite Question 2 being fully answered, in Chapter 2 we provide some explicit examples of ordinary singularities whose Tjurina number is strictly less than the Milnor number. Our examples are special cases of a more general class of curves given in [27], but there is a main difference: the approach used in [27] is analytical, while ours is entirely algebraic. We consider the family of curves

$$\mathcal{C}_{b,c} : x^m + y^m + x^b y^c = 0$$

with $b + c > m$, having an ordinary singularity at O and we compute the Tjurina number $\tau_O(\mathcal{C}_{b,c})$ using Gröbner basis. In particular, we prove in Theorem 2.3.6 that for $m \geq 5$ the curves $\mathcal{C}_{b,c}$ attain the lower bound in Theorem 3.

Symmetric schemes and tensors

As anticipated, the symmetric schemes inspired by Theorem 2 give a satisfying answer to Question 1. In Chapter 3 we start by generalising the definition of m-symmetric scheme and m-symmetric local complete intersection as follows.

Definition 2. (Definition 3.1.1) A 0-dimensional scheme X supported at one point $P \in \mathbb{P}^n$ is said to be

- m-*symmetric* if $\ell(X \cap L) = m$, for every line L passing through P;
- an m-*symmetric local complete intersection* (m-slci for short) if it is a local complete intersection of n hypersurfaces having multiplicity at P equal to m and whose tangent cones at P have no line in common.

The reason why m-symmetric schemes are good candidates to generalise fat points, is that m-fat points are the prime example of m-symmetric schemes and, moreover, any m-symmetric scheme supported at $P \in \mathbb{P}^n$ contains the fat point mP. In other words, fat points are the m-symmetric schemes which are minimal with respect to the schematic inclusion. In light of that, we found quite natural to ask the following questions:

Question 3. *Among all the m-symmetric schemes supported at the same point P, which are the maximal ones with respect to schematic inclusion?*

Question 4. *What is the maximum length of an m-symmetric scheme?*

Since these points are, in some sense, "fatter" than fat points, we call the maximal m-symmetric schemes m-*superfat points* and we answer to both questions thanks to the following theorem.

Theorem 4. (Theorem 3.1.9) *A scheme $X \subseteq \mathbb{P}^n$ is an m-superfat point supported at $P \in \mathbb{P}^n$ if and only if it is an m-slci. Thus, any m-superfat point in $\mathbb{P}^n$ has length m^n and it is a Gorenstein scheme.*

We also stress the existence of a special class of m-superfat points of $\mathbb{P}^n$, that of m-hypercubes, i.e. m-superfat points defined by an ideal of the form $(\ell_1^m, \ell_2^m, \dots, \ell_n^m)$ for $\ell_1, \dots, \ell_n \in \mathbb{C}[x_0, \dots, x_n]_1$ linearly independent linear forms. However, even though up to this moment we mentioned just m-fat points, m-superfat points and m-hypercubes, there are other schemes possessing m-symmetry; see, for instance, Remark 3.1.11.

This last remark shows how bad the situation can be in $\mathbb{P}^n$. For this reason, after we have given some general results on symmetric schemes, we narrow down to the case of $\mathbb{P}^2$, where the situation is easier to manage. The first noteworthy result in this direction is the coincidence of 2-superfat schemes of $\mathbb{P}^2$ with 2-squares, i.e. with the 2-hypercubes of $\mathbb{P}^2$.

Proposition 1. (Proposition 3.2.2) *Every 2-superfat scheme $X \subseteq \mathbb{P}^2$ is a 2-square, i.e. $\mathcal{I}_X$ can be written, up to some projectivity, as $\mathcal{I}_X = (x_1^2, x_2^2)$.*

This identification has nothing similar neither in higher dimension nor in higher degree; see Example 3.2.3. The other main result about superfat schemes in $\mathbb{P}^2$ is the following theorem, which allows us to relate fat points and superfat points.

Theorem 5. (Theorem 3.2.6) *For every $P \in \mathbb{P}^2$ and for any $m \geq 1$, the schematic union of all m-squares supported at P is the fat point $(2m-1)P$.*

A very classical issue when dealing with zero-dimensional schemes is the study of their postulation. For this reason, after analysing the aforementioned properties of 2-squares, we deemed appropriate to investigate the postulation of a generic union of 2-squares.

Question 5. *What is the postulation of a generic union of 2-squares in $\mathbb{P}^2$?*

In Chapter 4 we answer this question by showing that a generic union of 2-squares always has good postulation. The proof strategy we use is the "Horace method", introduced by J. Alexander and A. Hirschowitz in several papers, which we briefly recall in §1.6. We provide two different proofs, which agree for odd degrees but differ for the even ones. Indeed, the odd degree case can be solved using some simple specialisations, while the even one is more challenging.

In the first proof, we solve the problem by introducing a new specialisation: we collapse two 2-squares together, thus finding a new scheme that, with the help of differential Horace method, allows to bypass the arithmetic obstruction.

The idea of the second proof for even degrees is the following: we start by substituting one of the 2-squares with a double point contained in it, so obtaining a subscheme of the initial scheme and proving by induction that the number of conditions imposed on the degree d curves by this new scheme is one less than the expected number of conditions imposed by the initial scheme. After proving that, we conclude coming back to the original scheme and proving that when we pass from the double point to the 2-square, we actually impose one more condition.

As we have already mentioned, zero-dimensional schemes have proven to be very useful in the study of varieties parameterising tensors. For this reason, once enough tools to handle the 2-squares are obtained, it is quite natural to pose the following question:

Question 6. *Is it possible to obtain new information about tensors using our new class of symmetric schemes? If so, what kind of information?*

We partially answer this question for the special case of 2-squares but, as we will recall in the list of open problems at the end of this introduction, we reckon that a general insight of symmetric schemes can provide considerable information about symmetric and partially symmetric tensors. In our analysis, we consider some embeddings of 2-squares on Veronese and Segre-Veronese varieties, constructing a "bridge" between 2-squares and (partially) symmetric tensors. By doing so, we define new varieties, that we briefly describe here.

- $Q(V_{2,d})$ (Considered in §3.3)

  We define
  $$Q^0(V_{2,d}) := \bigcup_{Q \subseteq \mathbb{P}^2} L(\nu_{2,d}(Q)), \qquad Q(V_{2,d}) = \overline{Q^0(V_{2,d})}$$
  where the union is taken on all the 2-squares Q of $\mathbb{P}^2$. Even though we show that $Q(V_{2,d}) = \tau_2(V_{2,d})$, and thus $Q(V_{2,d})$ is an already known variety, this new way of defining it gives a more refined description of the forms in $\tau_2(V_{2,d})$; see Proposition 3.3.2. As a consequence, we can show that $\tau_2(V_{2,d})$ is always contained in $\sigma_4(V_{2,d})$; see Corollary 3.3.3.

- $QQ(V_{2,d})$ (Considered in §3.3)

  The description given by $Q(V_{2,d})$ highlights that the variety $\tau_2(V_{2,d})$ contains a 1-codimensional subvariety parameterising more particular forms, namely the ones that can be written (up to a projectivity in $\mathbb{P}^2$) as $y_0^{d-2}y_1y_2$. Let $d \geq 3$ and consider the morphism
  $$\begin{array}{rccc} \Phi: & \mathbb{P}(T_1) \times \mathbb{P}(T_1) \times \mathbb{P}(T_1) & \to & \tau_2(V_{2,d}) \subseteq \mathbb{P}(T_d) \\ & ([\ell_0], [\ell_1], [\ell_2]) & \mapsto & [\ell_0^{d-2}\ell_1\ell_2] \end{array}.$$
  The *cuckoo variety* $QQ(V_{2,d})$ of $V_{2,d}$ is defined to be the scheme theoretic image of Φ, that is,
  $$QQ(V_{2,d}) := \operatorname{Im} \Phi.$$
  We investigate the geometrical properties of $QQ(V_{2,d})$ in Proposition 3.3.7.

- $q_2(SV_{d,d})$ (Considered in §3.4)

  After considering Veronese varieties, we move on to Segre-Veronese varieties and, more precisely, we consider the (d,d)-embeddings
  $$\mathbb{P}^1 \times \mathbb{P}^1 \to SV_{d,d} \subseteq \mathbb{P}^{d^2-1}.$$
  Clearly, to do that we need to specify what we mean by a 2-square in $\mathbb{P}^1 \times \mathbb{P}^1$: given a point $P = [a_0, a_1; b_0, b_1]$ we call 2-square of $\mathbb{P}^1 \times \mathbb{P}^1$ supported at P the 0-dimensional subscheme $Q_P \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defined by the bihomogeneous ideal $(\ell_{1,0}^2, \ell_{0,1}^2) \subseteq \mathcal{R}$, where
  $$\ell_{1,0} = a_1x_{1,0} - a_0x_{1,1}, \quad \ell_t = b_1x_{2,0} - b_0x_{2,1}.$$
  The reason why we choose these subschemes is explained in Remark 3.4.1. At this point we can define, for any $d \geq 2$, the following varieties:
  $$q_2(SV_{d,d}) := \overline{\bigcup_{P \in \mathbb{P}^1 \times \mathbb{P}^1} L(sv_{d,d}(Q_P))}$$

whose points correspond to partially symmetric tensors of the form

$$(a_0s_0 + a_1s_1)^d(b_0t_0 + b_1t_1)^d.$$

We show that $q_2(SV_{d,d})$ and its secant variety $\sigma_2(q_2(SV_{d,d}))$ have the expected dimension for any $d \geq 2$; see Proposition 3.4.6 and Proposition 3.4.7. In particular, for $d = 2$ the secant variety $\sigma_2(q_2(SV_{d,d}))$ fills the whole $\mathbb{P}^8$, thus any partially symmetric tensor in $\mathbb{P}^8$ can be written as the sum of two partially symmetric tensors which depend only on four parameters each.

- $qq_2(SV_{(d,d)})$ (Considered in §3.4)

  Analogously to the cuckoo varieties $QQ(V_{2,d})$, we define the *cuckoo varieties* $qq_2(SV_{d,d})$ as the image of the morphism

$$\begin{array}{ccc} \mathbb{P}(\mathcal{T}_1^{(1)}) \times \mathbb{P}(\mathcal{T}_1^{(1)}) \times \mathbb{P}(\mathcal{T}_1^{(2)}) \times \mathbb{P}(\mathcal{T}_1^{(2)}) & \to & q_2(SV_{d,d}) \subseteq \mathbb{P}(\mathcal{T}_{d,d}) \\ ([m_s],[n_s],[m_t],[n_t]) & \mapsto & [m_s^{d-1}n_sm_t^{d-1}n_t] \end{array}.$$

  For $d = 2$, $qq_2(SV_{2,2})$ is the Segre Variety $S_{2,2}$, which is well-known to be 2-defective, i.e. $\dim \sigma_2(S_{2,2}) = 7$. This does not happen for $d \geq 3$, as shown in Proposition 3.4.13.

Zero-dimensional schemes on Veronese varieties

In Chapter 5, we change a bit our perspective and we consider the following problem related to the geometry of Veronese varieties:

Question 7. *What are the possible complete intersections lying on a Veronese variety $V_{n,d}$?*

There are several reasons that make this question interesting. Indeed, complete intersections and their algebraic counterpart, regular sequences, play a central role in Commutative Algebra and in Algebraic geometry. We have examples ranging from the more classical and still open Hartshorne conjecture to modern applications in the field of geometry of tensor. In fact, complete intersections have recently been shown to have unexpected applications. For example, in [18] and [25], the strength and the slice rank of polynomials are studied using complete intersections. For a more exhaustive overview on complete intersections, we advise to see [69].

Note that, for $d = 1$ and $n = 2$, the Veronese surface $V_{2,1}$ is the plane $\mathbb{P}^2$, so that our problem in this special case is exactly the Cramer-Euler problem, which consists in characterising the sets of points in $\mathbb{P}^2$ that are complete intersections. We answer Question 7 in the case of Veronese surfaces, showing that for $d > 2$ the only reduced complete intersections of $\mathbb{P}^{N_{n,d}}$ lying on $V_{2,d}$ are finite sets of either one or two points while, for the Veronese surface $V_{2,2} \subseteq \mathbb{P}^5$, one also has plane conics and their intersections with suitable hypersurfaces. More precisely, we prove the following theorem.

Theorem 6. (Theorem 5.3.5) *If $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^{N_{2,d}}$ is a reduced complete intersection of type $(a_1, \ldots, a_r)$, with $a_1 \le \cdots \le a_r$, then one of the following holds:*

1. $(d, r, (a_1, a_2, \ldots, a_r)) = (2,4,(1,1,1,2))$, *that is, $\mathbb{X}$ is a conic lying on $V_{2,2}$;*

2. $(d, r, (a_1, a_2, \ldots, a_r)) = (2,5,(1,1,1,2,a_5))$, *any $a_5 \in \mathbb{N}$, that is, $\mathbb{X}$ is a set of $2a_5$ complete intersection points of a conic lying on $V_{2,2}$ and a hypersurface of degree a_5;*

3. $(d, r, (a_1, a_2, \ldots, a_r)) = (d, N_{2,d}, (1,1,\ldots,1))$ *for any $d \ge 2$, that is, $\mathbb{X}$ is a reduced point;*

4. $(d, r, (a_1, a_2, \ldots, a_r)) = (d, N_{2,d}, (1,1,\ldots,1,2))$ *for any $d \ge 2$, that is, $\mathbb{X}$ is a set of two reduced points.*

In order to prove this theorem, we characterise the possible Hilbert functions of reduced subvarieties of Veronese varieties; see Theorem 5.1.11. Beyond their application to the proof of our theorem, Hilbert functions play a central role in Commutative Algebra and in Algebraic Geometry, for example see [24], [90] and [99]. In recent times Hilbert functions have also been used as tools in other fields, such as the study of Waring rank, that is the tensor rank for symmetric tensors, see [36], and the study of the identifiability of tensors.

In characterising these Hilbert functions, we generalise the notion of 0-sequences and of differentiable 0-sequences introduced in [67]. We give a more effective characterisation for the case of the rational normal curves $V_{1,d}$ in Theorem 5.1.7, thus recovering a classical result, and for the case of the surfaces $V_{2,d}$ in Theorem 5.2.4.

Moreover, in Theorem 5.3.3 we show that, except for the case $d = 2$, the only complete intersections lying on rational normal curves $V_{1,d}$ are the trivial ones, that is one single point or the set of two points. The case $V_{1,2}$, that is of a plane conic, is different. In fact, by cutting with any properly chosen curve, one will produce a complete intersection set of points. Inspired by this evidence, we formulate Conjecture 5.4.2: the only reduced complete intersections of $V_{n,d}$, $d \ge 3$, are finite sets of either one or two points, while for $d = 2$ one also has plane conics and their intersections with suitable hypersurfaces. We also checked the validity of the conjecture for $V_{3,2}$, see Proposition 5.4.1.

Open problems

We list here some open problems related to the topics of this thesis.

1. The Jacobian scheme of a plane curve whose singularities are just double and triple ordinary points is a zero-dimensional scheme whose components are

reduced points and 2-squares. What can be said about the freeness of the curve?

2. For $m = n = 2$, all the m-superfat points of $\mathbb{P}^n$ have maximal Hilbert function. This is not true for any other value of $m > 2$ and $n > 2$, but there is some evidence that the *generic* m-superfat point of $\mathbb{P}^n$ has maximal Hilbert function. Is this true?

3. Would it be possible to generalise the varieties $Q(V_{2,d}), QQ(V_{2,d}), q_2(SV_{d,d})$, $qq_2(SV_{22})$ by considering m-symmetric schemes more general than 2-squares? Clearly, this would require a deeper study of m-symmetric schemes.

4. Is it true that any generic union of m-hypercubes in $\mathbb{P}^n$ has good postulation? We just know that for $m = n = 2$.

5. Is it possible to find an "effective" characterisation of the Hilbert functions of subvarieties of $V_{n,d}$ for $n > 3$ similar to the one we found for $n = 2$?

6. Is it true that the only reduced complete intersections lying on a Veronese variety are the ones we listed in Conjecture 5.4.2?

Chapter 1

Preliminaries

If it is not different specified, we always work over the base field $\mathbb{C}$. This chapter is devoted to the introduction of some preliminary notions that will be widely used in this thesis. In particular, we describe the mathematical entities which are the objects of our research and we introduce the tools we will use to describe and analyse them.

1.1 Hilbert functions

In this section, we recall the notion of Hilbert function with a special focus on the Hilbert functions of standard graded algebras, that are strictly related to the geometry of projective varieties.

We start by setting some notation.

Notation 1.1.1. Let A be a $\mathbb{Z}$-graded commutative and unitary ring. Given a $\mathbb{Z}$-graded A-module M we denote by M_t its homogeneous summand of degree t, getting the following decomposition of Abelian groups.

$$M = \bigoplus_{t=-\infty}^{\infty} M_t.$$

In our case, the elements of the $\mathbb{Z}$-graded modules have non-negative degree, thus we write

$$M = \bigoplus_{t=0}^{\infty} M_t \text{ and } M_t = 0 \; \forall \; t < 0.$$

For any $d \in \mathbb{Z}$ we denote by $M(d)$ the $\mathbb{Z}$-graded A-module defined as

$$M(d)_t = M_{d+t}$$

and we call $M(d)$ the *d-th twist of* M. In particular, a homogeneous ideal $I \subseteq A$ is a $\mathbb{Z}$-graded A-module and we denote, in agreement with the previous notation, the degree t homogeneous part of I by I_t. In the following, we will refer to $\mathbb{Z}$-graded modules simply as *graded modules*. We will also need to use multigraded modules, but we prefer to postpone the introduction of the notations related to them until we begin using them. Finally, given $n \in \mathbb{N}$, we denote by R the ring $R := \mathbb{C}[x_0, \dots, x_n]$ with the standard grading.

Definition 1.1.2. If M is a finitely generated R-module, then the *Hilbert function* of M is the function

$$\begin{array}{rccc} H_M(t): & \mathbb{Z} & \to & \mathbb{N} \\ & t & \mapsto & \dim(M_t) \end{array}$$

where $\dim(M_t)$ is the dimension of M_t as $\mathbb{C}$-vector space.

The Hilbert function of a module can be defined in a much more general setting but, for the purposes of this thesis, we only need this more specific case. The following theorem is a first fundamental result on the Hilbert function of a graded module.

Theorem 1.1.3. *If M is a finitely generated R-module, then there exist a unique polynomial $HP_M(t) \in \mathbb{Q}[t]$, of degree less or equal than n, and $\bar{t} \in \mathbb{N}$ such that $H_M(t) = HP_M(t)$ for any $t \geq \bar{t}$.*

Proof See [57], Theorem 1.11 and Lemma 1.12. ■

Definition 1.1.4. With the notation of Theorem 1.1.3, we call $HP_M(t)$ the *Hilbert polynomial* of M and $\bar{t}$ the *regularity index* of M.

Now we want to consider the case where M is the algebra defining a scheme.

Notation 1.1.5. We denote by $\mathbb{P}^n$ the projective space of dimension n over the base field $\mathbb{C}$ and we use in $\mathbb{P}^n$ homogeneous coordinates $[x_0, \dots, x_n]$ so that R is the ring of coordinates of $\mathbb{P}^n$. For any scheme $\mathbb{X} \subseteq \mathbb{P}^n$, we denote by $\mathcal{I}(\mathbb{X})$, or equivalently by $\mathcal{I}_{\mathbb{X}}$, the only homogeneous and saturated ideal of R for which $\mathbb{X} = \mathrm{Proj}(R/\mathcal{I}(\mathbb{X}))$. Moreover, we denote by $R[\mathbb{X}] := R/\mathcal{I}(\mathbb{X})$ the homogeneous coordinate ring of $\mathbb{X}$.

The Hilbert function and the Hilbert polynomial of $R[\mathbb{X}]$ encode several geometrical information about the scheme $\mathbb{X}$ and about its embedding in the projective space. Let us be more precise by giving the following definition.

Definition 1.1.6. Given a closed subscheme $\mathbb{X} \subseteq \mathbb{P}^n$ we define:

- the *Hilbert function* of $\mathbb{X}$ to be $H_{\mathbb{X}}(t) := H_{R[\mathbb{X}]}(t)$;
- the *Hilbert polynomial* of $\mathbb{X}$ to be $HP_{\mathbb{X}}(t) := HP_{R[\mathbb{X}]}(t)$;
- the *regularity index* of $\mathbb{X}$ to be the regularity index of $H_{\mathbb{X}}(t)$.

Remark 1.1.7. For a given $\mathbb{X} \subseteq \mathbb{P}^n$ the value $H_{\mathbb{X}}(t)$ represents the "number of conditions" that the containment of $\mathbb{X}$ imposes to the degree t hypersurfaces. Indeed, we have

$$H_{\mathbb{X}}(t) = \dim(R/\mathcal{I}(\mathbb{X}))_t = \dim R_t - \dim(\mathcal{I}(\mathbb{X}))_t$$

and $\mathcal{I}(\mathbb{X})_t$ is exactly the linear system of the degree t hypersurfaces of $\mathbb{P}^n$ containing $\mathbb{X}$.

Usually, given a scheme $\mathbb{X}$, its dimension is defined to be the maximum of the lengths of chains of irreducible Zariski closed subsets of $\mathbb{X}$, while its degree is defined to be the number of points where a general linear space of the "right" codimension intersects $\mathbb{X}$. Using the Hilbert polynomial of $\mathbb{X}$ we can define, in a more algebraic way, both the dimension and the degree of $\mathbb{X}$ as follows.

Definition 1.1.8. Given a scheme $\mathbb{X}$, we call the degree d of $HP_{\mathbb{X}}(t)$ the *dimension* of $\mathbb{X}$ and we call the leading coefficient of $HP_{\mathbb{X}}(t)$ multiplied by $d!$ the *degree* of $\mathbb{X}$. We denote the dimension of $\mathbb{X}$ by $\dim \mathbb{X}$ and the degree of $\mathbb{X}$ by $\deg \mathbb{X}$. If $\dim \mathbb{X} = 0$, the degree of $\mathbb{X}$ is also called the *length* of $\mathbb{X}$ and it is denoted by $\ell(\mathbb{X})$.

It is easy to show that these algebraic definitions agree with the geometrical ones; more details can be found, for instance, in [59] Chapter III, §III.3.1, III.3.2, III.3.3 and in [73] Chapter I, §7.

Example 1.1.9. If $\mathbb{X} = \mathbb{P}^n$ then $R[\mathbb{X}] = R[\mathbb{P}^n] = R$ and thus

$$H_{\mathbb{P}^n}(t) = \dim R_t = \binom{n+t}{t}$$

and $HP_{\mathbb{P}^n}(t) = H_{\mathbb{P}^n}(t)$ for all $t \in \mathbb{N}$ agreeing with the fact that $\dim \mathbb{P}^n = n$ and $\deg \mathbb{P}^n = 1$.

Example 1.1.10. Let $\mathbb{X} \subseteq \mathbb{P}^n$ a degree e hypersurface. Clearly $\mathcal{I}(\mathbb{X}) = (F)$ and, for each $t \in \mathbb{N}$, we have the following short exact sequence

$$0 \longrightarrow R(-e)_t \xrightarrow{F} R_t \xrightarrow{\pi} (R/(F))_t \longrightarrow 0$$

where the first map is the multiplication by F and the second is the projection to the quotient. As a consequence, we get

$$H_{\mathbb{X}}(t) = \dim(R/(F))_t = \dim R_t - \dim R(-e)_t = \begin{cases} \binom{n+t}{n}, & \text{if } t \leq e-1 \\ \binom{n+t}{n} - \binom{n+t-e}{n}, & \text{if } t \geq e \end{cases}.$$

Hence, the Hilbert polynomial of $\mathbb{X}$ is

$$HP_{\mathbb{X}}(t) = \binom{n+t}{n} - \binom{n+t-e}{n} = \frac{e}{(n-1)!}t^{n-1} + \cdots$$

agreeing with the fact that $\dim \mathbb{X} = n-1$ and $\deg \mathbb{X} = e$.

Remark 1.1.11. If $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{P}^n$ are two schemes and $\mathbb{X} \subseteq \mathbb{X}'$, then $\mathcal{I}(\mathbb{X}') \subseteq \mathcal{I}(\mathbb{X})$ and thus $H_{\mathbb{X}}(t) \leq H_{\mathbb{X}'}(t)$ for any $t \geq 0$. Note that the equality $H_{\mathbb{X}}(t) = H_{\mathbb{X}'}(t)$ holds if and only if $\dim \mathcal{I}(\mathbb{X})_t = \dim \mathcal{I}(\mathbb{X}')_t$ or, equivalently, if and only if $\mathcal{I}(\mathbb{X})_t = \mathcal{I}(\mathbb{X}')_t$, that is, if and only if all the hypersurfaces of degree t containing $\mathbb{X}$ also contain $\mathbb{X}'$.

If $\mathbb{X}$ is a 0-dimensional scheme, its Hilbert polynomial is just a constant, but we can say something more on the behaviour of $H_{\mathbb{X}}(t)$.

Proposition 1.1.12. *If $\mathbb{X} \subseteq \mathbb{P}^n$ is a 0-dimensional scheme of length d and regularity index $\bar{t}$ then*

$$H_{\mathbb{X}}(t-1) < H_{\mathbb{X}}(t) \leq d \; \forall \; 0 \leq t \leq \bar{t} \text{ and } HP_{\mathbb{X}}(t) = d.$$

Proof See [59] Proposition III-59. ■

The Hilbert function of a 0-dimensional scheme $\mathbb{X}$ also allows to understand why the first cohomology of the ideal sheaf of $\mathbb{X}$ can be interpreted as a measure of the "speciality" of $\mathbb{X}$. To make that clearer, we fix some more notation.

Notation 1.1.13. Given a projective space $\mathbb{P}^n$, we denote by $\mathcal{O}_{\mathbb{P}^n}$ its structure sheaf and by $\mathcal{L}_d$ the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ of hypersurfaces in $\mathbb{P}^n$. If $\mathbb{X} \subseteq \mathbb{P}^n$ is a closed subscheme, we denote by $\mathfrak{I}_{\mathbb{X}}$ the ideal sheaf of $\mathbb{X}$ in $\mathcal{O}_{\mathbb{P}^n}$ and by $\mathcal{L}_d(\mathbb{X})$ the linear system $|\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathfrak{I}_{\mathbb{X}}|$ of hypersurfaces in $\mathbb{P}^n$ containing $\mathbb{X}$. In order to simplify some computations, when we write $\dim \mathcal{L}_d$ or $\dim \mathcal{L}_d(\mathbb{X})$ we refer to the vector dimension of the linear system; in particular we have

$$\dim \mathcal{L}_d = \binom{n+d}{d}.$$

Moreover, if $\mathbb{Y}$ is a closed subscheme of $\mathbb{P}^n$, we denote by $\mathfrak{I}_{\mathbb{X},\mathbb{Y}}$ the ideal sheaf of $\mathbb{X}$ restricted to $\mathbb{Y}$.

Remark 1.1.14. If $\mathbb{X}$ is a 0-dimensional subscheme of $\mathbb{P}^n$, we have

$$H_{\mathbb{X}}(t) = \dim R_t - \dim \mathcal{I}(\mathbb{X})_t = h^0(\mathcal{O}_{\mathbb{P}^n}(t)) - h^0(\mathfrak{I}_{\mathbb{X}}(t)) = \ell(\mathbb{X}) - h^1(\mathfrak{I}_{\mathbb{X}}(t))$$

so that

$$h^1(\mathfrak{I}_{\mathbb{X}}(t)) = \ell(\mathbb{X}) - H_{\mathbb{X}}(t)$$

is exactly the measure of how $\mathbb{X}$ fails to impose on hypersurfaces of degree t as many conditions as its points counted with multiplicity. Moreover, by this formula we get that

$$H_{\mathbb{X}}(t) = \ell(\mathbb{X}) \Leftrightarrow h^1(\mathfrak{I}_{\mathbb{X}}(t)) = 0.$$

Properties of Hilbert functions of schemes can also be studied, in a more algebraic way, through the so-called *standard graded algebras*.

Definition 1.1.15. Let $\mathbb{K}$ be a field and A be a graded $\mathbb{K}$-algebra. We say that A is a *standard graded* $\mathbb{K}$*-algebra* if $A_0 = \mathbb{K}$ and A is generated, as $\mathbb{K}$-algebra, by a finite number of elements of A_1.

Proposition 1.1.16. *A is a standard graded $\mathbb{K}$-algebra if and only if $A_0 = \mathbb{K}$ and there exist $n \in \mathbb{N}$ and an ideal $I \subseteq \mathbb{K}[x_0, \dots, x_n]$ such that $A \cong \mathbb{K}[x_0, \dots, x_n]/I$ as $\mathbb{K}$-algebras, where $\mathbb{K}[x_0, \dots, x_n]$ is endowed with the standard grading.*

Proof Since A is a standard graded $\mathbb{K}$-algebra, there exist $y_0, \dots, y_n \in A_1$ that generate A as $\mathbb{K}$-algebra. The map

$$\begin{array}{rccc} \varphi: & \mathbb{K}[x_0, \dots, x_n] & \to & A \\ & 1_{\mathbb{K}} & \mapsto & 1_A \\ & x_i & \mapsto & y_i \end{array},$$

extended in the obvious way, is a surjective homomorphism of $\mathbb{K}$-algebras and thus $A \cong \mathbb{K}[x_0, \dots, x_n]/\ker\varphi$. The vice versa is immediate. ■

Proposition 1.1.16 shows that studying Hilbert functions of closed schemes of $\mathbb{P}^n$ is equivalent to studying the Hilbert functions of standard graded $\mathbb{C}$-algebras. In the rest of the thesis we use these two points of view interchangeably.

1.1.1 Macaulay inequalities and 0-sequences

Proposition 1.1.12 is a special case of a much more general result on the behaviour of Hilbert functions. Indeed, the possible values that a Hilbert function can attain in t are in a range depending only on the value of the Hilbert function in $t-1$ and these ranges are called *Macaulay inequalities*. In order to define the Macaulay inequalities, we have first to introduce the *binomial expansion* of a non-negative number c.

Proposition 1.1.17. *For any non-negative integers i, c there exist $m_i, m_{i-1}, \dots, m_j$ non-negative integers with $m_i > m_{i-1} > \dots > m_j \geq j \geq 1$ such that*

$$c = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \dots + \binom{m_j}{j}$$

and the integers $m_i, \dots, m_j$ are unique.

Proof See [85] §1. ■

Definition 1.1.18. Given i and c positive integers, the expression

$$c = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \dots + \binom{m_j}{j}$$

is called the *i-binomial expansion* of c.

Note that Proposition 1.1.17 guarantees the well-posedness of Definition 1.1.18.

Definition 1.1.19. If c is a positive integer having i-binomial expansion as in Definition 1.1.18, we set

$$c^{<i>} := \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j+1}{j+1}.$$

Moreover, we set $0^{<i>} = 0$ for any $i \in \mathbb{N}$.

Example 1.1.20. If $c = 153$ and $i = 5$ then we have

$$153 = \binom{9}{5} + \binom{6}{4} + \binom{5}{3} + \binom{2}{1}$$

and thus we get

$$153^{<5>} = \binom{10}{6} + \binom{7}{5} + \binom{6}{4} + \binom{3}{2} = 249.$$

Definition 1.1.21. A sequence of natural numbers $(c_t)_{t\in\mathbb{Z}}$ is called a *0-sequence* if $c_t = 0$ for any $t < 0$, $c_0 = 1$ and $c_{t+1} \leq c_t^{<t>}$ $\forall\, t \geq 1$.

In order to state the next theorem in all its generality, we need to give the definition of *order ideal of monomials*.

Definition 1.1.22. Let $\mathbb{K}$ be a field and $M \subseteq \mathbb{K}[x_0,\ldots,x_n]$ a non-empty set of monomials in the variables $x_0,\ldots,x_n$. We say that M is an *order ideal of monomials* if the following condition holds:

$$m \in M, m' \mid m \Rightarrow m' \in M.$$

Note that, in general, an order ideal of monomials is not an ideal. As we mentioned above, these definitions allow us to characterise the Hilbert function via the following famous theorem, originally stated by Francis S. Macaulay in [88] and then rephrased by Richard P. Stanley in [99].

Theorem 1.1.23. *Let*

$$\begin{array}{cccc} h: & \mathbb{Z} & \to & \mathbb{N} \\ & t & \mapsto & h_t \end{array}$$

and let $\mathbb{K}$ be any field. The following conditions are equivalent:

1. *There exists a standard graded algebra A with $A_0 = \mathbb{K}$ and with Hilbert function h.*

2. *The sequence $(h_t)_{t\in\mathbb{Z}}$ is a 0-sequence.*

3. *$h_t = 0$ for any $t < 0$, $h_0 = 1$ and $h_{t+1} \leq h_t^{<t>}$ for any $t \geq 0$.*

4. *Let $s = h_1$ and, for each $t \geq 0$, let M_t be the set of the first h_t monomials in the variables $x_0, \dots, x_{s-1}$ with respect to the graded reverse lexicographic order. Then the set*
$$M := \bigcup_{t \geq 0} M_t$$
is an order ideal of monomials.

Proof See [99], Theorem 2.2. ■

Example 1.1.24. We consider the sequence $(h_t)_{t\in\mathbb{Z}}$ defined by the following table

t	0	1	2	3	4	$t \geq 5$
h_t	1	4	9	16	19	$4t+1$

Table 1.1: An example of 0-sequence

and we ask if there exists a standard graded algebra having $(h_t)_{t\in\mathbb{Z}}$ as Hilbert function. By Theorem 1.1.23 it is enough to check that $h_{t+1} \leq h_t^{<t>}$ and, in fact, we have

$$\begin{aligned}
h_1^{<1>} &= 4^{<1>} = 10 \geq 9 = h_2 \\
h_2^{<2>} &= 9^{<2>} = 16 \geq 16 = h_3 \\
h_3^{<3>} &= 16^{<3>} = 25 \geq 19 = h_4 \\
h_4^{<4>} &= 19^{<4>} = 26 \geq 21 = h_5.
\end{aligned}$$

Moreover, one can see that $h_{t+1} \leq h_t^{<t>}$ for any $t \geq 5$ and thus there exists a standard graded algebra with Hilbert function $(h_t)_{t\in\mathbb{Z}}$. The proof of Theorem 1.1.23 also gives us a way to explicitly construct such a standard graded algebra; we describe the construction just in this specific example without any presumption of generality, which would be outside the purposes of this thesis. First of all, since $h_1 = 4$, we work in the algebra $R = \mathbb{C}[x_0, x_1, x_2, x_3]$, although for this construction we could use any other base field. Consider the monomial ideal $I \subseteq R$ whose generators are defined as follows: for any $t \geq 1$ a monomial in R_t is a generator of I if and only if it is one of the last $\binom{3+t}{t} - h_t$ monomials of R_t with respect to the graded reverse lexicographic order. We describe more in details the first steps of the construction of I.

- $t = 1$
  We have
  $$R_1 =< x_0, x_1, x_3, x_4 >$$
  and $\binom{3+1}{1} - h_1 = 4 - 4 = 0$ so that I have no generators of degree 1.

- $t = 2$
  We have
  $$R_2 =< x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2, x_0x_3, x_1x_3, x_2x_3, x_3^2 >$$
  and $\binom{3+2}{2} - h_2 = 10 - 9 = 1$, so that the only generator of degree 2 of I is x_3^2.
- $t = 3$
  We have
  $$R_3 =< x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0^2x_2, x_0x_1x_2, x_1^2x_2, x_0x_2^2, x_1x_2^2, x_2^3, x_0^2x_3, \\ x_0x_1x_3, x_1^2x_3, x_0x_2x_3, x_1x_2x_3, x_2^2x_3, x_0x_3^2, x_1x_3^2, x_2x_3^2, x_3^3 >$$
  and $\binom{3+3}{3} - h_3 = 20 - 16 = 4$ so that we have to add to I the generators $x_0x_3^2, x_1x_3^2, x_2x_3^2$ and x_3^3.

By Dickson's Lemma I has a finite basis of monomials and thus the procedure ends, that is, there exists a degree $\bar{t}$ such that any generator of degree $t \geq \bar{t}$ can be obtained by the generators of smaller degrees. In particular, more computations show that is enough to reach the step $t = 7$ to determine I and finally one finds
$$I = (x_3^2, x_2^3x_3, x_1x_2^2x_3, x_0x_2^2x_3, x_1^2x_2x_3, x_0x_1x_2x_3, x_0^2x_2x_3, x_1^4x_3, x_0x_1^3x_3, \\ x_0^2x_1^2x_3, x_0^3x_1x_3, x_0^4x_3, x_2^6, x_1x_2^5, x_0x_2^5, x_1^3x_2^4)$$
and the standard graded algebra we were looking for is R/I.

1.1.2 The first difference function

In this subsection we analyse the first difference of a Hilbert function or, equivalently, of a 0-sequence.

Definition 1.1.25. If M is a finitely generated R-module then the *first difference* of its Hilbert function is the function $\Delta H_M(t) := H_M(t) - H_M(t-1)$.

For the first difference of the Hilbert function of a scheme $\mathbb{X}$ we use an analogue notation. We define the first difference of a 0-sequence as follows.

Definition 1.1.26. Given a 0-sequence $(c_t)_{t\in\mathbb{Z}}$ we define its *first difference* to be the sequence of integers $(\Delta c_t)_{t\in\mathbb{Z}}$ defined as $\Delta c_t = c_t - c_{t-1}$.

Remark 1.1.27. If A is a reduced standard graded $\mathbb{K}$-algebra, then there exists a non-zero-divisor $F \in A_1$ and the sequence
$$0 \longrightarrow A(-1)_t \xrightarrow{F} A_t \xrightarrow{\pi} (A/(F))_t \longrightarrow 0$$
is exact for any $t \in \mathbb{N}$. As a consequence, we get
$$\Delta H_A(t) = H_A(t) - H_A(t-1) = H_{A/(F)}(t)$$
so that, in particular, $\Delta H_A(t)$ is itself a Hilbert function.

The previous remark shows that a necessary condition for a 0-sequence $(c_t)_{t\in\mathbb{Z}}$ to be the Hilbert function of a reduced standard graded $\mathbb{K}$-algebra is that $(\Delta c_t)_{t\in\mathbb{Z}}$ is a 0-sequence but, actually, even more is true. For this purpose, we recall the definition of differentiable 0-sequence.

Definition 1.1.28. A 0-sequence $(c_t)_{t\in\mathbb{Z}}$ is called a *differentiable 0-sequence* if its first difference $(\Delta c_t)_{t\in\mathbb{Z}}$ is a 0-sequence.

A.V. Geramita, P. Maroscia and L. G. Roberts proved in [67] that being a differential 0-sequence is also a sufficient condition for a 0-sequence to be the Hilbert function of a reduced standard graded $\mathbb{K}$-algebra; in particular, they proved the following theorem.

Theorem 1.1.29. *Let $\mathbb{K}$ be an infinite field and let $(c_t)_{t\in\mathbb{Z}}$ be a differentiable 0-sequence with $c_1 = n+1$. There exists a radical ideal I in $\mathbb{K}[x_0,\ldots,x_n]$ such that $(c_t)_{t\in\mathbb{Z}}$ is the Hilbert function of $\mathbb{K}[x_0,\ldots,x_n]/I$.*

Proof See [67], Theorem 3.3. ■

Hence, differentiable 0-sequences completely characterise the Hilbert functions of reduced standard algebras.

Remark 1.1.30. The first difference of the Hilbert function of a scheme also has a geometrical interpretation. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a variety and let $\alpha : F = 0$ be a general hyperplane in $\mathbb{P}^n$. Since $\mathbb{X}$ is a reduced scheme, $R[\mathbb{X}]$ is a reduced $\mathbb{C}$-algebra and $[F]$, the equivalence class of F in $R[\mathbb{X}]$, is a non-zero-divisor. As a consequence, we have

$$\Delta H_{\mathbb{X}}(t) = H_{R[\mathbb{X}]/([F])}(t),$$

that is, $\Delta H_{\mathbb{X}}(t)$ is the Hilbert function of the generic hyperplane section of $\mathbb{X}$. Thus, if $\mathbb{X}$ is reduced, one can study $\Delta H_{\mathbb{X}}$ rather than $H_{\mathbb{X}}$. Indeed, $H_{\mathbb{X}}(t)$ can be totally recovered by $\Delta H_{\mathbb{X}}(t)$ through the formula

$$H_{\mathbb{X}}(t) = \sum_{i=0}^{t} \Delta H_{\mathbb{X}}(t)$$

but $\Delta H_{\mathbb{X}}(t)$ is the Hilbert function of a variety whose dimension is one less than $\mathbb{X}$ and embedded in a $\mathbb{P}^{n-1}$. In more algebraic terms, when we work with $\Delta H_{\mathbb{X}}(t)$, we have one variable less and the degree of the Hilbert polynomial is one less than the previous one.

1.1.3 Castelnuovo Functions

The study of Hilbert functions of 0-dimensional subschemes of $\mathbb{P}^2$ goes back to G. Castelnuovo, see for instance [42], and still plays an important role in Algebraic

Geometry. It is a very powerful tool allowing a deep understanding of the geometry, for example, of point configurations in $\mathbb{P}^2$ and it also provides several numerical invariants of 0-dimensional subschemes of $\mathbb{P}^2$. In this subsection we recall some results that we will need in the next chapters; for a more detailed treatise we refer to [50] and [83], this subsection being a short summary of these two papers. Let us start with some definitions.

Definition 1.1.31. Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{P}^2$. We say that a curve $\mathcal{C} \in \mathcal{L}_d$ (recall Notation 1.1.13) defined by $F \in R_d$ is a *fixed curve* of $\mathcal{L}_t(\mathbb{X})$ if every member of $\mathcal{L}_t(\mathbb{X})$ contains $\mathcal{C}$ or, equivalently, if $\mathcal{I}(\mathbb{X})_s \subseteq R_{s-d}F$ for every $s \leq t$. If $\mathcal{C}$ is the greatest common divisor of $\mathcal{L}_t(\mathbb{X})$ (viewed as a set of divisors) or, equivalently, if F is the greatest common divisor of the set of polynomials $\mathcal{I}(\mathbb{X})_t$, then we say that $\mathcal{C}$ is *the fixed curve* of $\mathcal{L}_t(\mathbb{X})$.

Definition 1.1.32. If $\mathbb{X}$ is a 0-dimensional subscheme of $\mathbb{P}^2$ we set:

- $\alpha(\mathbb{X}) := \min\{t \mid \mathcal{L}_t(\mathbb{X}) \neq \emptyset\}$;
- $\beta(\mathbb{X}) := \min\{t \mid \mathcal{L}_t(\mathbb{X})$ has no fixed curve$\}$;
- $\tau(\mathbb{X}) := \min\{t \mid H_{\mathbb{X}}(t) = \ell(\mathbb{X})\} = \min\{t \mid h^1(\mathfrak{I}_{\mathbb{X}}(t)) = 0\}$, that is $\tau(\mathbb{X})$ is the regularity index of $H_{\mathbb{X}}(t)$.

One has $\alpha(\mathbb{X}) \leq \beta(\mathbb{X}) \leq \tau(\mathbb{X}) + 1$, where the first inequality is immediate from the definition and the second one follows from the fact that $\mathcal{I}(\mathbb{X})$ is generated by forms whose degree is at most $\tau + 1$ (see [51] (3.7)). We are now ready to introduce the notion of Castelnuovo function and some of its properties.

Definition 1.1.33. If $\mathbb{X}$ is a 0-dimensional subscheme of $\mathbb{P}^2$, the first difference $\Delta H_{\mathbb{X}}(t)$ is called the *Castelnuovo function* of $\mathbb{X}$.

Remark 1.1.34. The Castelnuovo function of a 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{P}^2$ can also be interpreted as a measure of the trend of the speciality of $\mathbb{X}$ with respect to the curves of degree d. Indeed, we have

$$\Delta H_{\mathbb{X}}(t) = H_{\mathbb{X}}(t) - H_{\mathbb{X}}(t-1) = \ell(\mathbb{X}) - h^1(\mathfrak{I}_{\mathbb{X}}(t)) - (\ell(\mathbb{X}) - h^1(\mathfrak{I}_{\mathbb{X}}(t-1))) =$$
$$= h^1(\mathfrak{I}_{\mathbb{X}}(t-1)) - h^1(\mathfrak{I}_{\mathbb{X}}(t)).$$

The following theorem is a collection of results on the Castelnuovo function of a 0-dimensional scheme. They are mainly due to P. Dubreil (see, for instance, [56]), which is why we refer to the theorem as Dubreil Theorem, while the form in which we state the theorem is the same used by E. Davis in [50] and by E. Davis, A.V. Geramita and P. Maroscia in [51].

Theorem 1.1.35 (Dubreil)**.** *For any 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^2$ one has:*

- $\Delta H_{\mathbb{X}}(t) \geq 0$ *for any* $t \in \mathbb{Z}$ *and* $\Delta H_{\mathbb{X}}(t) \neq 0 \Leftrightarrow 0 \leq t \leq \tau(\mathbb{X})$*;*
- $\Delta H_{\mathbb{X}}(t) \leq t+1$ *for any* $t \in \mathbb{Z}$ *and* $\Delta H_{\mathbb{X}}(t) = t+1 \Leftrightarrow 0 \leq t \leq \alpha(\mathbb{X})-1$*;*
- $\Delta H_{\mathbb{X}}(t) \leq \Delta H_{\mathbb{X}}(t-1)$ *for any* $t \geq \alpha(\mathbb{X})$*;*
- $\Delta H_{\mathbb{X}}(t) < \Delta H_{\mathbb{X}}(t-1)$ *for any* $\beta(\mathbb{X}) \leq t \leq \tau(\mathbb{X})+1$*;*
- $\sum_{t=0}^{d} \Delta H_{\mathbb{X}}(t) \leq \ell(\mathbb{X})$ *and the equality holds if and only if* $d \geq \tau(\mathbb{X})$*.*

Proof See [50] (2.1). ∎

Actually, in [50] the author also proved other results on Castelnuovo functions that we recall in Proposition 1.1.37.

Definition 1.1.36. Given any function $f : \mathbb{Z} \to \mathbb{N}$ we denote by $f_d(t) := \min\{f(t), d\}$ the *truncation of f at d.*

Proposition 1.1.37. *If $\mathbb{X}$ is a 0-dimensional subscheme of $\mathbb{P}^2$, then the following hold:*

- *If $d \leq \alpha(\mathbb{X})$ and $\mathcal{C} \in \mathcal{L}_d$, then $\ell(\mathbb{X} \cap \mathcal{C}) \leq \ell_d(\mathbb{X})$. Moreover, if the equality holds, then $\Delta H_{\mathbb{X}\cap\mathcal{C}}(t) = \Delta H_{\mathbb{X}}(t)_d$ and $\mathcal{C}$ is a fixed curve of $\mathcal{L}_t(\mathbb{X})$ for every t such that $\Delta H_{\mathbb{X}}(t) \geq d$;*
- *If $\mathcal{C} \in \mathcal{L}_d$ is a fixed curve of $\mathcal{L}_t(\mathbb{X})$ and $t \geq \alpha(\mathbb{X})$ then $d \leq \Delta H_{\mathbb{X}}(t)$ and if the equality holds then $\ell(X \cap \mathcal{C}) = \ell_d(\mathbb{X})$;*
- *If $\Delta H_{\mathbb{X}}(t) = \Delta H_{\mathbb{X}}(t-1) \neq 0$, then $\beta(\mathbb{X}) > t \geq \alpha(\mathbb{X})$ and the fixed curve of $\mathcal{L}_t(\mathbb{X})$ has degree $\Delta H_{\mathbb{X}}(t)$;*
- *If $\alpha(\mathbb{X}) < \beta(\mathbb{X})$ and $\Delta H_{\mathbb{X}}(\beta(\mathbb{X})) = \Delta H_{\mathbb{X}}(\beta(\mathbb{X})-1)-1$, then the fixed curve of $\mathcal{L}_{\beta(\mathbb{X})-1}(\mathbb{X})$ has degree $\Delta H_{\mathbb{X}}(\beta(\mathbb{X})-1)$.*

Since Theorem 1.1.35 gives a complete characterisation of Castelnuovo functions, B. Kreuzer and M. Kreuzer gave, in [83], the definition of Castelnuovo function with assigned invariants α and τ. The definition below is equivalent to the one given in [83], but slightly different in order to preserve consistent notations with respect to Dubreil Theorem.

Definition 1.1.38. Let $\alpha_h \geq 1$ and $\tau_h \geq \alpha_h - 1$ be natural numbers and

$$\begin{array}{rccc} h: & \mathbb{Z} & \to & \mathbb{N} \\ & t & \mapsto & h(t) \end{array}$$

be such that

- $h(t) \geq 0$ for any $t \in \mathbb{Z}$ and $h(t) \neq 0 \Leftrightarrow 0 \leq t \leq \tau_h$;
- $h(t) \leq t+1$ for any $t \in \mathbb{Z}$ and $h(t) = t+1 \Leftrightarrow 0 \leq t \leq \alpha_h - 1$;
- $h(t) \leq h(t-1)$ for any $\alpha_h \leq t \leq \tau_h + 1$.

Then h is called a *Castelnuovo function* with invariants α_h and τ_h.

Given a Castelnuovo function h, in [83] the authors introduced some 0-dimensional subschemes of $\mathbb{P}^2$ associated to h and defined as follows.

Definition 1.1.39. Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. Let $h : \mathbb{Z} \to \mathbb{N}$ be a Castelnuovo function with invariants α_h and τ_h and let

$$\{s_0, s_1, \dots, s_{\tau_h}\} \subseteq \mathbb{K}, \quad \{t_0, t_1, \dots, t_{\alpha_h - 1}\} \subseteq \mathbb{K}$$

be sets of pairwise distinct elements. The reduced 0-dimensional subscheme of $\mathbb{P}^2$

$$\mathbb{X}(h) := \{[1, s_i, t_j] \in \mathbb{P}^2 \mid 0 \leq i+j \leq \tau_h,\ 0 \leq j \leq h(i+j)\}$$

is called the *Castelnuovo set* for h with parameters $s_0, \dots, s_{\tau_h}$ and $t_0, \dots, t_{\alpha_h-1}$.

Castelnuovo sets allow to construct 0-dimensional subschemes of $\mathbb{P}^n$ with a prescribed Castelnuovo function, as shown by the following theorem.

Theorem 1.1.40. *Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. If h is a Castelnuovo function with invariants α_h and τ_h and $s_0, s_1, \dots, s_{\tau_h} \in \mathbb{K}$ and $t_0, t_1, \dots, t_{\alpha_h-1} \in \mathbb{K}$ are pairwise distinct elements, then the Castelnuovo set $\mathbb{X}(h)$ for h with parameters $s_0, \dots, s_{\tau_h}$ and $t_0, \dots, t_{\alpha_h-1}$ has Castelnuovo function $\Delta H_{\mathbb{X}(h)} = h$. Moreover, if we set $m(i) = \min\{n \geq \alpha_h \mid h(n) \leq i\}$, we have*

$$\mathcal{I}(\mathbb{X}(h)) = (F_0, \dots, F_{\alpha_h})$$

where

$$F_i = \prod_{j=0}^{m(i)-i-1} (x_1 - t_j x_0) \prod_{j=0}^{i} (x_1 - s_j x_0).$$

Example 1.1.41. Let us consider, over $\mathbb{C}$, the Castelnuovo function $h : \mathbb{Z} \to \mathbb{N}$ defined by the following table

t	0	1	2	3	4	5	6	$t \geq 7$
$h(t)$	1	2	3	4	4	3	2	0

Table 1.2: An example of Castelnuovo function

We want to construct a Castelnuovo set for h above. Since $h(t) = t+1$ if and only if $0 \leq t \leq 3$ we have, by definition, that $\alpha_h = 4$ and since $h(t) \neq 0$ if and only if

$0 \leq t \leq 6$ we have, again by definition, that $\tau_h = 6$. To construct a Castelnuovo set associated to h we have to choose the two sets of pairwise distinct elements $\{s_0, s_1, \ldots, s_6\} \subseteq \mathbb{C}$ and $\{t_0, t_1, t_2, t_3\} \subseteq \mathbb{C}$ and the easiest way to do that is to choose $s_i = i$ for any $0 \leq i \leq 6$ and $t_i = i$ for any $0 \leq i \leq 3$. In doing so, the Castelnuovo set associated to h with respect to the chosen parameters is

$$\mathbb{X} := \{[1, i, j] \in \mathbb{P}^2 \mid 0 \leq i + j \leq 6,\ 0 \leq j \leq h(i + j)\}$$

and its affine representation is

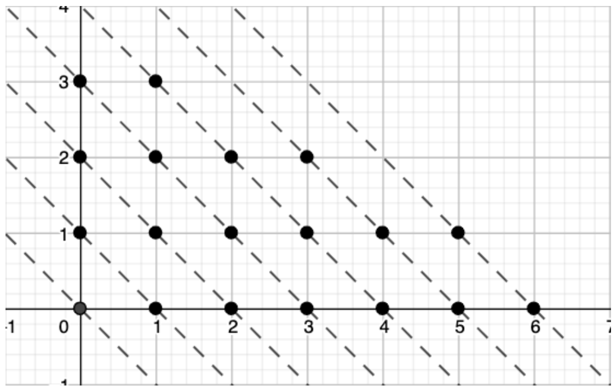


Figure 1.1: The Castelnuovo set associated to h

1.2 Fat Points

Among all the 0-dimensional schemes there is a very interesting family, that of fat points, which, due to its large number of interactions in different fields of Algebraic Geometry, deserves a more in-depth study. Historically, these 0-dimensional schemes were introduced in the study of linear systems of hypersurfaces of $\mathbb{P}^n$ with a fixed set of singularities; let us see this in more detail.

Notation 1.2.1. For the ideal of a reduced point we use a specific notation, that is, if P is a point of $\mathbb{P}^n$ we will denote by $\wp$ its ideal $\mathcal{I}(P)$.

Remark 1.2.2. We fix a point P in $\mathbb{P}^n$ defined by the ideal $\wp$ and, up to a projectivity, we can suppose $P = [1,0,\ldots,0]$ and $\wp = (x_1, x_2, \ldots, x_n)$. Any $F \in \wp_d$ can be written as

$$F = x_0^{d-1} f_1 + x_0^{d-2} f_2 + \cdots + x_0 f_{d-1} + f_d = \sum_{j=1}^{d} x_0^{d-j} f_j$$

with $f_j \in \mathbb{C}[x_1, \ldots, x_n]_j$. The hypersurface defined by F has a singularity at P if and only if

$$\frac{\partial F}{\partial x_i}(P) = 0 \text{ for any } i = 1, \ldots, n$$

and we have

$$\frac{\partial F}{\partial x_0} = \sum_{j=1}^{d-1}(d-j)x_0^{d-j-1}f_j \Rightarrow \frac{\partial F}{\partial x_0}(P) = \sum_{j=1}^{d-1}(d-j)f_j(0,\ldots,0) = 0$$

$$\frac{\partial F}{\partial x_i} = \sum_{j=1}^{d} x_0^{d-j}\frac{\partial f_j}{\partial x_i} \Rightarrow \frac{\partial F}{\partial x_i}(P) = \sum_{j=1}^{d}\frac{\partial f_j}{\partial x_i}(0,\ldots,0) = \frac{\partial f_1}{\partial x_i} \text{ for any } i = 1,\ldots,n$$

so that P is singular for the hypersurface defined by F if and only if

$$\frac{\partial f_1}{\partial x_i} = 0 \text{ for any } i = 1,\ldots,n \Leftrightarrow f_1 = 0 \Leftrightarrow F \in \wp^2.$$

This shows that $(\wp^2)_d$ is the linear system of the degree d hypersurfaces of $\mathbb{P}^n$ having a singularity at P or, equivalently, that all the hypersurfaces of $\mathbb{P}^n$ having a singularity at P contain the 0-dimensional scheme defined by $\wp^2$. A similar argument shows that, more in general, $(\wp^m)_d$ is the linear system of the degree d hypersurfaces of $\mathbb{P}^n$ having at P a point of multiplicity at least m. Note that all the ideals $\wp^m$ are $\wp$-primary, that is they are primary and $\sqrt{\wp^m} = \wp$. This discussion justifies the following definition.

Definition 1.2.3. Let $P \in \mathbb{P}^n$ be a point defined by the prime ideal $\wp \in R$. If m is any positive integer, then the subscheme $\mathrm{Proj}(R/\wp^m)$ of $\mathbb{P}^n$ is called the *fat point of* $\mathbb{P}^n$ *supported on* P *of multiplicity* m and it is denoted by mP.

From the point of view of the Hilbert function, a fat point of multiplicity m behaves like $\binom{n+m-1}{m-1}$ reduced points, as shown by the following lemma.

Lemma 1.2.4. Let $P \in \mathbb{P}^n$. Then $H_{mP}(t) = \min\left\{\binom{n+t}{t}, \binom{n+m-1}{m-1}\right\}$.

Proof We can suppose $P = [1,0,\ldots,0]$ so that $\wp = (x_1,\ldots,x_n)$. We start by computing $\dim(\wp^m)_t$ for any $t \in \mathbb{N}$. Clearly, if $t < m$ then we have $\dim(\wp^m)_t = 0$ and thus we can suppose, for the moment, $t \geq m$. The vector space $(\wp^m)_t$ can be written in the following way

$$(\wp^m)_t = \mathbb{C}[x_1,\ldots,x_n]_t \oplus x_0\mathbb{C}[x_1,\ldots,x_n]_{t-1} \oplus \cdots \oplus x_0^{t-m}\mathbb{C}[x_1,\ldots,x_n]_m$$

so, as a consequence, we have

$$\dim(\wp^m)_t = \sum_{i=m}^{t}\binom{n-1+i}{i} = \sum_{i=0}^{t}\binom{n-1+i}{i} - \sum_{i=0}^{m-1}\binom{n-1+i}{i} =$$
$$= \binom{n-1+t+1}{t} - \binom{n-1+m-1+1}{m-1} = \binom{n+t}{t} - \binom{n+m-1}{m-1}$$

where the second equality follows by the hockey-stick identity. Hence, we have

$$H_{mP}(t) = \dim R_t - \dim(\wp^m)_t = \begin{cases} \binom{n+t}{t} & \text{if } t < m \\ \binom{n+m-1}{m-1} & \text{if } t \geq m \end{cases}.$$

■

Clearly we can consider a scheme made up of more than one fat point.

Notation 1.2.5. If $\mathbb{X}$ and $\mathbb{Y}$ are two projective subschemes of $\mathbb{P}^n$, we denote by $\mathbb{X} + \mathbb{Y}$ their schematic union, that is, the closed subscheme of $\mathbb{P}^n$ defined by the ideal $\mathcal{I}(\mathbb{X}) \cap \mathcal{I}(\mathbb{Y})$.

Definition 1.2.6. Let $P_1, \dots, P_s$ be distinct points in $\mathbb{P}^n$ and let $m_1, \dots, m_s \in \mathbb{N}_{>0}$. The subscheme of $\mathbb{P}^n$

$$m_1 P_1 + m_2 P_2 + \cdots + m_s P_s$$

is called *a scheme of fat points in* $\mathbb{P}^n$.

Inspired by Lemma 1.2.4, one can wonder if s fat points in general position and of multiplicity $m_1, \dots, m_s$ in $\mathbb{P}^n$ have the same Hilbert function of

$$\sum_{i=1}^{s} \binom{n + m_i - 1}{m_i - 1}$$

reduced points in general position in $\mathbb{P}^n$. Unfortunately, however, the answer to this question is negative, and the situation is much more complicated, even though there are two notable results due to J. Alexander and A. Hirschowitz: the first one classifies, for $m_i \leq 2$ for any $i = 1, \dots, s$, all the cases when the Hilbert function is not the expected one, while the second one is an asymptotic result.

Theorem 1.2.7 (Alexander-Hirschowitz)**.** *Let $\mathbb{X}$ be a scheme of s double points in general position in $\mathbb{P}^n$. Then,*

$$H_{\mathbb{X}}(t) = \min\left\{ \binom{n+t}{n}, (n+1)s, \right\}$$

except in the following cases:

- $t = 2,\ 2 \leq s \leq n$ *and in this case* $H_{\mathbb{X}}(2) = \binom{n+2}{2} - \binom{n-s+2}{2}$;
- $n = 2,\ t = 4,\ s = 5$ *and in this case* $H_{\mathbb{X}}(4) = 14$;
- $n = 3,\ t = 4,\ s = 9$ *and in this case* $H_{\mathbb{X}}(4) = 34$;
- $n = 4,\ t = 3,\ s = 7$ *and in this case* $H_{\mathbb{X}}(3) = 34$;

- $n = 4,\ t = 4,\ s = 14$ *and in this case* $H_{\mathbb{X}}(4) = 69$.

Proof The original proof of the theorem can be found in [7], while more self-contained explanations and proofs of the theorem can be found in [26] and in [91].■

Unfortunately, up to our knowledge, there is no analogous result for higher multiplicities of the points. However, the following two theorems, again due to J. Alexander and A. Hirschowitz, give an asymptotic behaviour of the union of fat points of any multiplicity.

Theorem 1.2.8 (Alexander-Hirschowitz)**.** *For any* $m \geq 0$ *there exists* t_0 *such that any scheme of fat points*

$$\mathbb{X} = m_1P_1 + m_2P_2 + \cdots + m_sP_s$$

with $m_i \leq m$ *for any* $i = 1, \ldots, s$ *and the* P_i*'s in general position in* $\mathbb{P}^n$ *has Hilbert function*

$$H_{\mathbb{X}}(t) = \min\left\{\binom{n+t}{t}, \sum_{i=1}^{s}\binom{n+m_i-1}{m_i-1}\right\}\ \forall\, t \geq t_0.$$

Proof This theorem is a special case of the much more general Theorem 1.1 in [4]. ■

The previous theorem can also be stated in a more or less equivalent formulation as follows.

Theorem 1.2.9 (Alexander-Hirschowitz)**.** *For any* $m \geq 0$ *there exists an integer* ℓ *such that any scheme of fat points*

$$\mathbb{X} = m_1P_1 + m_2P_2 + \cdots + m_sP_s$$

with $m_i \leq m$ *for any* $i = 1, \ldots, s$*, the points* P_i*'s in general position in* $\mathbb{P}^n$ *and* $\ell(\mathbb{X}) \geq \ell$ *has Hilbert function*

$$H_{\mathbb{X}}(t) = \min\left\{\binom{n+t}{t}, \sum_{i=1}^{s}\binom{n+m_i-1}{m_i-1}\right\}\ \forall\, t \in \mathbb{Z}.$$

Proof This theorem is a special case of Corollary 1.2 in [4]. ■

These two theorems stress that the defective cases appear only for low degrees and low lengths of the scheme.

We conclude this section giving a way to graphically represent a fat point in $\mathbb{P}^2$ and, more in general, the 0-dimensional subschemes of $\mathbb{P}^2$ whose support is just one point and whose defining ideal is monomial.

Remark 1.2.10. Let us consider $\mathbb{X} = \mathrm{Proj}(R/\wp)$ a 0-dimensional subscheme of $\mathbb{P}^2$ supported at just one point which, up to projectivity, we can suppose to be $P = [1,0,0]$ and such that $\wp$ is monomial. Since $\mathbb{X}$ is supported at just one point, representing it is equivalent to representing $\underline{\mathbb{X}}$, the affinised scheme of $\mathbb{X}$ in the affine plane $\mathbb{A}^2$ with coordinate ring $\underline{R} = \mathbb{C}[x, y]$, where $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$. Since $\wp$ is monomial, its dehomogenised $\underline{\wp}$ with respect to x_0 is monomial too, that is there exists a set $A \subseteq \mathbb{N}^2$ such that

$$\underline{\wp} = (x^i y^j)_{(i,j)\in A} \subseteq \underline{R}$$

and, since $\underline{R}$ is Noetherian, we can suppose that A is finite, say

$$A = \{(i_1, j_1), \dots (i_r, j_r)\}.$$

Clearly, if $(i', j') \in A$ then any generator of the form $x^i y^j$ with $i \geq i'$ and $j \geq j'$ is redundant, thus can be removed and in this way we can suppose that $i_m \neq i_n$ and $j_m \neq j_n$ for any $m, n = 1, \dots, r$, $m \neq n$. At this point we can order A so that $i_1 < i_2 < \cdots < i_r$ (as a consequence, $j_1 > j_2 > \cdots > j_r$) and setting in the plane $\mathbb{N}^2$ all the points $(i_1, j_1), \dots (i_r, j_r)$ we will never find two distinct points among these having at least one of the two coordinate equal, that is, none of the lines joining two of the points will be vertical or horizontal. Finally, for any $m = 1, \dots, r-1$ we draw the segment joining (i_m, j_m) to (i_{m+1}, j_m) and that joining (i_{m+1}, j_m) to (i_{m+1}, j_{m+1}). In doing so we obtain a figure similar to a stair that can be interpreted as follows: the monomials contained in the stair are all and only the ones not contained in I and the number of squares inside the stair equals the length of $\mathbb{X}$.

Example 1.2.11. Let us consider the fat point $mP \in \mathbb{P}^2$ with $P = [1,0,0]$. Its defining ideal is $\wp^m = (x_1, x_2)^m$ whose dehomogenised with respect to x_0 is

$$\underline{\wp}^m = (x, y)^m = (x^m, x^{m-1}y, \dots, xy^{m-1}, y^m)$$

so that its graphic representation is

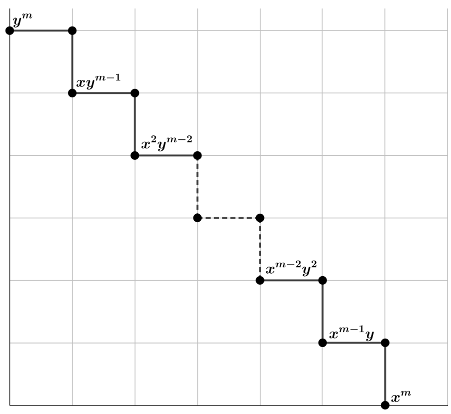


Figure 1.2: The graphic representation of $mP \in \mathbb{P}^2$

Note that the number of inner square of the stair is exactly

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2} = \binom{m+1}{2}$$

according to Lemma 1.2.4.

1.3 Apolarity Theory and Inverse Systems

A very powerful tool to study many questions concerning 0-dimensional subschemes of $\mathbb{P}^n$ is given by the theory of Apolarity and Inverse Systems, originally introduced by Macaulay and then rediscovered and applied to the study of 0-dimensional schemes by A. Iarrobino. In this section, we give a quick overview on this topic using as main reference [66]. We only treat the case when the base field is $\mathbb{C}$, but a much more general theory has been developed for an arbitrary base field.

Since in this and other sections we need to deal with different polynomial rings, some of which will be considered as coordinate rings of projective spaces, others as rings of polynomials and others as rings of derivations, we fix now some notations in order to avoid ambiguity and confusion.

Notation 1.3.1. For any $n \in \mathbb{N}$ we denote by T the ring $T = \mathbb{C}[y_0, \dots, y_n]$ and we think of the elements of T just as polynomials. We write a linear form $F \in T_1$ in the following form

$$F = x_0y_0 + x_1y_1 + \cdots + x_ny_n$$

so that the variables y_i are the indeterminates, the variables x_i are the coefficients, and we think of the projective space $\mathbb{P}^n$ having as coordinate ring $R = \mathbb{C}[x_0, \dots, x_n]$ as the projective space $\mathbb{P}(T_1)$ through the identification given by

$$\mathbb{P}(T_1) \ni [x_0y_0 + x_1y_1 + \cdots + x_ny_n] \leftrightarrow [x_0, x_1, \dots, x_n] \in \mathbb{P}^n.$$

For any $n, d \in \mathbb{N}_{>0}$ we denote by $N_{n,d} := \binom{n+d}{d} - 1$ the dimension of the projective space $\mathbb{P}(T_d) = \mathbb{P}(\mathbb{C}[y_0, \dots, y_n]_d)$ and by $\alpha_0, \dots, \alpha_{N_{n,d}}$ we denote the multi-indices of $\{0,1,\dots,d\}^{n+1}$ such that $|\alpha_j| = d$ ordered by the usual lexicographic order. Moreover, we denote by $\mathbb{N}_d^{n+1}$ the set of such multi-indices, that is

$$\mathbb{N}_d^{n+1} = \{(i_0, i_1, \dots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + i_1 + \cdots + i_n = d\} = \{\alpha_0, \alpha_1, \dots, \alpha_{N_{n,d}}\}.$$

We set

$$\underline{x} = (x_0, \dots, x_n)$$

and, if $\alpha_j = (\alpha_{0j}, \dots, \alpha_{nj})$, we set

$$\underline{x}^{\alpha_j} = x_0^{\alpha_{0j}} \dots x_n^{\alpha_{nj}}$$

and analogously for $\underline{y}$ and $\underline{y}^{\alpha_j}$. For any multi-index α_j we denote by $\alpha_j!$ the number

$$\alpha_j! = \prod_{i=0}^{n} \alpha_{ij}!$$

and by $\binom{d}{\alpha_j}$ the multinomial coefficient

$$\binom{d}{\alpha_j} = \binom{d}{\alpha_{0j}, \ldots, \alpha_{nj}} = \frac{d!}{\alpha_j!} = \prod_{i=0}^{n} \binom{\alpha_{0j} + \cdots + \alpha_{ij}}{\alpha_{ij}}.$$

A basis of the vector space T_d is given by

$$\left(\binom{d}{\alpha_j}\underline{y}^{\alpha_j}\right)_{j=0,\ldots,N_{n,d}}$$

so that we can write a form $G \in T_d$ in the following way

$$G = \sum_{j=0}^{N_{n,d}} z_j \binom{d}{\alpha_j}\underline{y}^{\alpha_j}$$

for some coefficients $z_j \in \mathbb{C}$. We think of $\mathbb{P}^{N_{n,d}}$ as the projective space $\mathbb{P}(T_d)$ with homogeneous coordinates $[z_0, \ldots, z_{N_{n,d}}]$ through the identification given by

$$\mathbb{P}(T_d) \ni \left[\sum_{j=0}^{N_{n,d}} z_j \binom{d}{\alpha_j}\underline{y}^{\alpha_j}\right] \leftrightarrow [z_0, \ldots, z_{N,d}] \in \mathbb{P}^{N_{n,d}}$$

and we denote by $S := \mathbb{C}[z_0, \ldots, z_{N_{n,d}}]$ the coordinate ring of $\mathbb{P}^{N_{n,d}}$. Finally, for any $n \in \mathbb{N}$ we denote by U the ring $U = \mathbb{C}[w_0, w_1, \ldots, w_n]$ and we think of its elements as derivations on T. We use the notations $\underline{w}$ and $\underline{w}^{\alpha_j}$ analogously to $\underline{x}$ and $\underline{x}^{\alpha_j}$

Remark 1.3.2. For any $n, d \in \mathbb{N}$ and for any $\beta = (\beta_0, \ldots, \beta_n) \in \mathbb{N}_d^{n+1}$ and $i \in \{1, \ldots, n\}$ we set

$$w_i \circ \underline{y}^{\beta} := \begin{cases} \frac{\partial \underline{y}^{\beta}}{\partial y_i} & \text{if } \beta_i > 0 \\ 0 & \text{otherwise} \end{cases}.$$

The map

$$\begin{array}{rccc} \circ : & U \times T & \to & T \\ & (F, G) & \mapsto & F \circ G \end{array}$$

obtained by extending in the obvious way the action of the variables $w_0, \ldots, w_n$ previously defined, is an action of U on T and makes T an U-module. Note that, since the action of U lowers the degree, T is not a finitely generated U-module. Moreover, the action $\circ$ respects the grading of U and T in the sense that for any $i, j \in \mathbb{N}$ it restricts to

$$\begin{array}{rccc} \circ : & U_i \times T_j & \to & T_{j-i} \\ & (F, G) & \mapsto & F \circ G \end{array}.$$

Definition 1.3.3. The map $\circ : U \times T \to T$ is called the *apolarity action of* U *on* T.

We introduce now a lemma, whose proof is trivial, that we will need in the following.

Lemma 1.3.4. *Let* $\beta, \gamma \in \mathbb{N}^{n+1}$ *(not necessarily with* $|\beta| = |\gamma|$*). If we set* $\beta = (\beta_0, \ldots, \beta_n)$ *and* $\gamma = (\gamma_0, \ldots, \gamma_n)$*, then*

$$\underline{w}^{\beta} \circ \underline{y}^{\gamma} = \begin{cases} 0 & \text{if } \beta \nmid \gamma \\ \frac{\gamma!}{(\beta-\gamma)!} \underline{y}^{\gamma-\beta} & \text{if } \beta \mid \gamma \end{cases}$$

Remark 1.3.5. Since $\circ$ makes T a U-module, the apolarity action induces a $\mathbb{C}$-bilinear pairing

$$U_t \times T_t \to \mathbb{C}$$

for any $t \in \mathbb{N}$. As a consequence we have two induced $\mathbb{C}$-linear maps:

$$\begin{array}{rccc} \phi : & U_t & \to & \mathrm{Hom}_{\mathbb{C}}(T_t, \mathbb{C}) \\ & F & \mapsto & \phi_F \end{array} \quad \text{where} \quad \begin{array}{rccc} \phi_F : & T_t & \to & \mathbb{C} \\ & G & \mapsto & F \circ G \end{array}$$

and

$$\begin{array}{rccc} \chi : & T_t & \to & \mathrm{Hom}_{\mathbb{C}}(U_t, \mathbb{C}) \\ & G & \mapsto & \chi_G \end{array} \quad \text{where} \quad \begin{array}{rccc} \chi_G : & U_t & \to & \mathbb{C} \\ & F & \mapsto & F \circ G \end{array}.$$

Proposition 1.3.6. *The* $\mathbb{C}$*-bilinear pairing* $U_t \times T_t \to \mathbb{C}$ *induced by the apolarity action is a perfect bilinear paring, i.e. the* $\mathbb{C}$*-linear maps* ϕ *and* χ *described in Remark 1.3.5 are isomorphisms of* $\mathbb{C}$*-vector spaces.*

Proof See [66] Proposition 2.3. ∎

We are now ready to give the definitions of perp and inverse system.

Definition 1.3.7. Let V be a vector subspace of U_t and W be a vector subspace of T_t. The *perp of* V *(with respect to* $\circ$*)* is the vector subspace of T_t defined as

$$V^{\perp} := \{G \in T_t \mid F \circ G = 0 \; \forall \; F \in V\} = \{G \in T_t \mid \chi_G(V) = \langle 0 \rangle\}$$

and analogously the *perp of* W *(with respect to* $\circ$*)* is the vector subspace of U_t defined as

$$W^{\perp} := \{F \in U_t \mid F \circ G = 0 \; \forall \; G \in W\} = \{F \in U_t \mid \phi_G(W) = \langle 0 \rangle\}.$$

Definition 1.3.8. Let I be a homogeneous ideal of the ring U. The *inverse system of* I, denoted by I^{-1}, is the U-submodule of T defined as

$$I^{-1} := \{G \in T \mid F \circ G = 0 \; \forall \; F \in I\}.$$

Remark 1.3.9. The inverse system of a homogeneous ideal $I \subseteq U$ is a graded submodule of T but, in general, it is not closed under multiplication, i.e. I^{-1} is not, in general, an ideal of T. Moreover, unfortunately, I^{-1} is not a finitely generated U-submodule of T, as the next easy example will show.

Example 1.3.10. Let us fix $n = 1$, so that $U = \mathbb{C}[w_0, w_1]$ and $T = \mathbb{C}[y_0, y_1]$ and let us consider the ideal $I = (w_0) \subseteq U$. By definition, we have

$$I^{-1} = \left\{ G \in T \;\middle|\; \frac{\partial G}{\partial y_0} = 0 \right\}$$

and easy computations show that

$$I^{-1} = \mathbb{C} \oplus \langle y_1 \rangle \oplus \langle y_1^2 \rangle \oplus \langle y_1^3 \rangle \oplus \cdots = \bigoplus_{i=0}^{\infty} \langle y_1^i \rangle.$$

In particular, I^{-1} is not finitely generated as U-submodule of T and it is not an ideal of T.

As their definition suggest, the perp and the inverse system are related, as shown by the following proposition. Also note that, since I^{-1} is graded, the proposition gives a way to construct, grade by grade, the entire submodule I^{-1}.

Proposition 1.3.11. *If I is a homogeneous ideal of U, then $(I^{-1})_t = I_t^{\perp}$.*

Proof See [66] Proposition 2.5. ∎

Remark 1.3.12. Let us consider a homogeneous ideal $I \subseteq U = \mathbb{C}[w_0, \dots, w_n]$. We have

$$\dim (I^{-1})_t = \dim I_t^{\perp} = N_{n,t} - \dim I_t = \dim R_t / I_t = H_{R/I}(t),$$

where the first equality follows from Proposition 1.3.11 and the second from the fact that $\circ : U_t \times T_t \to \mathbb{C}$ is a perfect pairing. Clearly, this can be used in two directions: on the one hand we can compute $H_{R/I}(t)$ by discussing the dimension of the inverse system of I, on the other hand we can compute the size of $(I^{-1})_t$ if we know $H_{R/I}(t)$.

Remark 1.3.13. Using Remark 1.3.12 we can find a necessary and sufficient condition for an ideal $I \subseteq U$ to have a finitely generated inverse system. Indeed, keeping in mind that the apolarity action lowers the degree of polynomials, we have that I^{-1} is finitely generated as U-submodule of T if and only if $\dim(I^{-1})_t = 0$ for all $t \gg 0$ and, by Remark 1.3.12, this happens if and only if $H_{R/I}(t) = 0$ for all $t \gg 0$. But this is true if and only if I is an artinian ideal, so that one has

$$I^{-1} \text{ is finitely generated as } U\text{-module} \Leftrightarrow I \text{ is an artinian ideal.}$$

Remark 1.3.14. We can use Proposition 1.3.11 to characterise the inverse system of a monomial ideal $I \subseteq U$. Since I is monomial, for any $t \in \mathbb{N}$ there exists $A_t \subseteq \mathbb{N}_t^{n+1}$ such that

$$I_t = \langle \underline{w}^{\alpha_j} \rangle_{\alpha_j \in A_t}$$

and by Lemma 1.3.4 we easily find that

$$I_t^{\perp} = \langle \underline{y}^{\alpha_j} \rangle_{\alpha_j \in \mathbb{N}_t^{n+1} \setminus A_t}$$

and thus, by Proposition 1.3.11, we get

$$I^{-1} = \bigoplus_{t=0}^{\infty} (I^{-1})_t = \bigoplus_{t=0}^{\infty} I_t^{\perp} = \bigoplus_{t=0}^{\infty} \langle \underline{y}^{\alpha_j} \rangle_{\alpha_j \in \mathbb{N}_t^{n+1} \setminus A_t} = \langle \underline{y}^{\beta} \mid \underline{w}^{\beta} \notin I \rangle.$$

The perp has a good behaviour with respect to the intersection and the sum of vector spaces, as shown by the following lemma.

Lemma 1.3.15. *Let W_1 and W_2 be two vector subspaces of T_t, or of U_t. Then the following hold:*

$$(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp} \quad \textit{and} \quad (W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}.$$

Proof See [66] Lemma 2.7. ∎

The inverse system of the intersection of two ideals can be computed using the following proposition, which follows immediately from Lemma 1.3.15.

Proposition 1.3.16. *Let I and J be ideals of U. Then*

$$(I \cap J)^{-1} = I^{-1} + J^{-1}.$$

We conclude this section describing the inverse system of a union of fat points. In order to preserve the coherence with the previous notations and with how we defined inverse systems, we consider fat points in the projective space $\mathbb{P}(U_1)$ thus defined by ideals of U. However, when in the following we will deal with fat points of $\mathbb{P}^n = \mathbb{P}(T_1)$ we will be able to apply all the results of this section using the trivial isomorphism $\mathbb{P}(U_1) \cong \mathbb{P}(T_1)$.

Theorem 1.3.17. *Let $P_1, \dots, P_s$ be points of $\mathbb{P}(U_1)$ with $\mathcal{I}(P_i) = \wp_i$ and let $m_1, \dots, m_s$ be positive integers. Suppose that $P_i = [p_{i0}, p_{i1}, \dots, p_{in}]$ and set*

$$L_{P_i} = p_{i0}y_0 + p_{i1}y_1 + \cdots + p_{in}y_n \in T_1$$

and $I = \wp_1^{m_1+1} \cap \cdots \cap \wp_s^{m_s+1} \in U$. Then we have

$$(I^{-1})_t = \begin{cases} T_j & \textit{for } j \leq \max\{m_i\} \\ L_{P_1}^{j-m_1} T_{m_1} + \cdots + L_{P_s}^{j-m_s} T_{m_s} & \textit{for } j \geq \max\{m_i + 1\} \end{cases}.$$

Proof See [66] Theorem 3.2. ■

Corollary 1.3.18. *Let* $\{L_1,\dots,L_s\}$ *be a general set of linear forms in* T_1*. Then, for any integer* t*, the vector space* $V = < L_1^t,\dots,L_s^t >$ *has the maximal possible dimension, i.e.*

$$\dim_{\mathbb{C}}(V) = \min\{s, \dim_{\mathbb{C}} T_t\}.$$

Proof See [66] Corollary 3.7. ■

1.4 Veronese varieties

As its name suggests, the Veronese surface was originally introduced and studied by G. Veronese in [105] and [106]. The definition was then extended to the more general class of Veronese varieties that are interesting thanks to several peculiar characteristics, that we discuss in this section.

Definition 1.4.1. For any $n, d \in \mathbb{N}_{>0}$ the (n,d)-*Veronese embedding* is the map

$$\begin{array}{rccc} \nu_{n,d}: & \mathbb{P}^n & \to & \mathbb{P}^{N_{n,d}} \\ & [x_0,\dots,x_n] & \mapsto & [\underline{x}^{\alpha_0},\dots,\underline{x}^{\alpha_{N_{n,d}}}] \end{array}$$

and the (n,d)-*Veronese variety* is $V_{n,d} := \nu_{n,d}(\mathbb{P}^n)$.

It is easy to show that $V_{n,d}$ is a variety and that $\nu_{n,d}$ is an isomorphism of algebraic varieties between $\mathbb{P}^n$ and $V_{n,d}$; see [97], §4.4, Example 1.28 for a detailed proof.

Remark 1.4.2. Given a linear form

$$F = x_0y_0 + x_1y_1 + \dots + x_ny_n \in T_1$$

by the multinomial theorem we have

$$F^d = \left(\sum_{i=0}^{n} x_iy_i\right)^d = \sum_{j=0}^{N_{n,d}} \binom{d}{\alpha_j} \underline{x}^{\alpha_j}\underline{y}^{\alpha_j} = \sum_{j=0}^{N_{n,d}} \underline{x}^{\alpha_j}\left(\binom{d}{\alpha_j}\underline{y}^{\alpha_j}\right).$$

Thus, using the identification mentioned in Notation 1.3.1, we have

$$\mathbb{P}(T_d) \ni F^d \leftrightarrow \underbrace{[\underline{x}^{\alpha_0},\dots,\underline{x}^{\alpha_{N_{n,d}}}]}_{\nu_{n,d}([x_0,\dots,x_n])} \in \mathbb{P}^{N_{n,d}}$$

so that another way to see the Veronese embedding is the following

$$\begin{array}{rccc} \nu_{n,d}: & \mathbb{P}(T_1) & \to & \mathbb{P}(T_d) \\ & [L] & \mapsto & [L^d] \end{array}$$

and Veronese varieties can be thought of as the varieties parameterising powers of linear forms.

This point of view on Veronese varieties immediately brings with it another one connected to tensors. Indeed, the polynomial algebra T is isomorphic as graded algebra to the symmetric algebra

$$\mathrm{Sym}(T_1) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(T_1)$$

and, more in particular, we have $T_d \cong \mathrm{Sym}^d(T_1)$ as vector spaces. In light of this, we could also see the Veronese embedding as

$$\begin{array}{rccc} \nu_{n,d}: & \mathbb{P}(\mathrm{Sym}^1(T_1)) & \to & \mathbb{P}(\mathrm{Sym}^d(T_1)) \\ & [v] & \mapsto & [\underbrace{v \otimes v \cdots \otimes v}_{d \text{ times}}] \end{array} .$$

We will mainly use the first or the second point of view; the purpose of introducing the last is just for the sake of completeness and coherence with Segre and Segre-Veronese varieties.

Remark 1.4.3. Let us consider a hypersurface $\mathcal{C}$ in $\mathbb{P}^n$ defined by the polynomial $F \in R_d$. If we write F as

$$F = \sum_{i=0}^{N_{n,d}} a_i \underline{x}^{\alpha_i}$$

we can construct the linear form

$$\mathrm{Lin}(F) := \sum_{i=0}^{N_{n,d}} a_i z_i \in S_1$$

defining a hyperplane $H_{\mathcal{C}}$ in $\mathbb{P}^{N_{n,d}}$ and an easy check shows that

$$\nu_{n,d}(\mathcal{C}) = H_{\mathcal{C}} \cap V_{n,d}.$$

In particular, if we now consider another hypersurface $\mathcal{D}$ in $\mathbb{P}^n$ defined by the polynomial $G \in R_d$ and we define the hyperplane $H_{\mathcal{D}}$ analogously to $H_{\mathcal{C}}$, we find that

$$\nu_{n,d}(\mathcal{C} \cap \mathcal{D}) = H_{\mathcal{C}} \cap H_{\mathcal{D}} \cap V_{n,d}.$$

Hence, the embedding $\nu_{n,d}$ allows to translate problems concerning intersection of hypersurfaces of degree d in $\mathbb{P}^n$ into problems concerning intersections of hyperplanes in $\mathbb{P}^{N_{n,d}}$ with the Veronese variety $V_{n,d}$.

A first important property of Veronese varieties is described by the following proposition.

Proposition 1.4.4. *Let $P_1, \dots, P_s \in V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ be distinct points in $V_{n,d}$. If $s \leq d+1$, then* $\dim L(P_1, \dots, P_s) = s-1$.

Proof It is enough to prove the statement for $s = d+1$. For any $s = 1, \dots, d+1$ we set $Q_s := \nu_{n,d}^{-1}(P_s) \subseteq \mathbb{P}^n$ and for any $s = 1, \dots, d$ let $F_s \in T_1$ be a linear form such that $F_s(Q_s) = 0$ and $F_s(Q_{s+1}) \neq 0$. The form

$$F := \prod_{s=1}^{d} F_s \in T_d$$

defines, as in Remark 1.4.3, a hyperplane of $\mathbb{P}^{N_{n,d}}$ containing the linear space $L(P_1, \dots, P_d)$, but not P_{d+1}, and this ends the proof. ■

We now state a useful proposition providing a link between Veronese embedding and Apolarity Theory; note that in this case, the considered apolarity action is the one of U on the ring R, which is obviously analogous to the apolarity action of U on T.

Notation 1.4.5. Given $\mathbb{X}$ a projective subscheme of $\mathbb{P}^n$, we denote by $L(\mathbb{X})$ the smallest linear projective subspace of $\mathbb{P}^n$ containing $\mathbb{X}$.

Proposition 1.4.6. *Let $\mathbb{X}$ be a projective subscheme of $\mathbb{P}^n$. Then*

$$L(\nu_{n,d}(\mathbb{X})) \cong \mathbb{P}(\mathcal{I}(\mathbb{X})_d^{\perp}) \subseteq \mathbb{P}(U_d)$$

and the isomorphism is natural.

Proof Given the scheme $\mathbb{X}$ and its ideal $\mathcal{I}(\mathbb{X}) = (F_1, \dots, F_s)$, we denote by r the number $r := |\{F \in \{F_1, \dots, F_s\}\ ,\ \deg(F) = d\}|$ and, up to a change of the order of the generators, we can suppose that $\deg(F_1) = \deg(F_2) = \cdots = \deg(F_r) = d$. We write the polynomials $F_1, \dots, F_r$ as

$$F_j = \sum_{i=0}^{N_{n,d}} a_{ij} \underline{x}^{\alpha_i}$$

and we set

$$H_i := \mathrm{Lin}(F_i) = \sum_{i=0}^{N_{n,d}} a_{ij} z_i$$

for any $j = 1, \dots, r$. Using the same argument of Remark 1.4.3, we get

$$\mathcal{I}(\nu_{n,d}(\mathbb{X}))_1 = \langle H_1, H_2, \dots, H_r \rangle \subseteq S_1$$

and, since $L(\nu_{n,d}(\mathbb{X})) = V(\mathcal{I}(\nu_{n,d}(\mathbb{X}))_1)$, we find

$$L(\nu_{n,d}(\mathbb{X})) = \left\{ [z_0, \dots, z_{N_{n,d}}] \in \mathbb{P}^{N_{n,d}} \mid \sum_{i=0}^{N_{n,d}} a_{ij} z_i = 0\ \forall\ j = 1, \dots, r \right\} =$$

$$= \left\{ \left[\sum_{i=0}^{N_{n,d}} z_i \binom{d}{\alpha_i} \underline{y}^{\alpha_i} \right] \in \mathbb{P}^{N_{n,d}} \mid \sum_{i=0}^{N_{n,d}} a_{ij} z_i = 0 \; \forall \, j = 1, \dots, r \right\} \subseteq \mathbb{P}^{N_{n,d}}.$$

Now we pass to describe the vector subspace $\mathcal{I}(\mathbb{X})_d^\perp \subseteq U_d$. We have

$$\mathcal{I}(\mathbb{X})_d = \langle F_1, \dots, F_r \rangle$$

and, by definition of perp and Lemma 1.3.15, we obtain

$$\mathcal{I}(\mathbb{X})_d^\perp = \bigcap_{j=1}^{r} (F_j)_d^\perp = \bigcap_{j=1}^{r} \{ G \in U_d \mid G \circ F_j = 0 \}.$$

A generic $G \in U_d$ can be written as (recall the notation $\alpha_i = (\alpha_{i0}, \dots, \alpha_{in})$)

$$G = \sum_{i=0}^{N_{n,d}} t_i \binom{d}{\alpha_i} \underline{w}^{\alpha_i}$$

and for any $j = 1, \dots, r$ we have

$$G \circ F_j = \sum_{i=0}^{N_{n,d}} t_i \binom{d}{\alpha_i} \underline{w}^{\alpha_i} \circ \sum_{k=0}^{N_{n,d}} a_{kj} \underline{x}^{\alpha_k} = \sum_{\substack{0 \le i \le N_{n,d} \\ 0 \le k \le N_{n,d}}} a_{kj} t_i \binom{d}{\alpha_i} \underline{w}^{\alpha_i} \circ \underline{x}^{\alpha_k} =$$

$$\sum_{\substack{0 \le i \le N_{n,d} \\ 0 \le k \le N_{n,d}}} a_{kj} t_i \binom{d}{\alpha_i} \alpha_k! \delta_{ik} = \sum_{i=0}^{N_{n,d}} a_{ij} \binom{d}{\alpha_i} \alpha_i! t_i = d! \sum_{i=0}^{N_{n,d}} a_{ij} t_i$$

where δ_{ik} is the usual Kronecker delta. Thus, we get

$$\mathcal{I}(\mathbb{X})_d^\perp \cong \left\{ \sum_{i=0}^{N_{n,d}} t_i \binom{d}{\alpha_i} \underline{w}^{\alpha_i} \in U_d \mid \sum_{i=0}^{N_{n,d}} a_{ij} t_i = 0 \; \forall \, j = 1, \dots, r \right\}.$$

Hence, the linear projective subspace $\mathbb{P}(\mathcal{I}(\mathbb{X})_d^\perp) \subseteq \mathbb{P}(U_d))$ is naturally isomorphic to $L(\nu_{n,d}(\mathbb{X}))$ via the trivial identification $y_i \leftrightarrow w_i$ and this concludes the proof. ■

Notation 1.4.7. In light of the proof of Proposition 1.4.6, we use in $\mathbb{P}(U_d)$ an identification analogous to the one we introduced for $\mathbb{P}(T_d)$. More precisely, we identify $\mathbb{P}(T_d)$ with $\mathbb{P}^{N_{n,d}}$ endowed with homogeneous coordinate $[t_0, \dots, t_{N_{n,d}}]$ through

$$\mathbb{P}(U_d) \ni \left[\sum_{j=0}^{N_{n,d}} t_j \binom{d}{\alpha_j} \underline{w}^{\alpha_j} \right] \leftrightarrow [t_0, \dots, t_{N,d}] \in \mathbb{P}^{N_{n,d}}.$$

In fact, we will need to work in $\mathbb{P}(U_d)$ only in Chapter 3, and there we will use the natural isomorphism given by Proposition 1.4.6. Hence, we will never again use homogeneous coordinates $[t_0, \dots, t_{N_{n,d}}]$ in $\mathbb{P}^{N_{n,d}}$, but we will always use $[z_0, \dots, z_{N_{n,d}}]$ as in Notation 1.3.1. (We gave this notation just to make things clearer and to stress the identification used in the proof of Proposition 1.4.6.)

1.4.1 Waring problem and secant varieties of $V_{n,d}$

In 1770 the British mathematician E. Waring, in his book *Meditationes Algebricae*, stated that every natural number is a sum of at most 9 positive cubes and of at most 19 fourth powers and supposed that, for every natural number $d \geq 2$, there existed a number $g(d)$ such that every natural number is a sum of at most $g(d)$ d^{th} powers of natural numbers. Waring did not prove any of these statements that, nevertheless, turned out to be true, as proved by D. Hilbert and other mathematicians.

We can follow Waring's footsteps and pose a similar question in the context of homogeneous polynomials.

Definition 1.4.8. Let $F \in T_d$ be a homogeneous polynomial of degree d. The *Waring rank*, or *symmetric rank*, of F, or simply the rank of F, is defined to be the number

$$\operatorname{srk}(F) = \min\left\{ r \in \mathbb{N} \mid \exists\, L_1, L_2, \ldots, L_r \in T_1 \text{ such that } F = \sum_{i=1}^{r} L_i^d \right\}.$$

Note that the well-posedness of the definition follows by Corollary 1.3.18. When no confusion can arise, we will refer to the Waring rank of a polynomial F just as the rank of F. The reason why the Waring rank is also called symmetric rank is explained by Remark 1.4.16.

Remark 1.4.9. Clearly, given $F \in T_d$ and $\lambda \in \mathbb{C}^*$, one has $\operatorname{srk}(F) = \operatorname{srk}(\lambda F)$ or, in other words, $\operatorname{srk}(F) = \operatorname{srk}(G)$ for any $G \in [F] \in \mathbb{P}(T_d) = \mathbb{P}^{N_{n,d}}$. Hence, in some sense, the notion of Waring rank is a "projective" notion, and when one deals with problems regarding the Waring rank, one can work in $\mathbb{P}^{N_{n,d}}$ rather than T_d.

At this point we can ask two questions:

- Does there exist a minimum integer $g(d)$ such that $\operatorname{srk}(F) \leq g(d)$ for all $[F] \in \mathbb{P}^{N_{n,d}}$?
- Does there exist a minimum integer $G(d)$ such that there exists a non-empty Zariski open subset W of $\mathbb{P}^{N_{n,d}}$ such that $\operatorname{srk}(F) \leq G(d)$ for all $[F] \in W$?

Clearly, one has $G(d) \leq g(d)$ and Corollary 1.3.18 guarantees that $g(d)$ exists and that $g(d) \leq \dim T_d$. Questions related to $G(d)$ and $g(d)$ or, more in general, to the Waring rank of homogeneous polynomials are all called *Waring problems*.

Now we want to highlight the link between Waring problems and Veronese varieties and the consequent usefulness of the Veronese variety geometry knowledge in the study of the Waring problems. This link is mainly given by secant and osculating varieties of Veronese varieties and by the fact that, as we have already said, the points of Veronese varieties can be seen of as powers of linear forms. Let us clarify what we mean.

Definition 1.4.10. Let $\mathbb{X} \subseteq \mathbb{P}^n$ a projective variety. We define

$$\sigma_s^0(\mathbb{X}) := \bigcup_{\substack{P_1,\dots,P_s \in \mathbb{X} \\ P_i \neq P_j \text{ for } i \neq j}} L(P_1, \dots, P_s)$$

and

$$\sigma_s(\mathbb{X}) := \overline{\sigma_s^0} \subseteq \mathbb{P}^n.$$

The variety $\sigma_s(\mathbb{X})$ is called s^{th}*-secant variety of* $\mathbb{X}$. Moreover, given a point $P \in \mathbb{P}^n$ we say that the $\mathbb{X}$-rank of P is the minimum integer r such that $P \in \sigma_r^0(\mathbb{X})$, and the $\mathbb{X}$-border rank of P is the minimum integer r' such that $P \in \sigma_{r'}(\mathbb{X})$.

As we will soon see, the notion of border rank is strictly necessary, since it can be different than the rank.

The s^{th}-secant variety of a variety $\mathbb{X}$ has an expected dimension obtained by a rough computation of the parameters, but it can happen that the true dimension of the secant variety is less than the expected one.

Definition 1.4.11. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a projective variety. The *expected dimension of* $\sigma_s(\mathbb{X})$ is

$$\operatorname{expdim}(\sigma_s(\mathbb{X})) := \min\{s(\dim(\mathbb{X}) + 1) - 1, n\}$$

and the number

$$\delta(\sigma_s(\mathbb{X})) := \operatorname{expdim}(\sigma_s(\mathbb{X})) - \dim(\sigma_s(\mathbb{X}))$$

is called the *defect* of $\sigma_s(\mathbb{X})$. If $\delta(\sigma_s(\mathbb{X})) > 0$, then the variety $\sigma_s(\mathbb{X})$ is called *defective.*

The effective dimension of a secant variety can be computed by using the famous Terracini Lemma. To recall this result, we first introduce the osculating spaces to a variety, which we will also use in the section on osculating varieties.

Definition 1.4.12. Let $\mathbb{X} \subseteq \mathbb{P}^n$ a projective variety and $P \in \mathbb{X}$. The k^{th}*-osculating space to* $\mathbb{X}$ *at* P is the projective linear space

$$\tau_{k,P}(\mathbb{X}) := L((k+1)P \cap \mathbb{X}).$$

Note that $\tau_{1,P}(\mathbb{X})$ is the classical tangent space to $\mathbb{X}$ at the point P; see [59] §III.2.4. for more details.

Lemma 1.4.13 (Terracini Lemma)**.** *Let* $\mathbb{X}$ *be a variety in* $\mathbb{P}^n$ *and let* $p_1, \dots, p_s \in \mathbb{X}$ *general points and* $z \in L(p_1, \dots, p_s)$ *a general point. Then*

$$\tau_{1,z}(\sigma_s(\mathbb{X})) = L(\tau_{1,p_1}(\mathbb{X}), \dots, \tau_{1,p_s}(\mathbb{X})).$$

Now we analyse in more detail the secant varieties of Veronese varieties.

Remark 1.4.14. Let us now consider the s^{th}-secant variety of the Veronese variety $V_{n,d}$. Keeping in mind that we can view $V_{n,d}$ as

$$V_{n,d} = \{[L^d] \mid L \in T_1\} \subseteq \mathbb{P}^{N_{n,d}}$$

we have

$$\sigma_s^0(V_{n,d}) = \bigcup_{[L_1^d],\dots,[L_s^d] \in V_{n,d}} L([L_1^d],\dots,[L_s^d]) =$$
$$= \left\{ [F] \in \mathbb{P}^{N_{n,d}} \mid [F] = \left[\sum_{j=1}^{s} a_j L_j^d \right] \text{ for some } a_j \in \mathbb{C}, L_j \in T_1 \right\} =$$
$$= \{[F] \in \mathbb{P}^{N_{n,d}} \mid \operatorname{srk}(F) \leq s\}$$

that is $\sigma_s^0(V_{n,d})$ is the set of the degree d form of T_d having Waring rank equal or less than s and in particular we have

$$\sigma_s^0 \setminus \sigma_{s-1}^0 = \{[F] \in \mathbb{P}^{N_{n,d}} \mid \operatorname{srk}(F) = s\}$$

and hence the Waring rank coincides with the $V_{n,d}$-rank. This characterisation of $\sigma_s^0(V_{n,d})$, together with Corollary 1.3.18, gives the following chain of inclusions

$$V_{n,d} = \sigma_1^0(V_{n,d}) \subsetneq \sigma_2^0(V_{n,d}) \subsetneq \cdots \subsetneq \sigma_r^0(V_{n,d}) = \mathbb{P}^{N_{n,d}}$$

for some $r \leq N_{n,d}$ (among all the possible values of r we choose the minimal one) and, consequently, there exists also $r' \in \mathbb{N}$ with $r' \leq r$ (we choose again the minimal r') such that

$$V_{n,d} = \sigma_1(V_{n,d}) \subsetneq \sigma_2(V_{n,d}) \subsetneq \cdots \subsetneq \sigma_{r'}(V_{n,d}) = \mathbb{P}^{N_{n,d}}.$$

The expected value of r' is

$$\min\{r \in \mathbb{N} \mid \operatorname{expdim}(\sigma_r(V_{n,d})) = N_{n,d}\}$$

and, as we will see, this expected value coincides with the actual value of r' except in some classified cases.

Unfortunately, $\sigma_s^0(V_{n,d})$ is not, in general, an algebraic variety so it is easier to work with its closure $\sigma_s(V_{n,d})$ but unfortunately, again, the points on the border of $\sigma_s(V_{n,d})$ can have a Waring rank greater than s. All this brings us to introduce another notion of Waring rank: the *border Waring rank*, which is nothing but the $V_{n,d}$-border rank.

Definition 1.4.15. Let $F \in T_d$ be a homogeneous polynomial of degree d. The *border Waring rank* of F, or simply the border rank of F, is defined to be the number

$$\overline{\operatorname{srk}}(F) = \min\{r \in \mathbb{N} \mid [F] \in \sigma_r(V_{n,d})\}.$$

Note that, clearly, it is always true that $\mathrm{srk}(F) \leq \overline{\mathrm{srk}}(F)$. As for the Waring rank, when no confusion arises, we will refer to the border Waring rank of F just as the border rank of F.

Remark 1.4.16. Given a symmetric tensor $T \in \mathrm{Sym}^d(T_1)$, its symmetric rank and its symmetric border rank are defined analogously to the Waring rank and the border Waring rank of a homogeneous polynomial through the secant varieties of Veronese varieties. Since we have a natural identification between homogeneous polynomials and symmetric tensors, we use for the symmetric rank and the border symmetric rank the same notations of the polynomial case, that is $\mathrm{srk}(T)$ and $\overline{\mathrm{srk}}(T)$.

In general, computing the rank or the border rank of a polynomial is very hard, see [76], but it is possible to determine, at least, the value of $G(d)$ for any d by using the secant varieties to Veronese varieties and Theorem 1.2.7. Before stating the theorem giving the exact value of $G(d)$ note that, with the notion of secant varieties, the numbers $g(d)$ and $G(d)$ can be redefined as follows:

- $g(d) = \min\{r \in \mathbb{N} \mid \sigma_r^0(V_{n,d}) = \mathbb{P}^{N_{n,d}}\}$;
- $G(d) = \min\{r \in \mathbb{N} \mid \sigma_r(V_{n,d}) = \mathbb{P}^{N_{n,d}}\}$.

Determining $g(d)$ is harder than determining $G(d)$, since $\sigma_r^0(V_{n,d})$ need not be closed, and, up to our knowledge, it is still an unsolved problem. To determine $G(d)$ we need instead to understand when $\mathrm{expdim}(\sigma_s(V_{n,d})) = \dim(\sigma_s(V_{n,d}))$. The key to do that is the following theorem.

Theorem 1.4.17. *Let $P_1, \ldots, P_s$ be generic points in $\mathbb{P}^n$ and*

$$\mathbb{X} = 2P_1 + 2P_2 + \cdots + 2P_s.$$

Then

$$\dim(\sigma_s(V_{n,d})) = H_{\mathbb{X}}(d) - 1.$$

Proof See [66] Theorem 6.1. ∎

Using this latter theorem and Theorem 1.2.7 one gets the following theorem, giving all the cases in which $\sigma_s(V_{n,d})$ is defective and thus all the values of $G(d)$.

Theorem 1.4.18. *For any $n, d, s \in \mathbb{N}$ it holds that $\mathrm{expdim}(\sigma_s(V_{n,d})) = \dim(\sigma_s(V_{n,d}))$ except in the following cases:*

- *$d = 2$, $2 \leq s \leq n$ and in this case $\delta(\sigma_s(V_{n,2})) = \frac{s(s-1)}{2}$;*
- *$n = 2, d = 4, s = 5$ and in this case $\delta(\sigma_5(V_{2,4})) = 1$;*

- *$n = 3, d = 4, s = 9$ and in this case $\delta(\sigma_9(V_{3,4})) = 1$;*
- *$n = 4, d = 3, s = 7$ and in this case $\delta(\sigma_7(V_{4,3})) = 1$;*
- *$n = 4, d = 4, s = 14$ and in this case $\delta(\sigma_{14}(V_{4,4})) = 1$.*

Note that the defectivity in the case $n = d = s = 2$ is a classical result due to Severi, who proved in [96] that the Veronese surface $V_{2,2} \subseteq \mathbb{P}^5$ is, up to projectivity, the only irreducible non-degenerate surface, not a cone, of $\mathbb{P}^5$ whose second secant variety is defective.

1.4.2 Generalised Waring problem and osculating varieties of $V_{n,d}$

Waring problems can be generalised as follows: given $F \in T_d$ and $k \leq d$, what is the minimum $r \in \mathbb{N}$ for which there exist $L_1, \dots, L_r \in T_1$ and $F_1, \dots, F_r \in T_k$ such that $F = L_1^{d-k}F_1 + \cdots + L_r^{d-k}F_r$? These kinds of problems are called *generalised Waring problems* and as well as the study of Waring problems can be carried out through the secant varieties of Veronese varieties, so the study of the generalised Waring problems can be reduced to the study of the secant varieties of the osculating varieties of $V_{n,d}$. Here we will be very concise; a more detailed treatise of this topic can be found, for instance, in [21] and [46].

Definition 1.4.19. Let $\mathbb{X} \subseteq \mathbb{P}^n$ a smooth projective variety. The k^{th}*-osculating variety* of $\mathbb{X}$ is the variety

$$\tau_k(\mathbb{X}) := \overline{\bigcup_{P \in \mathbb{X}} \tau_{k,P}(\mathbb{X})}.$$

As for secant varieties, we have an expected dimension for the osculating varieties of $V_{n,d}$:

$$\operatorname{expdim}(\tau_k(V_{n,d})) = \min\left\{N_{n,d}, n + \binom{n+k}{n} - 1\right\}.$$

In contrast to the case of secant varieties, the osculating varieties of $V_{n,d}$ are never defective, as stated in the following proposition.

Proposition 1.4.20. *The dimension of $\tau_k(V_{n,d})$ is always the expected one, that is*

$$\dim(\tau_k(V_{n,d})) = \min\left\{N_{n,d}, n + \binom{n+k}{n} - 1\right\}.$$

Proof See [21] Lemma 3.3. ∎

Remark 1.4.21. The relevance of the osculating varieties $\tau_k(V_{n,d})$ in the study of the generalised Waring problems is given by the following fact: if $P \in V_{n,d}$ is the point corresponding to $[L^d]$, with $L \in T_1$, then it can be shown that (see [46], §1)

$$\tau_{k,P}(V_{n,d}) = \{L^{d-k}F \mid F \in T_k\}$$

and, as a consequence, we have

$$\tau_k(V_{n,d}) = \{[F] \in \mathbb{P}^{N_{n,d}} \mid F = L^{d-k}F \text{ for some } L \in T_1, F \in T_k\}.$$

With this in mind, it is clear that determining the minimum r such that the generic $F \in T_d$ can be written as $F = L_1^{d-k}F_1 + \cdots + L_r^{d-k}F_r$ with $L_i \in T_1$ and $F_i \in T_k$, is equivalent to finding the minimum r such that $\sigma_r(\tau_k(V_{n,d})) = \mathbb{P}^{N_{n,d}}$.

The previous remark motivates the study of the osculating varieties of $V_{n,d}$ and of their secant varieties, which, unfortunately, looks more difficult than the secant varieties case. A concrete evidence comes from the fact that we do not have a complete classification of the defective cases, and there are some examples in which the defect of these secants variety is very high. To show how unfavourable the situation can be, we report in the following table the dimensions of $\sigma_s(\tau_4(V_{6,5})) \subseteq \mathbb{P}^{461}$ obtained from [21], §4, Example 4:

s	$\operatorname{expdim}(\sigma_s(\tau_4(V_{6,5})))$	$\dim(\sigma_s(\tau_4(V_{6,5})))$	$\delta(\sigma_s(\tau_4(V_{6,5})))$
2	431	345	86
3	461	417	44
4	461	452	9
≥ 5	461	461	0

See [21] §4 for a list of solved cases, conjectures, and open problems on the topic.

1.4.3 Catalecticant matrices and ideals of $V_{n,d}$

Catalecticant matrices were introduced by J. J. Sylvester in 1851 in [101], even though the author used this name to refer to them only the following year in [102] and [103]. They were originally investigated because the study of their determinant allows to determine the Waring rank of binary forms. Here, we give a very quick treatise following a more modern approach to catalecticant matrices.

Definition 1.4.22. For every $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_{>0}$ such that $i < j$ the $(i, j-i; n+1)$-*catalecticant matrix* $\operatorname{Cat}(i, j-i; n+1)$ is the $\binom{n+i}{n} \times \binom{n+j-i}{n}$ matrix with row and column indices respectively given by the multi-index sets $\{\alpha \in \mathbb{N}_i^{n+1}\}$ and $\{\beta \in \mathbb{N}_{j-i}^{n+1}\}$, whose (α, β)-entry is equal to $z_{\alpha+\beta}$.

Note that $\mathrm{Cat}(i, j-i; n+1) = \mathrm{Cat}(j-i, i; n+1)^t$ and $\mathrm{Cat}(i, j-i; n+1)$ is a square matrix if and only if $i = j - i$. For instance, the catalecticant matrix Cat(2,2; 3) is

$$\mathrm{Cat}(2{,}2;3) = \begin{pmatrix} z_{4,0,0} & z_{3,1,0} & z_{3,0,1} & z_{2,2,0} & z_{2,1,1} & z_{2,0,2} \\ z_{3,1,0} & z_{2,2,0} & z_{2,1,1} & z_{1,3,0} & z_{1,2,1} & z_{1,1,2} \\ z_{3,0,1} & z_{2,1,1} & z_{2,0,2} & z_{1,2,1} & z_{1,1,2} & z_{1,0,3} \\ z_{2,2,0} & z_{1,3,0} & z_{1,2,1} & z_{0,4,0} & z_{0,3,1} & z_{0,2,2} \\ z_{2,1,1} & z_{1,2,1} & z_{1,1,2} & z_{0,3,1} & z_{0,2,2} & z_{0,1,3} \\ z_{2,0,2} & z_{1,1,2} & z_{1,0,3} & z_{0,2,2} & z_{1,0,3} & z_{0,0,4} \end{pmatrix}.$$

Once one has defined the generic $\mathrm{Cat}(i, j-i; n+1)$, it is possible to associate to a homogeneous polynomial its catalecticant matrices.

Definition 1.4.23. Given $F \in T_d = \mathbb{C}[y_0, \ldots, y_n]_d$ and a positive integer $i < d$, its $(i, d-i; n+1)$-*catalecticant matrix* is the matrix $\mathrm{Cat}_F(i, d-i; n+1)$ obtained by substituting in the catalecticant matrix $\mathrm{Cat}(i, d-i; n+1)$ to z_γ the coefficient of $\underline{y}^\gamma$ divided by the number of occurrences of z_γ in $\mathrm{Cat}(i, d-i; n+1)$.

Example 1.4.24. Let us consider $F = y_0^4 + 2y_0^2 y_1 y_2 - 3y_0^2 + y_1^4 - y_2^4 + y_0 y_1^3 \in T_4$. If we want to construct $\mathrm{Cat}_F(2{,}2;3)$, following the definition, we have to substitute

$$z_{4,0,0} = 1,\ z_{2,1,1} = \frac{1}{2},\ z_{2,0,2} = -1,\ z_{0,0,4} = -1,\ z_{0,4,0} = 1,\ z_{1,3,0} = \frac{1}{2}$$

and $z_{i,j,k} = 0$ in all the other cases. Hence, we obtain

$$\mathrm{Cat}_F(2{,}2;3) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Lemma 1.4.25. *Given $F \in T_d = \mathbb{C}[y_0, \ldots, y_n]_d$ and a positive integer $i < d$ we have*

$$F = \begin{pmatrix} y_0^i & y_0^i y_1 & \cdots & y_n^i \end{pmatrix} \mathrm{Cat}_F(i, d-i; n+1) \begin{pmatrix} y_0^{d-i} & y_0^{d-i-1} y_1 & \cdots & y_n^{d-i} \end{pmatrix}^t$$

where $(y_0^i \quad y_0^i y_1 \quad \cdots \quad y_n^i)$ are all the monomials of T_i ordered by the lexicographic order and similarly for $(y_0^{d-i} \quad y_0^{d-i-1} y_1 \quad \cdots \quad y_n^{d-i})$.

Proof The proof is a straightforward check. ■

We will see more facts about catalecticant matrices in the following chapters; time being, we conclude this section presenting an important theorem that allows to compute the ideals of Veronese varieties through catalecticant matrices.

Theorem 1.4.26. *For any $n \geq 1$, $d \geq 2$ and $i = 1, \dots, d-1$ the ideal of $V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ has a system of generators given by the 2×2 minors of* $\mathrm{Cat}(i, d-i; n+1)$.

Proof See [92] Corollary 5.5. ∎

1.5 Segre varieties and Segre-Veronese varieties

In the same way one can parameterise symmetric tensors of symmetric rank 1 through the Veronese varieties, one can use the so-called Segre varieties to parameterise tensors having tensor rank 1, which are also called decomposable tensors. We start by recalling the definition of tensor rank and tensor border rank and then we give the definition of Segre varieties.

Definition 1.5.1. Let $V_1 \dots V_k$ be finitely generated $\mathbb{C}$-vector spaces and $T \in V_1 \otimes \cdots \otimes V_k$. The *tensor rank* of T is defined to be

$$\mathrm{rk}(T) := \min\{r \in \mathbb{N} \mid T = \sum_{i=1}^{r} v_{i1} \otimes \cdots \otimes v_{ik} \text{ for some } v_{ij} \in V_j\}$$

and the *tensor border rank* is defined to be

$$\overline{\mathrm{rk}}(T) := \min\{r \in \mathbb{N} \mid T = \lim_{t \to \infty} T_t \text{ with } \mathrm{rk}(T_t) \leq r \; \forall \, t\}.$$

Notation 1.5.2. Given $n_1, n_2, \dots, n_k$ with $k \geq 2$, we denote the homogeneous coordinates of $\mathbb{P}^{n_i}$ by $[x_{i0}, \dots, x_{in_i}]$ and the multihomogeneous coordinates of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$ by $[x_{10}, \dots, x_{1n_1}; \dots; x_{k0}, \dots, x_{kn_k}]$. We set

$$\mathcal{R}^{(i)} := \mathbb{C}[x_{i,0}, \dots, x_{i,n_i}] \text{ and } \mathcal{R} := \mathbb{C}[x_{10}, \dots, x_{1n_1}; \dots; x_{k0}, \dots, x_{kn_k}].$$

Moreover, on the set of all the variables x_{ij}, we use the lexicographic order with the variables ordered so that $x_{i_1 j_1} < x_{i_2 j_2}$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$.

Definition 1.5.3. Let $n_1, \dots, n_k$ be positive integers. The $(n_1, \dots, n_k)$-*Segre embedding* is the map

$$\begin{array}{cccc} s_{n_1,\dots,n_k}: & \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k} & \to & \mathbb{P}^{(n_1+1)(n_2+1)\dots(n_k+1)-1} \\ & [x_{10}, \dots, x_{1n_1}; \dots; x_{k0}, \dots, x_{kn_k}] & \mapsto & [\dots, x_{1i_1} x_{2i_2} \dots x_{ki_k}, \dots] \end{array}$$

where the products $x_{1i_1} x_{2i_2} \dots x_{ki_k}$ are all the possible ones, ordered according to the lexicographic order described in Notation 1.5.2. The image of $s_{n_i,\dots,n_k}$ is called the $(n_1, \dots, n_k)$-*Segre variety*, and it is denoted by $S_{n_1,\dots,n_k}$.

Notation 1.5.4. When we deal with $s_{n_1,\dots,n_k}$, we use in $\mathbb{P}^{(n_1+1)(n_2+1)\dots(n_k+1)-1}$ homogeneous coordinates $[u_{i_1,i_2,\dots,i_k}]$ with $i_j \in \{0,1,\dots,n_j\}$ for any $j = 1, \dots, k$. With this notation, we have that $S_{n_1,\dots,n_k}$ has parametric equations given by

$$u_{i_1,i_2,\dots,i_k} = x_{1i_1} x_{2i_2} \dots x_{ki_k}.$$

Remark 1.5.5. It is easy to see that the Segre embeddings can also be viewed in following way:

$$\begin{array}{rccc} s_{n_1,\dots,n_k}: & \mathbb{P}(\mathbb{C}^{n_1+1})\times\cdots\times\mathbb{P}(\mathbb{C}^{n_k+1}) & \to & \mathbb{P}(\mathbb{C}^{n_1+1}\otimes\cdots\otimes\mathbb{C}^{n_k+1}) \\ & ([v_1],[v_2],\dots,[v_k]) & \mapsto & [v_1\otimes\cdots\otimes v_k] \end{array}.$$

One can repeat all the discussion on the secant varieties of $V_{n,d}$, substituting $V_{n,d}$ with $S_{n_1,\dots,n_k}$ and in this way one obtains that the tensor rank and the border tensor rank respectively coincide with the $S_{n_1,\dots,n_k}$-rank and the $S_{n_1,\dots,n_k}$-border rank. For this reason, the secant varieties of Segre varieties are widely studied, but a complete classification of the defective cases is still missing. However, there are several partial results; see [23] for an exhaustive description of the state of the art.

Segre and Veronese varieties can be "combined", giving rise to the Segre-Veronese varieties, which parameterise the so-called decomposable partially symmetric tensors. We want to conclude this section with a quick overview on this topic.

Definition 1.5.6. Let $n_1,\dots,n_l,d_1,\dots,d_l$ be positive integers and set

$$\mathbf{n}:=(n_1,\dots,n_l),\quad \mathbf{d}:=(d_1,\dots,d_l),\quad n:=(n_1+1)+\cdots+(n_l+1),\quad d:=d_1+\cdots+d_l.$$

We say that a tensor $T\in(\mathbb{C}^n)^{\otimes d}$ is a $(\mathbf{n};\mathbf{d})$-*partially symmetric tensor* if

$$T\in\mathrm{Sym}^{d_1}(\mathbb{C}^{n_1+1})\otimes\mathrm{Sym}^{d_2}(\mathbb{C}^{n_2+1})\otimes\cdots\otimes\mathrm{Sym}^{d_l}(\mathbb{C}^{n_l+1})\subseteq(\mathbb{C}^n)^{\otimes d}$$

Definition 1.5.7. The *partially symmetric rank* of a $(\mathbf{n};\mathbf{d})$-partially symmetric tensor T is the number

$$\mathrm{psrk}(T):=\min\{r\in\mathbb{N}\mid T=\sum_{i=1}^{r}v_{i,1}^{\otimes d_1}\otimes\cdots\otimes v_{i,l}^{\otimes d_l}\text{ for some }v_{i,j}\in\mathbb{C}^{n_j+1}\}$$

and the *partially symmetric border rank* of T is the number

$$\overline{\mathrm{psrk}}(T):=\min\{r\in\mathbb{N}\mid T=\lim_{t\to\infty}T_t\text{ with }\mathrm{psrk}(T_t)\le r\ \forall\, t\}.$$

In the same way that symmetric tensors can be identified with homogeneous polynomials, partially symmetric tensors can be identified with multihomogeneous polynomials. To clarify this concept, we need to introduce some cumbersome, but necessary, notations; we imitate Notation 1.3.1 in a multigraded setting.

Notation 1.5.8. Given $n_1,n_2,\dots,n_l,d_1,d_2,\dots,d_l$ positive integers we set

$$\mathbf{n}:=(n_1,\dots,n_l),\quad \mathbf{d}:=(d_1,\dots,d_l).$$

We denote by $\mathcal{T}^{(i)}:=\mathbb{C}[y_{i,0},\dots,y_{i,n_i}]$ and we think of the elements of $\mathcal{T}^{(i)}$ just as polynomials. We write a linear form $F^{(i)}\in\mathcal{T}^{(i)}$ in the following way

$$F^{(i)}=x_{i,0}y_{i,0}+x_{i,1}y_{i,1}+\cdots+x_{i,n_i}y_{i,n_i}$$

so that the y_{i,k_i} are the indeterminates, the x_{i,k_i} are the coefficients and we think of the projective space $\mathbb{P}^{n_i}$ with coordinate ring $\mathcal{R}^{(i)}$ as the projective space $\mathbb{P}(\mathcal{T}_1^{(i)})$ through the identification given by

$$\mathbb{P}(\mathcal{T}_1^{(i)}) \ni [x_{i,0}y_{i,0} + x_{i,1}y_{i,1} + \cdots + x_{i,n_i}y_{i,n_i}] \leftrightarrow [x_{i,0}, x_{i,1}, \ldots, x_{i,n_i}] \in \mathbb{P}^{n_i}.$$

At this point we consider the ring

$$\mathcal{T} := \mathbb{C}[y_{1,0}, \ldots, y_{1,n_1}; \ldots; y_{l,0}, \ldots, y_{l,n_l}]$$

endowed with the multigrading induced by $\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(l)}$. The multihomogeneous part of $\mathcal{T}$ with multidegree $(d_1, \ldots, d_l)$ is denoted by $\mathcal{T}_{d_1,\ldots,d_l}$ and it has dimension

$$\binom{n_1 + d_1}{n_1} \cdots \binom{n_l + d_l}{n_l},$$

and we set $N_{\mathbf{n};\mathbf{d}} := \dim \mathcal{T}_{d_1,\ldots,d_l} - 1$. An analogous notation is used for the multihomogeneous parts of a multihomogeneous ideals of $\mathcal{T}$. A basis of the vector space $\mathcal{T}_{d_1,\ldots,d_l}$ is given by

$$\left(\prod_{j=1}^{l} \binom{d_j}{\alpha_{i_j}} \underline{y_j}^{\alpha_{i_j}} \right)_{(\alpha_{i_1},\ldots,\alpha_{i_l}) \in \mathbb{N}_{d_1}^{n_1+1} \times \cdots \times \mathbb{N}_{d_l}^{n_l+1}},$$

so that we can write a form $G \in \mathcal{T}_{d_1,\ldots,d_l}$ in the following way

$$G = \sum_{(\alpha_{i_1},\ldots,\alpha_{i_l}) \in \mathbb{N}_{d_1}^{n_1+1} \times \cdots \times \mathbb{N}_{d_l}^{n_l+1}} v(\alpha_{i_1}, \ldots, \alpha_{i_l}) \prod_{j=1}^{l} \binom{d_j}{\alpha_{i_j}} \underline{y_j}^{\alpha_{i_j}}$$

for some coefficients $v(\alpha_{i_1}, \ldots, \alpha_{i_l}) \in \mathbb{C}$ depending on G and on $\alpha_{i_1}, \ldots, \alpha_{i_l}$. Given any multi-index $\alpha = (a_0, a_1, \ldots, a_n) \in \mathbb{N}_d^{n+1}$ we can associate to it a d-ple $(i_1, \ldots, i_d)$ as follows:

$$(a_0, a_1, \ldots, a_n) \xrightarrow{*} (\underbrace{0,0,\ldots,0}_{a_0 \text{ times}}, \underbrace{1,1,\ldots,1}_{a_1 \text{ times}}, \ldots, \underbrace{n, n, \ldots, n}_{a_n \text{ times}}) =: *((a_0, a_1, \ldots, a_n)).$$

In what follows, it will be useful to write

$$v(\alpha_{i_1}, \ldots, \alpha_{i_l}) = v_{(i_{11},\ldots,i_{1d_1}),\ldots,(i_{l1},\ldots,i_{ld_l})}$$

where $(i_{j1}, \ldots, i_{jd_j})$ is obtained by α_{i_j} using the association $*$. Note that

$$\left\{ *(\alpha_{i_j}) \mid \alpha_{i_j} \in \mathbb{N}_{d_j}^{n_j+1} \right\} = \left\{ (i_{j1}, \ldots, i_{jd_j}) \mid 0 \le i_{j1} \le i_{j2} \le \cdots \le i_{jn_j} \le n_j \right\}.$$

We think of the projective space $\mathbb{P}^{N_{\mathbf{n};\mathbf{d}}}$ as the projective space $\mathbb{P}(\mathcal{T}_{d_1,\dots,d_l})$ with homogeneous coordinates $[v_{(i_{11},\dots,i_{1d_1}),\dots,(i_{l1},\dots,i_{ld_l})}]$ through the identification given by

$$\mathbb{P}(\mathcal{T}_{d_1,\dots,d_l}) \ni \left[\sum v_{(i_{11},\dots,i_{1d_1}),\dots,(i_{l1},\dots,i_{ld_l})} \prod_{j=1}^{l} \binom{d_j}{\alpha_{i_j}} \underline{y_j}^{\alpha_{i_j}}\right] \leftrightarrow [v_{(i_{11},\dots,i_{1d_1}),\dots,(i_{l1},\dots,i_{ld_l})}] \in \mathbb{P}^{N_{\mathbf{n};\mathbf{d}}}$$

and we denote by $\mathcal{S} := \mathbb{C}[v_{(i_{11},\dots,i_{1d_1}),\dots,(i_{l1},\dots,i_{ld_l})}]$ the coordinate ring of $\mathbb{P}^{N_{\mathbf{n};\mathbf{d}}}$. Finally, we denote by $\mathcal{U}$ the ring

$$\mathcal{U} := \mathbb{C}[w_{1,0},\dots,w_{1,n_1};\dots;w_{l,0},\dots,w_{l,n_l}]$$

and we think of its elements as derivations on $\mathcal{T}$. For the multigraded apolar action of $\mathcal{U}$ on $\mathcal{T}$ we refer to [63]. We will sometimes identify the projective space $\mathbb{P}(\mathcal{U}_{d_1,d_2,\dots,d_l})$ with $\mathbb{P}^{N_{\mathbf{n};\mathbf{d}}}$ through identifications analogous to the one we introduced for $\mathbb{P}(\mathcal{T}_{d_1,d_2,\dots,d_l})$.

Remark 1.5.9. Let T be a $(\mathbf{n},\mathbf{d})$-partially symmetric tensor, which means that

$$T \in \mathrm{Sym}^{d_1}(\mathbb{C}^{n_1+1}) \otimes \mathrm{Sym}^{d_2}(\mathbb{C}^{n_2+1}) \otimes \cdots \otimes \mathrm{Sym}^{d_l}(\mathbb{C}^{n_l+1}).$$

Via the isomorphisms

$$\mathrm{Sym}^{d_j}(\mathbb{C}^{n_j+1}) \cong \mathcal{T}^{(j)}_{d_j}$$

we get

$$\begin{array}{ccc} \mathrm{Sym}^{d_1}(\mathbb{C}^{n_1+1}) \otimes \mathrm{Sym}^{d_2}(\mathbb{C}^{n_2+1}) \otimes \cdots \otimes \mathrm{Sym}^{d_l}(\mathbb{C}^{n_l+1}) & \cong & \mathcal{T}_{d_1,\dots,d_l} \\ (F^{(1)},\dots,F^{(l)}) & \leftrightarrow & F^{(1)}\cdots F^{(l)} \end{array}$$

and thus we can identify partially symmetric tensors with multihomogeneous polynomials.

We can now give the definition of Segre-Veronese embeddings.

Definition 1.5.10. Let $n_1,\dots,n_l,d_1,\dots,d_l$ be positive integers and set

$$\mathbf{n} := (n_1,\dots,n_l), \quad \mathbf{d} := (d_1,\dots,d_l).$$

The $(\mathbf{n};\mathbf{d})$-*Segre-Veronese embedding* is

$$\begin{array}{rccc} sv_{\mathbf{n};\mathbf{d}}: & \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_l} & \to & \mathbb{P}^{N_{\mathbf{n},\mathbf{d}}} \\ & [x_{10},\dots,x_{1n_1};\dots;x_{l0},\dots,x_{ln_l}] & \mapsto & [\dots,x_{10}^{i_{10}}\cdots x_{1n_1}^{i_{1n_1}}\cdots x_{l0}^{i_{l0}}\cdots x_{ln_l}^{i_{ln_l}},\dots] \end{array}$$

where the products $x_{10}^{i_{10}}\cdots x_{1n_1}^{i_{1n_1}}\cdots x_{l0}^{i_{l0}}\cdots x_{ln_l}^{i_{ln_l}}$ are all the possible ones such that $i_{j0}+\cdots+i_{jn_l} = d_j$ for any $j=1,\dots,l$, ordered according to the lexicographic order introduced in Notation 1.5.2. The image of the embedding $sv_{\mathbf{n};\mathbf{d}}$ is called the $(\mathbf{n};\mathbf{d})$-*Segre-Veronese variety* and is denoted by $SV_{\mathbf{n};\mathbf{d}}$.

Remark 1.5.11. Using notation 1.5.8, we have that $SV_{\mathbf{n};\mathbf{d}}$ has parametric equations given by

$$v_{(i_{11},\dots,i_{1d_1}),\dots,(i_{l1},\dots,i_{ld_l})} = x_{1i_{11}} \cdots x_{1i_{1d_1}} \cdots x_{li_{l1}} \cdots x_{li_{ld_l}}.$$

Remark 1.5.12. Using Remark 1.5.9 and the identifications that we introduced in Notation 1.5.8, it is easy to see that the Segre-Veronese embeddings can also be viewed as

$$\begin{array}{rccc} sv_{\mathbf{n};\mathbf{d}}: & \mathbb{P}(\mathcal{T}_1^{(1)}) \times \cdots \times \mathbb{P}(\mathcal{T}_1^{(l)}) & \to & \mathbb{P}(\mathcal{T}_{d_1,\dots,d_l}) \\ & ([L^{(1)}] \times \cdots \times [L^{(l)}]) & \mapsto & [(L^{(1)})^{d_1} \cdots (L^{(l)})^{d_l}] \end{array}$$

or equivalently as

$$\begin{array}{rccc} sv_{\mathbf{n};\mathbf{d}}: & \mathbb{P}(\mathbb{C}^{n_1+1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{n_l+1}) & \to & \mathbb{P}(\mathrm{Sym}^{d_1}(\mathbb{C}^{n_1+1}) \otimes \cdots \otimes \mathrm{Sym}^{d_l}(\mathbb{C}^{n_l+1})) \\ & ([v_1] \times \cdots \times [v_l]) & \mapsto & [v_1^{\otimes d_1} \otimes \cdots \otimes v_l^{\otimes d_l}] \end{array}.$$

In particular, $SV_{\mathbf{n};\mathbf{d}}$ parameterises the partially symmetric tensors having partially symmetric rank equal to 1. Hence, in a totally analogous way to Veronese and Segre varieties, one finds that the partially symmetric rank and the partially symmetric border rank respectively coincide with the $SV_{\mathbf{n};\mathbf{d}}$-rank and the $SV_{\mathbf{n};\mathbf{d}}$-border rank. For a description of the state of the art on the secant varieties of Segre-Veronese varieties, see [2].

1.6 Méthode d'Horace and Méthode d'Horace différentielle

The Horace method, or Méthode d'Horace, and its stronger differential version were introduced and developed by J. Alexander and A. Hirschowitz in several papers having as main purpose the study of the postulation of fat points and other 0-dimensional schemes; see [4], [5], [6], [7] and [77]. The strength of this method lies in its use to make an induction start in order to compute the postulation or, equivalently, the Hilbert function of a scheme; in particular, the method works very well under the assumption of generality of the scheme because in this case it can be combined with specialisation techniques. Here, we do not introduce the Horace method and its differential version in all their generality and we just focus on what we will need in this thesis; in order to give a more friendly exposition of the topic, our main reference for this section is [22].

Definition 1.6.1. Let $H \subset \mathbb{P}^n$ be a reduced hypersurface and $\mathbb{X}$ a closed subscheme of $\mathbb{P}^n$. The closed subscheme of H given by the schematic intersection

$$H \cap \mathbb{X}$$

and defined by the ideal sheaf $\mathfrak{I}_{\mathbb{X}} \otimes \mathcal{O}_H$ is called *the trace of* $\mathbb{X}$ *on* H and it is denoted by $\mathrm{Tr}_H(\mathbb{X})$. The closed subscheme of $\mathbb{P}^n$ defined by the ideal sheaf $\mathfrak{I}_{\mathbb{X}} : \mathcal{O}_{\mathbb{P}^n}(-H)$ is called *the residual of* $\mathbb{X}$ *with respect to* H and it is denoted by $\mathrm{Res}_H(\mathbb{X})$.

The canonical exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_H(\mathbb{X})}(-H) \longrightarrow \mathfrak{I}_{\mathbb{X}} \longrightarrow \mathfrak{I}_{\mathrm{Tr}_H(\mathbb{X}),H} \longrightarrow 0$$

is called the *residual (or Castelnuovo) exact sequence of* $\mathbb{X}$ *with respect to* H.

Lemma 1.6.2 (Lemme d'Horace)**.** *Let d be a positive integer. If $\mathbb{X}$ is a closed subscheme of $\mathbb{P}^n$ and H is a reduced hypersurface of $\mathbb{P}^n$ such that $h^0(\mathfrak{I}_{\mathrm{Tr}_H(\mathbb{X}),H}(d)) = 0$, then*

$$h^0(\mathfrak{I}_{\mathbb{X}}) = h^0(\mathfrak{I}_{\mathrm{Res}_H(X)}(d - \deg(H))).$$

Proof The statement follows immediately from taking the long cohomology exact sequence induced by the residual sequence of $\mathbb{X}$ with respect to H tensorised by $\mathcal{O}_{\mathbb{P}^n}(d)$. ■

Definition 1.6.3. Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{P}^n$. We say that $\mathcal{L}_d(\mathbb{X})$ has the expected dimension if $\mathbb{X}$ imposes as many conditions as possible to the degree d hypersurfaces of $\mathbb{P}^n$, i.e. if

$$\dim(\mathcal{L}_d(\mathbb{X})) = \mathrm{expdim}(\mathcal{L}_d(\mathbb{X})) = \max\{0, \dim(\mathcal{L}_d) - \ell(\mathbb{X})\}.$$

If $\dim(\mathcal{L}_d(\mathbb{X})) = \mathrm{expdim}(\mathcal{L}_d(\mathbb{X}))$ for every $d \in \mathbb{N}$ we say that $\mathbb{X}$ *has good postulation*, otherwise we say that $\mathbb{X}$ *has bad postulation*.

Remark 1.6.4. Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{P}^n$ (we are not interested in the case $\dim(\mathbb{X}) > 0$) and suppose that we want to prove that the postulation of $\mathbb{X}$ is good. The ingredients that make the Horace method work are the following:

1. By the semicontinuity theorem (see [73], Chapter III, Theorem 12.8) the function h^0 is upper semicontinuous so that $h^0(\mathbb{X}) \le h^0(\mathbb{X}')$ for any $\mathbb{X}'$ obtained by specialising $\mathbb{X}$. In particular

$$h^0(\mathfrak{I}_{\mathbb{X}'}(d)) = \mathrm{expdim}(\mathcal{L}_d(\mathbb{X})) \Rightarrow h^0(\mathfrak{I}_{\mathbb{X}}(d)) = \mathrm{expdim}(\mathcal{L}_d(\mathbb{X}));$$

2. If we find a specialisation $\mathbb{X}'$ and a reduced hypersurface H such that

$$h^0(\mathfrak{I}_{\mathrm{Tr}_H(\mathbb{X}'),H}(d)) = 0$$

then, by Lemma 1.6.2, we have

$$h^0(\mathfrak{I}_{\mathbb{X}'}) = h^0(\mathfrak{I}_{\mathrm{Res}_H(X')}(d - \deg(H)))$$

and, by induction on d and on $\ell(X)$, we can suppose that $h^0(\mathfrak{I}_{\mathrm{Res}_H(X')}(d - \deg(H)))$ is the expected one.

Note that, from a more geometric point of view, when we specialise $\mathbb{X}$ to $\mathbb{X}'$ in order to have $h^0(\mathfrak{I}_{\mathrm{Tr}_H(\mathbb{X}'),H}(d)) = 0$, we are considering a configuration of our points in such a way that H is a fixed component of the linear system $\mathcal{L}_d(\mathbb{X}')$ so that the number of the degree d hypersurfaces passing through $\mathbb{X}'$ equals the number of the degree $d - \deg(H)$ hypersurfaces passing through the part of $\mathbb{X}'$ not lying on H.

Even though the Horace method is a powerful tool, it can be impeded by arithmetic obstructions. Indeed, it could be impossible to specialise $\mathbb{X}$ without "wasting" conditions, that is, without imposing on the trace more conditions than what we need. In some cases, when $\ell(\mathbb{X}) > \dim \mathcal{L}_d$, this could not be a problem but in other ones, the so-called *cas rangé*, when $\ell(\mathbb{X}) = \dim \mathcal{L}_d$ and all conditions imposed by $\mathbb{X}$ are necessary for the good postulation, this can cause some problems. Luckily, this problem is solved by the differential version of the Horace method, that allows us to take just one layer of the 0-dimensional component of $\mathbb{X}$ that we are specialising. Let us be more precise.

Definition 1.6.5. In the algebra of formal functions $\mathbb{C}[[\mathbf{x}, y]]$, where

$$\mathbf{x} = (x_1, \ldots, x_{n-1}),$$

a *vertically graded ideal with respect to y* is an ideal of the form:

$$I = I_0 \oplus I_1 y \oplus \cdots \oplus I_{m-1} y^{m-1} \oplus (y^m)$$

where for $i = 0, \ldots, m-1$, $I_i \subseteq \mathbb{C}[[\mathbf{x}]]$ is an ideal.

Definition 1.6.6. Let H be a smooth irreducible hypersurface of $\mathbb{P}^n$. We say that $\mathbb{X} \subseteq \mathbb{P}^n$ is a *vertically graded subscheme* of $\mathbb{P}^n$ with base H and support $P \in H$, if $\mathbb{X}$ is a 0-dimensional scheme with support at the point P and there is a regular system of parameters $(\mathbf{x}, y)$ at P such that $y = 0$ is a local equation for H and the ideal of $\mathbb{X}$ in $\widehat{\mathcal{O}}_{\mathbb{P}^n,P} \cong \mathbb{C}[[\mathbf{x}, y]]$ is vertically graded with respect to y.

We can now introduce the differential trace and residual.

Definition 1.6.7. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a vertically graded subscheme of $\mathbb{P}^n$ with base H and $p \geq 0$ a fixed integer. The p^{th} *differential residual of $\mathbb{X}$ with respect to H* is the closed subscheme of $\mathbb{P}^n$ denoted by $\mathrm{Res}^p_H(\mathbb{X})$ and defined by the ideal sheaf

$$\mathfrak{I}_{Res^p_H(\mathbb{X})} := \mathfrak{I}_{\mathbb{X}} + (\mathfrak{I}_{\mathbb{X}} : \mathfrak{I}_H^{p+1})\mathfrak{I}_H^p.$$

The p^{th} *differential trace of $\mathbb{X}$ on H* is the closed subscheme of $\mathbb{P}^n$ denoted by $\mathrm{Tr}^p_H(\mathbb{X})$ and defined by the ideal sheaf

$$\mathfrak{I}_{\mathrm{Tr}^p_H(\mathbb{X}),H} := (\mathfrak{I}_{\mathbb{X}} : \mathfrak{I}_H^p) \otimes \mathcal{O}_H.$$

Note that $\mathrm{Res}^p_H(\mathbb{X})$ is obtained by removing from $\mathbb{X}$ the $(p+1)^{th}$ "slice" of $\mathbb{X}$, while $\mathrm{Tr}^p_H(\mathbb{X})$ is exactly the $(p+1)^{th}$ slice. Moreover, for $p = 0$ we obtain the standard trace and residual.

Notation 1.6.8. *Let $\mathbb{X}_1, \dots, \mathbb{X}_r \subseteq \mathbb{P}^n$ be closed vertically graded subschemes with base H, $\mathbb{X} = \mathbb{X}_1 + \cdots + \mathbb{X}_r$ and $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$. We set*

$$\mathrm{Tr}^{\mathbf{p}}_H(\mathbb{X}) := \mathrm{Tr}^{p_1}_H(\mathbb{X}_1) + \cdots + \mathrm{Tr}^{p_r}_H(\mathbb{X}_r), \qquad \mathrm{Res}^{\mathbf{p}}_H(\mathbb{X}) := \mathrm{Res}^{p_1}_H(\mathbb{X}_1) + \cdots + \mathrm{Res}^{p_r}_H(\mathbb{X}_r)$$

We are finally ready to state the Horace differential lemma.

Lemma 1.6.9 (Lemme d'Horace différentielle)**.** *Let H be a hyperplane in $\mathbb{P}^n$ and let $\mathbb{X}$ be a 0-dimensional closed subscheme of $\mathbb{P}^n$. Let $\mathbb{Y}_1, \dots, \mathbb{Y}_r, \mathbb{Y}'_1, \dots, \mathbb{Y}'_r$ be 0-dimensional irreducible subschemes of $\mathbb{P}^n$ such that $\mathbb{Y}_i \cong \mathbb{Y}'_i$ for $i = 1, \dots, r$, $\mathbb{Y}'_i$ has support on H and is vertically graded with base H, and the supports of $\mathbb{Y} = \mathbb{Y}_1 + \cdots + \mathbb{Y}_r$ and $\mathbb{Y}' = \mathbb{Y}'_1 + \cdots + \mathbb{Y}'_r$ are generic in their respective Hilbert schemes. Let $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$. If*

1. $h^0(\mathfrak{I}_{\mathrm{Tr}_H(\mathbb{X}) + \mathrm{Tr}^{\mathbf{p}}_H(\mathbb{Y}'), H}(d)) = 0$ *and*
2. $h^0(\mathfrak{I}_{\mathrm{Res}_H(\mathbb{X}) + \mathrm{Res}^{\mathbf{p}}_H(\mathbb{Y}')}(d-1)) = 0$,

then

$$h^0(\mathfrak{I}_{\mathbb{X}+\mathbb{Y}}(d)) = 0.$$

Proof See [4] Proposition 9.1. ∎

In some sense, the Horace differential Lemma tells us that we can differentially specialise some of our points by taking only of their layers, in order to satisfy the condition on the trace without "wasting" conditions.

We conclude now the section giving the definition of d-jets.

Definition 1.6.10. Let $\mathbb{X}$ be a 0-dimensional scheme of $\mathbb{P}^n$ with support at the point $P \in \mathbb{P}^n$. We say that $\mathbb{X}$ is a *d-jet* if there exists a line $L \subseteq \mathbb{P}^n$ such that $\mathcal{I}(\mathbb{X}) = \mathcal{I}(P)^d + \mathcal{I}(L)$.

1.7 Singularities of plane algebraic curves

In this last section we recall some basic facts about the singularities of plane algebraic curves, with a special focus on the Jacobian (or Tjurina) and Milnor algebras, that are one of our subjects of study in the next chapters. Our main references for this part are [70] and [71].

Notation 1.7.1. Given $\mathcal{C} : F(x_0, x_1, x_2) = 0$ a reduced curve of degree d in $\mathbb{P}^2$, we denote by $\operatorname{Sing}\mathcal{C}$ the union of the singular points of $\mathcal{C}$ and if P is a point of multiplicity m for $\mathcal{C}$, we write $m_P(\mathcal{C}) = m$. For the partial derivatives, we use the following short notation:

$$\partial_i F := \frac{\partial F}{\partial x_i}, \quad i = 0,1,2.$$

Moreover, if $\mathbb{X}$ is a subscheme of $\mathbb{P}^2$ and P is an isolated point of its support, $\mathbb{X}_P$ denotes the component of $\mathbb{X}$ supported on P.

Definition 1.7.2. Given $\mathcal{C} : F(x_0, x_1, x_2) = 0$ a reduced curve of degree d in $\mathbb{P}^2$, the degree $d-1$ curves

$$\mathcal{C}_i : \partial_i F = 0, \quad i = 0,1,2$$

are called *the derivative curves of* $\mathcal{C}$.

Now we can give the definition of Jacobian ideal and Jacobian scheme.

Definition 1.7.3. Given $\mathcal{C} : F(x_0, x_1, x_2) = 0$ a reduced curve in $\mathbb{P}^2$, the *(projective) Jacobian scheme* $\mathbb{X}(\mathcal{C})$ *of* $\mathcal{C}$ is the 0-dimensional subscheme of $\mathbb{P}^2$ defined by the homogeneous (but maybe not saturated) ideal

$$\mathbb{J}(\mathcal{C}) := (\partial_0 F, \partial_1 F, \partial_2 F)$$

called *the (projective) Jacobian ideal of* $\mathcal{C}$. The length of $\mathbb{X}(\mathcal{C})$ is called *the global Tjurina number* $\tau(\mathcal{C})$ *of* $\mathcal{C}$.

Note that the support of the Jacobian scheme is $\operatorname{Sing}\mathcal{C}$, which consists of a finite number of points since the curve is reduced and thus the definition of $\tau(\mathcal{C})$ is well-posed.

Lemma 1.7.4. *Let* $\mathcal{C} : F(x_0, x_1, x_2) = 0$ *be a reduced curve in* $\mathbb{P}^2$ *and* $P \in \mathcal{C}$ *with* $m_P(\mathcal{C}) = m \geq 2$. *Then*

a) *The curve* $\mathcal{C}$ *contains* $\mathbb{X}(\mathcal{C})$*;*

b) $m_P(\mathcal{C}_i) \geq m-1$ *and for at least one of the* $\mathcal{C}_i$ *the multiplicity at* P *is exactly* $m-1$*;*

c) *In particular,* $\mathbb{X}_P \supseteq (m-1)P$ *and* $\mathbb{X}_P \not\supseteq mP$.

Proof

a) The Euler relation $d \cdot F = \sum x_i(\partial_i F)$ implies that $F \in \mathbb{J}(\mathcal{C})$, that is, $\mathcal{C} \supseteq \mathbb{X}(\mathcal{C})$.

b) We have that
$$m_P(\mathcal{C}) = m \Leftrightarrow \frac{\partial^{m-1} F}{\partial x_0^j x_1^h x_2^{m-1-j-h}}\Big|_P = 0$$
for $0 \leq j, h \leq m-1$ and at least one of the derivatives
$$\frac{\partial^{m} F}{\partial x_0^j x_1^h x_2^{m-j-h}}\Big|_P$$
is not 0. Hence, for each $i = 0,1,2$ and $0 \leq j, h \leq m-2$ one has
$$\frac{\partial^{m-2} (\partial_i F)}{\partial x_0^j x_1^h x_2^{m-2-j-h}}\Big|_P = 0$$
and for at least one i, one j and one h we have
$$\frac{\partial^{m-1} (\partial_i F)}{\partial x_0^j x_1^h x_2^{m-1-j-h}}\Big|_P \neq 0.$$

c) It is enough to recall that for a curve $\mathcal{D}$ we have: $m_P(\mathcal{D}) = k \Leftrightarrow \mathcal{D} \supseteq kP$ and $\mathcal{D}$ does not contain $(k+1)P$.

∎

When we want to locally analyse a curve, it is better to work with affine coordinates. For this reason we now introduce the Jacobian scheme in its affine version and a new scheme, called Milnor scheme of the curve.

Notation 1.7.5. In $\mathbb{A}^2$ we use affine coordinates (x, y). Let $\mathcal{C} : f(x, y) = 0$ be a reduced curve of degree d in $\mathbb{A}^2$; we use the same notation as in $\mathbb{P}^2$, i.e. $\operatorname{Sing} \mathcal{C}$ denotes the union of the singular points of $\mathcal{C}$, and if P is a point of multiplicity m for $\mathcal{C}$, we write $m_P(\mathcal{C}) = m$. Moreover, we set
$$f_x := \frac{\partial f}{\partial x}, \quad f_y := \frac{\partial f}{\partial y}$$
and we denote by $\mathcal{C}_x, \mathcal{C}_y$ the degree $d-1$ derivative curves of $\mathcal{C}$:
$$\mathcal{C}_x : f_x = 0, \quad \mathcal{C}_y : f_y = 0.$$

Definition 1.7.6. Given $\mathcal{C} : f(x, y) = 0$ a reduced curve in $\mathbb{A}^2$, the *(affine) Jacobian scheme $X(\mathcal{C})$ of $\mathcal{C}$* is the 0-dimensional subscheme of $\mathbb{A}^2$ defined by the ideal, called *the (affine) Jacobian ideal of $\mathcal{C}$*,
$$J(\mathcal{C}) := (f, f_x, f_y).$$

Note that, since the curve is reduced, $\operatorname{Sing}\mathcal{C}$ consists of a finite number of points $P_1, \dots, P_r$, and $X(\mathcal{C})$ is the union of the 0-dimensional schemes $X(\mathcal{C})_{P_1}, \dots, X(\mathcal{C})_{P_r}$. This guarantees that the following definition is well-posed.

Definition 1.7.7. Given $\mathcal{C} : f(x, y) = 0$ a reduced curve in $\mathbb{A}^2$, *the Tjurina number of* $\mathcal{C}$ *at a singular point* P is $\tau_P(\mathcal{C}) := \ell(X_P)$. If no confusion arises, i.e. when we work locally, looking at the curve $\mathcal{C}$ at the point P, we just write τ instead of $\tau_P(C)$.

It is easy to see that, if $\bar{\mathcal{C}}$ is a curve in $\mathbb{P}^2$, U_0 is the affine chart $x_0 \neq 0$, and $\mathcal{C} = \bar{\mathcal{C}} \cap U_0$, then $\mathbb{X}(\bar{\mathcal{C}}) \cap U_0 = X(\mathcal{C})$. Hence, $\mathbb{X}(\bar{\mathcal{C}})_P = X(\mathcal{C})_P$ for any $P \in \operatorname{Sing}\mathcal{C}$.

Definition 1.7.8. Given $\mathcal{C} : f(x, y) = 0$ a reduced curve in $\mathbb{A}^2$, the *(affine) Milnor scheme* $Z(\mathcal{C})$ *of* $\mathcal{C}$ is the subscheme of $\mathbb{A}^2$ defined by the ideal, called *the (affine) Milnor ideal of* $\mathcal{C}$,

$$M(\mathcal{C}) := (f_x, f_y).$$

Note that, by [70] Lemma 2.3 p.113, if P is a singular, necessarily isolated (the curve being reduced), point of $\mathcal{C}$, then P is an isolated point of $Z(\mathcal{C})$. This guarantees that the following definition is well-posed.

Definition 1.7.9. Given $\mathcal{C} : f(x, y) = 0$ a reduced curve in $\mathbb{A}^2$, the *Milnor number of* $\mathcal{C}$ *at a singular point* P is $\mu_P(\mathcal{C}) := \ell(Z_P)$. If no confusion arises, i.e. when we work locally, looking at the curve $\mathcal{C}$ at the point P, we just write μ instead of $\mu_P(C)$.

For completeness, we remark that some authors use a different notation than ours and call Tjurina ideal the one defined in Definition 1.7.3 and Jacobian ideal the one in Definition 1.7.8.

Now we briefly recall the notion of analytic, or right, equivalence and the analytic classification of double points, which we will use in the rest of this thesis. The discussion could be carried out more in general, but for our purposes it is enough to restrict the treatise just to plane algebraic curves. For more details, see [70], Chapter I, §1 and §2.

Notation 1.7.10. We denote by $\mathbb{C}\{x, y\}$ the ring of convergent power series in the two variables x, y.

Definition 1.7.11. Let $\mathcal{C} : f = 0$ and $\mathcal{D} : g = 0$ be reduced curves in $\mathbb{A}^2$ and let $P = (p_1, p_2) \in \mathcal{C}$ and $Q = (q_1, q_2) \in \mathcal{D}$. We say that the germs of $\mathcal{C}$ at P and of $\mathcal{D}$ at Q, which we respectively denote by $(\mathcal{C}, P)$ and $(\mathcal{D}, Q)$, are *analytically (or right) equivalent* if there exists a germ of a biholomorphic function $\varphi \in \mathbb{C}\{x - p_1, y - p_2\}$ such that $\varphi(f) = g(x+q_1-p_1, y+q_2-p_2)$, and in this case we write $(\mathcal{C}, P) \sim (\mathcal{D}, Q)$.

Note that in [70], Definition 2.9, pag. 118, the definition of contact equivalence is also given, but the notions of right equivalence and contact equivalence coincide in the case of double points of reduced hypersurfaces, and thus in particular for the plane curves. At this point we can describe the classification, up to analytic equivalence, of the double points of reduced plane curves.

Theorem 1.7.12. *Let $\mathcal{C}$ be a reduced curve in $\mathbb{A}^2$ and let $P \in \mathcal{C}$ be such that $m_P(\mathcal{C}) = 2$. Then $(\mathcal{C}, P) \sim (A_k, O)$ where $O = (0,0)$ and $A_k : y^2 - x^{k+1} = 0$ for a certain $k \geq 1$ and we say that P is an A_k-singularity. Moreover, if k is even we say that P is a cuspidal point of $\mathcal{C}$ while if k is odd we say that P is a nodal point of $\mathcal{C}$.*

Proof See [70], Chapter I, Theorem 2.48. ∎

Chapter 2

The Jacobian scheme of a plane algebraic curve

This chapter is based on a joint work with A. Gimigliano and M. Idà, see [34], and it is mainly devoted to study the Jacobian scheme of a plane algebraic curve and to introduce the notion of symmetric scheme, which we will widely study in next chapters.

If $\mathcal{C} : F = 0$ is a reduced curve of $\mathbb{P}^2$ passing through a point P, then to say that $\mathcal{C}$ has in P a singular point of multiplicity m means that $\mathcal{C} \supseteq mP$ and $\mathcal{C} \not\supseteq (m+1)P$. This is a very rough information, but there are others 0-dimensional schemes contained in $\mathcal{C}$ which could characterise the singularity more carefully. For example, if P is an A_n singularity, then P is a nodal-type singularity if and only if for any $\ell \geq 1$ there is a curvilinear scheme supported at P of length ℓ contained in $\mathcal{C}$, while P is a cuspidal singularity A_{2r} if and only if for any $\ell \leq 2r+1$ there is a curvilinear scheme supported at P of length ℓ contained in $\mathcal{C}$, and no curvilinear scheme supported at P of length $> 2r+1$ is contained in $\mathcal{C}$ (see [68], Theorem 2.3).

So, if we want to study a singularity, one of the possible approaches is to answer to the following question: which kind of "maximal" 0-dimensional schemes supported at P is contained in $\mathcal{C}$? But the curve being 1-dimensional, in many cases it will not be possible to bound the length of these schemes, due to the curvilinear schemes contained in $\mathcal{C}$. However, there is a very interesting 0-dimensional subscheme of the curve that gives information on the singularity, the Jacobian scheme, which, together with the Tjurina and Milnor numbers, has been an intensive object of study in recent years, e.g. see [71], [8], [3], [107], [75], [65].

In this chapter, we focus our attention mainly on ordinary singularities. It is precisely the study of these singularities that will lead us to introduce k-symmetric schemes, which will be widely studied in the rest of the thesis. The structure of the chapter is as follows.

In §2.1 we show that, if $\mathfrak{b} := (g_1, \dots, g_t) \subseteq \mathbb{C}[x, y]$ is the ideal of a 0-dimensional scheme with the origin O contained in its support, and $\mathfrak{m} := (x, y)$, then there is a canonical isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[x, y]_{\mathfrak{m}}/\mathfrak{b}\mathbb{C}[x, y]_{\mathfrak{m}} \cong \mathbb{C}\{x, y\}/\mathfrak{b}\mathbb{C}\{x, y\}.$$

This allows us to state in a more algebraic way the Mather-Yau Theorem in [89] saying that, if $V : f = 0$ and $W : g = 0$ are germs of hypersurfaces in $\mathbb{C}^{n+1}$ with isolated singularities at O, then (V, O) is analytically equivalent to (W, O) if and only if

$$\frac{\mathbb{C}\{x_1, \dots, x_{n+1}\}}{\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n+1}}\right)} \cong \frac{\mathbb{C}\{x_1, \dots, x_{n+1}\}}{\left(g, \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}}\right)}$$

as $\mathbb{C}$-algebras.

In §2.2, in order to study the geometry of the Jacobian schemes at ordinary m-multiple points, we introduce the notion of k-symmetric scheme, i.e. a scheme supported at a point P and intersecting each line through P with the same length k. In Theorem 2.2.7 we show that, if $\mathcal{C} : f = 0$ is a plane reduced curve having an ordinary multiple point at P, the Jacobian scheme of $\mathcal{C}$ at P is $(m-1)$-symmetric and its length τ is at most $(m-1)^2$. After that we give in Theorem 2.2.10 the sharp bounds for the Tjurina number of an ordinary singularity, which is an immediate consequence of more general results in [27] and [87].

In §2.3 we apply the theory of Gröbner basis to compute the Tjurina number at the origin of the family of curves

$$\mathcal{C}_{b,c} : x^m + y^m + x^b y^c = 0$$

with $b + c > m$, having an ordinary singularity at O. In this way, we give a large family of examples of curves having an ordinary singularity with $\tau < \mu$, thus partially recovering some results of [87]. In particular, we prove in Theorem 2.3.6 that for $m \geq 5$ the curves $\mathcal{C}_{b,c}$ allow to obtain the minimum expected by Theorem 2.2.10.

In §2.4 we show that, in the case of double points, the characterisation through the Jacobian scheme is very easy: a double point is of type A_n if and only if the Jacobian scheme of the curve at P is a curvilinear scheme of length n.

Finally, in the last section we prove a result on the global Tjurina number of a curve in $\mathbb{P}^2$. Namely, if $\mathcal{C}$ is an irreducible curve of degree d and geometric genus g, with no infinitely near points, then $\mathcal{C}$ has only nodes if and only if $\tau(C) = \binom{d-1}{2} - g$.

2.1 The Mather-Yau Theorem for algebraic curves

In [89] the authors proved that the germs of two hypersurfaces in $\mathbb{C}^{n+1}$ at one of their isolated singularities are analytically equivalent if and only if their Jacobian (or Tjruina) algebras are isomorphic as $\mathbb{C}$-algebras. In this section, we want to translate this theorem in a more algebraic language. In particular, as a consequence of [89], we prove that the analytic germs at O of two reduced algebraic plane curves $\mathcal{C}$ and $\mathcal{D}$, which we may assume to be in $\mathbb{A}^2$, are analytically equivalent if and only if their algebraic Jacobian schemes $X(\mathcal{C})_O$ and $X(\mathcal{D})_O$ are isomorphic.

Notation 2.1.1. If A is a ring or a $\mathbb{C}$-algebra, with $\dim A$ we denote the Krull dimension of A. If A is a finite $\mathbb{C}$-algebra, i.e. A is finitely generated as $\mathbb{C}$-vector space, the dimension of the $\mathbb{C}$-vector space A is the length of A.

Lemma 2.1.2. *A Noetherian $\mathbb{C}$-algebra A of dimension 0 is a finite $\mathbb{C}$-algebra.*

Proof On the one hand, a ring is an Artin ring if and only if it is Noetherian and of dimension 0, see [13], Theorem 8.5; on the other hand, if A is a finitely generated $\mathbb{K}$-algebra for a certain field $\mathbb{K}$, then A is an Artin ring if and only if A is a finite $\mathbb{K}$-algebra, see [13] Exercise 8.3. ■

Lemma 2.1.3. *Let $\mathfrak{m} := (x, y)$ in $\mathbb{C}[x, y]$, and consider the injective morphisms of $\mathbb{C}$-algebras*

$$\mathbb{C}[x, y] \overset{j}{\hookrightarrow} \mathbb{C}[x, y]_{\mathfrak{m}} \overset{\varphi}{\hookrightarrow} \mathbb{C}\{x, y\}$$

and set

$$A := \mathbb{C}[x, y]_{\mathfrak{m}}, \quad B := \mathbb{C}\{x, y\}$$

The rings A and B are local rings with maximal ideals respectively $\mathfrak{m}A$ and $\mathfrak{m}B$, and the map φ induces an isomorphism on the completions: $\hat{\varphi} : \hat{A} \overset{\cong}{\to} \hat{B}$.

Moreover, (A, B) is a flat couple, which means that B is a flat A-module and for any ideal $\mathfrak{a}$ of A the following holds:

$$\mathfrak{a}B \cap A = \mathfrak{a}$$

Proof See [95] Proposition 3 for the first statement and [95] Proposition 22 and Proposition 28 for the second one. ■

Theorem 2.1.4. *Let $g_1, \dots, g_t \in \mathbb{C}[x, y]$. Assume that the subscheme of $\mathbb{A}^2$ associated to the ideal $\mathfrak{b} := (g_1, \dots, g_t)$ is 0-dimensional with the origin O contained in its support, and let $\mathfrak{m} := (x, y)$. Then, there is an isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[x, y]_{\mathfrak{m}}/\mathfrak{b}\mathbb{C}[x, y]_{\mathfrak{m}} \cong \mathbb{C}\{x, y\}/\mathfrak{b}\mathbb{C}\{x, y\}$$

Proof Let $\mathfrak{b} = (g_1, \dots, g_t) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ be a minimal primary decomposition, with $\mathfrak{m}_1 = \sqrt{\mathfrak{q}_1}, \dots, \mathfrak{m}_n = \sqrt{\mathfrak{q}_n}$ maximal ideals; since O is in the support of the subscheme defined by $\mathfrak{b}$, we may assume $\mathfrak{m}_1 = (x, y)$; set $\mathfrak{q} = \mathfrak{q}_1$, $\mathfrak{m} = \mathfrak{m}_1$.

Since $\sqrt{\mathfrak{q}} = (x, y)$, there exist $n, m > 0$ such that $x^n \in \mathfrak{q}, y^m \in \mathfrak{q}$; hence, every homogeneous polynomial of degree $\geq n+m-1$ is in $\mathfrak{q}$, i.e. $\mathfrak{q} \supseteq (x, y)^{n+m-1}$.

Note that

$$\mathfrak{b}\mathbb{C}[x,y]_{\mathfrak{m}} = (\mathfrak{q} \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n)\mathbb{C}[x,y]_{\mathfrak{m}} = \mathfrak{q}\mathbb{C}[x,y]_{\mathfrak{m}} \cap \mathfrak{q}_2\mathbb{C}[x,y]_{\mathfrak{m}} \cap \cdots \cap \mathfrak{q}_n\mathbb{C}[x,y]_{\mathfrak{m}}$$

and $\mathfrak{q}_j\mathbb{C}[x,y]_{\mathfrak{m}} = \mathbb{C}[x,y]_{\mathfrak{m}}$ for $j = 2, \ldots, n$, since $\mathfrak{q}_j \nsubseteq \mathfrak{m}$ for $j = 2, \ldots, n$. Hence $\mathfrak{b}\mathbb{C}[x,y]_{\mathfrak{m}} = \mathfrak{q}\mathbb{C}[x,y]_{\mathfrak{m}}$, so that

$$\mathfrak{b}\mathbb{C}\{x,y\} = \mathfrak{q}\mathbb{C}\{x,y\}. \tag{$\circ$}$$

From now on we use the notation of Lemma 2.1.3, and we set $\mathfrak{a} := \mathfrak{b}A$.

The ideal $\mathfrak{a}B = \mathfrak{b}B$ is the ideal of B generated by $(\varphi \circ j)(g_1), \ldots, (\varphi \circ j)(g_t)$, that is, $\mathfrak{a}B = (g_1, \ldots, g_t)B = \{\sigma_1 g_1 + \cdots + \sigma_t g_t, \sigma_1, \ldots, \sigma_t \in B\}$. Now consider the map induced by φ

$$\begin{array}{rccc} \Phi: & A/\mathfrak{a} & \to & B/\mathfrak{a}B \\ & f + \mathfrak{a} & \mapsto & \varphi(f) + \mathfrak{a}B \end{array}$$

which is well-defined, since $f - g \in \mathfrak{a} \Rightarrow \varphi(f) - \varphi(g) = \varphi(f-g) \in \varphi(\mathfrak{a}) \subseteq \mathfrak{a}B$, and is a morphism of $\mathbb{C}$-algebras. Moreover, the map Φ is injective: indeed, if $\varphi(f) \in \mathfrak{a}B$ then, using Lemma 2.1.3, we have $f \in \varphi^{-1}(\mathfrak{a}B) = \mathfrak{a}B \cap A = \mathfrak{a}$.

Now we want to prove that Φ is surjective. Since $A/\mathfrak{a}$ is a Noetherian $\mathbb{C}$-algebra of dimension 0, by Lemma 2.1.2 the $\mathbb{C}$-vector space $A/\mathfrak{a}$ is finite dimensional and thus there are $f_1, \ldots, f_s \in A$ such that $f_1 + \mathfrak{a}, \ldots, f_s + \mathfrak{a}$ is a basis for $A/\mathfrak{a}$. Since we have the injection

$$\mathbb{C}[x,y] \overset{j}{\hookrightarrow} A$$

the following is true:

$$\forall\, q \in \mathbb{C}[x,y] \;\; \exists\, a_1, \ldots, a_s \in \mathbb{C} \text{ such that } q - (a_1 f_1 + \cdots + a_s f_s) \in \mathfrak{a}. \tag{$*$}$$

If we prove that $\varphi(f_1) + \mathfrak{a}B, \ldots, \varphi(f_s) + \mathfrak{a}B$ is a basis for the $\mathbb{C}$-vector space $B/\mathfrak{a}B$ we are done. The injectivity of the map φ gives the linear independence over $\mathbb{C}$ of $\varphi(f_1) + \mathfrak{a}B, \ldots, \varphi(f_s) + \mathfrak{a}B$. Now we want to prove that for any $\sigma \in B$ there are $a_1, \ldots, a_s \in \mathbb{C}$ such that

$$\sigma + \mathfrak{a}B = \sum_{i=1}^{s} a_i(\varphi(f_i) + \mathfrak{a}B).$$

We can write

$$\sigma = \sum_{0 \leq i+j < n+m-1} a_{i,j}x^i y^j + \sum_{i+j \geq n+m-1} a_{i,j}x^i y^j$$

and, since the series

$$\sum_{i+j \geq n+m-1} a_{i,j}x^i y^j$$

is uniformly convergent around O, we can rearrange its terms, so we get

$$\sum_{i+j\geq n+m-1} a_{i,j}x^i y^j = x^n \left(\sum_{\substack{i+j\geq n+m-1\\ i\geq n}} a_{i,j}x^{i-n}y^j \right) + y^m \left(\sum_{\substack{i+j\geq n+m-1\\ i<n}} a_{i,j}x^i y^{j-m} \right) \in \mathfrak{q}B.$$

By $(\circ)$ we have $\mathfrak{a}B = \mathfrak{b}B = \mathfrak{q}B$, so that

$$\sigma + \mathfrak{a}B = \sum_{0\leq i+j<n+m-1} a_{i,j}x^i y^j + \mathfrak{a}B$$

and we conclude applying $(*)$ to the polynomial

$$\sum_{0\leq i+j<n+m-1} a_{i,j}x^i y^j.$$

■

Corollary 2.1.5. *Let $\mathcal{C} : f = 0$ be a reduced curve in $\mathbb{A}^2$ with a singular point at O and let J be its Jacobian ideal and M be its Milnor ideal. Let us consider the analytic germ of $\mathcal{C}$ at O and let J^{an}, respectively M^{an}, denote the ideals generated by f, f_x, f_y, respectively f_x, f_y, in $\mathbb{C}\{x, y\}$. Then there are canonical isomorphisms of $\mathbb{C}$-algebras:*

$$\mathbb{C}[x, y]_{(x,y)}/J\mathbb{C}[x, y]_{(x,y)} \cong \mathbb{C}\{x, y\}/J^{an}$$
$$\mathbb{C}[x, y]_{(x,y)}/M\mathbb{C}[x, y]_{(x,y)} \cong \mathbb{C}\{x, y\}/M^{an}.$$

In particular, the Tjurina number τ, respectively the Milnor number μ, of $\mathcal{C}$ at O are the dimensions of the analytic algebras $\mathbb{C}\{x, y\}/(f, f_x, f_y)\mathbb{C}\{x, y\}$, respectively $\mathbb{C}\{x, y\}/(f_x, f_y)\mathbb{C}\{x, y\}$.

Hence, the Theorem in [89] gives

Theorem 2.1.6. *Let $\mathcal{C} : f = 0$, $\mathcal{D} : g = 0$ be reduced algebraic curves in $\mathbb{A}^2$ with a singular point at O. Then the analytic germs of $\mathcal{C}$ and $\mathcal{D}$ at O are analytically equivalent if and only if their (algebraic) Jacobian schemes at O are isomorphic as schemes over $\mathbb{C}$.*

2.2 Jacobian schemes at ordinary singularities

In this section we want to study some geometrical properties of the Jacobian schemes at ordinary singularities; in order to proceed with our investigation, we give the definition of the k-symmetric schemes.

Notation 2.2.1. In the following, given a polynomial g, we always denote with g_k its homogeneous component of degree k.

First of all, we want to stress that a curve $\mathcal{C}$ having a multiple ordinary point at P can have derivative curves with non-ordinary singularities at P, as shown in the following remark.

Remark 2.2.2. Let $\mathcal{C} : f(x, y) = 0$ be a reduced curve of degree d in $\mathbb{A}^2$ with $m_O(\mathcal{C}) = m$; it is not restrictive to assume that the line $x = 0$ is not a principal tangent at O, hence writing f as the sum of its homogeneous components we have

$$f = f_m + \cdots + f_d, \qquad f_m = y^m + \alpha_{m-1}xy^{m-1} + \cdots + \alpha_0 x^m$$

and O is an ordinary singularity of $\mathcal{C}$ if and only if f_m is the product of m distinct linear factors, i.e. if and only if the discriminant $\Delta(g)$ is not zero, where $g(t) := f_m(t,1)$. Anyhow, the derivative curves

$$\mathcal{C}_x : (f_m)_x + \cdots + (f_d)_x = 0, \quad \mathcal{C}_y : (f_m)_y + \cdots + (f_d)_y = 0$$

may have a non-ordinary singularity at O. For instance, if we take

$$f = \frac{1}{4}y^4 - \frac{(2a+b)}{3}xy^3 + \frac{(2ab+a^2)}{2}x^2y^2 - a^2bx^3y$$

with $\Delta(f_4(1, y)) = \Delta(f(1, y)) \neq 0$, the curve $\mathcal{C}$ has an ordinary point of multiplicity 4 at O but

$$f_y = y^3 - (2a+b)xy^2 + (2ab+a^2)x^2y - a^2bx^3 = (y-ax)^2(y-bx).$$

Hence, $\mathcal{C}_y$ has a non-ordinary triple point at O.

Before giving the definition of k-symmetric scheme, we recall that if $\mathcal{D}$ and $\mathcal{E}$ are two plane curves and P is a point such that $m_P(\mathcal{D}) = k, m_P(\mathcal{E}) = k$, $k \geq 1$, then $(\mathcal{D} \cdot \mathcal{E})_P \geq k^2$, with equality if and only if they do not have common tangents; see for example [61] property (5) in §3.3 or [70] p.190. We also recall the definition of local complete intersection.

Definition 2.2.3. A closed subscheme $\mathbb{Y}$ of $\mathbb{P}^n$, respectively $\mathbb{A}^n$, is said to be a local complete intersection if there exists a complete intersection $\mathbb{X} \subseteq \mathbb{P}^n$, respectively $\mathbb{X} \subseteq \mathbb{A}^n$, such that $\mathbb{Y}$ is an irreducible component of $\mathbb{X}$.

Definition 2.2.4. Let Y be a 0-dimensional scheme supported at one point $P \in \mathbb{P}^2$ or $\mathbb{P} \in \mathbb{A}^2$. We say that Y is *k-symmetric* if, for every line r passing through P, $\ell(Y \cap r) = k$. We say that Y is a *k-symmetric local complete intersection* (k-slci for short) if it is a local complete intersection of two curves $\mathcal{D}, \mathcal{E}$ with no tangent in common at P and such that $m_P(\mathcal{D}) = k, m_P(\mathcal{E}) = k$, this implying $\ell(Y) = k^2$.

It is immediate to see that a k-slci scheme is k-symmetric.

Lemma 2.2.5. *If Y is a k-slci scheme, then Y is k-symmetric.*

Proof Since our schemes are supported at one point, we can work in $\mathbb{A}^2$. Let Y denote the 0-dimensional component of the scheme defined by the ideal (ϕ, ψ), where

$$\phi(x,y) = \prod_{j=1}^{k} l_j + \phi_{k+1} + \dots \phi_p, \quad \psi(x,y) = \prod_{j=1}^{k} h_j + \psi_{k+1} + \dots \psi_q$$

with $l_1 = 0, \dots, l_k = 0, h_1 = 0, \dots, h_k = 0$ lines through O, $l_j \neq h_i$ for $i, j = 1, \dots, k$. The curves $\phi = 0$ and $\psi = 0$ have no common irreducible component at O (they may obviously have common components away from O), since their tangent cones have no common lines. Let r be a line through O. Choosing coordinates we may assume that $r : y = 0$. We have

$$\tilde{\phi}(x) := \phi(x,0) = a_k x^k + a_{k+1} x^{k+1} + \dots + a_s x^s$$

$$\tilde{\psi}(x) := \psi(x,0) = b_k x^k + b_{k+1} x^{k+1} + \dots + b_t x^t$$

where at least one between a_k and b_k is not zero since, by assumption, if r is one of the tangents of the curve $\phi = 0$ at O, r cannot be in the tangent cone of $\psi = 0$. Let us say $a_k \neq 0$. There exists a polynomial $f(x)$ such that $f(0) \neq 0$ and $\tilde{\phi}(x) = x^k f(x)$. Moreover, if $\tilde{\psi}(x) \neq 0$, then there exist a polynomial $g(x)$ with $g(0) \neq 0$ and an $n \geq k$ such that $\tilde{\psi}(x) = x^n g(x)$. Hence

$$(\mathbb{C}[x,y]/(\phi, \psi, y))_{(x,y)} \cong \left(\mathbb{C}[x]/(\tilde{\phi}, \tilde{\psi})\right)_{(x)} \cong \mathbb{C}[x]/(x^k).$$

■

Remark 2.2.6. We want now to stress some facts on the relationship between k-symmetric schemes, k-slci and Jacobian schemes.

1. A k-symmetric scheme needs not to be a k-slci, indeed there are a lot of k-symmetric schemes supported on P, and the smallest one is kP. For example, if $P = O$, all the monomial schemes of ideal I with

   $$(x^k, y^k) \subseteq I \subseteq (x,y)^k$$

   are k-symmetric; the above inclusions give $\binom{k+1}{2} \leq \ell(\mathbb{C}[x,y]/I) \leq k^2$.

2. If P is a multiple ordinary point of multiplicity $m \leq 3$ for a plane curve $\mathcal{C}$, then the Jacobian scheme of $\mathcal{C}$ at P is an $(m-1)$-slci. If $m = 2$ this follows by Theorem 2.4.1. If $m = 3$, by Theorem 2.51 p.152 in [70], the germ of $\mathcal{C}$ at P is analytically equivalent to the germ of any union of three distinct lines meeting at O, for example $\mathcal{D} : x^3 - y^3 = 0$; since the Jacobian ideal of $\mathcal{D}$ is (x^2, y^2), the conclusion follows by Theorem 2.1.6.

3. For any m there exist curves with a multiple ordinary point of multiplicity m at P and such that their Jacobian scheme is a $(m-1)$-slci. In fact, let us consider the curve $\mathcal{C} : x^m - y^m = 0$, which is the union of the m distinct lines $(x-\eta_j y) = 0$ where $\eta_1, \dots, \eta_m$ are the m^{th} roots of unity. The curve $\mathcal{C}$ has a multiple ordinary point at O of multiplicity m, and no other singularities. Its Jacobian ideal is

$$J = (mx^{m-1}, my^{m-1}, x^m - y^m) = (x^{m-1}, y^{m-1})$$

and thus its Jacobian scheme is a $(m-1)$-slci.

Theorem 2.2.7. *Let P be a multiple ordinary point of multiplicity m for a plane curve $\mathcal{C}$ in $\mathbb{P}^2$ and let Z_P be its Milnor scheme at P and X_P be its Jacobian scheme at P. Then:*

1. *the tangent cones of the curves $\mathcal{C}_x, \mathcal{C}_y$ have no lines in common, hence $Z_P = (\mathcal{C}_x \cap \mathcal{C}_y)_P$ is a $(m-1)$-slci, so that $\mu = \ell(Z_P) = (m-1)^2$;*
2. *X_P is a $(m-1)$-symmetric scheme and $\tau = \ell(X_P) \leq (m-1)^2$;*
3. *in particular, if $\mathcal{C}$ is an union of m distinct lines through P, then $X_P = Z_P$, so that $\ell(X_P) = (m-1)^2$.*

Proof

1. Without loss of generality, we may assume $P = O$. Let $\mathcal{C} : f = 0$, $f = f_m + \cdots + f_d$; since f_m is a homogeneous polynomial in x and y of degree m, $(f_m)_x$ and $(f_m)_y$ are homogeneous in x and y of degree $m-1$. If $(f_m)_x = 0$, we have $f_m = ay^m$ against the assumption that P is an ordinary singularity, and analogously for f_y, hence $m_P(\mathcal{C}_x) = m_P(\mathcal{C}_y) = m-1$. The polynomials $(f_m)_x$ and $(f_m)_y$ are products of $m-1$ linear factor each, and no factors of $(f_m)_x$ divides $(f_m)_y$ and vice versa. Indeed, assume they do have a linear factor l in common
$$(f_m)_x = ll_1 \dots l_{m-2}, \quad (f_m)_y = lh_1 \dots h_{m-2}.$$
By Euler formula
$$x(f_m)_x + y(f_m)_y = m\, f_m \;\Rightarrow\; l|f_m \;\Rightarrow\; f_m = lg \;\Rightarrow\; (f_m)_x = l_x g + l g_x \;\Rightarrow$$
$$\Rightarrow\; l|l_x g \;\Rightarrow\; l|g \;\Rightarrow\; l^2|f_m$$
against the assumption that the tangent cone of $\mathcal{C}$ at P is the union of m distinct lines. Hence the curves $\mathcal{C}_x$ and $\mathcal{C}_y$ have no tangent in common, and we conclude that $Z_P = (\mathcal{C}_x \cap \mathcal{C}_y)_P$ is a $(m-1)$-slci of length $\ell(Z_P) = (m-1)^2$.

2. Let $r : h = 0$ be a line passing through O; by 1. and Lemma 2.2.5 we have $\ell(Z_P \cap r) = m - 1$. We want to prove that $\ell(X_P \cap r) = m - 1$; we have $(f, f_x, f_y) + (h) = (f, h) + (f_x, f_y, h)$, that is, $X_P \cap r = (\mathcal{C} \cap r) \cap (Z_P \cap r)$. Since P has multiplicity m for $\mathcal{C}$, we have $\ell(\mathcal{C} \cap r) \geq m$, so that $X_P \cap r$ is obtained intersecting the subscheme of r of length $m-1$ supported on P with a subscheme of r of length $\geq m$ supported on P, and the thesis follows. Moreover, since $Z_P \supseteq X_P$, we have that $\ell(X_P) \leq \ell(Z_P)$.

3. Let $\mathcal{C} : f = 0$; since f is homogeneous of degree m, we have $mf = xf_x + yf_y$, hence $(f, f_x, f_y) = (f_x, f_y)$, i.e. $X_P = Z_P$.

■

Remark 2.2.8. In Remark 2.2.6 we noted that the Jacobian ideal of $x^m - y^m$ is (x^{m-1}, y^{m-1}) and the associated Jacobian scheme is a $(m-1)$-slci. At this point one can wonder if all the k-slci are isomorphic to $\operatorname{Spec}\left(\mathbb{C}[x,y]/(x^k, y^k)\right)$, but Theorem 2.2.7 gives us a quick way to show that this is not true. For example, let $\mathcal{C}$ be the union of the lines $x = 0, y = 0, x + y = 0, x + 2y = 0$ and let $\mathcal{D}$ be the union of the lines $x = 0, y = 0, x + y = 0, x + 3y = 0$ respectively; then, the germs of $\mathcal{C}$ and $\mathcal{D}$ at O are not analytically equivalent (see [70] p.157) so that, by Theorem 2.1.6, their Jacobian schemes at O are not isomorphic. On the other hand, their Jacobian schemes are 3-slci by Theorem 2.2.7, hence we found two 3-slci which cannot be both isomorphic to $\operatorname{Spec}(\mathbb{C}[x,y]/(x^3, y^3))$.

Remark 2.2.9. Let $\mathcal{C}$ be a plane curve and assume that $P \in \operatorname{Sing} C$ is a multiple ordinary point of multiplicity $m \geq 2$. If X_P is the component of $X = X(\mathcal{C})$ at P, then, by Lemma 1.7.4, we have $X_P \supseteq (m-1)P$ and it is almost immediate to see that this inclusion is strict, so that $\ell(X_P) > \binom{m}{2}$. However, this bound can be much improved: indeed, Theorem 3.2 in [8] says that, for any isolated plane singularity, $\tau > \frac{3}{4}\mu$. Hence, keeping in mind that for $m \geq 3$ one has $\binom{m}{2} < \frac{3}{4}(m-1)^2$ and that, by Theorem 2.2.7, $\mu_P(\mathcal{C}) = (m-1)^2$, we get a better lower bound:

$$\frac{3}{4}(m-1)^2 < \tau_P(\mathcal{C}) \leq (m-1)^2.$$

Moreover, the upper bound is sharp because we have already seen in Remark 2.2.6 that the case $\ell(X_P) = (m-1)^2$ actually occurs, for example if $\mathcal{C} : x^m - y^m = 0$. However, the lower bound is not sharp but can be made so by applying some results of [27] and [87], which we collect in the following theorem.

Theorem 2.2.10. *Let $\mathcal{C}$ be a plane algebraic curve and assume that $P \in \operatorname{Sing} \mathcal{C}$ is a multiple ordinary point of multiplicity $m \geq 2$. Then*

$$\left\lfloor \frac{3m^2 - 2m - 4}{4} \right\rfloor \leq \tau_P(\mathcal{C}) \leq (m-1)^2.$$

Moreover, the bounds are sharp and all the values of $\tau_P(\mathcal{C})$ occur.

Proof The upper bound follows by Remark 2.2.9 and the lower bound can be computed using the formulas in Proposition 5.14 and Corollary 5.15 of [87] or of Proposition p. 550 and Tableau 1, p. 543 in [27]. Finally, Theorem 5.2 of [87] guarantees that all the values occur and thus that the bounds are sharp. ∎

2.3 Ordinary singularities with $\tau < \mu$

In this section we compute the Tjurina number at the origin of the curves $\mathcal{C}_{b,c}$ defined below, in order to give a large class of examples of ordinary singularities having $\tau < \mu$. At the end of the computation of these Tjurina numbers, we partially recover a more general result presented in [87], where the curves $x^a + y^b = 0$ and their miniversal μ-constant families are analysed. The main purpose of this section is to present a more algebraic way to carry out these computations through Gröbner basis.

Throughout this section a, b, c denote non-negative integers, and we set

$$f = f_{b,c} := x^a + y^a + x^b y^c, \qquad a \geq 2, \quad b + c > a, \quad b \geq c$$

For any fixed $a \geq 2$, we study the family of curves

$$\mathcal{C}_{b,c}: \quad x^a + y^a + x^b y^c = 0, \qquad b + c > a$$

having an ordinary singularity of multiplicity a at $O = (0{,}0)$, with the further assumption $b \geq c$. Clearly, the corresponding results for $b < c$ can be deduced by symmetry.

In the following we set $\mathfrak{m} := (x, y)$ in $\mathbb{C}[x, y]$ and $J = J(\mathcal{C}_{b,c})$. For generalities about Gröbner basis, we refer to [49] Chapter 2, §7. With "grlex order" we mean the graduate lexicographic order.

Lemma 2.3.1. *We have $x^a, y^a \in J$. In particular, we have that the Jacobian scheme $X(\mathcal{C}_{b,c})$ is supported at $O = (0{,}0)$, the Jacobian ideal J is primary with radical $\mathfrak{m}$ and $\tau = (\mathcal{C}_{b,c}) = \tau_O(\mathcal{C}_{b,c})$.*

Proof It is enough to notice that:

$$x^a = \frac{1}{(a-b)(a-c) - bc} \left((a-c)(x f_x - bf) + b(y f_y - cf) \right)$$

$$y^a = \frac{1}{(a-b)(a-c) - bc} \left(c(x f_x - bf) + (a-b)(y f_y - cf) \right).$$

∎

From now on, and for the rest of the section, we set $\tau = \tau_O(\mathcal{C}_{b,c})$.

Proposition 2.3.2. *If $b \geq a$ then $J = (x^{a-1}, y^{a-1})$ and $\tau = (a-1)^2 = \mu$.*

Proof We have

$$f_x = x^{a-1}(a + bx^{b-a}y^c), \quad f_y = ay^{a-1} + (cx^{b-a+1}y^{c-1})x^{a-1}$$

$$f = x^{a-1}(x + x^{b-a+1}y^c) + y^{a-1}(y)$$

then, in the local ring $\mathbb{C}[x,y]_{\mathfrak{m}}$ we have

$$J\mathbb{C}[x,y]_{\mathfrak{m}} = (x^{a-1}(a + bx^{b-a}y^c), f_y,\ f)\mathbb{C}[x,y]_{\mathfrak{m}} = (x^{a-1}, f_y,\ f)\mathbb{C}[x,y]_{\mathfrak{m}} =$$

$$= (x^{a-1}, y^{a-1},\ f)\mathbb{C}[x,y]_{\mathfrak{m}} = (x^{a-1}, y^{a-1})\mathbb{C}[x,y]_{\mathfrak{m}}.$$

In a minimal primary decomposition of the Jacobian ideal J, the primary ideal with radical $\mathfrak{m}$ is the contraction of the ideal $J\mathbb{C}[x,y]_{\mathfrak{m}}$ (see [13], Proposition 4.8), that is, J being primary, the ideal (x^{a-1}, y^{a-1}). ■

Lemma 2.3.3. *Assume $b < a$ and set*

$$f^{(1)} = f_x = bx^{b-1}y^c + ax^{a-1}, \quad f^{(2)} = f_y = cx^by^{c-1} + ay^{a-1}, \quad f^{(3)} = x^a, \quad f^{(4)} = y^a$$

$$f^{(5)} = x^{a-b-1}y^{a-1}, \quad f^{(6)} = x^{a-1}y^{a-c-1}, \quad f^{(7)} = x^{a-b}y^{a-1}.$$

Then a reduced Gröbner basis of the Jacobian ideal J, up to normalization of the leading term and with respect to the grlex order, is given by the following table (the cases marked by "-" cannot occur under our assumptions):

		A	***B***	***C***
		$b = \frac{a+1}{2}$	$\frac{a+1}{2} < b < a-1$	$b = a-1$
1	$c < \frac{a-1}{2}$	-	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(7)}$	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(7)}$
2	$c = \frac{a-1}{2}$	-	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(7)}$	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(7)}$
3	$c = \frac{a}{2}$	-	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(5)}$	$f^{(1)}, f^{(3)}$ $f^{(5)}, f^{(6)}$
4	$c = \frac{a+1}{2}$	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(5)}, f^{(6)}$	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(5)}, f^{(6)}$	$f^{(1)}, f^{(3)}$ $f^{(5)}, f^{(6)}$
5	$\frac{a+1}{2} < c < a-1$	-	$f^{(1)}, f^{(2)}, f^{(3)}$ $f^{(4)}, f^{(5)}, f^{(6)}$	$f^{(1)}, f^{(3)}$ $f^{(5)}, f^{(6)}$
6	$c = a-1$	-	-	$f^{(5)}, f^{(6)}$

Table 2.1: Gröbner basis of J.

Proof Given $g, h \in \mathbb{C}[x, y]$, let $\tilde{S}(g, h)$ denote the S-polynomial of g and h as defined in [49], Chapter 2, §6, Definition 4. We set

$$S(g, h) := \alpha\beta\, \tilde{S}(g, h)$$

where α and β are the leading coefficients of g and h (see [49], Chapter 2, §2, Definition 7). For each case, let us denote by $\mathcal{G}$ the set of elements appearing in a single cell of Table 1. We denote by $\overline{g}^{\mathcal{G}}$ the remainder of the division of g by $\mathcal{G}$ (see [49], Chapter 2, §3, Theorem 3). It is immediate to see that for monomials p, q one has $\overline{S(p, q)}^{\mathcal{G}} = 0$. We prove that $\mathcal{G}$ is a reduced Gröbner basis using the Buchberger's Criterion (see [49], Chapter 2, §6, Theorem 6), i.e. showing first that $J = (\mathcal{G})$, and then that $\overline{S(f^{(i)}, f^{(j)})}^{\mathcal{G}} = 0\ \forall f^{(i)}, f^{(j)} \in \mathcal{G}$. We consider the cases summarised in Table 1, depending on the values of b and c.

- **B1)** $c < \frac{a-1}{2}, \frac{a+1}{2} < b < a-1$.
  By Lemma 2.3.1, in order to verify that $J = (\mathcal{G})$ is it enough to show that $f \in (\mathcal{G})$ and $f^{(7)} \in J$. We have:

$$f = \frac{x}{b} f^{(1)} + \left(1 - \frac{a}{b}\right) f^{(3)} + f^{(4)} \Rightarrow f \in (\mathcal{G})$$

$$f^{(7)} = \frac{1}{a}(x^{a-b} f^{(2)} - c y^{c-1} f^{(3)}) \Rightarrow f^{(7)} \in J.$$

  Moreover,

$$S(f^{(1)}, f^{(2)}) = acx^a - aby^a = acf^{(3)} - abf^{(4)}, \quad S(f^{(1)}, f^{(3)}) = ax^{2a-b} = ax^{a-b} f^{(3)}$$

$$S(f^{(1)}, f^{(4)}) = ax^{a-1}y^{a-c} = y^{a-c} f^{(1)} - bx^{b-1} f^{(4)}$$

$$S(f^{(1)}, f^{(7)}) = ax^{a-1}y^{a-c-1} = \frac{a}{c} x^{a-b-1} y^{a-2c} f^{(2)} - \frac{a^2}{c} x^{a-b-1} y^{a-2c-1} f^{(4)}$$

$$S(f^{(2)}, f^{(3)}) = ax^{a-b}y^{a-1} = x^{a-b} f^{(2)} - cy^{c-1} f^{(3)}$$

$$S(f^{(2)}, f^{(4)}) = ay^{2a-c} = ay^{a-c} f^{(4)}, \quad S(f^{(2)}, f^{(7)}) = ay^{2a-c-1} = ay^{a-c-1} f^{(4)}$$

$$S(f^{(3)}, f^{(4)}) = S(f^{(3)}, f^{(7)}) = S(f^{(4)}, f^{(7)}) = 0.$$

  Thus we have $\overline{S(f^{(i)}, f^{(j)})}^{\mathcal{G}} = 0\ \forall f^{(i)}, f^{(j)} \in \mathcal{G}$, hence $\mathcal{G}$ is a Gröebner basis and it is easy to check that it is reduced.

- The cases **C1)** $c < \frac{a-1}{2}, b = a-1$, **B2)** $c = \frac{a-1}{2}, \frac{a+1}{2} < b < a-1$ and **C2)** $c = \frac{a-1}{2}, b = a-1$ are analogous to the case **B1**.

- **B3)** $c = \frac{a}{2}, \frac{a+1}{2} < b < a-1$.
To prove that $J = (\mathcal{G})$ we have just to check that $f^{(5)} \in J$:
$$f^{(5)} = -\frac{c}{a^2}\left(y^{c-1}f^{(1)} - \left(\frac{b}{a}x^{b-1} + \frac{a}{c}x^{a-b-1}\right)f^{(2)} + \frac{bc}{a}x^{2b-a-1}y^{c-1}f^{(3)}\right).$$
As in case **B1**, one can see that
$\overline{S(f^{(1)}, f^{(2)})}^{\mathcal{G}} = \overline{S(f^{(1)}, f^{(3)})}^{\mathcal{G}} = \overline{S(f^{(1)}, f^{(4)})}^{\mathcal{G}} = \overline{S(f^{(2)}, f^{(3)})}^{\mathcal{G}} = \overline{S(f^{(2)}, f^{(4)})}^{\mathcal{G}} = 0.$
Moreover we have:
$$S(f^{(1)}, f^{(5)}) = ax^{a-1}y^{a-c-1} = 2cx^{a-1}y^{c-1} = 2x^{a-b-1}f^{(2)} - 2af^{(5)}$$
$$S(f^{(2)}, f^{(5)}) = ay^{2a-c-1} = ay^{a-c-1}f^{(4)}$$
$$S(f^{(3)}, f^{(4)}) = S(f^{(3)}, f^{(5)}) = S(f^{(4)}, f^{(5)}) = 0$$
so that $\overline{S(f^{(i)}, f^{(j)})}^{\mathcal{G}} = 0,\ \forall f^{(i)}, f^{(j)} \in \mathcal{G}$. Hence $\mathcal{G}$ is a Gröbner basis and again it is easy to check that it is reduced.

- **C3)** $c = \frac{a}{2}, b = a-1$.
The proof that $(f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}, f^{(5)})$ is a Gröbner basis is analogous to the previous one. However, this is not a reduced Gröbner basis. Indeed, we have $f^{(5)} = x^{a-b-1}y^{a-1} = y^{a-1}$ so that we can remove $f^{(4)}$ and replace $f^{(2)}$ by
$$f^{(2)} - af^{(5)} = x^b y^{c-1} = x^{a-1}y^{a-c-1} = f^{(6)}.$$
Hence the reduced Gröbner basis is $\mathcal{G} = (f^{(1)}, f^{(3)}, f^{(5)}, f^{(6)})$.

- **B5)** $\frac{a+1}{2} < c < a-1, \frac{a+1}{2} < b < a-1$.
In order to prove that $J = (\mathcal{G})$ it is enough to verify that $f^{(5)}, f^{(6)} \in J$. In fact, we have:
$$f^{(5)} = \frac{1}{a^2}\left(-cy^{c-1}f^{(1)} + ax^{a-b-1}f^{(2)} + bcx^{b-1}y^{2c-a-1}f^{(4)}\right)$$
$$f^{(6)} = \frac{1}{a^2}\left(ay^{a-c-1}f^{(1)} - bx^{b-1}f^{(2)} + bcx^{2b-a-1}y^{c-1}f^{(3)}\right).$$
As in case **B1**, one can see that
$\overline{S(f^{(1)}, f^{(2)})}^{\mathcal{G}} = \overline{S(f^{(1)}, f^{(3)})}^{\mathcal{G}} = \overline{S(f^{(1)}, f^{(4)})}^{\mathcal{G}} = \overline{S(f^{(2)}, f^{(3)})}^{\mathcal{G}} = \overline{S(f^{(2)}, f^{(4)})}^{\mathcal{G}} = 0.$
Moreover we have:
$$S(f^{(1)}, f^{(5)}) = ax^{a-1}y^{a-c-1} = af^{(6)}, \quad S(f^{(1)}, f^{(6)}) = ax^{2a-b-1} = ax^{a-b-1}f^{(3)}$$
$$S(f^{(2)}, f^{(5)}) = ay^{2a-c-1} = ay^{a-c-1}f^{(4)}, \quad S(f^{(2)}, f^{(6)}) = ax^{a-b-1}y^{a-1} = af^{(5)}$$
$$S(f^{(3)}, f^{(4)}) = S(f^{(3)}, f^{(5)}) = S(f^{(3)}, f^{(6)}) = 0$$
$$S(f^{(4)}, f^{(5)}) = S(f^{(4)}, f^{(6)}) = S(f^{(5)}, f^{(6)}) = 0.$$
Hence $\mathcal{G}$ is a Gröbner basis and, again, it is easy to check that it is reduced.

- The cases **A4)** $c = \frac{a+1}{2}, b = \frac{a+1}{2}$ and **B4)** $c = \frac{a+1}{2}, \frac{a+1}{2} < b < a-1$ are analogous to the case **B5**.

- **C5)** $\frac{a+1}{2} < c < a-1, b = a-1$.
The proof that $(f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}, f^{(5)}, f^{(6)})$ is a Gröbner basis is analogous to the case **B5**, but this is not a reduced one. To show that a reduced Gröbner basis is $\mathcal{G} = (f^{(1)}, f^{(3)}, f^{(5)}, f^{(6)})$ one can proceed as in case **C3**.

- **C4)** $c = \frac{a+1}{2}, b = a-1$.
It is analogous to the case **C5**.

- **C6)** $c = a-1, b = a-1$.
This case follow easily by case **C5** noting that $f^{(5)} = y^{a-1}, f^{(6)} = x^{a-1}$. ■

Corollary 2.3.4. *Assume $b < a$; then a system of generators for the leading terms ideal $(LT(J))$ of J is given by the following table:*

		A	***B***	***C***
		$b = \frac{a+1}{2}$	$\frac{a+1}{2} < b < a-1$	$b = a-1$
1	$c < \frac{a-1}{2}$	-	$x^{b-1}y^c, x^by^{c-1}$ $x^a, y^a, x^{a-b}y^{a-1}$	$x^{b-1}y^c, x^by^{c-1}$ $x^a, y^a, x^{a-b}y^{a-1}$
2	$c = \frac{a-1}{2}$	-	$x^{b-1}y^c, x^by^{c-1}$ $x^a, y^a, x^{a-b}y^{a-1}$	$x^{b-1}y^c, x^by^{c-1}$ $x^a, y^a, x^{a-b}y^{a-1}$
3	$c = \frac{a}{2}$	-	$x^{b-1}y^c, x^by^{c-1}$ $x^a, y^a, x^{a-b-1}y^{a-1}$	$x^{b-1}y^c, x^{a-b-1}y^{a-1}$ $x^a, x^{a-1}y^{a-c-1}$
4	$c = \frac{a+1}{2}$	$x^{b-1}y^c, x^by^{c-1}, x^a, y^a$ $x^{a-b-1}y^{a-1}, x^{a-1}y^{a-c-1}$	$x^{b-1}y^c, x^by^{c-1}, x^a, y^a$ $x^{a-b-1}y^{a-1}, x^{a-1}y^{a-c-1}$	$x^{b-1}y^c, x^{a-b-1}y^{a-1}$ $x^a, x^{a-1}y^{a-c-1}$
5	$\frac{a+1}{2} < c < a-1$	-	$x^{b-1}y^c, x^by^{c-1}, x^a, y^a$ $x^{a-b-1}y^{a-1}, x^{a-1}y^{a-c-1}$	$x^{b-1}y^c, x^{a-b-1}y^{a-1}$ $x^a, x^{a-1}y^{a-c-1}$
6	$c = a-1$	-	-	$x^{a-b-1}y^{a-1}$ $x^{a-1}y^{a-c-1}$

Table 2.2: Generators for $(LT(J))$.

Proof It follows by the definition of Gröbner basis and by Lemma 2.3.3. ■

Proposition 2.3.5. *Assume $b < a$ and set*

$$\ell_1(a,b,c) = b(a-1)+c(a+1)-bc-a+1, \quad \ell_2(a,b,c) = b(a-1)+c(a+1)-bc-a$$

$$\ell_3(a,b,c) = (a-1)^2, \quad \ell_4(a,b,c) = b(a-1)+c(a-1)-bc$$

Let Y be the scheme associated to $(LT(J))$; then $\ell(Y)$ is given by the following table:

		A	***B***	***C***
		$b = \frac{a+1}{2}$	$\frac{a+1}{2} < b < a-1$	$b = a-1$
1	$c < \frac{a-1}{2}$	-	$\ell_1(a,b,c)$	$\ell_1(a,b,c)$
2	$c = \frac{a-1}{2}$	-	$\ell_1(a,b,c)$	$\ell_1(a,b,c)$
3	$c = \frac{a}{2}$	-	$\ell_2(a,b,c)$	$\ell_3(a,b,c)$
4	$c = \frac{a+1}{2}$	$\ell_4(a,b,c)$	$\ell_4(a,b,c)$	$\ell_3(a,b,c)$
5	$\frac{a+1}{2} < c < a-1$	-	$\ell_4(a,b,c)$	$\ell_3(a,b,c)$
6	$c = a-1$	-	-	$\ell_3(a,b,c)$

Table 2.3: Length of Y.

Proof The length of a scheme given by a monomial ideal can be easily computed if we know a system of generators for the ideal, for example using its graphic representation, which we described in Remark 1.2.10 (see [49], Chapter 9, §2, Example 1 for more details). In our case, the generators of $(LT(J))$ are given by Corollary 2.3.4, and it is enough to prove cases **B1**, **B3**, **B5**, **C5** and **C6** because the other ones are analogous.

- **B1**) $c < \frac{a-1}{2}$, $\frac{a+1}{2} < b < a-1$.
  Since $(LT(J)) = (x^{b-1}y^c, x^by^{c-1}, x^a, y^a, x^{a-b}y^{a-1})$, the graphic representation of Y is

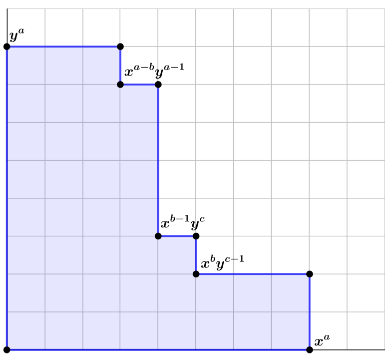


Figure 2.1: Graphic representation of Y in case **B1**.

and

$$\ell(Y) = (a-b)a+(2b-a-1)(a-1)+c+(a-b)(c-1) = b(a-1)+c(a+1)-bc-a+1.$$

- **B3**) $c = \frac{a}{2}$, $\frac{a+1}{2} < b < a-1$.
  Since $(LT(J)) = (x^{b-1}y^c, x^by^{c-1}, x^a, y^a, x^{a-b-1}y^{a-1})$, the graphic representation of Y is

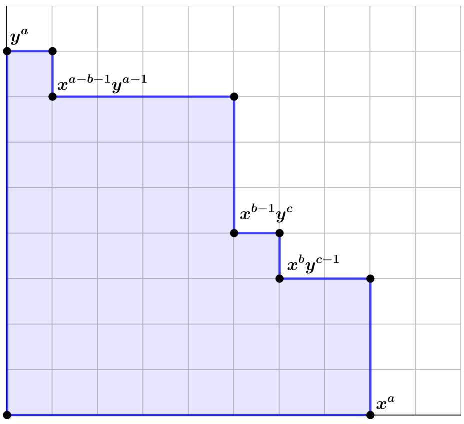


Figure 2.2: Graphic representation of Y in case **B3**.

and

$$\ell(Y) = (a-b-1)a+(2b-a)(a-1)+c+(a-b)(c-1) = b(a-1)+c(a+1)-bc-a.$$

- **B5**) $\frac{a+1}{2} < c < a-1$, $\frac{a+1}{2} < b < a-1$.
  Since $(LT(J)) = (x^{b-1}y^c, x^by^{c-1}, x^a, y^a, x^{a-b-1}y^{a-1}, x^{a-1}y^{a-c-1})$, the graphic representation of Y is

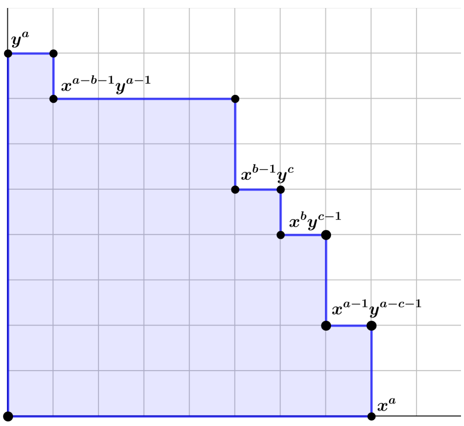


Figure 2.3: Graphic representation of Y in case **B5**.

and

$$\ell(Y) = (a-b-1)a+(2b-a)(a-1)+c+(a-1-b)(c-1)+a-c-1 = b(a-1)+c(a-1)-bc.$$

- **C5** $\frac{a+1}{2} < c < a-1, b = a-1$.
  Since $(LT(J)) = (x^{a-2}y^c, x^a, y^{a-1}, x^{a-1}y^{a-c-1})$, the graphic representation of Y is

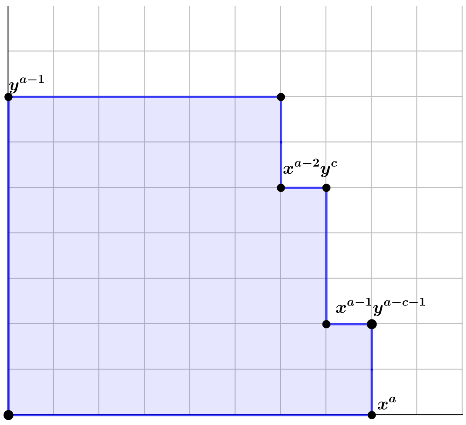


Figure 2.4: Graphic representation of Y in case **C5**.

and
$$\ell(Y) = (a-2)(a-1) + c + a - c - 1 = (a-1)^2.$$

- **C6**) $c = a-1, b = a-1$.
  Since $(LT(J)) = (x^{a-1}, y^{a-1})$ then $\ell(Y) = (a-1)^2$.

■

Theorem 2.3.6. *Let $a \in \mathbb{N}$ with $a \geq 2$ and $b, c \in \mathbb{N}$ with $b+c > a$. Then the curve*

$$\mathcal{C}_{b,c}: \quad x^a + y^a + x^b y^c = 0$$

has an ordinary singularity of multiplicity a at O, and, if $b \geq c$, then the possible values for its Tjurina number τ at O are:

$$\tau = \begin{cases} (a-1)^2 & if\ b \geq a\ or\ b = a-1,\ c \geq \frac{a}{2} \\ b(a-1)+c(a+1)-bc-a+1 & if\ \frac{a+1}{2} < b \leq a-1,\ c \leq \frac{a-1}{2} \\ b(a-1)+c(a+1)-bc-a & if\ \frac{a+1}{2} < b < a-1,\ c = \frac{a}{2} \\ b(a-1)+c(a-1)-bc & if\ \frac{a+1}{2} \leq b < a-1,\ \frac{a+1}{2} \leq c < a-1 \end{cases}.$$

The minimum value reached by the Tjurina number τ of $\mathcal{C}_{b,c}$ at O is

$$\tau_a := \min_{b,c\in\mathbb{N}, b+c>a}\{\tau(\mathcal{C}_{b,c})_O\} = \left\lfloor \frac{3a^2-2a-4}{4} \right\rfloor .$$

Proof If Y denotes the scheme associated to the leading terms ideal $(LT(J))$, then the affine Hilbert function of $\mathbb{C}[x,y]/J$ is equal to the affine Hilbert function of $\mathbb{C}[x,y]/(LT(J))$ (see [49], Chapter 9, §3, Proposition 4), so that, Y being supported uniquely at O, $\tau = \ell(Y)$. Hence, the first statement follows from Lemma 2.3.2 and Proposition 2.3.5.

For the second statement, we prove, under the assumption $b \geq c$, that $\tau_a = \frac{3a^2-2a-4}{4}$ if a is even and $\tau_a = \frac{3a^2-2a-5}{4}$ if a is odd, and once this is done we can drop the assumption $b \geq c$ just exchanging the variables x and y.

If P is a multiple ordinary point of multiplicity $a \leq 3$ for any plane curve $\mathcal{D}$, we have already noticed in Remark 2.2.6 that the Jacobian scheme of $\mathcal{D}$ at P is an $(a-1)$-slci, hence $\tau = 1$ if $a = 2$, $\tau = 4$ if $a = 3$, and in fact $\tau_2 = 1$ and $\tau_3 = 4$. Hence in the following we may assume $a \geq 4$.

We use the notation of Proposition 2.3.5; recall that

$$\ell_1 = b(a-1)+c(a+1)-bc-a+1, \quad \ell_2 = b(a-1)+c(a+1)-bc-a$$

$$\ell_4 = b(a-1)+c(a-1)-bc.$$

Assume a is even. To prove the statement, it is enough to minimise ℓ_1, ℓ_2 and ℓ_4 in their definition domains. We find the minimum of ℓ_1, ℓ_2 and ℓ_4 and finally the minimum between these three minima. Let us start with ℓ_1. Its domain is defined by the following inequalities:

$$b+c \geq a+1, \quad \frac{a+1}{2} < b \leq a-1, \quad c \leq \frac{a-1}{2} \quad (\text{hence } \ b \geq c)$$

but, since a is even and $a, b, c \in \mathbb{N}$, we can refine these inequalities to the following ones:

$$b+c \geq a+1, \quad \frac{a}{2}+1 \leq b \leq a-1, \quad 2 \leq c \leq \frac{a}{2}-1.$$

The plane region corresponding to these inequalities is the triangle T as shown by Figure 2.5

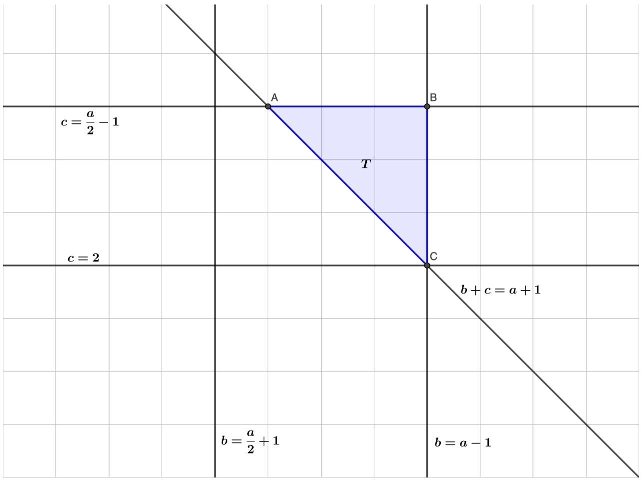


Figure 2.5: The domain of ℓ_1.

where $A = (\frac{a}{2} + 2, \frac{a}{2} - 1), B = (a - 1, \frac{a}{2} - 1)$ and $C = (a - 1,2)$.
We have $\nabla \ell_1(b, c) = (a - 1 - c, a + 1 - b)$, so that

$$\nabla \ell_1(b, c) = (0,0) \Leftrightarrow (b, c) = (a + 1, a - 1) \notin T.$$

Hence the minimum of ℓ_1 is along the boundary of T. We have:

$$\ell_1(b, c)|_{AB} = \ell_1 \left(b, \frac{a}{2} - 1\right) = \frac{ab + a^2 - 3a}{2} \Rightarrow \min_{AB} \ell_1 = \ell_1 \left(\frac{a}{2} + 2, \frac{a}{2} - 1\right) = \frac{3a^2 - 2a}{4}$$

$$\ell_1(b, c)|_{BC} = \ell_1(a - 1, c) = 2c + a^2 - 3a + 2 \Rightarrow \min_{BC} \ell_1 = \ell_1(a - 1,2) = a^2 - 3a + 6$$

$$\ell_1(b, c)|_{CA} = \ell_1(b, a + 1 - b) = b^2 + (-a - 3)b + a^2 + a + 2 \Rightarrow$$

$$\Rightarrow \min_{CA} \ell_1 = \ell_1 \left(\frac{a}{2} + 2, \frac{a}{2} - 1\right) = \frac{3a^2 - 2a}{4}.$$

Under our assumptions it easy to check that

$$\min\{\frac{3a^2 - 2a}{4}, a^2 - 3a + 6\} = \frac{3a^2 - 2a}{4}$$

for $a \neq 5$, and since we are assuming a even, we conclude that $\min \ell_1 = \frac{3a^2-2a}{4}$.

Now we find the minimum of ℓ_2. The domain of ℓ_2 is defined by the following inequalities:

$$\frac{a+1}{2} < b < a - 1, \quad c = \frac{a}{2}, \quad b + c \geq a + 1 \quad \text{(hence } b \geq c)$$

that, under our assumptions, can be refined as follows:

$$\frac{a}{2} + 1 \leq b \leq a - 2, \quad c = \frac{a}{2}.$$

Hence, we get:

$$\ell_2(b,c) = \ell_2\left(b, \frac{a}{2}\right) = \frac{(a-2)b + a^2 - a}{2} \Rightarrow \min \ell_2 = \ell_2\left(\frac{a}{2}+1, \frac{a}{2}\right) = \frac{3a^2 - 2a - 4}{4}.$$

Finally, we find the minimum of ℓ_4. The domain of ℓ_4 is defined by the following inequalities:

$$\frac{a+1}{2} \leq b < a-1, \quad \frac{a+1}{2} \leq c < a-1, \quad b + c \geq a+1, \quad b \geq c$$

that, under our assumptions, can be refined as follows:

$$\frac{a}{2} + 1 \leq b \leq a-2, \quad \frac{a}{2} + 1 \leq c \leq a-2, \quad b + c \geq a+1, \quad b \geq c.$$

The plane region corresponding to these inequalities is the triangle T' as shown by Figure 2.6:

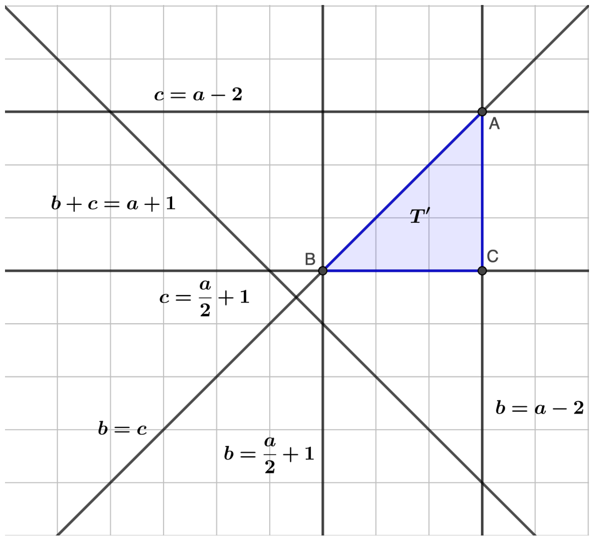


Figure 2.6: The domain of ℓ_4.

where $A = (a-2, a-2)$, $B = (\frac{a}{2}+1, \frac{a}{2}+1)$ and $C = (a-2, \frac{a}{2}+1)$.
Proceeding as we did for ℓ_1 it easy to check that the minimum of ℓ_4 is along the boundary of T' and it is $\min \ell_4 = \frac{3a^2-12}{4}$.
Finally, an easy computation shows that

$$\min \left\{ \frac{3a^2 - 2a}{4}, \frac{3a^2 - 2a - 4}{4}, \frac{3a^2 - 12}{4} \right\} = \frac{3a^2 - 2a - 4}{4}$$

and hence the result is proved for a even. If a is odd the proof is analogous (but it is easier, since it is not necessary to consider ℓ_2). ■

2.4 Jacobian schemes at double points

We now consider the Jacobian scheme at a double, not necessarily ordinary, point. Recall that, if $k \geq 1$, a double point of type A_{2k-1}, respectively A_{2k}, for a plane curve $\mathcal{C}$ is a 2-branched, respectively 1-branched, double point, which needs k successive blow ups to be smoothed (hence A_1 is a node, A_2 an ordinary cusp, A_3 a tacnode and so on). For any n, even or odd, the normal form for an A_n singularity is: $y^2 - x^{n+1} = 0$; in other words, the germ of $\mathcal{C}$ at O is analytically equivalent to the germ of the curve $y^2 - x^{n+1} = 0$ at O.

Theorem 2.4.1. *A point P is a double point of type A_n for a plane curve D if and only if the Jacobian scheme of D at P is a curvilinear scheme of length n. Hence, a double point P for a plane curve D is of type A_n if and only if $\tau(D)_P = n$.*

Proof Let us consider the curve $\mathcal{C}_n : y^2 - x^{n+1} = 0$; $\mathcal{C}$ has a double point of type A_n at O, and no other singularities in the affine plane; its Jacobian ideal is

$$J = (y^2 - x^{n+1}, 2y, (n+1)x^n) = (y, x^n)$$

hence its Jacobian scheme is a curvilinear scheme of length n. The conclusion follows by Theorem 2.1.6. ■

Remark 2.4.2. The case of double points could induce to think that τ at an ordinary singularity is always smaller than τ at a non-ordinary singularity, but the following example shows that this is not always the case. Let us consider the curve $\mathcal{C} : xy(x-y)(x+y)^2 + x^6 + y^6 = 0$, which has a 5-ple non-ordinary point at O. A computation with CoCoA shows that $\tau = 15$, while for the curve $\mathcal{D} : x^5 - y^5 = 0$ we have $\tau = 16$.

2.5 A remark on the Tjurina number

In this short section we give a condition for a curve $\mathcal{C}$ to have only nodes, using the global Tjurina number $\tau(\mathcal{C})$.

Proposition 2.5.1. *Let $\mathcal{C} : f(x,y) = 0$ be a plane curve. If $P \in \mathcal{C}$, $m = m_P(\mathcal{C}) \geq 2$ and P is not a node, then X_P contains properly the 0-dimensional scheme $(m-1)P$, so that*

$$\ell(X_P) > \binom{m}{2}$$

Proof First assume that P is an ordinary singularity with $m \geq 3$; then, $X_P \supseteq (m-1)P$ by Lemma 1.7.4, and by Remark 2.2.9 we have $\ell(X_P) = \tau > \frac{3}{4}(m-1)^2$. Since for $m \geq 3$ one has $\binom{m}{2} < \frac{3}{4}(m-1)^2$, we get $X_P \supsetneq (m-1)P$.

If $m \geq 2$ and P is not an ordinary singularity, we can assume that $P = (0,0)$ and the tangent cone contains the double line supported on the x-axis, so that

$$f = y^2 h_1 \cdots h_{m-2} + \phi$$

where $h_1, \ldots, h_{m-2}$ are linear forms and ϕ is the sum of forms of degree $\geq m+1$. We have

$$f_x = \underbrace{y^2(h_1 \cdots h_{m-2})_x}_{\deg m-1} + \phi_x, \qquad f_y = \underbrace{2y(h_1 \cdots h_{m-2}) + y^2(h_1 \cdots h_{m-2})_y}_{\deg m-1} + \phi_y$$

with $\phi_x = 0$ or $\deg \phi_x \geq m$, and $\phi_y = 0$ or $\deg \phi_y \geq m$. Hence the curves

$$\mathcal{C}_x : f_x = 0, \;\; \mathcal{C}_y : f_y = 0$$

have a singularity at P with $m_P(\mathcal{C}_x) = m-1$, $m_P(\mathcal{C}_y) = m-1$, and the two curves have a common tangent at P, i.e. the x-axis, so that $\mathcal{C}_x \cap \mathcal{C}_y$ contains the 0-dimensional scheme Y union of $(m-1)P$ and of the curvilinear scheme of length m supported on the x-axis, i.e. $\mathcal{I}_Y = (x,y)^{m-1} \cap (x^m, y) = (x^m, x^{m-2}y, \ldots, xy^{m-2}, y^{m-1})$; we have $\ell(Y) = \binom{m}{2} + 1$. Moreover, $\mathcal{I}_Y \supsetneq (x,y)^m$, that is, $Y \subsetneq mP$.

The Jacobian scheme at P is the schematic intersection $\mathcal{C} \cap \mathcal{C}_x \cap \mathcal{C}_y$ at P, and $\mathcal{C} \supsetneq mP \supsetneq Y$, so we get that $Y \subseteq X_P$, this giving $\ell(X_P) \geq \binom{m}{2} + 1$. ■

Proposition 2.5.2. *Let $\mathcal{C}$ be an irreducible curve of degree d and geometric genus g in $\mathbb{P}^2$, with no infinitely near points. Then $\mathcal{C}$ has only nodes if and only if $\tau(\mathcal{C}) = \binom{d-1}{2} - g$.*

Proof Let $P_1, \ldots, P_r$ be the singular points of $\mathcal{C}$, of multiplicity $m_1, \ldots, m_r$, and let $\mathbb{X}$ be its Jacobian scheme. Assume $\tau(\mathcal{C}) = \binom{d-1}{2} - g$. Since there are no singular infinitely near points, $g = p_a(\mathcal{C}) - \sum_{i=1}^r \binom{m_i}{2}$, that is

$$\binom{d-1}{2} - g = \sum_{i=1}^r \binom{m_i}{2}.$$

Hence the assumption $\binom{d-1}{2} - g = \tau(\mathcal{C}) = \sum_{i=1}^r \ell(\mathbb{X}_{P_i})$ gives:

$$\sum_{i=1}^r \binom{m_i}{2} = \sum_{i=1}^r \ell(\mathbb{X}_{P_i}).$$

If P_i is a node, then $\ell(\mathbb{X}_{P_i}) = 1 = \binom{m_i}{2}$, while if P_i is not a node we have $\ell(\mathbb{X}_{P_i}) > \binom{m_i}{2}$ by Proposition 2.5.1, hence all the singular points must be nodes. The vice versa is immediate. ■

Chapter 3

Superfat points and associated tensors

This chapter is based on a joint work with M. V. Catalisano, A. Gimigliano and M. Idà (see [35]). In all this chapter we will use the identifications defined in Notation 1.3.1 and in Remark 1.4.2 so that we will interchangeably think of $\nu_{n,d}$ as a map from $\mathbb{P}^n$ to $\mathbb{P}^{N_{n,d}}$ or as a map from $\mathbb{P}(T_1)$ to $\mathbb{P}(T_d)$.

The ideas for this chapter sprang from the symmetric schemes we encountered in Chapter 2 during the study of the Jacobian schemes of ordinary singularities (see Definition 2.2.4). We found surprising that in the wide panorama of studies among 0-dimensional schemes the following simple (and, to us, quite natural) questions had not been asked:

- What are the possible structures of a 0-dimensional scheme supported at one point P which are *symmetric*, i.e. that give the same length m when intersected with any line through P?
- Given m, how many points can sit "symmetrically" on a point P of the space?

The formulation of the last question deliberately recalls the well known one: "How many angels can stand on the tip of a needle?" which has become a sort of metaphor for "useless logic argument","needless point", even though it is related to Middle Ages scholastic theology and to its way of debating similar questions (e.g. see [12] for problems in angelology and [100] or [93] for a discussion about the story and possible educational use of this kind of questions). We hope that our questions about 0-dimensional schemes are not so abstruse!

In this chapter, we start by generalising the definition of m-symmetric scheme and m-slci scheme from $\mathbb{P}^2$ to $\mathbb{P}^n$. After that we give the definition of m-superfat point as a symmetric scheme having the property of maximality with respect to the inclusion and we show the coincidence of m-slci and m-superfat points.

At this point the aim of the chapter is to begin the study of such 0-dimensional schemes, also related to the varieties they can generate via Veronese or Segre-Veronese embeddings of the space (either projective or multi-projective) where they are embedded; this aspect could be of interest also for possible applications to tensor decomposition (e.g. to products of symmetric W-states in evaluating tensor rank as a measure of entanglement).

The plan of the chapter is the following: in §3.1 we give the main definition and first properties related to symmetric 0-dimensional schemes. In particular, we show that every m-symmetric scheme is contained in an m-slci and we deduce that m-slci are m-symmetric schemes which are maximal with respect to the inclusion.

In §3.2 we study in more detail the case of points in the plane. In particular, we show that the 2-superfat points of $\mathbb{P}^2$ coincide with the 2-squares and we prove that the schematic union of all the m-superfat points supported at $P \in \mathbb{P}^2$ is the fat point $(2m-1)P$. After that we stress some intuition-baffling properties of symmetric schemes.

Finally, in §3.3 and in §3.4 we point our attention to the kind of symmetric and partially symmetric tensors which are parameterised by points in the span of the image of such schemes via Veronese or Segre-Veronese embeddings and we study the varieties that they define, determining the defectivity of some secant varieties thereof.

3.1 Symmetric and superfat points in $\mathbb{P}^n$

We start generalising the definition of m-symmetric scheme and m-slci, which we originally gave only for $\mathbb{P}^2$ in Definition 2.2.4.

Definition 3.1.1. A 0-dimensional scheme X supported at one point $P \in \mathbb{P}^n$ is said to be

- *m-symmetric* if $\ell(X \cap L) = m$, for every line L passing through P;
- an *m-symmetric local complete intersection* (m-slci for short) if it is a local complete intersection of n hypersurfaces having multiplicity at P equal to m and whose tangent cones at P have no line in common.

Remark 3.1.2. It is easy to see that an m-slci is m-symmetric, and the proof is analogous to the one of Lemma 2.2.5. Moreover, by [62], Corollary 12.4. one finds that the length of an m-slci of $\mathbb{P}^n$ is m^n.

As for the case $n = 2$, an m-fat point is an m-symmetric scheme in any $\mathbb{P}^n$. Actually, the m-fat points are the m-symmetric schemes of smallest possible length and their peculiarity with respect to m-symmetry is illustrated by the following lemma.

Notation 3.1.3. We will use in the affine space $\mathbb{A}^n$ coordinates $(x_1, \ldots, x_n)$ and we will think of it as the affine chart $\{x_0 \neq 0\}$ of $\mathbb{P}^n$.

Lemma 3.1.4. *Let X be an m-symmetric scheme supported at $P \in \mathbb{P}^n$ and $\mathcal{I}_X = (G_1, \ldots, G_s)$, its defining ideal, where $G_1, \ldots, G_s$ is a minimal set of generators. Then all the hypersurfaces $G_i = 0$ have multiplicity at least m at P, and at least n of them have multiplicity exactly m. Moreover, there is no line common to all the tangent cones of the hypersurfaces $G_i = 0$ which have multiplicity exactly m at P.*

Proof If there were a hypersurface $G_i = 0$ having multiplicity at P less than m then a line L passing through P and not contained in the tangent cone of $\{G_i = 0\}$, would locally intersect $\{G_i = 0\}$ with length $m' < m$, hence we would have $\ell(L \cap X) \leq m' < m$, thus getting a contradiction.

Now let us suppose that there are r hypersurfaces in $\mathcal{I}_X$ with multiplicity exactly m at P, say $G_1, \ldots, G_r$ and, by contradiction, that $r < n$. Let $\{F_1 = 0\}, \ldots, \{F_r = 0\}$ be their tangent cones at P and consider the scheme $Y \subseteq \mathbb{P}^n$ with $\mathcal{I}_Y = (F_1, \ldots, F_r)$. Since Y is a cone and $\dim(Y) \geq 1$, then there is a line $L \subseteq Y$. By construction, L passes through P and $\ell(L \cap X) > m$ against the hypothesis that X is m-symmetric.

Finally, if there were a line L common to all the tangent cones $\{F_1 = 0\}, \ldots, \{F_r = 0\}$, then we would have $\ell(L \cap X) > m$ again contradicting our hypothesis of m-symmetry. ∎

Remark 3.1.5. As an immediate consequence of the previous lemma, we have that every m-symmetric scheme $X \subseteq \mathbb{P}^n$ supported at P contains the m-fat point mP. Hence fat points are (with respect to inclusion) the smallest m-symmetric schemes; in particular, the length reaches its minimum, i.e. for every m-symmetric scheme X we have $\ell(X) \geq \binom{m+n-1}{n}$, with equality if and only if $X = mP$.

Now we want to find out "how fat can an m-symmetric point be" i.e. we want to consider the following questions:

- Among all the m-symmetric schemes supported on the same point P, which are the maximal ones (with respect to schematic inclusion)?
- What is the maximum length of an m-symmetric scheme?

One can think of this problem as a problem of "packaging of points": given m, we want to "fit together" infinitesimal points over a point P in such a way that $m-$symmetry holds, and we want to know how many of them we can "keep packaging together" without violating m-symmetry.

Remark 3.1.6. The ideal $(x_1^m, \ldots, x_n^m) \subseteq \mathbb{C}[x_0, \ldots, x_n]$ defines a projective scheme $X \subseteq \mathbb{P}^n$ of length $\ell(X) = m^n$, and it is easy to check that it satisfies m-symmetry, hence the answer to the second question above is at least m^n, i.e. the maximal length for an m-symmetric scheme in $\mathbb{P}^n$ is at least m^n.

In Theorem 3.1.9 we show that in fact that the symmetric schemes which are maximal with respect to the inclusion are exactly the symmetric local complete intersections and thus they all have the same length and the maximum number of infinitesimal points we can fit over a point P is m^n. Before doing that we give two definition.

Definition 3.1.7. An m-symmetric scheme in $\mathbb{P}^n$ which is maximal with respect to the inclusion is called an *m-superfat point*, or just a superfat point if we do not need to specify m.

Definition 3.1.8. An m-symmetric scheme whose ideal is of type $(\ell_1^m, \ell_2^m, \ldots, \ell_n^m)$ for $\ell_i \in \mathbb{C}[x_0, ..., x_n]_1$, with $\ell_1, \ldots \ell_n$ linearly independent, is called an *m-hypercube.*

Theorem 3.1.9. *A scheme $X \subseteq \mathbb{P}^n$ is an m-superfat point supported at $P \in \mathbb{P}^n$ if and only if it is an m-slci. Thus, any $m-$superfat point in $\mathbb{P}^n$ has length m^n and it is a Gorenstein scheme.*

Proof By Remark 3.1.2, proving the statement is equivalent to proving that any $m-$symmetric scheme X supported at P is contained in an m-slci.

We assume $P = [1{,}0, ...{,}0]$ and we work in $\mathbb{A}^n$ using affine coordinates $(x_1, ..., x_n)$. If we set $\mathcal{I}_X = (G_1, ..., G_s) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ then, by Lemma 3.1.4, we have that all G_i's have multiplicity at least m at P, and at least n of them have multiplicity exactly m. Let $G_1, \ldots, G_r$, $n \leq r \leq s$ be the ones that have tangent cone of degree m at P, and let $F_1, \ldots, F_r \in \mathbb{C}[x_1, ..., x_n]_m$ be the degree m summands of $G_1, \ldots, G_r$, i.e. the equations defining their tangent cones. If the F_i's are linearly dependent, and for example $F_1 = a_2F_2 + \ldots. + a_rF_r$, then G_1 can be replaced by $G_1 - (a_2G_2 + \ldots. + a_rG_r)$, which has tangent cone of degree $> m$; hence we can assume that the F_i's are linearly independent, and, again by Lemma 3.1.4, we have that $F_1, ..., F_r$ do not have any common line.

We want to show that there are n polynomials in $\langle F_1, \ldots F_r\rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]_m$ which have no common lines (actually we will find a regular sequence $H_1, \ldots, H_n$). This is obvious if $r = n$, so we can assume $r \geq n + 1$. Consider $H_1 := F_1$ and a generic linear combination of $F_1, \ldots, F_r$

$$a_{21}F_1 + a_{22}F_2 + \ldots + a_{2r}F_r =: H_2.$$

We want to check that $\dim\{H_1 \cap H_2\} = n - 2$. Let $C_1, \ldots C_k$, $k \leq m$, be the irreducible components of $\{H_1 = 0\}$ and for any $i = 1, \ldots, k$ let $P_i \in C_i \setminus P$. In order to have that no C_i is contained in $\{H_2 = 0\}$ is enough that $H_2(P_i) \neq 0$ for any $i = 1, \dots, k$ and this is true for the genericity of the linear combination H_2: indeed, for each i, we can view

$$x_{21}F_1(P_i) + x_{22}F_2(P_i) + \ldots + x_{2r}F_r(P_i) = 0$$

as a hyperplane in $\mathbb{P}^{r-1}$, with respect to homogeneous coordinates $[x_{21}, \dots, x_{2r}]$, so it is enough to choose a point $[a_{21}, a_{22} \dots, a_{2r}]$ not lying on these hyperplanes. Hence $\{H_1 \cap H_2\}$ has dimension equal to $n-2$. Now we repeat this procedure by defining a generic linear combination

$$a_{31}F_1 + a_{32}F_2 + \ldots + a_{3r}F_r =: H_3$$

such that $\dim\{H_1 \cap H_2 \cap H_3\} = n-3$ and so on in order to get $H_1, \dots, H_n$ that form a regular sequence and their intersection is only supported at P.

Now, let $K_1 := G_1$ and

$$K_i := \sum_{j=1}^{r} a_{ij}G_j, \quad \forall\, i = 2, \dots, n$$

so K_i has H_i as tangent cone at P. Since

$$\bigcap_{i=1}^{n} H_i = \{P\}$$

the scheme Y defined by the ideal $(K_1, \dots, K_n)$ is 0-dimensional at P and Y_P is locally complete intersection of n hypersurfaces with multiplicity m at P and whose tangent cones have no common lines. We have that $(K_1, \dots, K_n) \subseteq \mathcal{I}_X$, hence $X \subseteq Y$ and this concludes the proof. ■

Remark 3.1.10. Two hypercubes of $\mathbb{P}^n$ given by the ideals $I = (\ell_1^m, ..., \ell_n^m)$ and $J = (h_1^m, ..., h_n^m)$ with the same support are different, provided that $\{l_1, ...l_n\} \neq \{h_1, ...h_n\}$. In fact, $h_j^m \in I_m$ if and only if there exists i such that $h_j = l_i$, since the forms $l_1^m, ...l_n^m, h_j^m$, viewed as points of the Veronese variety $V_{n,m}$, are in general position.

Remark 3.1.11. Even though, up to this moment, we mentioned just hypercubes, fat and superfat points, there are other schemes possessing m-symmetry. As an example, consider $X \subseteq \mathbb{P}^2$ defined by the ideal $(x_1^3, x_2^3, x_1^2x_2^2)$: this is 3-symmetric and $\ell(X) = 8$. This gives the opportunity of pointing out a few peculiar behaviours of the 0-dimensional schemes which sometimes baffle our intuition. Let $P = [1,0,0]$, $X = 2P$ and Y the hypercube of $\mathbb{P}^2$ having ideal (x_1^2, x_2^2). We can observe that:

- Even though the linear sections $X \cap L = Y \cap L$ coincide for any line L through P, we have $X \neq Y$.

- If J_L denotes the 2-jet supported on P and contained in L, we have

$$Y \cap (\bigcup_{L \ni P} L) = Y \cap \mathbb{P}^2 = Y$$

and

$$\bigcup_{L \ni P} (Y \cap L) = \bigcup_{L \ni P} J_L = 2P = X$$

where L varies on the set of the lines passing through P. Hence the schematic unions and intersections does not commute, while they do so if we consider just the support of the schemes.

As we will see, there are other unexpected properties of symmetric schemes.

In defining $m-$symmetry we have used lines through the support point, but the following result shows that this is equivalent to using smooth curves.

Proposition 3.1.12. *A 0-dimensional scheme X, supported at one point $P \in \mathbb{P}^n$, is m-symmetric if and only if $\ell(X \cap \mathcal{C}) = m$ for every curve $\mathcal{C}$ smooth at P.*

Proof Let $X \subseteq \mathbb{P}^n$ be an $m-$symmetric scheme with support at $P \in \mathbb{P}^n$, and $\mathcal{C} \subseteq \mathbb{P}^n$ be a curve smooth at P. We have $mP \subseteq X$, and $\ell(mP \cap \mathcal{C}) = m$, so $\ell(X \cap \mathcal{C}) \geq m$. Since $\mathcal{C}$ is smooth, it is locally a complete intersection, i.e. there are polynomials $F_1, \dots, F_{n-1}$ such that they are smooth at P, the ideal $(F_1, \dots F_{n-1})$ defines $\mathcal{C}$ at P, and the intersection of their tangent cones at P is the tangent line $\tau_{1,P}(\mathcal{C})$. If we consider any $F \in \mathcal{I}_X$, whose tangent cone has multiplicity m and does not contain $\tau_{1,P}(C)$ (it must exist since X is $m-$symmetric), then the length of the projective scheme defined by the ideal $(F, F_1, \dots, F_{n-1})$ is m by [62] Corollary 12.4, and $(F, F_1, ..., F_{n-1}) \subseteq \mathcal{I}_X + \mathcal{I}_{\mathcal{C}} \subseteq \mathcal{I}_{X \cap \mathcal{C}}$ hence $\ell(X \cap \mathcal{C}) \leq m$. ∎

Recall that 0-dimensional schemes in $\mathbb{P}^n$ for $n \geq 3$ are not all smoothable, i.e. obtained by collapsing simple points; nevertheless, the $m-$hypercubes are all smoothable.

Proposition 3.1.13. *Let $X \subseteq \mathbb{P}^n$ be an $m-$hypercube, then X is smoothable.*

Proof Modulo a projectivity, the ideal of any $m-$hypercube X can be put in the form $\mathcal{I}_X = (x_1^m, \dots, x_n^m)$ and such an ideal can be seen as $\lim_{t \to 0} \mathcal{I}_t$ where

$$\mathcal{I}_t = (F_1(x_1, t), \dots F_n(x_n, t))$$

and

$$F_i(x_i, t) = x_i(x_i + t)(x_i + 2t) \cdots (x_i + (m-1)t)$$

for each $i = 1, \dots, n$. The statement follows by the fact that $\mathcal{I}_t$ is actually the ideal of m^n simple points arranged on a hypercube for any t. ∎

The above proposition leads the way to its generalisation: also $m-$superfat points are not among the "bad 0-dimensional schemes" which are not smoothable in $\mathbb{P}^n$ for $n \geq 3$, i.e. we have the following proposition.

Proposition 3.1.14. *Let $X \subseteq \mathbb{P}^n$ be an $m-$superfat point. Then X is smoothable, $\forall\, m, n \in \mathbb{N}$.*

Proof The fact is actually known since every 0-dimensional locally complete intersection is smoothable (e.g. see [80] Theorem 4.36) and m-superfat points are locally complete intersection by Proposition 3.1.9. We just sketch the idea here: if we have, locally, $\mathcal{I}_X = (F_1, ..., F_n)$, consider the schemes X_t defined, locally, by $\mathcal{I}_{X_t} = (F_1 + tG_1, ..., F_n + tG_n)$, where the G_i's are generic forms of the same degree as F_i. We will have that, locally, X_t is given by m^n simple points, and, as $t \to 0$, $X_t \to X$, so X is smoothable. ∎

3.2 Superfat and m-symmetric points in $\mathbb{P}^2$

In this section we consider the case $n = 2$, where more detailed results are easier to get.

Notation 3.2.1. In the case $n = 2$ we use the notation *m-squares* instead of m-hypercubes.

We start showing that in the case $m = 2$, we have that actually $2-$superfat points are $2-$squares.

Proposition 3.2.2. *Every 2-superfat scheme $X \subseteq \mathbb{P}^2$ is a 2-square, i.e. $\mathcal{I}_X$ can be written, modulo projectivity, as $\mathcal{I}_X = (x_1^2, x_2^2)$.*

Proof Let $P = [1,0,0]$ be the support of X. By Theorem 3.1.9 X is a local complete intersection of two conics $\mathcal{C}_1 : F = 0$ and $\mathcal{C}_2 : G = 0$ such that $m_P(\mathcal{C}_1) = m_P(\mathcal{C}_2) = 2$ and their tangent cones at P have no common line; in other words $\mathcal{C}_1$ and $\mathcal{C}_2$ are both the union of two distinct lines meeting in P and the four lines are pairwise distinct. Now, let Y be the 0-dimensional scheme defined by the ideal (F, G). Since $\ell(Y) = 4 = \ell(X)$ and $X \subseteq Y$ we have $X = Y$ and in particular we get

$$\mathcal{I}_X = (F, G).$$

Let $F = L_1L_2$, $G = L_3L_4$, where all $L_i \in \mathbb{C}[x_1, x_2]_1$. In the pencil

$$\{aL_1L_2 + bL_3L_4\}$$

there will always be two conics of rank 1, since such pencil gives a line in $\mathbb{P}(\mathbb{C}[x_1, x_2]_2) \cong \mathbb{P}^2$ which will intersect in two points the conic representing the forms of rank 1 (i.e. the 2-Veronese embedding of $\mathbb{P}^1$, parameterising squares of linear forms). Note that our pencil cannot be represented by a tangent line to the conic, since such lines represent pencils of conics with a common linear factor. Hence, the ideal of X can be written, modulo a projectivity, as $\mathcal{I}_X = (x_1^2, x_2^2)$. ■

Example 3.2.3. The coincidence of 2-superfat points of $\mathbb{P}^2$ and 2-square has nothing similar neither in higher dimension nor in higher degree. Let us see some examples.

- In $\mathbb{P}^3$ the 2-superfat point of ideal $(x_0x_2^2, x_1x_2^2, x_2^3, x_0x_1x_2 + x_2^2x_3, x_0^2, x_1^2)$ is not a 2-hypercube because it has generic Hilbert function while a 2-hypercube does not.

- Even in $\mathbb{P}^2$ the situation for $m = 3$ is not similar to the case $m = 2$, i.e. a 3-superfat scheme X is not always a complete intersection of 2 cubics. Of course any ideal of type
$$(L_1L_2L_3, M_1M_2M_3)$$
where $L_i, M_j \in \mathbb{C}[x_1, x_2]_1$ and $L_i \neq \alpha M_j$, for all $i, j \in \{1,2,3\}$, gives a 3-superfat point, but they are not all. For instance, consider the ideal
$$\mathcal{I}_X = ((x_0 - x_1)^3, x_1^3x_2 + x_0^2x_1^2, x_0x_1^3, x_1^4).$$
It can be seen that it defines a scheme of length 9 which is 3-symmetric, i.e. a 3-superfat point. Nevertheless, X is not a complete intersection because its ideal generation is the generic one for a scheme of length 9: one cubic and three quartics. Anyway, if we consider the intersection of its two first generators at $P = [0,0,1]$, we get a scheme of length 9, which has to be X; in other words, the scheme X is the local complete intersection of two curves with a triple point at P and with no common tangent, in accord with Theorem 3.1.9.

Looking at several examples leads to the following conjecture.

Conjecture 3.2.4. *For every $m \geq 2$ there exist an $m-$superfat point in $\mathbb{P}^2$ having generic Hilbert function.*

Let us note that since the $0-$dimensional schemes with maximal Hilbert function form an open subset in $\mathcal{H}ilb^r(\mathbb{P}^2)$, if one can prove that the $m-$symmetric points form an irreducible subscheme (or at least a subscheme with only one component of maximal dimension) in $\mathcal{H}ilb_P^{m^2}(\mathbb{P}^2) \subseteq \mathcal{H}ilb^{m^2}(\mathbb{P}^2)$, where $\mathcal{H}ilb_P^{m^2}(\mathbb{P}^2)$ is the Hilbert scheme parameterising 0-dimensional subschemes of $\mathbb{P}^2$ of length m^2 and supported at a point $P \in \mathbb{P}^2$, then Conjecture 3.2.4 would imply that the generic $m-$superfat point has maximal Hilbert function.

In the following sections, we will be considering m-squares on Veronese and Segre-Veronese surfaces. To this aim it is useful to check what happens when we consider all the m-squares supported at the same point P, so we want to find out what the schematic union of all m-squares supported at one point P is.

In the sequel, we need the following combinatorial result on binomials (which we think is also interesting *per se*). It might be already known, but for lack of a reference we prove it here.

Lemma 3.2.5. *In $\mathbb{Z}^\infty$, $\forall\, m, i \geq 1$ consider the two vectors*

$$v_m = \left(\binom{m-1}{0}, -\binom{m}{1}, \binom{m+1}{2}, -\binom{m+2}{3}, \dots, (-1)^j\binom{m+j-1}{j}, \dots\right)$$

$$w_{i,m} = \left(\binom{m}{i}, \binom{m}{i-1}, \dots, \binom{m}{2}, \binom{m}{1}, \binom{m}{0}, 0, \dots, 0, \dots\right).$$

Then $v_m \cdot w_{i,m} = 0$.

Proof Note that, for $i \geq m+1$, $w_{i,m}$ has $(i-m)$ initial 0's. We have that:

- For $m = 1$ one has $v_1 = (1, -1, 1, -1, \dots)$ and $w_{i,1} = (0, \dots, 0, 1, 1, 0, \dots)$, with $(i-1)$ initial 0's;
- For $i = 1$, we have $w_{1,m} = (m, 1, 0, \dots)$.

Hence, for all i, we have $v_1 \cdot w_{i,1} = 0$, and for all m, we have $v_m \cdot w_{1,m} = 0$.

Now we assume $m > 1$ and $i > 1$ and we work by induction on $m + i$. By the identity

$$\binom{k}{\alpha} = \binom{k-1}{\alpha} + \binom{k-1}{\alpha-1}$$

we get $v_m = v_{m-1} + v'_m$, where

$$v'_m = \left(0, -\binom{m-1}{0}, \binom{m}{1}, -\binom{m+1}{2}, \dots, (-1)^i\binom{m+i-2}{i-1}, \dots\right)$$

and

$$w_{i,m} = w_{i,m-1} + w_{i-1,m-1}.$$

Hence

$$v_m \cdot w_{i,m} = (v_{m-1} + v'_m) \cdot w_{i,m} = v_{m-1} \cdot w_{i,m} + v'_m \cdot w_{i,m} =$$

$$= v_{m-1} \cdot w_{i,m-1} + v_{m-1} \cdot w_{i-1,m-1} + v'_m \cdot w_{i,m}.$$

Since $v'_m \cdot w_{i,m} = -v_m \cdot w_{i-1,m}$, then the three summands above are zero by the induction hypothesis, and we are done. ∎

Theorem 3.2.6. *For every $P \in \mathbb{P}^2$ and for any $m \geq 1$, we have that the schematic union of all m-squares supported at P is the fat point $(2m-1)P$.*

Proof Without loss of generality, we can work in the affine case and consider the case $P = (0,0)$. Let $A := \{(\ell_1, \ell_2) \in \mathbb{C}[x_1, x_2]_1 \times \mathbb{C}[x_1, x_2]_1 \mid \ell_1 \nmid \ell_2\}$. What we have to prove is that

$$\bigcap_{(\ell_1,\ell_2)\in A} (\ell_1^m, \ell_2^m) = (x_1, x_2)^{2m-1}.$$

First let us check that

$$(x_1, x_2)^{2m-1} \subseteq \bigcap_{(\ell_1,\ell_2)\in A} (\ell_1^m, \ell_2^m).$$

Actually, for any choice of (ℓ_1, ℓ_2) in A, every generator of $(x_1, x_2)^{2m-1}$ can be written as

$$a_0\ell_1^{2m-1}+a_1\ell_1^{2m-2}\ell_2+...+a_{m-1}\ell_1^{m}\ell_2^{m-1}+a_{m+1}\ell_1^{m-1}\ell_2^{m}+...+a_{2m-2}\ell_1\ell_2^{2m-2}+a_{2m-1}\ell_2^{2m-1},$$

of course with different coefficients $a_i \in \mathbb{C}$ if we change our choice of (ℓ_1, ℓ_2). Since in every term of this polynomial either ℓ_1 or ℓ_2 appears with power at least m, we get

$$(x_1, x_2)^{2m-1} \subseteq \bigcap_{(\ell_1,\ell_2)\in A} (\ell_1^m, \ell_2^m).$$

Now, in order to complete the proof, we have to prove that no form of $\mathbb{C}[x_1, x_2]_d$, with $d \leq 2m-2$, belongs to

$$\bigcap_{(\ell_1,\ell_2)\in A} (\ell_1^m, \ell_2^m)$$

and of course it is enough to prove that this happens for $d = 2m-2$. Since the statement is trivially true for $m = 1$, we assume $m \geq 2$, i.e. $2m-2 \geq m$. We choose $2m-1$ particular m-squares supported at P, and we prove that the intersection of their ideals has no form of $\mathbb{C}[x_1, x_2]_{2m-2}$. More precisely, we will prove that the following ideal

$$I := (x_1^m, \ell_1^m)\cap(x_1^m, \ell_2^m)\cap\ldots\cap(x_1^m, \ell_{m-1}^m)\cap(x_1^m, x_2^m)\cap(\ell_1^m, x_2^m)\cap(\ell_2^m, x_2^m)\cap\ldots\cap(\ell_{m-1}^m, x_2^m),$$

has no form of degree $2m-2$, where the ℓ_i's are distinct linear forms different from x_1 and x_2.

In order to prove our result, we study first the ideal $(x_1^m, (x_1 + ax_2)^m)$ in degree $2m-2$. Since

$$(x_1 + ax_2)^m = \sum_{i=0}^{m} \binom{m}{i} x_1^{m-i} a^i x_2^i$$

we get that $(x_1^m, (x_1 + ax_2)^m)_{2m-2}$ is generated by the following $2m-2$ forms:

$$x_1^{2m-1-j}x_2^{j-1},\ \forall\, j = 1, \ldots, m-1$$

$$\sum_{i=j}^{m} \binom{m}{i} a^i x_1^{m-1-i+j} x_2^{m-1-j+i},\ \forall\, j = 1, \ldots, m-1$$

We identify these forms with the following points in $\mathbb{P}(\mathbb{C}[x_1,x_2]_{2m-2})$

$$P_0=[1,0,\dots,0], P_1=[0,1,0,\dots,0],\cdots, P_{m-2}=[0,0,\dots,0,1,0,\dots,0]$$
$$P_{2m-2-j}=[\underbrace{0,\dots,0}_{m-1\text{ times}}, \binom{m}{j}a^j, \binom{m}{j+1}a^{j+1},\dots,\binom{m}{m}a^m, \underbrace{0,\dots,0}_{j-1\text{ times}}],\ \forall\, j=1,\dots,m-1.$$

Clearly these points are linearly independent, so they span a single hyperplane. If $\mathbb{C}[z_0,\dots,z_{2m-2}]$ is the coordinate ring of $\mathbb{P}^{2m-2}$, then the hyperplane spanned by the P_i's is

$$\pi: \binom{m-1}{0}a^{m-1}z_{m-1}-\binom{m}{1}a^{m-2}z_m+\binom{m+1}{2}a^{m-3}z_{m+1}+\dots+(-1)^{m-1}\binom{2m-2}{m-1}z_{2m-2}=0,$$

in fact, obviously, $P_0,\dots,P_{m-2}\in\pi$, and $P_{m-1},\dots,P_{2m-3}\in\pi$ by Lemma 3.2.5. Analogously, if we consider the ideal $(x_2^m,(x_1+ax_2)^m)$ in degree $2m-2$, we get that the forms of degree $2m-2$ correspond to points in $\mathbb{P}^{2m-2}$ which span the hyperplane

$$(-1)^{m-1}\binom{2m-2}{m-1}a^{m-1}z_0+\dots+\binom{m+1}{2}a^2z_{m-3}-\binom{m}{1}az_{m-2}+\binom{m-1}{0}z_{m-1}=0.$$

Note that if we start from the m-square (x_1^m,x_2^m), i.e. for $a=0$, we get the hyperplane $z_{m-1}=0$. Let $\pi_{x_1,i}$, $\pi_{x_2,i}$ and π_{x_1,x_2} be the $2m-1$ hyperplanes corresponding to the m-squares (x_1^m,ℓ_i^m), (x_2^m,ℓ_i^m) and (x_1^m,x_2^m), respectively.

Now, in order to prove that the ideal I has no form of degree $2m-2$ it is enough to prove that the intersection of the hyperplanes $\pi_{y,i}$, $\pi_{x,y}$ and $\pi_{x,i}$ is empty. We get a homogeneous linear system, whose $(2m-1)\times(2m-1)$ matrix is the following:

$$\begin{pmatrix}
| & \dots & \dots & \dots & | & 0 & 0 & 0 & 0\\
| & \dots & A & \dots & | & 0 & 0 & 0 & 0\\
| & \dots & \dots & \dots & | & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & | & \dots & \dots & \dots & |\\
0 & 0 & 0 & 0 & | & \dots & A' & \dots & |\\
0 & 0 & 0 & 0 & | & \dots & \dots & \dots & |
\end{pmatrix}$$

The $(m-1)\times(m)$ block A (in the first $m-1$ rows) is:

$$A=\begin{pmatrix}
(-1)^{m+1}\binom{2m-2}{m-1}a_1^{m-1} & (-1)^m\binom{2m-3}{m-2}a_1^{m-2} & \dots & \binom{m+1}{2}a_1^2 & -\binom{m}{1}a_1 & 1\\
(-1)^{m+1}\binom{2m-2}{m-1}a_2^{m-1} & (-1)^m\binom{2m-3}{m-2}a_2^{m-2} & \dots & \binom{m+1}{2}a_2^2 & -\binom{m}{1}a_2 & 1\\
(-1)^{m+1}\binom{2m-2}{m-1}a_3^{m-1} & (-1)^m\binom{2m-3}{m-2}a_3^{m-2} & \dots & \binom{m+1}{2}a_3^2 & -\binom{m}{1}a_3 & 1\\
\vdots & & \vdots & & \vdots & \\
(-1)^{m+1}\binom{2m-2}{m-1}a_{m-1}^{m-1} & (-1)^m\binom{2m-3}{m-2}a_{m-1}^{m-2} & \dots & \binom{m+1}{2}a_{m-1}^2 & -\binom{m}{1}a_{m-1} & 1
\end{pmatrix}$$

The block A' is the same as A, but with the columns and the powers of a in reverse order. These two blocks have maximal rank $m-1$, since they can be viewed as a Cauchy-Vandermonde matrix where each column is multiplied by a constant. Since the two blocks lies in the first and in the last $m-1$ columns of the matrix and the middle row of the matrix is $(0,...,0,1,0,...,0)$, the rank of the matrix is $2m-1$ and we are done. ■

Remark 3.2.7. Let us note that the fat point $2mP$ can never be obtained as union of squares. Let us show that in the cases $m=1$ and $m=2$:

- $m = 1$
  Since any 2-square supported at P properly contains the fat point $2P$, then $2P$ should be obtained as a union of 1-squares, but this is impossible, P being the only 1-square supported at P.

- $m = 2$
  Since any 4-square supported at P properly contains the fat point $4P$, then $4P$ should be obtained as a union of 1-squares, 2-squares and 3-squares supported at P. Since the unique 1-square is P and it is contained in any 2-square and in any 3-square supported at P, we can suppose that $4P$ is a union of 2-squares and 3-squares supported at P. Let
  $$A_2 = \{Q \mid Q \text{ is a 2-square supported at } P\}$$
  $$A_3 = \{Q \mid Q \text{ is a 3-square supported at } P\}$$
  and suppose, by contradiction, that there exist $A_2' \subseteq A_2$ and $A_3' \subseteq A_3$ such that
  $$4P = \bigcup_{Q \in A_2' \cup A_3'} Q.$$
  If r is a line through P, we have
  $$4P \setminus r^2 = \bigcup_{Q \in A_2' \cup A_3'} (Q \setminus r^2),$$
  where r^2 is the double line supported at r and we are considering the schematic difference. If $Q \in A_2$, then either $Q \setminus r^2 = \emptyset$ or $Q \setminus r^2 = P$. If $Q \in A_3$, then either $Q \setminus r^2$ is a 2-square or $Q \setminus r^2$ is a 3-jet. Thus, $4P \setminus r^2 = 2P$ should be a union of 2-squares and 3-jets, and this is a contradiction.

The proof for higher multiplicities is analogous to the case $m = 2$.

Remark 3.2.8. Let us note that something quite different can happen if we do not consider all the pairs of lines as we did in Theorem 3.2.6. For example, consider $P = [1,0,0] \in \mathbb{P}^2$ and the union of the 2-squares supported at P that are defined via two lines which are "perpendicular" with respect to the apolar action of $\mathbb{C}[w_1, w_2]$ on $\mathbb{C}[x_1, x_2]$; in this case we do not get the entire fat point $3P$. In fact, if we set $A_P = \{(\ell_1, \ell_2) \in (\mathcal{I}_P)_1 \times (\mathcal{I}_P)_1 | \ \ell_1 \perp \ell_2$ and (ℓ_1^2, ℓ_2^2) is a 2-square $\}$ then

$$\bigcap_{(\ell_1,\ell_2)\in A_P} (\ell_1^2, \ell_2^2) = (x_1^2 + x_2^2, x_1^3, x_1^2 x_2).$$

It is quite immediate that each ideal (ℓ_1^2, ℓ_2^2) contains the ideal $(\ell_1, \ell_2)^3 = (x_1, x_2)^3$; if moreover $\ell_1 \perp \ell_2$, we can write $\ell_1 = ax_1 - bx_2$, $\ell_2 = bx_1 + ax_2$ and thus (ℓ_1^2, ℓ_2^2) contains both $a^2x_1^2 + b^2x_2^2 - 2abx_1x_2$, $b^2x_1^2 + a^2x_2^2 + 2abx_1x_2$ and we have that $(a^2 + b^2)(x_1^2 + x_2^2) \in (\ell_1^2, \ell_2^2)$, i.e. $(x_1^2 + x_2^2)$ is contained in any ideal (ℓ_1^2, ℓ_2^2) with $(\ell_1, \ell_2) \in A_P$, and the thesis follows.

Note that it is actually enough to intersect two of those ideals to obtain the total intersection ideal. Note also that $(a^2 + b^2) \neq 0$, because the pairs (a, b) for which it is zero correspond to the only two particular lines through P, namely $\{x_1 \pm ix_2 = 0\}$ which we have to exclude among the pairs of lines in A_P , because they are isotropic, i.e. "perpendicular to themselves" and thus the ideal

$$((x_1 \pm ix_2)^2, (ix_1 \mp x_2)^2) = ((x_1 \pm ix_2)^2)$$

is not the ideal of a $2-$square point but of a double line.

It is also interesting to observe that the scheme

$$Z = \bigcup_{(\ell_1,\ell_2)\in A_P} Q_{\ell_1\ell_2}$$

where $\mathcal{I}_{Q_{\ell_1\ell_2}} = (\ell_1^2, \ell_2^2)$, is not 2-symmetric, even if it is an (infinite) union of 2-symmetric schemes. Indeed, $Z \cap L$ has length 2 for all lines L, except for the two lines $x_1 \pm ix_2 = 0$ which meet it with length 3. On the other hand, the fat point $3P$ is 3- and not 2- symmetric, although it is a union of 2-symmetric schemes by Theorem 3.2.6.

Finally, observe that the scheme Z considered above *is* 2-symmetric if we consider it over the reals. Hence Theorem 3.1.9 does not hold over $\mathbb{R}$, in fact $(x_1^2 + x_2^2, x_1^2x_2, x_1^3)$ defines a 2-symmetric scheme in $\mathbb{P}^2_{\mathbb{R}}$ of length 5.

For the moment we stop our treatise on 2-squares and we pass to consider their embedding on Veronese and Segre-Veronese varieties, but we will come back to them in Chapter 4, where we will deal with the interpolation problem for 2-squares.

3.3 2-squares on Veronese surfaces

Now we want to begin to see how the m-squares can give, with their immersions on Veronese surfaces $V_{2,d}$ (see Chapter 1, §1.4), parameterisations of structured symmetric tensors. We will start by considering only 2-squares, which at the moment are the ones we know the best. Recall Notation 1.3.1.

Remark 3.3.1. If we have a subscheme $\mathbb{X} \subseteq \mathbb{P}^2$, then, by Proposition 1.4.6, we know that $L(\nu_{n,d}(\mathbb{X})) \subseteq \mathbb{P}^{N_{n,d}}$ is naturally isomorphic to $\mathbb{P}((\mathcal{I}_{\mathbb{X}})_d^{\perp}) \subseteq \mathbb{P}(U_d)$. In the following we will identify $\mathbb{P}((\mathcal{I}_{\mathbb{X}})_d^{\perp})$ to $L(\nu_{n,d}(\mathbb{X}))$. For instance, consider a 2-square $Q \subseteq \mathbb{P}^2$. Up to a linear change of coordinates we can suppose that $\mathcal{I}_Q = (x_0^2, x_1^2)$ whose perp in degree d is

$$(\mathcal{I}_Q)_d^{\perp} =< w_0 w_1 w_2^{d-2}, w_0 w_2^{d-1}, w_1 w_2^{d-2}, w_2^d > \subseteq \mathbb{P}(U_d).$$

With our identification we have

$$\mathbb{P}((\mathcal{I}_Q)_d^{\perp}) = L(\nu_{2,d}(Q)) = L([y_0 y_1 y_2^{d-2}], [y_0 y_2^{d-1}], [y_1 y_2^{d-2}], [y_2^d]) \subseteq \mathbb{P}(T_d) = \mathbb{P}^{N_{2,d}}.$$

In other words we are doing nothing but considering the apolarity action of T on R.

Now we want to consider the variety spanned by all the possible schemes $\nu_{2,d}(Q_P)$, on the surface $V_{2,d}$.

Proposition 3.3.2. *Let*

$$Q^0(V_{2,d}) := \bigcup_{Q \subseteq \mathbb{P}^2} L(\nu_{2,d}(Q)), \qquad Q(V_{2,d}) = \overline{Q^0(V_{2,d})}$$

where the union is made on all the 2-squares Q of $\mathbb{P}^2$. Then we have

$$Q(V_{2,d}) = \tau_2(V_{2,d}).$$

Moreover, if a point $[F]$ of $\mathbb{P}^{N_{2,d}}$ lies on $\tau_2(V_{2,d})$, the form $F \in T_d$ can be written, modulo a change of variables in $\mathbb{P}^2$, either as

$$F = y_2^{d-2}(a_0 y_0 y_2 + a_1 y_1 y_2 + a_2 y_2^2 + a_3 y_0 y_1), \text{ if } [F] \in Q^0(V_{2,d})$$

or as

$$F = y_2^{d-2}(a_0 y_0^2 + a_1 y_1 y_2), \text{ if } [F] \in Q(V_{2,d}) \setminus Q^0(V_{2,d}).$$

Proof By Theorem 3.2.6 we know that the union of all 2-squares supported at the same point P in $\mathbb{P}^2$ is the fat point $3P$ so we have

$$Q(V_{2,d}) = \overline{\bigcup_{Q \subseteq \mathbb{P}^2} L(\nu_{2,d}(Q))} = \overline{\bigcup_{P \in \mathbb{P}^2} \bigcup_{Q_P} L(\nu_{2,d}(Q_P))} = \overline{\bigcup_{P \in \mathbb{P}^2} L(\bigcup_{Q_P} \nu_{2,d}(Q_P))} =$$

$$= \overline{\bigcup_{P \in \mathbb{P}^2} L(\nu_{2,d}(\bigcup_{Q_P} Q_P)} = \overline{\bigcup_{P \in \mathbb{P}^2} L(\nu_{2,d}(3P))} = \tau_2(V_{2,d})$$

where Q_P varies on the set of 2-squares supported at P.

By Remark 1.4.21 we know that if $[F] \in \tau_2(V_{2,d})$ then F can be written as $F = y_2^{d-2} G$, where G is a conic. We write

$$G = a_0 y_0 y_2 + a_1 y_1 y_2 + a_2 y_2^2 + H(y_0, y_1)$$

with $H \in \mathbb{C}[y_0, y_1]_2$ and we distinguish two cases:

- $H(y_0, y_1)$ is not a square.
  In this case there exist $\ell_0, \ell_1 \in \mathbb{C}[y_0, y_1]_1$ such that $H = \ell_0\ell_1$ and $\ell_0 \nmid \ell_1$. As a consequence, there exist $b_0, b_1 \in \mathbb{C}$ such that
  $$a_0y_0 + a_1y_1 = b_0\ell_0 + b_1\ell_1$$
  and thus we get
  $$G = y_2(a_0y_0 + a_1y_1 + a_2y_2) + H(y_0, y_1) = y_2(b_0\ell_0 + b_1\ell_1 + a_2y_2) + \ell_0\ell_1$$
  and
  $$F = y_2^{d-2}(b_0\ell_0y_2 + b_1\ell_1y_2 + a_2y_2^2 + \ell_0\ell_1).$$
  Note that in this case $[F] \in L(\nu_{2,d}(Q))$, with $\mathcal{I}_Q = (\ell_0^2, \ell_1^2)$, thus $[F] \in Q^0(V_{2,d})$.

- $H(y_0, y_1)$ is a square
  In this case we have $H(y_0, y_1) = \ell_0^2$ for some $\ell_0 \in \mathbb{C}[y_0, y_1]_1$ and we get
  $$F = y_2^{d-2}(y_2(a_0y_0 + a_1y_1 + a_2y_2) + \ell_0^2)$$
  that, modulo a linear change of coordinate of $\mathbb{P}^2$, we can write as
  $$F = y_2^{d-2}(b_0y_0^2 + b_1y_1y_2).$$
  Note that in this case $[F] \notin L(\nu_{2,d}(Q))$, for any 2-squares $Q \subseteq \mathbb{P}^2$. Nevertheless, we know that $[F] \in \tau_2(V_{2,d}) = Q(V_{2,d})$ and thus $[F] \in Q(V_{2,d}) \setminus Q^0(V_{2,d})$. To see how in this case F is the limit of forms that lie in $Q^0(V_{2,d})$ consider the linear form $\ell_\epsilon = y_0 + \epsilon y_1$, and the 2-square Q_ε with $\mathcal{I}_{Q_\varepsilon} = (\ell_\epsilon^2, y_0^2)$. Then we have
  $$L(\nu_{2,d}(Q_\varepsilon)) = \mathbb{P}((\mathcal{I}_{Q_\varepsilon})_d^\perp) = L([y_0y_2^{d-1}], [y_0y_2^{d-2}\ell_\varepsilon], [y_2^{d-1}\ell_\varepsilon], [y_2^d])$$
  so
  $$\lim_{\epsilon\to 0} L(\nu_{2,d}(Q_\varepsilon)) = L([y_0^2y_2^{d-2}], [y_0y_2^{d-1}], [y_2^d])$$
  and any $[F]$ in there is such that F can be written, modulo projectivities of the plane, as
  $$y_2^{d-2}(a_0y_0^2 + a_1y_1y_2).$$

Since these are all the possible cases the proof is concluded. ■

Note that Proposition 3.3.2 gives, in some sense, a more refined way to distinguish the form lying on $\tau_2(V_{2,d})$. Moreover, it has a noteworthy consequence which, for $d = 4$, is Lemma 4.1 in [20].

Corollary 3.3.3. *The second osculating variety $\tau_2(V_{2,d})$ of a Veronese surface $V_{2,d} \subseteq \mathbb{P}^{N_{2,d}}$ is contained in the secant variety $\sigma_4(V_{2,d})$. Thus, for every $[F] \in \tau_2(V_{2,d})$, we have $\overline{\mathrm{srk}}(F) \leq 4$.*

Proof We have to prove that if $[F] \in \tau_2(V_{2,d})$ then $[F] \in \sigma_4(V_{2,d})$. By Proposition 3.3.2 we know that

$$\tau_2(V_{2,d}) = Q^0(V_{2,d}) \cup (Q(V_{2,d}) \setminus Q^0(V_{2,d})).$$

Thus, if $[F] \in \tau_2(V_{2,d})$ then either $[F] \in Q^0(V_{2,d})$ or $[F] \in Q(V_{2,d}) \setminus Q^0(V_{2,d})$. We distinguish two cases:

- $[F] \in Q^0(V_{2,d})$
  By Proposition 3.3.2 there exist a 2-square $Q \subseteq \mathbb{P}^2$ such that $[F] \in L(\nu_{2,d}(Q))$. Hence, since $\ell(Q) = 4$ and Q is smoothable by Proposition 3.1.13, we have that $[F]$ is a $\mathbb{P}^3$ which is the limit of a family of $\mathbb{P}^3$'s which are 4-secant to $V_{2,d}$ and thus $[F] \in \sigma_4(V_{2,d})$.

- $[F] \in Q(V_{2,d}) \setminus Q^0(V_{2,d})$
  By Proposition 3.3.2 we know that, up to a projectivity of $\mathbb{P}^2$, F can be written as
  $$F = y_2^{d-2}(a_0 y_0^2 + a_1 y_1 y_2).$$
  If $a_0 = 0$, then $F = a_1 y_1 y_2^{d-1}$ and thus
  $$[F] \in \tau_1(V_{2,d}) \subseteq \sigma_2(V_{2,d}) \subseteq \sigma_4(V_{2,d})$$
  and we are done. If $a_1 = 0$, then $F = a_0 y_0^2 y_2^{d-2}$ and thus
  $$[F] \in \tau_2(\mathcal{C}_d) \subseteq \sigma_3(V_{2,d}) \subseteq \sigma_4(V_{2,d})$$
  for some rational normal curve $\mathcal{C}_d \subseteq V_{2,d}$ and we are done. So we can suppose $a_0, a_1 \neq 0$. Consider the scheme $Z \subseteq \mathbb{P}^2$, with $\mathcal{I}_Z = (x_0^3, x_0 x_1, x_1^2)$. We have $\ell(Z) = 4$ and, since all the 0-dimensional schemes of $\mathbb{P}^2$ are smoothable, $L(\nu_{2,d}(Z)) \subseteq \sigma_4(V_{2,d})$. More precisely, we have
  $$L(\nu_{2,d}(Z)) = \mathbb{P}((x_1^2, x_0 x_1, x_0^3)_d^{\perp}) = L([y_0^2 y_2^{d-2}], [y_0 y_2^{d-1}], [y_1 y_2^{d-1}], [y_2^d])$$
  So that $[F] \in L(\nu_{2,d}(Z))$ and thus $[F] \in \sigma_4(V_{2,d})$.

This concludes the proof. ∎

Corollary 3.3.4. *Every form in $\mathbb{C}[y_0, y_1, y_2]_d$ can be written as a sum of $s = \lceil \frac{d^2+3d+2}{16} \rceil$, polynomials of the form described in Proposition 3.3.2 with the only exception of the case $d = 4$, for which such s is 3 and not 2.*

Proof By Proposition 1.4.20 we know that the osculating varieties of Veronese varieties always have the expected dimension, in particular we get $\dim(\tau_2(V_{2,d})) = 7$. Moreover, by [14], it is known that $\sigma_s(\tau_2(V_{2,d}))$ has always the expected dimension, except for the case of $\sigma_2(\tau_2(V_{2,4})) \subseteq \mathbb{P}^{14}$, which should fill up its ambient space, but it is a hypersurface. This implies that, apart from the case $d = 4$, for which we need $\sigma_3(\tau_2(V_{2,4}))$ to fill up $\mathbb{P}^{14}$, in all other cases $\dim(\sigma_s(\tau_2(V_{2,d}))) = 7s + s - 1 = 8s - 1$, hence we get that the first s for which $\sigma_s(\tau_2(V_{2,d}))$ is the whole ambient space is for $8s - 1 \geq \binom{d+2}{2} - 1$, i.e. $s = \lceil \frac{d^2+3d+2}{16} \rceil$. ■

Remark 3.3.5. We can use the language of catalecticant matrix (see Chapter 1, §1.4.3) to restate Proposition 3.3.2. A general $F \in \mathbb{C}[y_0, y_1, y_2]_d$, $d \geq 3$, has a $(2, d-2; 3)$-catalecticant matrix of the form

$$\mathrm{Cat}(2, d-2; 3) = \begin{pmatrix} z_{d,0,0} & z_{d-1,1,0} & \cdots & z_{2,0,d-2} \\ z_{d-1,1,0} & z_{d-2,2,0} & \cdots & z_{1,1,d-2} \\ z_{d-1,0,1} & z_{d-2,1,1} & \cdots & z_{1,0,d-1} \\ z_{d-2,2,0} & z_{d-3,3,0} & \cdots & z_{0,2,d-2} \\ z_{d-2,1,1} & z_{d-3,2,1} & \cdots & z_{0,1,d-1} \\ z_{d-2,0,2} & z_{d-3,1,2} & \cdots & z_{0,0,d} \end{pmatrix}.$$

By Proposition 3.3.2 we know that the generic $[F] \in \tau_2(V_{2,d})$ can be written, modulo projectivities of $\mathbb{P}^2$, as

$$F = y_2^{d-2}(a_0 y_0 y_2 + a_1 y_1 y_2 + a_2 y_2^2 + a_3 y_0 y_1)$$

and thus its catalecticant matrix $\mathrm{Cat}_F(2, d-2; 3)$ can be written in such a way that only five columns have some non-zero entries (for the case $d = 4$, see also [20], Theorem 4.4 (2)). By the way, there is a mistake in [20], where it is stated that such polynomials can be written as $y_0^2 y_1 y_2$ via a Gauss elimination on $\mathrm{Cat}_F(2, d-2; 3)$: this is false since that Gauss elimination does not correspond to a projectivity in $\mathbb{P}^2$. We will analyse which polynomials are of that monomial type in Proposition 3.3.7.

3.3.1 The cuckoo varieties $QQ(V_{2,d})$

The variety $\tau_2(V_{2,d})$ contains a 1-codimensional subvariety parameterising more particular forms, namely the ones that can be written (modulo a projectivity in $\mathbb{P}^2$) as $y_0^{d-2} y_1 y_2$. In this section we want to investigate such a subvariety.

Definition 3.3.6. Let $d \geq 3$ and consider the morphism

$$\begin{array}{cccc} \Phi: & \mathbb{P}(T_1) \times \mathbb{P}(T_1) \times \mathbb{P}(T_1) & \to & \tau_2(V_{2,d}) \subseteq \mathbb{P}(T_d) \\ & ([\ell_0], [\ell_1], [\ell_2]) & \mapsto & [\ell_0^{d-2} \ell_1 \ell_2] \end{array}.$$

The *cuckoo variety* $QQ(V_{2,d})$ of $V_{2,d}$ is defined to be the scheme theoretic image of Φ, that is

$$QQ(V_{2,d}) := \operatorname{Im} \Phi.$$

Clearly the map Φ can also be thought as a map from $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ to $\tau_2(V_{2,d}) \subseteq \mathbb{P}^{N_{n,d}}$ through the identifications we defined in Notation 1.3.1 and Remark 1.4.2. Note that for $d = 2$ one would have $QQ(V_{2,2}) = \mathbb{P}^{N_{2,2}} = \mathbb{P}^5$ while for $d \geq 3$ one has $\dim QQ(V_{2,d}) = 6$.

Proposition 3.3.7. *Let $d \geq 3$. The following hold:*

1. *if $[F] \in QQ(V_{2,d})$, then $\operatorname{srk}(F) \in \{1, d-1, d, 2d-2\}$ for $d \neq 3$ and $\operatorname{srk}(F) \in \{1,3,4\}$ for $d = 3$. In both cases, the generic point $[F]$ in $QQ(V_{2,d})$ is such that $\operatorname{srk}(F) = 2d-2$;*

2. *$\forall\, P \in \mathbb{P}^2$ we have that*
$$QQ(V_{2,d}) \cap \tau_{2,\nu_{2,d}(P)}(V_{2,d}) \cong \sigma_2(V_{2,2}) = \tau_1(V_{2,2});$$

3. *for any 2-square $Q_P \in \mathbb{P}^2$ supported at $P \in \mathbb{P}^2$ one has*
$$QQ(V_{2,d}) \cap L(\nu_{2,d}(Q_P)) \cong \tau_{1,\nu_{2,d}(P)}(V_{2,d}) \cup \mathcal{Q}_{Q_P}$$
where $\mathcal{Q}_{Q_P} \subseteq L(\nu_{2,d}(Q_P)) \cong \mathbb{P}^3$ is a smooth quadric and we have
$$\tau_{1,\nu_{2,d}(P)}(V_{2,d}) = \tau_{1,\nu_{2,d}(P)}(\mathcal{Q}_{Q_P})$$
i.e. $V_{2,d}$ and $\mathcal{Q}_{Q_P}$ have the same tangent plane at $\nu_{2,d}(P)$.

Proof Let us consider the degenerate cases first: let $[F] \in \mathbb{P}^{N_{2,d}}$ be the point such that $F = \ell_0^{d-2}\ell_1\ell_2$; if the form is of type ℓ_0^d, then $\operatorname{srk}(F) = 1$ and $[F] \in V_{2,d}$; if the form is of type $\ell_0^{d-1}\ell_1$, then $[F] \in \tau_1(V_{2,d})$, and more precisely, since it can be written in two variables, there is a rational normal curve $\mathcal{C}_d \subseteq V_{2,d}$ such that $[F] \in \tau_1(\mathcal{C}_d)$. Eventually, if the form is of type $\ell_0^{d-2}\ell_1^2$, then $[F] \in \tau_2(\mathcal{C}_d)$ for some rational normal curve $\mathcal{C}_d \subseteq V_{2,d}$. It is known (e.g. see [20], Remark 24 or [36], Proposition 3.1) that
$$\operatorname{srk}(\ell_0^{d-1}\ell_1) = \max\{2, d\} = d$$
and
$$\operatorname{srk}(\ell_0^{d-2}\ell_1^2) = \max\{3, d-1\} = d-1$$
unless for $d = 3$, when it is $\operatorname{srk}(\ell_0\ell_1\ell_2) = 3$. Of course, all these $[F]$'s of the degenerate kind constitute a closed subset D of $QQ(V_{2,d})$. When the form is of type $\ell_0^{d-2}\ell_1\ell_2$, with $\ell_i \nmid \ell_j$ $i \neq j \in \{0,1,2\}$, then $\operatorname{srk}(F) = 2d-2$ (e.g. see [36]). Thus, the part regarding the symmetric rank is proved.

Now we fix $P \in \mathbb{P}^2$ corresponding to the linear form ℓ_0 and we consider the osculating space $\tau_{2,\nu_{2,d}(P)}(V_{2,d}) \cong \mathbb{P}^5$. We know that the points of this osculating space are of the form $[\ell_0^{d-2}G]$ for some $G \in T_2$ and have that the points of type $\ell_0^{d-2}\ell_1^2$ are the image under Φ of the points $([\ell_0], [\ell_1], [\ell_1])$, i.e. of $\{[\ell_0]\} \times \Delta$, where $\Delta \cong \mathbb{P}^2$

is the diagonal of $(\mathbb{P}^2) \times (\mathbb{P}^2)$, so that they form a subvariety V of $\tau_{2,\nu_{2,d}(P)}(V_{2,d})$ isomorphic to $V_{2,2}$, the Veronese surface in $\mathbb{P}^5$. Now, we have two ways to check that

$$QQ(V_{2,d}) \cap \tau_{2,\nu_{2,d}(P)}(V_{2,d}) = \sigma_2(V).$$

First, if we consider two distinct points $[\ell_0^{d-2}m_1^2]$ and $[\ell_0^{d-2}m_2^2]$ on V but not on $V_{2,d}$, we have that the line joining them parameterises all the forms that can be written as

$$\ell_0^{d-2}(\alpha^2 m_1^2 - \beta^2 m_2^2) = \ell_0^{d-2}(\alpha m_1 - \beta m_2)(\alpha m_1 + \beta m_2).$$

Since any two lines in a pencil can be projectively transformed in other two lines of the pencil, any form $\ell_0^{d-2}\ell_1'\ell_2'$, with $\ell_1' \nmid \ell_2'$ can be projectively transformed into one of the form $\ell_0^{d-2}(\alpha m_1 - \beta m_2)(\alpha m_1 + \beta m_2)$ and thus we get that

$$QQ(V_{2,d}) \cap \tau_{2,\nu_{2,d}(P)}(V_{2,d}) = \sigma_2(V).$$

Otherwise, and more simply, it suffices to consider that $\sigma_2(V_{2,2}) = \tau_1(V_{2,2})$, hence $\tau_1(V)$ parameterises the forms of type $\ell_0^{d-2}\ell_1\ell_2$. So, *ii)* is proved.

Now consider a 2−square $Q \in \mathbb{P}^2$, with $\mathcal{I}_Q = (\ell_1^2, \ell_2^2)$. We have seen in Proposition 3.3.2 that the forms in $L(\nu_{2,d}(Q)\rangle$ can be written as

$$\ell_0^{d-2}(a_0\ell_0^2 + a_1\ell_0\ell_1 + a_2\ell_0\ell_2 + a_3\ell_1\ell_2)$$

so that if

$$[F] \in QQ(V_{2,d}) \cap L(\nu_{2,d}(Q_P)) \cong \tau_{1,\nu_{2,d}(P)}(V_{2,d}) \cup \mathcal{Q}_{Q_P}$$

then there exist $\ell_0, \ell_1', \ell_2' \in T_1$ such that

$$F = \ell_0^{d-2}(a_0\ell_0^2 + a_1\ell_0\ell_1 + a_2\ell_0\ell_2 + a_3\ell_1\ell_2) = \ell_0^{d-2}\ell_1'\ell_2'.$$

We write

$$\ell_1' = \alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2, \quad \ell_2' = \beta_0 y_0 + \beta_1 y_1 + \beta_2 y_2$$

and, up to projectivity of the plane we can suppose $\ell_i = x_i$ for $i = 1,2,3$. At this point, by imposing the equality, we get the system

$$\begin{cases} \alpha_0\beta_0 = a_0 \\ \alpha_0\beta_1 + \alpha_1\beta_0 = a_1 \\ \alpha_0\beta_2 + \alpha_2\beta_0 = a_2 \\ \alpha_1\beta_2 + \alpha_2\beta_1 = a_3 \\ \alpha_1\beta_1 = 0 \\ \alpha_2\beta_2 = 0 \end{cases}$$

which has solution if and only if $a_3 = 0$ or $a_0a_3 - a_1a_2 = 0$. The forms with $a_3 = 0$ are exactly those in $\tau_{1,\nu_{2,d}(P)}(V_{2,d})$ while the equations $a_0a_3 - a_1a_2 = 0$ defines the smooth quadric $\mathcal{Q}_Q$ that we were looking for. Note that if $[F] \in \mathcal{Q}_Q$ and $a_3 = 0$ then

$$F = \ell_0^{d-1}(a_0\ell_0 + a_1\ell_1 + a_2\ell_2) \in \tau_{1,\nu_{2,d}(P)}(V_{2,d})$$

and either $a_1 = 0$ or $a_2 = 0$. Hence $\tau_{1,\nu_{2,d}(P)}(V_{2,d}) \cap \mathcal{Q}_{Q_P}$ is given by two lines, and so the tangent plane to $V_{2,d}$ is also tangent to $\mathcal{Q}_{Q_P}$. ■

Note that if we knew the equations defining $QQ(V_{2,d})$, we would be able to check if a given form $F \in \mathbb{C}[y_0, y_1, y_2]_d$ can be written as a monomial $y_0^{d-2}y_1y_2$ modulo a linear change of coordinates. Hence it would be interesting to solve the following problem.

Problem 3.3.8. *Find equations defining (even just set-theoretically) the variety* $QQ(V_{2,d})$.

Remark 3.3.9. The above problem could be interesting for applications, since we have also that a symmetric tensor in $\mathbb{P}^{N_{2,d}}$ describes what in quantum information theory is called a $d-$ *qutrits* symmetric state, which is not entangled if it is on $V_{2,d}$. The generic elements in $QQ(V_{2,d})$ would represent entangled states; see [16] Lemma 2.1 and [28] for more details on the relationship between symmetric tensors and quantum information.

3.4 2-squares on Segre-Veronese surfaces

In this section we consider 2-squares in $\mathbb{P}^1 \times \mathbb{P}^1$ and their embedding on Segre-Veronese surfaces $SV_{(1,1;d,d)}$ (see Definition 1.5.10). Since we will just deal with this kind of Segre-Veronese varieties, in this section we write $SV_{d,d}$ instead of $SV_{(1,1;d,d)}$ and analogously $sv_{d,d}$ instead of $sv_{(1,1;d,d)}$.

Remark 3.4.1. As we said, we want to consider in $\mathbb{P}^1 \times \mathbb{P}^1$ a kind of schemes similar to 2-squares but, up to this moment, we know what a 2-square is just if we are in $\mathbb{P}^2$, so we have to make things clearer. What we mean by 2-square in $\mathbb{P}^1 \times \mathbb{P}^1$ is the following: consider the affine chart $U_{1,1} = \{x_{1,1} \neq 0, x_{2,1} \neq 0\}$ of $\mathbb{P}^1 \times \mathbb{P}^1$, with coordinates (x_1', x_2'), where

$$x_1' = \frac{x_{1,0}}{x_{1,1}}, \quad x_2' = \frac{x_{2,0}}{x_{2,1}}.$$

For any point $P = [a_0, a_1; b_0, b_1] \in U_{1,1}$ we can consider the 0-dimensional subscheme $Q_P \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ supported at P and defined by the bihomogeneous ideal $(\ell_{1,0}^2, \ell_{0,1}^2) \subseteq \mathcal{R}$, where

$$\ell_{1,0} = a_1x_{1,0} - a_0x_{1,1}, \quad \ell_{0,1} = b_1x_{2,0} - b_0x_{2,1}.$$

Since $bd \neq 0$ we can look at Q_P in the affine chart $U_{1,1}$ and here we have

$$\mathcal{I}_{Q_P} = \left(\left(x_1' - \frac{a_0}{a_1} \right)^2, \left(x_2' - \frac{b_0}{b_1} \right)^2 \right).$$

Hence, Q_P is a 2-square in the alternative compactification of $U_{1,1}$ as a chart in $\mathbb{P}^2$ with homogeneous coordinates $[x_0, x_1, x_2]$, when we view

$$(x_1', x_2') = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \right).$$

This will be the kind of 2-squares we are going to consider in multi-projective environment. Note that there are other structures which are 2-squares when considered in an affine chart, but the bidegree of their generators is higher and we are not going to consider them.

3.4.1 The variety $q_2(SV_{2,2})$

Now we consider the Segre-Veronese embedding $sv_{2,2} : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^8$. To visualise what kind of structured tensors are parameterised by this variety, let us make a brief detour about 2^4-tensors.

General 2^4-tensors

If we consider general tensors of this format, those of tensor rank 1 are parameterised by the Segre variety $S_{1,1,1,1}$ given by the embedding (see Definition 1.5.3)

$$s_{1,1,1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{15}.$$

Recall Notation 1.5.4: in this $\mathbb{P}^{15}$ we have homogeneous coordinates $[u_{i_1,i_2,i_3,i_4}]$ with $i_1, i_2, i_3, i_4 \in \{0,1\}$. These general tensors in $\mathbb{P}^{15}$ can be viewed as in Fig 3.1, and the equations of $S_{1,1,1,1}$ are given by all the 2×2-minors of all flattenings of the tensor in question.

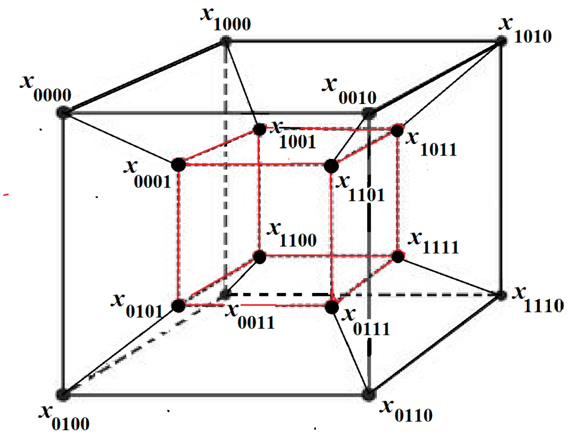


Figure 3.1: The 2^4-tensors.

Partially Symmetric 2^4-tensors

The Segre-Veronese variety $SV_{2,2} \subseteq \mathbb{P}^8$ can be viewed as given first by the Veronese embedding $\nu_{1,2}$ of both the $\mathbb{P}^1$-factors into $\mathbb{P}^2$, followed by the Segre embedding $s_{1,1} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$. Moreover, this space $\mathbb{P}^8$ is actually a subspace of the space $\mathbb{P}^{15}$ of general 2^4-tensors, and it parameterises 2^4-tensors which are symmetric on the first two indices and on the second two, i.e. the (1,1; 2,2)-partially symmetric tensors (see Definition 1.5.6). More precisely, this $\mathbb{P}^8$ is the subspace of $\mathbb{P}^{15}$ defined by the equations

$$u_{i_1,i_2,i_3,i_4} = u_{\sigma(i_1,i_2),\tau(i_3,i_4)} \text{ for all } \sigma, \tau \in \mathfrak{S}_2$$

where $\mathfrak{S}_2$ is the symmetric group on two elements. In this $\mathbb{P}^8$ the variety $SV_{2,2}$ parameterise exactly the (1,1; 2,2)-partially symmetric tensors having partial symmetric rank 1. As we said in Notation 1.5.8, in this $\mathbb{P}^8$ we use homogeneous coordinates

$$\begin{array}{ccc} v_{(0,0),(0,0)}, & v_{(0,0),(0,1)}, & v_{(0,0),(1,1)}, \\ v_{(0,1),(0,0)}, & v_{(0,1),(0,1)}, & v_{(0,1),(1,1)}, \\ v_{(1,1),(0,0)}, & v_{(1,1),(0,1)}, & v_{(1,1),(1,1)}. \end{array}$$

We can view the partially symmetric 2^4-tensors in figure 3.2. Note that the four 2×2 faces moving from left to right are symmetric and so are the four 2×2 faces joining the "big cube" to the "small one" in the direction "perpendicular to the paper"

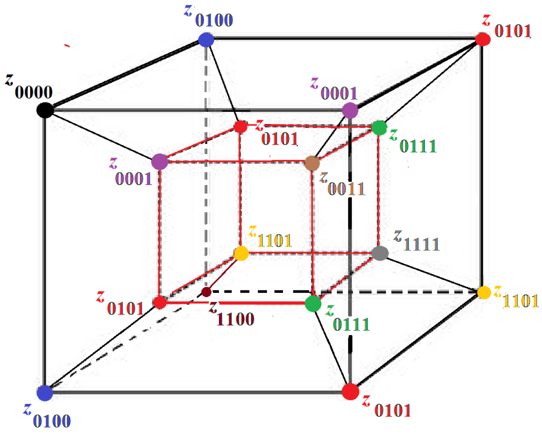


Figure 3.2: The Partially symmetric 2^4-tensors.

We have that $SV_{2,2}$ is a Del Pezzo surface, of degree 8 in $\mathbb{P}^8$ (e.g. see [44]), its ideal is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} v_{(0,0),(0,0)} & v_{(0,0),(0,1)} & v_{(0,1),(0,0)} & v_{(0,1),(0,1)} \\ v_{(0,0),(0,1)} & v_{(0,0),(1,1)} & v_{(0,1),(0,1)} & v_{(0,1),(1,1)} \\ v_{(0,1),(0,0)} & v_{(0,1),(0,1)} & v_{(1,1),(0,0)} & v_{(1,1),(0,1)} \\ v_{(0,1),(0,1)} & v_{(0,1),(1,1)} & v_{(1,1),(0,1)} & v_{(1,1),(1,1)} \end{pmatrix}.$$

Moreover, the 3×3-minors of the matrix above generate the ideal of the secant variety $\sigma_2(SV_{2,2})$, which has the expected dimension 5, while its determinant defines $\sigma_3(SV_{2,2})$, which is defective, because its expected dimension was 8 (see again [44]).

Symmetric 2^4-tensors

Finally, if we want to consider symmetric 2^4-tensors, those are given by the subspace, in the space $\mathbb{P}^{15}$ parameterising all tensors, which is made of all points defined by the equations

$$x_{ijkl} = x_{\sigma(ijkl)} \text{ for all } \sigma \in \mathfrak{S}_4.$$

This subspace is a $\mathbb{P}^4$ and, according to Notation 1.3.1, in this $\mathbb{P}^4$ we use homogeneous coordinates $[z_0, z_1, z_2, z_3]$; clearly this $\mathbb{P}^4$ is a subspace of the $\mathbb{P}^8$ containing the (1,1; 2,2)-partially symmetric tensors. The variety parameterising tensor of symmetric rank 1 is the Veronese variety $V_{1,4}$ (a rational normal quartic curve), which can also be viewed as what we obtain when in the Segre map $s_{1,1,1,1}$ we identify the four copies of $\mathbb{P}^1$. The tensors we are considering now can be viewed in Figure 3.3

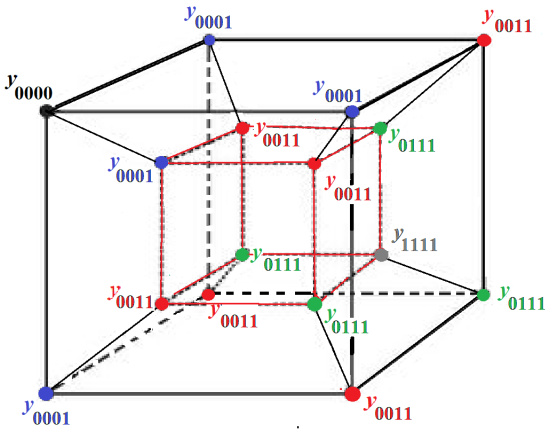


Figure 3.3: The symmetric 2^4-tensors, the colours of the dots signal equal coordinates.

As stated by Theorem 1.4.26, the ideal of $V_{1,4}$ can be obtained by the 2×2-minors of certain catalecticant matrices.

Now let us go back to the 2-square schemes we defined in $\mathbb{P}^1 \times \mathbb{P}^1$.

Notation 3.4.2. For every point $P = [a_0, a_1; b_0, b_1]$ we denote by Q_P the scheme defined by the ideal

$$\mathcal{I}_{Q_P} = (a_1x_{1,0} - a_0x_{1,1}, b_1x_{2,0} - b_0x_{2,1}).$$

The image under the Segre-Veronese embedding $sv_{2,2}$ of Q_P is such that

$$L(sv_{2,2}(Q_P)) \cong \mathbb{P}^3.$$

Here we want to consider the variety

$$q_2(SV_{2,2}) := \overline{\bigcup_{P \in \mathbb{P}^1 \times \mathbb{P}^1} L(sv_{2,2}(Q_P))}$$

and to analyse its secant varieties. Before doing that, we understand what $L(sv_{2,2}(Q_P))$ is by using the multigraded apolar action of $\mathcal{U}$ on $\mathcal{R}$ (see [63] for more details on multigraded apolarity).

Lemma 3.4.3. *Let $P = [a_0, a_1; b_0, b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ and let $m_{1,0} := a_0w_{1,0} + a_1w_{1,1} \in \mathcal{U}_{1,0}$ and $m_{0,1} := b_0w_{2,0} + b_1w_{2,1} \in \mathcal{U}_{0,1}$. Then,*

$$(\mathcal{I}_{Q_P})^{\perp}_{d,d} = m_{1,0}^{d-1} m_{0,1}^{d-1} \mathcal{U}_{1,1}$$

where the perp is computed with respect to the apolar action of $\mathcal{U}$ *on* $\mathcal{R}$*. Moreover, the point* $sv_{d,d}(Q_P) \subseteq SV_{d,d}$ *corresponds to the form* $(a_0y_{1,0}+a_1y_{1,1})^d(b_0y_{2,0}+b_1y_{2,1})^d$ *and*

$$(a_0y_{1,0}+a_1y_{1,1})^{d-1}(b_0y_{2,0}+b_1y_{2,1})^{d-1}\mathcal{T}_{1,1} = L(sv_{d,d}(Q_P)) \cong \mathbb{P}((\mathcal{I}_{Q_P})^{\perp}_{2,2})$$

and the isomorphism is canonical.

Proof Recall that we defined

$$\mathcal{I}_{Q_P} = (\ell_{1,0}^2, \ell_{0,1}^2) \subseteq \mathcal{R}$$

where $\ell_{1,0} = a_1x_{1,0} - a_0x_{1,1}$ and $\ell_{0,1} = b_1x_{2,0} - b_0x_{2,1}$ and note that $\ell_{1,0} \perp m_{1,0}$ and $\ell_{0,1} \perp m_{0,1}$. Since

$$(\mathcal{I}(Q_P))_{d,d} = \langle \ell_{1,0}^i \ell_{0,1}^j \mathcal{R}_{d-i,d-j} \rangle_{\substack{2\leq i\leq d\\ 2\leq j\leq d}}$$

it is clear that $m_{1,0}^{d-1}m_{0,1}^{d-1}\mathcal{U}_{1,1} \subseteq (\mathcal{I}_{Q_P})^{\perp}_{d,d}$. On the other hand, if $m_{1,0}^{d-1} \nmid G \in \mathcal{U}_{d,d}$ then $G \circ \ell_{1,0}^d \neq 0$ and thus $m_{1,0}^{d-1}G \notin (\mathcal{I}_{Q_P})^{\perp}_{d,d}$ and similarly if $m_{0,1}^{d-1} \nmid G$. Hence

$$m_{1,0}^{d-1}m_{0,1}^{d-1}\mathcal{U}_{1,1} = (\mathcal{I}_{Q_P})^{\perp}_{d,d}.$$

The fact that $sv_{d,d}(Q_P) \subseteq SV_{d,d}$ corresponds to the form

$$(a_0y_{1,0}+a_1y_{1,1})^d(b_0y_{2,0}+b_1y_{2,1})^d$$

is a trivial check using our identifications. Finally, the second part of the statement is analogous to the proof of Proposition 1.4.6 and the canonical isomorphism is given just by changing the name of the variables from $w_{i,j}$ to $y_{i,j}$. ■

Example 3.4.4. Let us consider for example $P = [0,1;0,1]$, then $\mathcal{I}_{Q_P} = (x_{1,0}^2, x_{2,0}^2)$ and

$$(\mathcal{I}_{Q_P}^{\perp})_{(2,2)} = \langle w_{1,0}w_{1,1}w_{2,0}w_{2,1}, w_{1,0}w_{1,1}w_{2,1}^2, w_{1,1}^2w_{2,0}w_{2,1}, w_{1,1}^2w_{2,1}^2 \rangle$$

and thus the tensors in any $L(sv_{2,2}(Q_P))$ can be written, modulo a bilinear change of coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$, as described in Figure 3.4

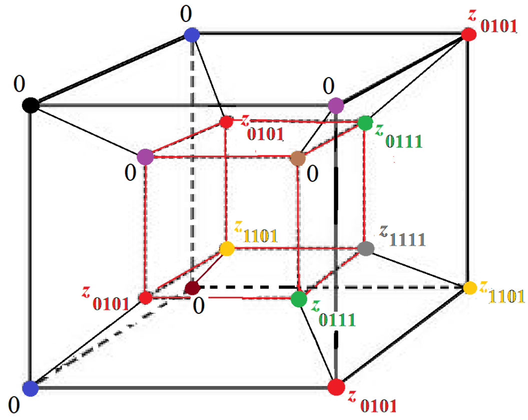


Figure 3.4: Tensors in $L(sv_{2,2}(Q_P))$

Hence, $L(sv_{2,2}(Q_P))$ is defined by equations $v_{(i_0,i_1),(j_0,j_1)} = 0$, for all $v_{(i_0,i_1),(j_0,j_1)}$ where either both i's or both j's are 0.

In the following, by virtue of Lemma 3.4.3, we will consider the forms $m_{1,0}, m_{0,1}$ in $\mathcal{T}$, instead of $\mathcal{U}_{1,1}$, with the trivial identifications.

Remark 3.4.5. Using the same notation of Lemma 3.4.3, note that since $L(sv_{2,2}(Q_P)) = \mathbb{P}(m_{1,0}m_{0,1}\mathcal{T}_{1,1})$ and

$$\tau_{1,sv_{2,2}(P)}(SV_{2,2}) = \mathbb{P}(m_{1,0}m_{0,1}(m_{1,0}\mathcal{T}_{0,1} + m_{0,1}\mathcal{T}_{1,0}))$$

we have $\tau_1(SV_{2,2}) \subseteq q_2(SV_{2,2})$.

Now we consider the secant variety of $q_2(SV_{2,2})$ and we want to prove the following proposition.

Proposition 3.4.6. *We have that* $\dim q_2(SV_{2,2}) = 5$ *and* $\sigma_2(q_2(SV_{2,2})) = \mathbb{P}^8$*, as expected. Hence the generic partially symmetric tensor in* $\mathbb{P}^8$ *can be written as the sum of two p.s. tensors which depends only on four parameters each (and can be written, not at the same time, as in Figure 3.4).*

Proof Recall that $SV_{2,2} \subseteq \mathbb{P}^8$ and this $\mathbb{P}^8$ can be identified with $\mathbb{P}(\mathcal{T}_{2,2})$. By Lemma 3.4.3, to give a point in $L(sv_{2,2}(Q_P)) = \mathbb{P}^3 \subseteq \mathbb{P}^8$ amounts to choosing a form $m_{1,0}m_{0,1}m_{1,1}$, with $m_{1,0} \in \mathcal{T}_{1,0}$, $m_{0,1} \in \mathcal{T}_{0,1}$, $m_{1,1} \in \mathcal{T}_{1,1}$. Hence, in order to find the tangent space to $q_2(SV_{2,2})$ at the point $m_{1,0}m_{0,1}m_{1,1}$, we have to consider another generic point $\ell_{1,0}\ell_{0,1}\ell_{1,1} \in q_2(SV_{2,2})$, and then compute (e.g. see [45], [46]):

$$\lim_{\lambda \to 0} \frac{d}{d\lambda} [(m_{1,0} + \lambda\ell_{1,0})(m_{0,1} + \lambda\ell_{0,1})(m_{1,1} + \lambda\ell_{1,1})] =$$
$$= \ell_{1,0}m_{0,1}m_{1,1} + m_{1,0}\ell_{0,1}m_{1,1} + m_{1,0}m_{0,1}m_{1,1} \subseteq \mathcal{T}_{2,2}.$$

As $\ell_{1,0}, \ell_{0,1}, \ell_{1,1}$ vary, we get that the affine cone on the tangent space that we considered is

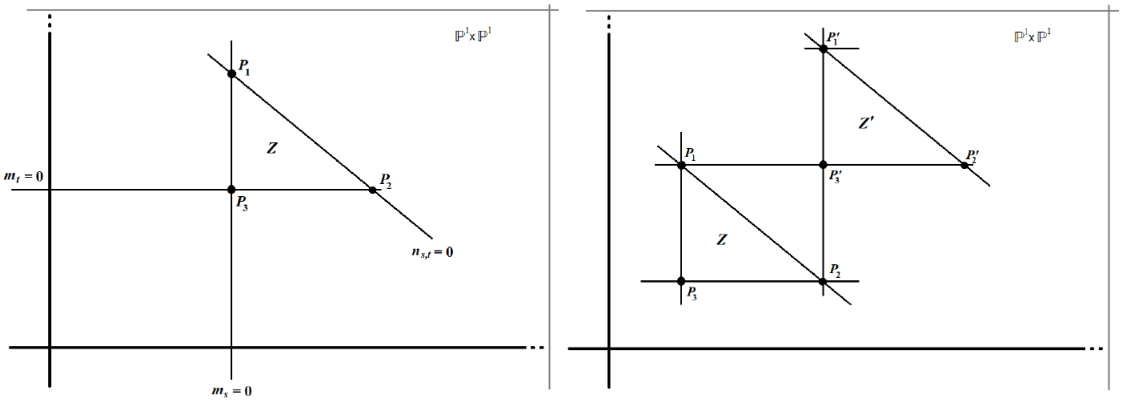
$$W = m_{1,0}m_{0,1}\mathcal{T}_{1,1} + m_{1,0}m_{1,1}\mathcal{T}_{0,1} + m_{0,1}m_{1,1}\mathcal{T}_{1,0}.$$

The vector dimension of W is 6 and thus $\dim(q_2(SV_{2,2})) = 5$, as expected.

If we consider the ideal

$$I = (m_{1,0}m_{0,1}, m_{1,0}m_{1,1}, m_{0,1}m_{1,1})$$

we have that $W = I_{2,2}$, and I is the ideal of three points $P_1, P_2, P_3 \in \mathbb{P}(\mathcal{T}^{(1)}) \times \mathbb{P}(\mathcal{T}^{(2)})$. Since they can be respectively defined by the ideals $(m_{1,0}, m_{1,1})$, $(m_{0,1}, m_{1,1})$ and $(m_{1,0}, m_t)$, the three of them are not contained in a fibre, but P_1, P_3 and P_2, P_3 are (see Figure 3.5 a).



(a) **(b)**

Figure 3.5: **(a)** P_1, P_2, P_3 in $\mathbb{P}^1 \times \mathbb{P}^1$; **(b)** $\mathbb{X} \cup \mathbb{X}'$ specialised.

Now we want to use Terracini Lemma (see Lemma 1.4.13) to compute the dimension of $\sigma_2(q_2(SV_{2,2}))$. We have to consider two tangent spaces to $q_2(SV_{2,2})$ at two generic points of $q_2(SV_{2,2})$ and compute the projective subspace generated by them. Let us say that the two generic points are given by $m_{1,0}m_{0,1}m_{1,1}$ and $\ell_{1,0}\ell_{0,1}\ell_{1,1}$. If their affine cones are W and W', by Terracini Lemma $\mathbb{P}(W + W')$ is the tangent cone at a generic point of $\sigma_2(q_2(SV_{2,2}))$. Since $W = (\mathcal{I}_{\mathbb{X}})_{2,2}$ and $W' = (\mathcal{I}_{\mathbb{X}'})_{2,2}$, where $\mathbb{X}$ and $\mathbb{X}'$ are given by two configuration of three simple points as in Figure 3.5 a, we have that

$$W \cap W' = (\mathcal{I}_{\mathbb{X} \cup \mathbb{X}'})_{2,2}.$$

Now we want to prove that if $\mathbb{X} = P_1 + P_2 + P_3$ and $\mathbb{X}' = P_1' + P_2' + P_3'$ are two schemes of three points, both positioned as in Figure 3.5 a, then they impose independent conditions to forms of bidegree (2,2), i.e. $\dim(I_{\mathbb{X} \cup \mathbb{X}'})_{2,2} = 3$. To prove that we can specialise $\mathbb{X}'$ so that the line $\ell_{1,0} = 0$ contains P_3', P_1' and also a point (say P_2) of $\mathbb{X}$ (see Figure 3.5 b). This forces the forms in $(\mathcal{I}_{\mathbb{X}\cup\mathbb{X}'})_{2,2}$ to be of type $\ell_{1,0}F$, with $F \in \mathcal{T}_{1,2}$; now we have also that since $\{F = 0\}$ contains P_1 and P_2', then F has to be of type $F = \ell_{0,1}G$, with $G \in \mathcal{T}_{1,1}$ and $P_3 \in \{G = 0\}$. Hence $\dim(\mathcal{I}_{\mathbb{X}\cup\mathbb{X}'})_{2,2} = \dim(\mathcal{I}_{P_3})_{1,1} = 3$. Note that we have, in some sense, used a multigraded version of the residual exact sequence.

By that we get $\dim(W \cap W') = \dim(I_{\mathbb{X}\cup\mathbb{X}'})_{2,2} = 3$, so

$$\dim(W + W') = \dim W + \dim W' - \dim(W \cap W') = 6 + 6 - 3 = 9$$

and $\sigma_2(q_2(SV_{2,2})) = \mathbb{P}^8$. ∎

3.4.2 The varieties $SV_{d,d}$ and their $q_2(SV_{d,d})$, $d \geq 3$.

Now we want to generalise what we did to the case of Segre-Veronese varieties $SV_{d,d} = sv_{d,d}(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^{(d+1)^2-1}$, with $d \geq 3$. Also in this case $SV_{d,d}$ parameterises partially symmetric $2^{2d}-$tensors and we can define a variety $q_2(SV_{d,d})$ as before, that is

$$q_2(SV_{d,d}) := \overline{\bigcup_{P \in \mathbb{P}^1 \times \mathbb{P}^1} L(sv_{d,d}(Q_P))}.$$

Proposition 3.4.7. *For all $d \geq 3$, $\dim q_2(SV_{d,d}) = 5$ and $\dim \sigma_2(q_2(SV_{d,d})) = 11$, as expected.*

Proof By Lemma 3.4.3 we know that

$$L(sv_{d,d}(Q_P)) = m_{1,0}^{d-1}m_{0,1}^{d-1}\mathcal{T}_{1,1} = (m_{1,0}^{d-1}m_{0,1}^{d-1})_{d,d}.$$

In order to find the tangent space to $q_2(SV_{d,d})$ at the point corresponding to $m_{1,0}^{d-1}m_{0,1}^{d-1}m_{1,1}$ (remember the identification of $\mathbb{P}^{(d+1)^2-1}$ with $\mathcal{T}_{d,d}$), we have to consider another generic point $\ell_{1,0}^{d-1}\ell_{0,1}^{d-1}\ell_{1,1} \in q_2(SV_{d,d})$, and then to compute

$$\lim_{\lambda \to 0} \frac{d}{d\lambda}\left[(m_{1,0} + \lambda\ell_{1,0})^{d-1}(m_{0,1} + \lambda\ell_{0,1})^{d-1}(m_{1,1} + \lambda\ell_{1,1})\right] =$$

$$= (d-1)m_{1,0}^{d-2}\ell_{1,0}m_{0,1}^{d-1}m_{1,1} + (d-1)m_{0,1}^{d-2}\ell_{0,1}m_{1,0}^{d-1}m_{1,1} + m_{1,0}^{d-1}m_{0,1}^{d-1}\ell_{1,1}.$$

This, as $\ell_{1,0}, \ell_{0,1}, \ell_{1,1}$ vary, gives the space

$$W = (m_{1,0}^{d-1}m_{0,1}^{d-1}, m_{1,0}^{d-1}m_{0,1}^{d-2}m_{1,1}, m_{1,0}^{d-2}m_{0,1}^{d-1}m_{1,1})_{d,d}.$$

If we let $\mathbb{X}$ be the scheme of $\mathbb{P}(\mathcal{T}^{(1)}) \times \mathbb{P}(\mathcal{T}^{(2)})$ defined by

$$\mathcal{I}_{\mathbb{X}} = (m_{1,0}^{d-1}m_{0,1}^{d-1}, m_{1,0}^{d-2}m_{0,1}^{d-2}m_{1,1})$$

we have $W \subseteq (\mathcal{I}_{\mathbb{X}})_{d,d}$. Note that $\mathbb{X}$ is the scheme which is made of the two lines $m_{1,0} = 0$, $m_{0,1} = 0$, both with multiplicity $d-2$, plus two $(d-1)$-jets on

$\{m_{1,1} = 0\}$, supported at the points respectively defined by the ideals $(m_{1,0}, m_{1,1})$ and $(m_{1,1}, m_{0,1})$ (see Figure 3.6).

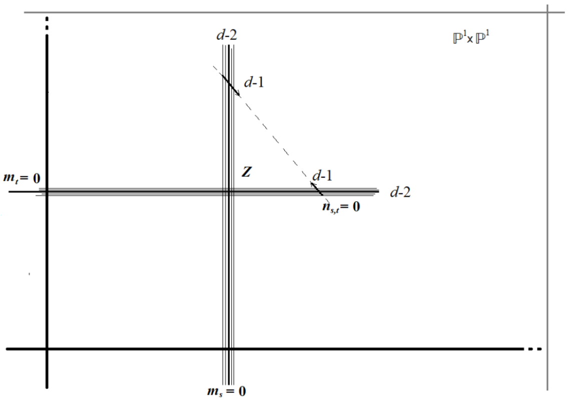


Figure 3.6: The scheme given by two $(d-2)$-ple lines with two $(d-1)$-jets sprouting from them.

We have that $\dim W = 6$ and thus $\dim q_2(SV_{d,d}) = 5$, as expected, while $\dim(\mathcal{I}_{\mathbb{X}})_{d,d} = 7$: indeed, all forms in $(\mathcal{I}_{\mathbb{X}})_{d,d}$ are of type $m_{1,0}^{d-2} m_{0,1}^{d-2} F$, where F is a (2,2)$-$form passing through the two points where the jets are, hence $\dim(\mathcal{I}_{\mathbb{X}})_{d,d} = 9 - 2 = 7$.

By Terracini Lemma, to compute the dimension of $\sigma_2(q_2(SV_{d,d}))$ we have to consider two tangent spaces to $q_2(SV_{d,d})$ at two generic points of $q_2(SV_{d,d})$ and we have to compute the projective subspace generated by them. More precisely, if the affine cones at the two generic points are W and W', the space $\mathbb{P}(W + W')$ will be the tangent space to a generic point of $\sigma_2(q_2(SV_{d,d}))$.

Since $W \subseteq (\mathcal{I}_{\mathbb{X}})_{d,d}$ and $W' \subseteq (\mathcal{I}_{\mathbb{X}'})_{d,d}$, where $\mathbb{X}$ and $\mathbb{X}'$ are made as in Figure 3.6, i.e. two lines of multiplicity $d-2$ with two $(d-1)$-jets, we have

$$W \cap W' \subseteq (\mathcal{I}_{\mathbb{X}})_{d,d} \cap (\mathcal{I}_{\mathbb{X}'})_{d,d} = (\mathcal{I}_{\mathbb{X} \cup \mathbb{X}'})_{d,d}.$$

We want to check that $(\mathcal{I}_{\mathbb{X} \cup \mathbb{X}'})_{d,d} =< 0 >$. We set

$$\mathcal{I}_{\mathbb{X}'} = (m'^{d-1}_{1,0} m'^{d-1}_{0,1}, m'^{d-2}_{1,0} m'^{d-2}_{0,1} m'_{1,1})$$

and we distinguish two cases:

- $d = 3$.
  Since the forms in $(\mathcal{I}_{\mathbb{X} \cup \mathbb{X}'})_{3,3}$ should contain the factor $m_{1,0} m_{0,1} m'_s m'_t \in \mathcal{T}_{2,2}$, and no form in $\mathcal{T}_{1,1}$ passes through the four points which are the support of the four jets, we get $(I_{\mathbb{X} \cup \mathbb{X}'})_{3,3} =< 0 >$.

- $d \geq 4$
  The forms in $(\mathcal{I}_{\mathbb{X}\cup\mathbb{X}'})_{d,d}$ should contain the factor $m_{1,0}^{d-2} m_{0,1}^{d-2} m_{1,0}'^{d-2} m_{0,1}'^{d-2}$, which is impossible for $d \geq 5$, while for $d = 4$ there would be only this form, which does not contain the four $d-1$ jets at the four points $m_{1,0} \cap m_{1,1}$, $m_{0,1} \cap m_{1,1}$, $m'_s \cap n'_{s,t}$ and $m'_t \cap n'_{s,t}$, hence also in this case $(I_{\mathbb{X}\cup\mathbb{X}'})_{3,3} = \{0\}$.

Now, from Grassmann equality, we get

$$\dim(W + W') = \dim W + \dim W' - \dim(W \cap W') = 6 + 6 - 0$$

hence $\dim(W + W') = 12$ and $\dim \sigma_2(q_2(V_{3,3})) = 11$. ∎

3.4.3 The cuckoo varieties $qq_2(SV_{d,d})$

As we have done in the Veronese case, here too we are going to define a *"cuckoo variety"* inside $q_2(SV_{d,d})$, which could also have some interest in relation to Quantum Entanglement. If we consider the Hilbert space of a composite quantum system, then this is the tensor product of the Hilbert spaces of the constituent systems, and tensor rank is a natural measure of the entanglement for the corresponding quantum states. The Hilbert space of a k-body system is obtained as the tensor product of k copies of the single particle Hilbert space $\mathcal{H}_1$. In the case of indistinguishable bosonic particles, the totally symmetric states under particle exchange are physically relevant, which amounts to restricting the attention to the subspace $H_s = \mathrm{Sym}^N(\mathcal{H}_1) \subseteq \mathcal{H}_1^{\otimes N}$ of symmetric tensors. See [17], [19] and [28] for more details on this topic.

In case we have $k \geq 2$ different species of indistinguishable bosonic particles, the relevant Hilbert space is

$$\mathrm{Sym}^{N_1}(\mathcal{H}_1) \otimes \mathrm{Sym}^{N_2}(\mathcal{H}_2) \otimes \ldots \otimes \mathrm{Sym}^{N_k}(\mathcal{H}_k).$$

Of particular interest, in physics literature, are the so-called W-states, i.e. quantum entangled states that can be expressed in Dirac notation as:

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{n}}(|100...0\rangle + |010...0\rangle + ... + |00...01\rangle).$$

Which, if $\mathcal{H}_1 = \mathbb{C}^2$ with coordinates (x, y), can be written as:

$$W_d = y \otimes x \otimes \ldots \otimes x + x \otimes y \otimes x \otimes \ldots \otimes x + x \otimes x \otimes \ldots \otimes y.$$

When treating with bosonic particles, hence with symmetric tensors, W_d can be represented simply as a monomial $x^{d-1}y$, hence in the study of entanglement of k different d-body systems (made of different species of indistinguishable bosonic particles, like photons), we can consider the product of k W_d states: $W_d \otimes \ldots \otimes W_d$, where each $W_d \cong x^{d-1}y \in \mathrm{Sym}^d \mathbb{C}^2 \subseteq (\mathbb{C}^2)^{\otimes d}$ (e.g. see [16]).

In [16], Lemma 2.1, it is proved that $W_d \otimes \ldots \otimes W_d \in L(sv_{(1,\ldots,1);(d,\ldots,d)}(\mathbb{X}))$, where $\mathbb{X}$ is a $2-$hypercube in $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$, and this is expressed by saying that the cactus rank of $W_d^{\otimes k}$ is 2^k and it is realised by $\mathbb{X}$ (the partially symmetric cactus rank of a tensor $T \in \mathbb{P}^{(d+1)^k-1}$ is the minimum length of a $0-$dimensional scheme $\mathbb{X} \subseteq SV_{(1,\ldots,1);(2,\ldots,2)}$ such that $T \in L(\mathbb{X})$). We can improve a bit that lemma in this setting (in [16] also the case of $W_{d_1} \otimes \ldots \otimes W_{d_k}$ is considered, with different d_i's).

Corollary 3.4.8. *Let $T \in \mathbb{P}^{(d+1)^k-1}$ parameterise $W_d \otimes \ldots \otimes W_d$; then the smoothable rank of T is 2^k.*

Proof The only difference between smoothable rank and cactus rank is that the smoothable rank of T is r, if and only if there is a smoothable $0-$dimensional scheme $\mathbb{X} \subseteq SV_{(1,\ldots,1);(d,\ldots,d)}$ such that $T \in L(\mathbb{X})$ and $\ell(\mathbb{X}) = r$ and there is none with $\ell(\mathbb{X}) < r$. Since, by Proposition 3.1.13, any $2-$hypercube is smoothable, the statement is an immediate consequence of this and of Lemma 2.1 of [16], since for any tensor the smoothable rank is greater or equal than the cactus rank. ■

Now we focus again on the case $k = 2$. We know that $q_2(SV_{d,d})$ parameterises partially symmetric tensors in the spaces

$$L(sv_{d,d}(Q_P)) = (m_{1,0}^{d-1} m_{0,1}^{d-1})_{d,d}$$

and thus for any partially symmetric tensor of type $W_d \otimes W_d = m_{1,0}^{d-1} a_{1,0} m_{0,1}^{d-1} b_t$, $m_{1,0}, a_{1,0} \in \mathcal{T}_{1,0}$, $m_{0,1}, b_t \in \mathcal{T}_{0,1}$, we have $W_d \otimes W_d \in L(sv_{d,d}(Q_P))$, for some $P \in \mathbb{P}^1 \times \mathbb{P}^1$. More specifically, let us consider the subvariety which parameterises exactly the tensors of type $W_d \otimes W_d$.

Definition 3.4.9. The *cuckoo variety* $qq_2(SV_{d,d}) \subseteq q_2(SV_{d,d})$ is the image of the morphism

$$\begin{array}{ccc} \mathbb{P}(\mathcal{T}_1^{(1)}) \times \mathbb{P}(\mathcal{T}_1^{(1)}) \times \mathbb{P}(\mathcal{T}_1^{(2)}) \times \mathbb{P}(\mathcal{T}_1^{(2)}) & \to & q_2(SV_{d,d}) \subseteq \mathbb{P}(\mathcal{T}_{d,d}) \\ ([m_{1,0}], [n_{1,0}], [m_{0,1}], [n_{0,1}]) & \mapsto & [m_{1,0}^{d-1} n_{1,0} m_{0,1}^{d-1} n_{0,1}] \end{array}.$$

Clearly the map can also be thought as a map from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to $q_2(SV_{d,d}) \subseteq \mathbb{P}^8$ through the usual identifications. Note that $qq_2(SV_{d,d})$ has dimension 4, and, via a multilinear change of coordinates, every form parameterised by a point in $qq_2(SV_{d,d})$ can be written as a monomial $y_{1,0}^{d-1} y_{1,1} y_{2,0}^{d-1} y_{2,1}$ (for results on the various ranks of such monomials see [47],[63] and [16]).

Proposition 3.4.10. *The cuckoo variety $qq_2(SV_{d,d})$ is such that:*

1. *$\forall\ P \in \mathbb{P}^1 \times \mathbb{P}^1$, $qq_2(SV_{d,d}) \cap L(sv_{d,d}(Q_P)) \cong \mathcal{Q}_P$, where $\mathcal{Q}_P$ is a smooth quadric in $L(sv_{d,d}(Q_P)) \cong \mathbb{P}^3$.*

2. *$\forall P \in \mathbb{P}^1 \times \mathbb{P}^1$, we have $\tau_{1,sv_{d,d}(P)}(\mathcal{Q}_P) = \tau_{1,sv_{d,d}(P)}(SV_{d,d})$.*

3. $\mathrm{Sing}(qq_2(SV_{d,d}))$ *is the locus of forms of type* $m_{1,0}^d m_{0,1}^{d-1} n_{0,1}$ *or* $m_{1,0}^{d-1} n_{1,0} m_{0,1}^d$.

Proof By Lemma 3.4.3, if $sv_{d,d}(P) = m_{1,0}^d m_{0,1}^d$, then

$$L(sv_{d,d}(Q_P)) = (m_{1,0}^{d-1} m_{0,1}^{d-1})_{d,d} = m_{1,0}^{d-1} m_{0,1}^{d-1} \mathcal{T}_{1,1}.$$

We chose $n_{1,0}, n_{0,1}$ such that

$$\mathcal{T}_{1,0} = \langle m_{1,0}, n_{1,0} \rangle, \quad \mathcal{T}_{0,1} = \langle m_{0,1}, n_{0,1} \rangle, \quad \mathcal{T}_{1,1} = \langle m_{1,0} m_{0,1}, m_{1,0} n_{0,1}, n_{1,0} m_{0,1}, n_{1,0} n_{0,1} \rangle.$$

and thus, any point in $L(sv_{d,d}(Q_P))$ is a form of type

$$m_{1,0}^{d-1} m_{0,1}^{d-1} (a_0 m_{1,0} m_{0,1} + a_1 m_{1,0} n_{0,1} + b_0 n_{1,0} m_{0,1} + b_1 n_{1,0} n_{0,1}).$$

The forms of $L(sv_{d,d}(Q_P))$ are all and only those such that

$$a_0 m_{1,0} m_{0,1} + a_1 m_{1,0} n_{0,1} + b_0 n_{1,0} m_{0,1} + b_1 n_{1,0} n_{0,1} = \ell_{1,0} \ell_{0,1}$$

for some $\ell_{1,0} \in \mathcal{T}_{1,0}$ and $\ell_{0,1} \in \mathcal{T}_{0,1}$. This condition is satisfied if and only if $a_0 b_1 - a_1 b_0 = 0$ and this equation defines a quadric $\mathcal{Q}_P \subseteq L(sv_{d,d}(Q_P))$. This proves the part *1*.

To prove part *2* note that $\tau_{1,sv_{d,d}(P)}(SV_{d,d})$ is given by the forms in

$$m_{1,0}^{d-1} m_{0,1}^{d-1} (m_{1,0} \mathcal{T}_{0,1} + m_{0,1} \mathcal{T}_{1,0}).$$

It follows that

$$\mathcal{Q}_P \cap \tau_{1,\nu_{d,d}(P)}(V_{d,d}) = \{m_{1,0}^d m_{0,1}^{d-1}(m_{0,1} + \alpha_{0,1}) \mid \alpha_{0,1} \in \mathcal{T}_{0,1}\} \cup \{m_{1,0}^{d-1} m_{0,1}^d (m_{1,0} \alpha_{1,0}) \mid \alpha_{1,0} \in \mathcal{T}_{1,0}\}.$$

Thus $\mathcal{Q}_P \cap \tau_{1,\nu_{d,d}(P)}(V_{d,d})$ is the union of two lines in $\tau_{1,\nu_{d,d}(P)}(V_{d,d})$ and hence this is the tangent plane to $\mathcal{Q}_P$ in $sv_{d,d}(Q_P)$.

In order to prove *3*, let us consider the affine cone W over the tangent space of $qq_2(SV_{d,d})$ at one of its points, say the one associated to $m_{1,0}^{d-1} n_{1,0} m_{0,1}^{d-1} n_{0,1}$. If we consider another point $\ell_{1,0}^{d-1} r_{1,0} \ell_{0,1}^{d-1} r_{0,1} \in qq_2(SV_{d,d})$, and we compute

$$\begin{aligned}\lim_{\lambda \to 0} \frac{d}{d\lambda} \Big[(m_{1,0} + \lambda \ell_{1,0})^{d-1} (n_{1,0} + \lambda r_{1,0}) (m_{0,1} + \lambda \ell_{0,1})^{d-1} (n_{0,1} + \lambda r_{0,1}) \Big] = \\ = (d-1) m_{1,0}^{d-2} \ell_{1,0} n_{1,0} m_{0,1}^{d-1} n_{0,1} + m_{1,0}^{d-1} r_{1,0} m_{0,1}^{d-1} n_{0,1} + \\ + (d-1) m_{1,0}^{d-1} n_{1,0} m_{0,1}^{d-2} \ell_{0,1} n_{0,1} + m_{1,0}^{d-1} n_{1,0} m_{0,1}^{d-1} r_{0,1}\end{aligned}$$

we find, as $\ell_{1,0}, r_{1,0}, \ell_{0,1}, r_{0,1}$ vary, that

$$W = \langle m_{1,0}^{d-2} m_{0,1}^{d-2} \left(m_{1,0} m_{0,1} (n_{0,1} \mathcal{T}_{1,0} + n_{1,0} \mathcal{T}_{0,1}) + n_{1,0} n_{0,1} (m_{0,1} \mathcal{T}_{1,0} + m_{1,0} \mathcal{T}_{0,1}) \right) \rangle \subseteq \mathcal{T}_{d,d}.$$

Generically, we have $\dim(a_{0,1} \mathcal{T}_{1,0} + a_{1,0} \mathcal{T}_{0,1}) = 3$, since they have $\langle a_{1,0} a_{0,1} \rangle$ in common, hence W is the sum of two subspaces of vector dimension 3, which have $\langle m_{1,0}^{d-1} n_{1,0} m_{0,1}^{d-1} n_{0,1} \rangle$ in common, so $\dim W = 5$, as expected. The locus $\mathrm{Sing}(qq_2(SV_{d,d}))$ is given by the points where $\dim W < 5$, and it is easy to check that this happens for either $m_{1,0} = n_{1,0}$ or $m_{0,1} = n_{0,1}$, and this proves *3*. ■

Remark 3.4.11. There is another way to view the variety $qq_2(SV_{d,d})$: consider the embedding $sv_{d,d}(\mathbb{P}^1 \times \mathbb{P}^1)$ as the composition

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^d \times \mathbb{P}^d \to \mathbb{P}^{d^2+2d}$$

where the first arrow is $\nu_{2,d} \times \nu_{2,d}$ and the second is the Segre embedding $s_{d,d}$. If the image of the first map is $\mathcal{C}_d^{(1)} \times \mathcal{C}_d^{(2)}$, where $\mathcal{C}_d^{(1)}, \mathcal{C}_d^{(2)}$ are the rational normal curves having as coordinate rings $\mathcal{R}_d^{(1)}$ and $\mathcal{R}_d^{(2)}$ respectively, we can consider the product of their tangential varieties $\tau(\mathcal{C}_d^{(1)}) \times \tau(\mathcal{C}_d^{(2)}) \subseteq \mathbb{P}^d \times \mathbb{P}^d$, parameterising pairs of forms like $(m_{1,0}^{d-1}a_{1,0}, m_{0,1}^{d-1}a_{0,1})$ and we get

$$s_{d,d}(\tau(\mathcal{C}_d^{(1)}) \times \tau(\mathcal{C}_d^{(2)})) = qq_2(SV_{d,d}).$$

We know that the singular locus of the tangential surface to a rational normal curve is the rational normal curve itself, hence

$$\mathrm{Sing}(\tau(\mathcal{C}_d^{(1)}) \times \tau(\mathcal{C}_d^{(2)})) = (\tau(\mathcal{C}_d^{(1)}) \times \mathcal{C}_d^{(2)}) \cup (\mathcal{C}_d^{(1)} \times \tau(\mathcal{C}_d^{(2)}))$$

in correspondence with what we saw in Proposition 3.4.10, *3*.

Remark 3.4.12. Note that for $d = 2$, we have

$$qq_2(SV_{2,2}) = s_{2,2}(\tau(\mathcal{C}_2^{(1)}) \times \tau(\mathcal{C}_2^{(2)})) = s_{2,2}(\mathbb{P}^2 \times \mathbb{P}^2)$$

hence $qq_2(SV_{2,2})$ is just the Segre variety $S_{2,2} \subseteq \mathbb{P}^8$ which is well-known to be 2-defective (it is the variety of 3×3 matrices of rank 2), i.e. $\dim \sigma_2(S_{2,2}) = 7$.

We want to check that this does not happen for $d \geq 3$.

Proposition 3.4.13. *For* $d \geq 3$, $\dim \sigma_2(qq_2(SV_{2,2})) = 9$, *as expected.*

Proof By Terracini Lemma, the dimension of the affine tangent cone at a generic point of $\sigma_2(qq_2(SV_{d,d}))$ will be $\dim W_1 + W_2$, where W_1, W_2 are the affine tangent cones at two generic points of $qq_2(SV_{d,d})$. Thus, in order to prove our statement, we have to show that $\dim(W_1 + W_2) = 10$ that is, since $\dim W_1 = \dim W_2 = 5$ by Proposition 3.4.10, that $W_1 \cap W_2 =< 0 >$.

In the proof of Proposition 3.4.10, *3*, we have computed the affine tangent cone W at a generic point of $qq_2(SV_{2,2})$, hence if we pick two generic points given by forms: $m_{1,0}^{d-1}n_{1,0}m_{0,1}^{d-1}n_{0,1}$ and $\ell_{1,0}^{d-1}r_{1,0}\ell_{0,1}^{d-1}r_{0,1}$, we will have:

$$W_1 = m_{1,0}^{d-2}m_{0,1}^{d-2}\,(m_{1,0}m_{0,1}n_{0,1}\mathcal{T}_{1,0} + m_{1,0}n_{1,0}m_{0,1}\mathcal{T}_{0,1} + n_{1,0}m_{0,1}n_{0,1}\mathcal{T}_{1,0} + m_{1,0}n_{1,0}n_{0,1}\mathcal{T}_{0,1})$$

$$W_2 = \ell_{1,0}^{d-2}\ell_{0,1}^{d-2}\,(\ell_{1,0}\ell_{0,1}r_{0,1}\mathcal{T}_{1,0} + \ell_{1,0}r_{1,0}\ell_{0,1}\mathcal{T}_{0,1} + r_{1,0}\ell_{0,1}r_{0,1}\mathcal{T}_{1,0} + \ell_{1,0}r_{1,0}r_{0,1}\mathcal{T}_{0,1})$$

and in particular $W_1 \subseteq (m_{1,0}^{d-2}m_{0,1}^{d-2})\mathcal{T}_{2,2}$, and $W_2 \subseteq (\ell_{1,0}^{d-2}\ell_{0,1}^{d-2})\mathcal{T}_{2,2}$. When $d \geq 5$ it is immediate to check that $W_1 \cap W_2 = \{0\}$, so we are done.

If $d = 4$, then

$$(m_{1,0}^{d-2} m_{0,1}^{d-2})\mathcal{T}_{2,2} \cap (\ell_{1,0}^{d-2} \ell_{0,1}^{d-2})\mathcal{T}_{2,2} = \langle m_{1,0}^2 m_{0,1}^2 \ell_{1,0}^2 \ell_{0,1}^2 \rangle$$

and it is easy to check that $m_{1,0}^2 m_{0,1}^2 \ell_{1,0}^2 \ell_{0,1}^2 \notin W_1 \cap W_2$ (it suffices to consider $m_{1,0} = y_{1,0}, m_{0,1} = y_{2,0}, n_{1,0} = y_{1,1}, n_{0,1} = y_{2,1}$ and note that $y_{1,i}^2 y_{2,j}^2 \notin W_1$, while these monomials will appear in $\ell_{1,0}^2 \ell_{0,1}^2$).

Eventually, for $d = 3$, let

$$W_1 = y_{1,0} y_{2,0} \langle y_{1,0}^2, y_{1,0} y_{1,1} y_{2,0} y_{2,1}, y_{1,0} y_{1,1} y_{2,0}^2, y_{1,1}^2 y_{2,0} y_{2,1}, y_{1,0} y_{1,1} y_{2,1}^2 \rangle.$$

If there is something not 0 in $W_1 \cap W_2$ it should be of the form $y_{1,0} y_{2,0} \ell_{1,0} \ell_{0,1} m_{1,1}$, with $m_{1,1} \in \mathcal{T}_{1,1}$, but since $\ell_{1,0}, \ell_{0,1}$ are generic, say

$$\ell_{1,0} \ell_{0,1} = (a_{1,0} y_{1,0} + a_{0,1} y_{1,1})(a_{2,0} y_{2,0} + a_{2,1} y_{2,1}),$$

in $\ell_{1,0} \ell_{0,1} m_{1,1}$ the monomials $y_{1,i}^2 y_{2,j}^2$ should appear, and this is impossible since they are not in

$$\langle y_{1,0}^2, y_{1,0} y_{1,1} y_{2,0} y_{2,1}, y_{1,0} y_{1,1} y_{2,0}^2, y_{1,1}^2 y_{2,0} y_{2,1}, y_{1,0} y_{1,1} y_{2,1}^2 \rangle.$$

Hence, $W_1 \cap W_2 = \{0\}$ also for $d = 3$ and we are done. ∎

Chapter 4

Postulation of 2-squares

In this chapter, we prove that a general union of 2-squares in $\mathbb{P}^2$ has good postulation (see Definition 1.6.3). In order to do that, we use the Horace method and, in one of the lemmata needed for the induction, the differential Horace method (see §1.6). Note that, for each degree d, it will be enough to prove the good postulation for a general union of s 2-squares for only two values of s:

- $s = s_*(d) := \left\lfloor \frac{\binom{d+2}{2}}{4} \right\rfloor$, i.e., the largest number of 2-squares that we expect to impose independent conditions on the space of degree-d plane curves;
- $s = s^*(d) := \left\lceil \frac{\binom{d+2}{2}}{4} \right\rceil$, i.e., the smallest number of 2-squares that we expect to admit no degree-d plane curve passing through them.

We will prove the theorem in two different ways: both of them are based on an Horace type argument, but with different specialisation; in particular, the second proof avoids the differential Horace.

In order to simplify computations, we would like to use lines as divisors for the residual exact sequences. Thus, given $\mathbb{X} = Z_1 + Z_2 + \cdots + Z_s$ a general union of s 2-squares, a general line r and $d \in \mathbb{N}$, we look for a specialisation $\mathbb{X}'$ of $\mathbb{X}$ such that $h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}')}) = 0$. Unfortunately, as soon as one tries to do that, one immediately notes that there is an arithmetic obstruction. Indeed, the intersection of a 2-square with any line r passing through its support has length 2. Therefore, whenever $d \equiv 0 \mod 2$, in order to get $h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}')})$, we would need to specialise the support of $d/2+1$ components of $\mathbb{X}$ to lie on the line r, that means that we put $d+2$ conditions on the line r. However, since $h^0(\mathcal{O}_r(d)) = d+1$, this is more than needed. This can cause problems whenever $\binom{d+2}{2}/4$ is an integer because if we "waste" conditions on $\mathrm{Tr}_r(\mathbb{X}')$ then we would be left with too few conditions on $\mathrm{Res}_r(\mathbb{X}')$. Indeed, in this case we should prove that $\dim \mathcal{L}_d(\mathbb{X}') = 0$, but with such a specialisation we would have

$$\ell(\mathrm{Res}_r(\mathbb{X}')) = \binom{d+2}{2} - (d+2) = \frac{(d+2)(d-1)}{2} = \binom{d+1}{2} - 1$$

and, therefore, $\dim \mathcal{L}_d(\mathbb{X}') = \dim \mathcal{L}_{d-1}(\mathrm{Res}_r(\mathbb{X}')) > 0$. The two proofs that we present will differ only in the way in which this problem is solved; in particular, they will differ only for d even.

4.1 First proof

In light of the arithmetic obstruction we have noticed, we need to specialise differently our 2-squares on our line r. We introduce in the following remark two different kinds of specialisation, which we will use for the first proof.

Remark 4.1.1. We can suppose, without loss of generality, that the line r has equation $r : x_1 = 0$.

1. Note that we can specialise a 2-square Z on r in two ways:

   (a) we can specialise Z in order to have

   $$\mathcal{I}(Z) = (x_0^2, x_1^2)$$

   (b) we can specialise Z in order to have

   $$\mathcal{I}(Z) = ((x_0 - x_1)^2, (x_0 + x_1)^2)$$

   In both cases, $\mathrm{Res}_r(Z)$ is a 0-dimensional scheme of length 2, and the difference between the two ways is that:

   - in (a), we have $(x_0^2, x_1^2) : (x_1) = (x_0^2, x_1)$ so that $\mathrm{Res}_r(Z) \subseteq r$ and in particular
   $$\ell(\mathrm{Tr}_r(\mathrm{Res}_r(Z))) = 2;$$
   - in (b), we have $((x_0 - x_1)^2, (x_0 + x_1)^2) : (x_1) = (x_0, x_1^2)$ so that
   $$\ell(\mathrm{Tr}_r(\mathrm{Res}_r(Z))) = 1.$$

2. Given two 2-squares with different support, we may collapse them together. Namely, for $t \in (0{,}1]$ let

   $$\mathcal{I}(Z_t) = (x_0^2, x_1^2) \cap ((x_0 + tx_2)^2, (x_1 + tz)^2) =$$
   $$\left((x_0 - x_1)^3, (x_0 - x_1)(x_0 + x_1)(x_0 + x_1 + 2tx_2), x_1^2(x_1 + tx_2)^2\right).$$

   Then, we get a 0-dimensional scheme Z_0 such that

   $$\mathcal{I}(Z_0) = \lim_{t \to 0} \mathcal{I}(Z_t) = ((x_0 - x_1)^3, (x_0 - x_1)(x_0 + x_1)^2, x_1^4).$$

Here is the key: the deformation (2) produces a 0-dimensional scheme having a slice of length 3 which should make the arithmetic work. Let us now show more in detail how the residues of Z_0, obtained using successively the line r as divisor, are. We can write the ideal $\mathcal{I}(Z_0)$ in the following way

$$\mathcal{I}(Z_0) = ((x_0 - x_1)^3, (x_0^2 - x_0x_1)x_1, (x_0 - x_1)^2x_1^2, (x_0 - x_1)x_1^3, x_1^4)$$

and thus we have that:

- $\mathcal{I}(Z_0) + (x_1) = (x_0^3, x_1)$ so that $\ell(\mathrm{Tr}_r(Z_0)) = 3$ and $Z_0' := \mathrm{Res}_r(Z_0)$ is defined by the ideal
$$\mathcal{I}(Z_0') = \mathcal{I}(Z_0) : (x_1) = (x_0^2 - x_0x_1, x_0x_1^2, x_1^3);$$
- $\mathcal{I}(Z_0') + (x_1) = (x_0^2, x_1)$ so that $\ell(\mathrm{Tr}_r(Z_0')) = 2$ and $Z_0'' := \mathrm{Res}_r(Z_0')$ is defined by the ideal
$$\mathcal{I}(Z_0'') = \mathcal{I}(Z_0') : (x_1) = (x_0, x_1)^2$$
and in particular $Z_0'' = 2P$, where P is the point [0,0,1].

Notation 4.1.2. When in what follows we use a specialisation of one of the types introduced in Remark 4.1.1, we will refer to them respectively as specialisation of type (1.a), (1.b) or (2).

Now we can use these specialisations to generate an inductive argument.

Lemma 4.1.3. *Let $d \in \mathbb{N}$ such that d is odd and consider a 0-dimensional scheme $\mathbb{X} = \mathbb{X}_1 + \mathbb{X}_2$ where:*

- *$\mathbb{X}_1 = Z_1 + Z_2 + \cdots + Z_{(d+1)/2}$, where all Z_i's are 2-squares with support on a line r in such a way that Z_1 is specialised as type (1.b) and $Z_2, Z_3, \ldots, Z_{d+1/2}$ are specialised as type (1.a);*
- *$\mathbb{X}_2$ does not intersect r.*

Then

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-2}(\mathbb{X}_2 + P)$$

where P is the support of Z_1.

Proof We have the residual exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_r(\mathbb{X})}(d-1) \longrightarrow \mathfrak{I}_{\mathbb{X}}(d) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}),r}(d) \longrightarrow 0$$

and

$$\ell(\mathrm{Tr}_r(\mathbb{X})) = 2\frac{d+1}{2} = d+1$$

so that $h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}),r}(d)) = d+1-(d+1) = 0$ and $h^0(\mathfrak{I}_{\mathbb{X}}(d)) = h^0(\mathfrak{I}_{\mathrm{Res}_r(\mathbb{X})}(d-1))$. Now we are left with

$$\mathbb{X}' := \mathrm{Res}_r(\mathbb{X}) = \mathrm{Res}_r(\mathbb{X}_1) + \mathrm{Res}_r(\mathbb{X}_2) = Z'_1 + Z'_2 + \cdots + Z'_{(d+1)/2} + \mathbb{X}_2$$

where $Z'_i := \mathrm{Res}_r(Z_i)$ for $i = 1, \ldots, \frac{d+1}{2}$ and we have the residual exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_r(\mathbb{X}')}(d-2) \longrightarrow \mathfrak{I}_{\mathbb{X}'}(d-1) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}'),r}(d-1) \longrightarrow 0 \ .$$

Thanks to the way we have specialised our points we get

$$\ell(\mathrm{Tr}_r(\mathbb{X}')) = \sum_{i=1}^{(d+1)/2} \ell(\mathrm{Tr}_r(Z'_i)) = 1 + 2\left(\frac{d+1}{2} - 1\right) = d$$

$$\mathrm{Res}_r(\mathbb{X}') = \mathbb{X}_2 + P$$

where $P := \mathrm{Res}_r(Z'_1)$ is a point on r. Thus

$$h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}'),r}(d-1)) = d - d = 0$$

and

$$h^0(\mathfrak{I}_{\mathbb{X}'}(d-1)) = h^0(\mathfrak{I}_{\mathbb{X}_2+P}(d-2)).$$

Hence, we finally have

$$\dim \mathcal{L}_d(\mathbb{X}) = h^0(\mathfrak{I}_{\mathbb{X}}(d)) = h^0(\mathfrak{I}_{\mathbb{X}'}(d-1)) = h^0(\mathfrak{I}_{\mathbb{X}_2+P}(d-2)) = \dim \mathcal{L}_{d-2}(\mathbb{X}_2 + P)$$

and this concludes the proof. ∎

Lemma 4.1.4. *Let $Z_1, Z_2, \ldots, Z_s$ be general 2-squares in $\mathbb{P}^2$, $\mathbb{X} = Z_1 + Z_2 + \cdots + Z_s$ and let $d \in \{2,4\}$. Then*

$$H_{\mathbb{X}}(d) = \min\left\{\binom{d+2}{2}, 4s\right\}$$

or, equivalently,

$$\dim \mathcal{L}_d(\mathbb{X}) = \binom{d+2}{2} - \min\left\{\binom{d+2}{2}, 4s\right\}.$$

Proof For each degree d, it is enough to prove the Lemma just for $s = s^*(d)$ and for $s = s_*(d)$ so we have to analyse four cases:

- $d = 2$ and $s = 1$

  The proof in this case is immediate because we already know that the Hilbert function of a 2-square X is $H_{\mathbb{X}}(1) = 3$ and $H_{\mathbb{X}}(d) = 4$ for any $d \geq 2$.

- $d = 2$ and $s = 2$

  In this case we have $\mathbb{X} = Z_1 + Z_2$ and, since by Remark 1.6.4 it is enough to prove the statement for a specialisation of $\mathbb{X}$, we can suppose that $\mathbb{X}$ is obtained by collapsing the two 2-squares Z_1 and Z_2 as in specialisation (2). At this point the result follows from the fact that, by Remark 4.1.1, the ideal of $\mathbb{X}$ does not have generators of degree 2.

- $d = 4$ and $s = 3$

  In this case we have $\mathbb{X} = Z_1 + Z_2 + Z_3$, but again we can suppose that $\mathbb{X}$ is specialised as $\mathbb{X}$=$\mathbb{Y}$+Z_3, where $\mathbb{Y}$ and Z_3 have support on the line r, $\mathbb{Y}$ is the specialisation of Z_1 and Z_2 as in specialisation (2) and Z_3 is specialised as in specialisation (1.a). We have the residual exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_r(\mathbb{X})}(3) \longrightarrow \mathfrak{I}_{\mathbb{X}}(4) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}),r}(4) \longrightarrow 0$$

  and

$$\ell(\mathrm{Tr}_r(\mathbb{X})) = 3 + 2 = 5$$

  so that $h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}),r}(4)) = 5 - 5 = 0$ and $h^0(\mathfrak{I}_{\mathbb{X}}(4)) = h^0(\mathfrak{I}_{\mathrm{Res}_r(\mathbb{X})}(3))$. At this point we are left with

$$\mathbb{X}' := \mathrm{Res}_r(\mathbb{X}) = \mathbb{Y}' + Z_3'$$

  where $\mathbb{Y}' := \mathrm{Res}_r(\mathbb{Y})$ and $Z_3' := \mathrm{Res}_r(Z_3)$ and we have the exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_r(\mathbb{X}')}(2) \longrightarrow \mathfrak{I}_{\mathbb{X}'}(3) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}'),r}(3) \longrightarrow 0 \ .$$

  Because of the way we have specialised our points we get

$$\ell(\mathrm{Tr}_r(\mathbb{X}')) = \ell(\mathrm{Tr}_r(\mathbb{Y}')) + \ell(\mathrm{Tr}_r(Z_3')) = 2 + 2 = 4$$

$$\mathrm{Res}_r(\mathbb{X}') = \mathrm{Res}_r(\mathbb{Y}') = 2P$$

  where $2P$ is a double point of $\mathbb{P}^2$ whose support is on r. In particular, we get $h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}'),r}(3)) = 0$ and thus

$$\dim \mathcal{L}_4(\mathbb{X}) = h^0(\mathfrak{I}_{\mathbb{X}}(4)) = h^0(\mathfrak{I}_{\mathbb{X}'}(3)) = h^0(\mathfrak{I}_{2P}(2)) = 3.$$

- $d = 4$ and $s = 4$ In this case we have $\mathbb{X} = Z_1 + Z_2 + Z_3 + Z_4$ and we can specialise again Z_1, Z_2 and Z_3 on a line r and Z_4 away from r. Repeating the same passages of the previous case, we find that

$$h^0(\mathfrak{I}_{\mathbb{X}}(4)) = h^0(\mathfrak{I}_{2P+Z_4}(2)).$$

  If $I(Z_4) = (\ell_1^2, \ell_2^2)$, the only conics that could contain $2P$ and Z_4 would be ℓ_1^2 and ℓ_2^2 but we can suppose that P does not lie neither on $\ell_1 = 0$ nor on $\ell_2 = 0$ so that we get $h^0(\mathfrak{I}_{2P+Z_4}(2)) = 0$.

■

Lemma 4.1.5. *Let $d \in \mathbb{N}$ be such that $d \geq 6$ and d is even and consider a 0-dimensional scheme $\mathbb{X}$=$\mathbb{X}_1 + \mathbb{X}_2 + \mathbb{X}_3$ where:*

- $\mathbb{X}_1 = \mathbb{Y} + Z_2 + Z_3 + \cdots + Z_{d/2}$ *where all the Z_i's are 2-squares with support on a general line r in such a way that $Z_2, \ldots, Z_{d/2}$ are specialised as type (1.a) and $\mathbb{Y}$ has support on r and it is obtained by collapsing two 2-squares as in specialisation (2);*
- $\mathbb{X}_2 = Z_{d/2+1} + \cdots + Z_{d-1}$ *where all the Z_i's are 2-squares away from r, with support on a general line m such that the support of $\mathbb{Y}$ is not on m and such that $Z_{d/2+1}$ and $Z_{d/2+2}$ are specialised as type (1.b) and $Z_{d/2+3}, \ldots, Z_{d-1}$ are specialised as type (1.a);*
- $\mathbb{X}_3$ *is a 0-dimensional scheme not intersecting neither r nor m.*

Then

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2)$$

where P_1, P_2 are the supports of $Z_{d/2+1}$ and of $Z_{d/2+2}$.

Proof We have the residual exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_r(\mathbb{X})}(d-1) \longrightarrow \mathfrak{I}_{\mathbb{X}}(d) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}),r}(d) \longrightarrow 0$$

and

$$\ell(\mathrm{Tr}_r(\mathbb{X})) = 2\left(\frac{d}{2} - 1\right) + 3 = d + 1$$

and thus $h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}),r}(d)) = d + 1 - (d+1) = 0$ and $h^0(\mathfrak{I}_{\mathbb{X}}(d)) = h^0(\mathfrak{I}_{\mathrm{Res}_r(\mathbb{X})}(d-1))$. Now we are left with

$$\mathbb{X}' := \mathrm{Res}_r(\mathbb{X}) = \mathrm{Res}_r(\mathbb{X}_1) + \mathrm{Res}_r(\mathbb{X}_2) + \mathrm{Res}_r(\mathbb{X}_3) = \mathbb{Y}' + Z_2' + \cdots + Z_{d/2}' + \mathbb{X}_2 + \mathbb{X}_3$$

where $\mathbb{Y}' := \mathrm{Res}_r(\mathbb{Y})$ and $Z_i' := \mathrm{Res}_r(Z_i)$ for $i = 2, \ldots, \frac{d}{2}$ and we have the residual exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_r(\mathbb{X}')}(d-2) \longrightarrow \mathfrak{I}_{\mathbb{X}'}(d-1) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}'),r}(d-1) \longrightarrow 0 \ .$$

Thanks to the way we have specialised our points, we get

$$\ell(\mathrm{Tr}_r(\mathbb{X}')) = \ell(\mathrm{Tr}_r(\mathbb{Y}')) + \sum_{i=2}^{d/2} \ell(\mathrm{Tr}_r(Z_i')) = 2 + 2\left(\frac{d}{2} - 1\right) = d$$

$$\mathrm{Res}_r(\mathbb{X}') = \mathbb{X}_2 + \mathbb{X}_3 + 2P$$

where $2P := \mathrm{Res}_r(\mathbb{Y}')$ is a double point whose support is on r but not on m. Thus, we have
$$h^0(\mathfrak{I}_{\mathrm{Tr}_r(\mathbb{X}'),r}(d-1)) = d - d = 0$$
and
$$h^0(\mathfrak{I}_{\mathbb{X}'}(d-1)) = h^0(\mathfrak{I}_{\mathbb{X}_2+\mathbb{X}_3+2P}(d-2)).$$
Now we apply the Horace differentiélle using as divisor the line m and specialising differentially the double point $2P$ to m. We take
$$\mathbf{p} = (0{,}0{,}1)$$
so that
$$\mathrm{Tr}^{\mathbf{p}}_m(\mathbb{X}_2 + \mathbb{X}_3 + 2P) = \mathrm{Tr}_m(\mathbb{X}_2) + \mathrm{Tr}_m(\mathbb{X}_3) + \mathrm{Tr}^1_m(2P) = \mathrm{Tr}_m(\mathbb{X}_2) + \mathrm{Tr}^1_m(2P)$$
and thus
$$\ell(\mathrm{Tr}^{\mathbf{p}}_m(\mathbb{X}_2 + \mathbb{X}_3 + 2P)) = 2\left(d - 1 - \frac{d}{2} - 1 + 1\right) + 1 = d - 1 = h^0(\mathcal{O}_m(d-2))$$
so it is enough to prove the vanishing of $h^0(\mathcal{O}_m(d-2))$. Now, keeping in mind that the only point differentially specialised was $2P$, we are left with the differential residue
$$\mathbb{X}'' := \mathrm{Res}^{\mathbf{p}}_m(\mathbb{X}_2 + \mathbb{X}_3 + 2P) = \mathrm{Res}_m(\mathbb{X}_2) + \mathbb{X}_3 + \mathrm{Res}^1_m(2P) = \sum_{i=d/2+1}^{d-1} Z''_i + \mathbb{X}_3 + J$$
where $Z''_i := \mathrm{Res}_m(Z_i)$ for $i = d/2+1, \ldots, d-1$ and J is a 2-jet on the line m and, by Lemma 1.6.9, we have
$$h^0(\mathfrak{I}_{\mathbb{X}_2+\mathbb{X}_3+2P}(d-2)) = h^0(\mathfrak{I}_{\mathbb{X}''}(d-3)).$$
We can now use the residual exact sequence
$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_m(\mathbb{X}'')}(d-4) \longrightarrow \mathfrak{I}_{\mathbb{X}''}(d-3) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_m(\mathbb{X}''),m}(d-3) \longrightarrow 0\ .$$
Thanks to the way we have specialised our points we get
$$\ell(\mathrm{Tr}_m(\mathbb{X}'')) = \sum_{i=d/2+1}^{d-1} \ell(Z''_i) + \ell(J) = 1 + 1 + 2\left(d - 1 - \frac{d}{2} - 3 + 1\right) + 2 = d - 2$$
$$\mathrm{Res}_m(\mathbb{X}'') = \mathbb{X}_3 + P_1 + P_2$$
where $P_1 := \mathrm{Res}_m(Z''_{d/2+1})$ and $P_2 := \mathrm{Res}_m(Z''_{d/2+2})$ are two points on m. So, we get $h^0(\mathfrak{I}_{\mathrm{Tr}_m(\mathbb{X}''),m}(d-3)) = d - 2 - (d-2) = 0$ and
$$h^0(\mathfrak{I}_{\mathbb{X}''}(d-3)) = h^0(\mathfrak{I}_{\mathbb{X}_3+P_1+P_2}(d-4)).$$

Hence, we have
$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2)$$
and this ends the proof. ■

We now have all the elements needed to provide the proof of our theorem. For the sake of clarity, we sketch, in the following remark, the steps of the proof and we summarise them in the diagram of Figure 4.1.

Remark 4.1.6. Let $\mathbb{Y}^{(d)}$ be a general union of s 2-squares, with $s, d \geq 2$ and $s \geq s_*(d)$. We fix the statement
$$A(d) = \{\mathbb{Y}^{(d)} \text{ has good postulation in degree } d\}.$$

We distinguish two cases:

- If $d \equiv 0 \mod 2$ and $d \geq 6$ we specialise $\mathbb{Y}^{(d)}$ to
$$\mathbb{X}^{(d)} := \mathbb{X}_1^{(d)} + \mathbb{X}_2^{(d)} + \mathbb{X}_3^{(d)},$$
where $\mathbb{X}_1^{(d)}$ and $\mathbb{X}_2^{(d)}$ are as in Lemma 4.1.5 and $\mathbb{X}_3^{(d)}$ is a general union of $s-d+1$ 2-squares. We denote by $\mathbb{X}^{(d-4)}$ the scheme
$$\mathbb{X}^{(d-4)} := \mathbb{X}_3^{(d)} + P_1 + P_2,$$
where P_1 and P_2 are general points.

- If $d \equiv 1 \mod 2$ we specialise $\mathbb{Y}^{(d)}$ to
$$\mathbb{X}^{(d)} := \mathbb{X}_1^{(d)} + \mathbb{X}_2^{(d)},$$
where $\mathbb{X}_1^{(d)}$ is as in Lemma 4.1.3 and $\mathbb{X}_2^{(d)}$ is a general union of $s - \frac{d+1}{2}$ 2-squares. We denote by $\mathbb{X}^{(d-2)}$ the scheme
$$\mathbb{X}^{(d-2)} := \mathbb{X}_2^{(d)} + P,$$
where P is a general point.

$$B(d) = \{\mathbb{X}^{(d)} \text{ has good postulation in degree } d\}.$$

The sketch of the proof is resumed in the following diagram:

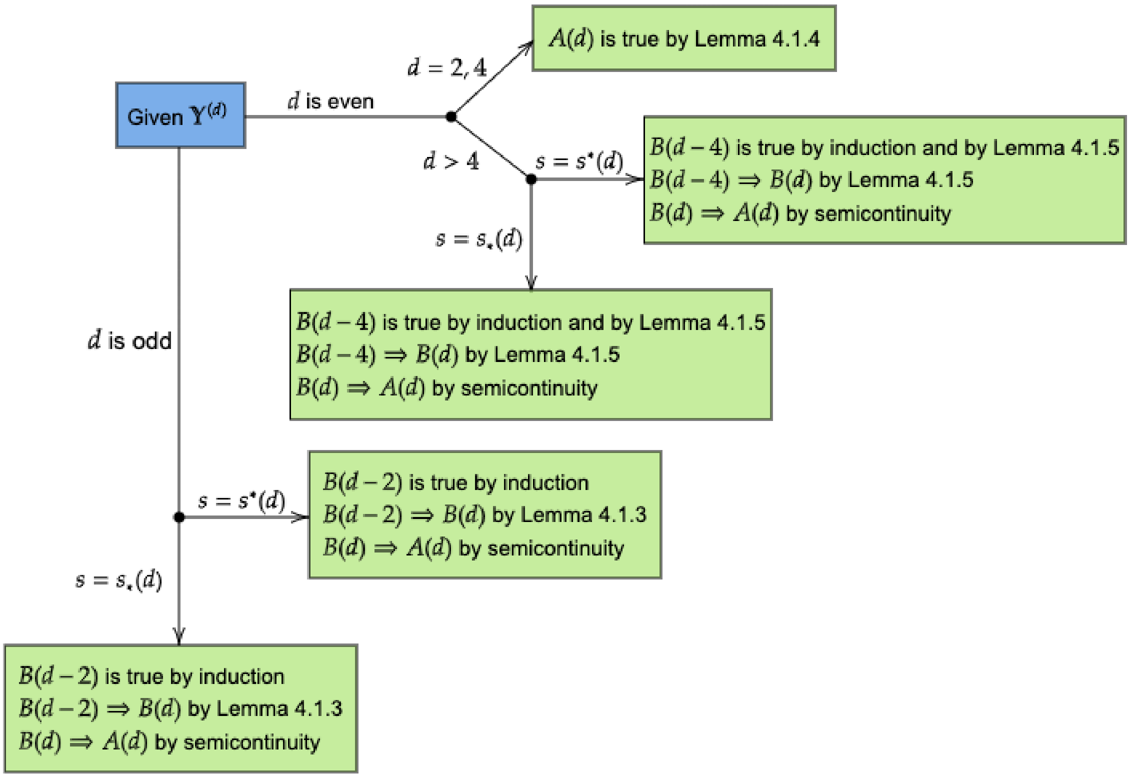


Figure 4.1: Sketch of the proof of Theorem 4.1.7.

Finally, we are ready to give the first proof of our theorem.

Theorem 4.1.7. *If $\mathbb{X} = Z_1 + Z_2 + \cdots + Z_s \subseteq \mathbb{P}^2$ is a general union of s 2-squares then $\mathbb{X}$ has good postulation, that is*

$$H_{\mathbb{X}}(d) = \min\{\dim R_d, 4s\}$$

or, equivalently,

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim R_d - \min\{\dim R_d, 4s\}.$$

Proof Remember that, for each degree d, it is enough to prove the theorem for $s = s^*(d)$ and for $s = s_*(d)$. The proof is by induction on d: we prove that the statement for $d-2$, respectively $d-4$, implies the statement for d in case d odd, respectively d even. The base case $d = 1$ is trivial, while the base cases $d = 2,4$ are true by Lemma 4.1.4.

Now we distinguish four cases, according to the parity of d and $s = s_*(d)$ or $s = s^*(d)$. Before of analysing each case we note that a straightforward check shows that for any $d \in \mathbb{N}$ there exists $\varepsilon' \in \{0,1,2,3\}$ such that

$$s = s_*(d) = \left\lfloor \frac{\binom{d+2}{2}}{4} \right\rfloor = \left\lfloor \frac{(d+2)(d+1)}{8} \right\rfloor = \frac{(d+2)(d+1) - 2\varepsilon'}{8}$$

and there exists $\varepsilon'' \in \{0,1,2,3\}$ such that

$$s = s^*(d) = \left\lceil \frac{\binom{d+2}{2}}{4} \right\rceil = \left\lceil \frac{(d+2)(d+1)}{8} \right\rceil = \frac{(d+2)(d+1)+2\varepsilon''}{8}.$$

Now we start to prove our theorem in each of the four cases. Note that, by semi-continuity of h^0, it is enough to prove, in each case, the good postulation for a specialisation of $\mathbb{X}$; see Remark 1.6.4.

- Case 1: $d \equiv 1 \mod 2$ and $s = s_*(d)$

  We have
  $$s = s_*(d) = \frac{(d+2)(d+1)-2\varepsilon}{8}$$
  and thus we have to prove that
  $$H_{\mathbb{X}}(d) = \min\{\dim R_d, 4s\} = \min\left\{\binom{d+2}{2}, \frac{(d+2)(d+1)-2\varepsilon}{2}\right\} =$$
  $$\min\left\{\binom{d+2}{2}, \binom{d+2}{2} - \varepsilon\right\} = \binom{d+2}{2} - \varepsilon$$
  or, equivalently, that
  $$\dim \mathcal{L}_d(\mathbb{X}) = \varepsilon.$$
  If we suppose that $\mathbb{X}$ is specialised as in Lemma 4.1.3 we have that
  $$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-2}(\mathbb{X}_2 + P)$$
  where $\mathbb{X}_2$ is a general union of $s - \frac{d+1}{2} = \frac{(d-2)(d+1)-2\varepsilon}{8}$ 2-squares and P is a general point. By induction, we have
  $$H_{\mathbb{X}_2}(d-2) = \min\left\{\binom{d}{2}, 4\left(\frac{(d-2)(d+1)-2\varepsilon}{8}\right)\right\} =$$
  $$\min\left\{\binom{d}{2}, \binom{d}{2} - \varepsilon - 1\right\} = \binom{d}{2} - \varepsilon - 1$$
  and thus
  $$\dim \mathcal{L}_{d-2}(\mathbb{X}_2 + P) = \binom{d}{2} - \left(\binom{d}{2} - \varepsilon - 1\right) - 1 = \varepsilon.$$

- Case 2: $d \equiv 1 \mod 2$ and $s = s^*(d)$

  We have
  $$s = s^*(d) = \frac{(d+2)(d+1)+2\varepsilon}{8}$$

and, since if $\varepsilon = 0$ then $s_*(d) = s^*(d)$ and the discussion is analogous to Case 1, in this case we can assume that $\varepsilon \in \{1,2,3\}$. We have to prove that

$$H_{\mathbb{X}}(d) = \min\{\dim S_d, 4s\} = \min\left\{\binom{d+2}{2}, \frac{(d+2)(d+1)+2\varepsilon}{2}\right\} =$$

$$= \min\left\{\binom{d+2}{2}, \binom{d+2}{2} + \varepsilon\right\} = \binom{d+2}{2}$$

or, equivalently, that

$$\dim \mathcal{L}_d(\mathbb{X}) = 0.$$

If $\mathbb{X}$ is specialised as in Lemma 4.1.3 we have that

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-2}(\mathbb{X}_2 + P)$$

where $\mathbb{X}_2$ is a general union of $s - \frac{d+1}{2} = \frac{(d-2)(d+1)+2\varepsilon}{8}$ 2-squares and P is a general point. By induction, we have

$$H_{\mathbb{X}_2}(d-2) = \min\left\{\binom{d}{2}, 4\left(\frac{(d-2)(d+1)+2\varepsilon}{8}\right)\right\} = \min\left\{\binom{d}{2}, \binom{d}{2} + \varepsilon - 1\right\} = \binom{d}{2}$$

and thus

$$\dim \mathcal{L}_{d-2}(\mathbb{X}_2 + P) = \binom{d}{2} - \binom{d}{2} = 0.$$

- Case 3: $d \equiv 0 \mod 2$ and $s = s_*(d)$

  Since the cases $d = 2$ and $d = 4$ are already solved by Lemma 4.1.4, we can suppose $d \geq 6$. We have

  $$s = s_*(d) = \frac{(d+2)(d+1) - 2\varepsilon}{8}$$

  and, as in Case 1, we have to prove that

  $$\dim \mathcal{L}_d(\mathbb{X}) = \varepsilon.$$

  If $\mathbb{X}$ is specialised as in 4.1.5 we have that

  $$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2)$$

  where $\mathbb{X}_3$ is a general union of $s - d = \frac{d^2-5d+2-2\varepsilon}{8}$ 2-squares and P_1, P_2 are general points. By induction, we have

  $$H_{\mathbb{X}_3}(d-4) = \min\left\{\binom{d-2}{2}, 4\left(\frac{d^2-5d+2-2\varepsilon}{8}\right)\right\} =$$

$$= \min\left\{\binom{d-2}{2}, \binom{d-2}{2} - \varepsilon - 2\right\} = \binom{d-2}{2} - \varepsilon - 2$$

and thus

$$\dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2) = \binom{d-2}{2} - \left(\binom{d-2}{2} - \varepsilon - 2\right) - 2 = \varepsilon.$$

- Case 4: $d \equiv 0 \mod 2$ and $s = s_*(d)$

  As in the previous case, we can suppose again that $d \geq 6$. We have

$$s = s^*(d) = \frac{(d+2)(d+1) + 2\varepsilon}{8}$$

  and, as in Case 2, we have to prove that

$$\dim \mathcal{L}_d(\mathbb{X}) = 0.$$

  If $\mathbb{X}$ is specialised as in Lemma 4.1.5 we have that

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2)$$

  where $\mathbb{X}_3$ is a general union of $s - d = \frac{d^2-5d+2+2\varepsilon}{8}$ 2-squares and P_1, P_2 are general points. By induction, we have

$$H_{\mathbb{X}_3}(d-4) = \min\left\{\binom{d-2}{2}, 4\left(\frac{d^2 - 5d + 2 + 2\varepsilon}{8}\right)\right\}$$

$$= \min\left\{\binom{d-2}{2}, \binom{d-2}{2} + \varepsilon - 2\right\}$$

  and thus, keeping in mind that $\varepsilon \in \{0,1,2,3\}$, we get

$$0 \leq \dim \mathcal{L}_{d-4}(\mathbb{X}_3) \leq 2.$$

  Hence, for any $\varepsilon \in \{0,1,2,3\}$ we obtain

$$\dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2) = 0$$

  and this concludes the proof of the theorem.

■

4.2 Second proof

The second proof is again based on the Horace method and coincides with the first one when the degree d of the curves we are considering is odd. In fact, the main difference is in the case in which d is even: indeed, in this case we will use some specialisations but without collapsing 2-squares. The proof starts by substituting one of the 2-squares with a double point contained in it, so obtaining a subscheme of the initial scheme and proving by induction that the number of conditions imposed on the degree d curves by this new scheme is one less than the expected number of conditions imposed by the initial scheme. After proving that, we conclude coming back to the original scheme and proving that when we pass from the double point to the 2-square, we actually impose one more condition.

Since, like we have just said, the proof is different just for even d's we will show it just for even d's. Moreover, remember that the cases $d = 2$ and $d = 4$ are already proved in Lemma 4.1.4.

Lemma 4.2.1. *Let $d \in \mathbb{N}$ such that d is even and $d \geq 6$ and consider a 0-dimensional scheme $\mathbb{X} = \mathbb{X}_1 + \mathbb{X}_2 + \mathbb{X}_3$ where:*

- *$\mathbb{X}_1 = Z_1 + Z_2 + \cdots + Z_{d/2}$ where $Z_1, \ldots, Z_{(d/2)}$ are 2-squares with support on a general line r in such a way that Z_1 is specialised as type (1.b) and $Z_2, \ldots, Z_{(d/2)}$ are specialised as type (1.a);*
- *$\mathbb{X}_2 = Z_{(d/2)+1} + \cdots + Z_d$ where $Z_{(d/2)+1}, \ldots, Z_d$ are 2-squares away from r and with support on a general line l in such a way that $Z_{(d/2)+1}$ and $Z_{(d/2)+2}$ are specialised as type (1.b) and $Z_{(d/2)+3}, \ldots, Z_d$ are specialised as type (1.a);*
- *$\mathbb{X}_3$ is a 0-dimensional scheme not intersecting neither r nor l.*

Then

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2)$$

where P_1 and P_2 are general points.

Proof In order to prove the lemma, we can prove that

$$\dim \mathcal{L}_d(\mathbb{X} + P) = \dim \mathcal{L}_{d-4}(\mathbb{X}_3 + P_1 + P_2 + P_3)$$

where $P = r \cap l$ and P_3 is a general point. We set $\mathbb{Y} = \mathbb{X} + P$ and we prove this second equality. We have the residual exact sequence

$$0 \longrightarrow \mathfrak{I}_{\mathrm{Res}_{r\cup l}(\mathbb{Y})}(d-2) \longrightarrow \mathfrak{I}_{\mathbb{Y}}(d) \longrightarrow \mathfrak{I}_{\mathrm{Tr}_{r\cup l}(\mathbb{Y}), r\cup l}(d) \longrightarrow 0$$

and we have

$$\ell(\mathrm{Tr}_{r\cup l}(\mathbb{Y})) = \ell(\mathrm{Tr}_{r\cup l}(\mathbb{X}_1)) + \ell(\mathrm{Tr}_{r\cup l}(\mathbb{X}_2)) + \ell(\mathrm{Tr}_{r\cup l}(\mathbb{X}_3)) + \ell(\mathrm{Tr}_{r\cup l}(P)) =$$

$$= d + d + 0 + 1 = 2d + 1$$

so that

$$h^0(\mathfrak{I}_{\mathrm{Tr}_{r\cup l}(\mathbb{Y}),r\cup l}(d)) = 2d + 1 - (2d + 1) = 0$$

and $h^0(\mathfrak{I}_{\mathbb{Y}}(d)) = h^0(\mathfrak{I}_{\mathrm{Res}_{r\cup l}(\mathbb{Y})}(d-2))$ and we are left with

$$\mathbb{Y}' := \mathrm{Res}_{r\cup l}(\mathbb{Y}) = \mathrm{Res}_{r\cup l}(\mathbb{X}_1) + \mathrm{Res}_{r\cup l}(\mathbb{X}_2) + \mathrm{Res}_{r\cup l}(\mathbb{X}_3) = Z_1' + Z_2' + \dots Z_d' + \mathbb{X}_3$$

where $Z_i' := \mathrm{Res}_{r\cup l}(Z_i)$ for $i = 1, \dots d$. By the way we specialised our scheme, we have

$$\deg(\mathrm{Tr}_r(\mathbb{Y}')) = \sum_{i=1}^{d/2} \deg(\mathrm{Tr}_r(Z_i')) = 1 + 2\left(\frac{d}{2} - 1\right) = d - 1$$

$$\deg(\mathrm{Tr}_l(\mathbb{Y}')) = \sum_{i=\frac{d}{2}+1}^{d} \deg(\mathrm{Tr}_l(Z_i')) = 1 + 1 + 2\left(\frac{d}{2} - 2\right) = d - 2$$

so that using first the residual exact sequence of $\mathbb{Y}'$ with respect to r and then the residual exact sequence of $\mathrm{Res}_r(\mathbb{Y}')$ with respect to l we find

$$h^0(\mathfrak{I}_{\mathbb{Y}'}(d-2)) = h^0(\mathfrak{I}_{\mathbb{Y}''}(d-4))$$

where

$$\mathbb{Y}'' = \mathrm{Res}_r(Z_1') + \mathrm{Res}_l(Z'_{(d/2)+1}) + \mathrm{Res}_l(Z'_{(d/2)+2}) + \mathbb{X}_3.$$

The fact that $\mathrm{Res}_r(Z_1'), \mathrm{Res}_l(Z'_{(d/2)+1})$ and $\mathrm{Res}_l(Z'_{(d/2)+2})$ are 3 general points ends the proof. ■

To give the second proof we need the following lemma.

Lemma 4.2.2. *Let $d \in \mathbb{N}$ with $d \geq 6$. Set $s \geq s_*(d)$ and consider $P_1, \dots, P_s$ points in general position in $\mathbb{P}^2$. Then*

$$\dim \mathcal{L}_d(3P_1 + 3P_2 + \dots + 3P_s) = 0.$$

Proof It is known that a general union of triple points in the plane has good postulation with respect to the degree d curves for any $d \geq 9$ and, for $d \leq 6 < 9$, the only exceptions are 5 triple points for $d = 6{,}7$ (see [77] and [78]). However, we have

$$\frac{\binom{6+2}{2}}{4} = 7, \qquad \frac{\binom{7+2}{2}}{4} = 9$$

so that under our hypothesis the triple points $3P_1, \dots, 3P_s$ always have good postulation. Thus, if we set $\mathbb{X} = 3P_1 + \dots + 3P_s$ we have

$$H_{\mathbb{X}}(d) = \min\left\{\binom{d+2}{2}, 6s_*(d)\right\} = \binom{d+2}{2}$$

and this concludes the proof. ■

Now we are ready to give an alternative proof of Theorem 4.1.7 for even degrees.

Proposition 4.2.3. *If* $\mathbb{X} = Z_1 + Z_2 + \cdots + Z_s \subseteq \mathbb{P}^2$ *is a general union of* s *2-squares then*

$$H_{\mathbb{X}}(d) = \min\{\dim S_d, 4s\}$$

or, equivalently,

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim S_d - \min\{\dim S_d, 4s\}$$

for any even $d \in \mathbb{N}$.

Proof The cases $d = 2,4$ are already proved in Lemma 4.1.4, so we have to prove the theorem for $d \geq 6$. We use induction on d and, as usual, we have to prove the theorem only for $s = s^*(d)$ and for $s = s_*(d)$.

- Case 1: $s = s_*(d)$
  As we said in the proof of Theorem 4.1.7, there exists $\varepsilon \in \{0,1,2,3\}$ such that

  $$s = s_*(d) = \frac{(d+2)(d+1) - 2\varepsilon}{8}$$

  and we have to prove that

  $$H_{\mathbb{X}}(d) = \binom{d+2}{2} - \varepsilon$$

  or, equivalently, that

  $$\dim \mathcal{L}_d(\mathbb{X}) = \varepsilon.$$

  Set P_i the support of Z_i for $i = 1, \ldots, s$ and

  $$\mathbb{X}' := Z_1 + \cdots + Z_{s-1} + 2P_s.$$

  For $d \geq 6$ we have $s_*(d) \geq d+1$ so that we can apply Lemma 4.2.1 and we get

  $$\dim \mathcal{L}_d(\mathbb{X}') = \dim \mathcal{L}_{d-4}(\mathbb{X}'_3 + Q_1 + Q_2)$$

  where $\mathbb{X}'_3$ is a general union of $s - d - 1$ 2-squares and a double point and Q_1, Q_2 are two general points. Since any 2-square contains the double point with the same support, $\mathbb{X}'_3$ is a subscheme of a scheme of $s - d$ 2-squares which, by induction, has good postulation in degree $d - 4$ and thus $\mathbb{X}'_3$ has in turn good postulation in degree $d - 4$ and we get

  $$H_{\mathbb{X}'_4} = \min\left\{\binom{d-2}{2}, 4(s-d-1)+3\right\} = \min\left\{\binom{d-2}{2}, \binom{d-2}{2} - \varepsilon - 3\right\} =$$

  $$= \binom{d-2}{2} - \varepsilon - 3.$$

As a consequence we have

$$\dim \mathcal{L}_d(\mathbb{X}') = \dim \mathcal{L}_{d-4}(\mathbb{X}'_3 + Q_1 + Q_2) = \varepsilon + 1.$$

Now, suppose by contradiction that $\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_d(\mathbb{X}')$. Since we are considering general 2-squares, claiming that $\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_d(\mathbb{X}')$ is equivalent to claiming that the $\varepsilon+1$ curves passing through $\mathbb{X}'$ also contain all the 2-squares having support at P_s. In particular, the $\varepsilon + 1$ curves contain the union of all the 2-squares having support at P_s which, by Theorem 3.2.6, is exactly $3P_s$ and thus they have a triple point at P_s. By symmetry, the same argument shows that the $\varepsilon+1$ curves have a triple point at each of $P_1, P_2, \ldots, P_s$ and this is a contradiction by Lemma 4.2.2. Hence $\dim \mathcal{L}_d(\mathbb{X}) \neq \dim \mathcal{L}_d(\mathbb{X}')$ and, since $\ell(\mathbb{X}) = \ell(\mathbb{X}) + 1$, we get

$$\dim \mathcal{L}_d(\mathbb{X}) = \dim \mathcal{L}_d(\mathbb{X}') - 1 = \varepsilon + 1 - 1 = \varepsilon.$$

- Case 2: $s = s^*(d)$
  This time there exists $\varepsilon \in \{0,1,2,3\}$ such that

$$s = s^*(d) = \frac{(d+2)(d+1) + 2\varepsilon}{8}$$

and we have to prove that

$$H_{\mathbb{X}}(d) = \binom{d+2}{2}$$

or, equivalently, that

$$\dim \mathcal{L}_d(\mathbb{X}) = 0.$$

We can suppose $\varepsilon \neq 0$ and setting again

$$\mathbb{X}' := Z_1 + \cdots + Z_{s-1} + 2P_s$$

and proceeding as in the previous case one finds that $\dim \mathcal{L}_d(\mathbb{X}') = 0$ and thus $\dim \mathcal{L}_d(\mathbb{X}) = 0$. ■

Chapter 5

Complete Intersections on Veronese Surfaces

This chapter is based on a joint work with E. Carlini (see [33]).

Complete intersection subvarieties are both a classical and a modern topic of study in Algebraic Geometry. Indeed, in [60] Euler asked when a set of points in the plane is the intersection of two curves, that is, using the modern terminology, when a set of points in the plane is a complete intersection. In the same period, Cramer asked similar questions so that this type of questions is presently known as the *Cramer-Euler problem*. The Euler solution in the case of nine points in the plane gave rise to what are now known as the Cayley-Bacharach Theorems, see [58].

Complete intersections and their algebraic counterpart, regular sequences, play a central role in Commutative Algebra and in Algebraic geometry. Consider, for example, the well-known Hartshorne conjecture, stated for the first time by Hartshorne in [74] and still open, which is probably one of the most studied problems regarding complete intersections. More recently, complete intersections have shown to have unexpected applications. For example, in [18] and [25], the strength and the slice rank of polynomials are studied using complete intersections. See also [41] for an application in proving the existence of special families of vector bundles on quartic surfaces of $\mathbb{P}^3$. For a more exhaustive overview on complete intersections, we advise seeing [69].

In this chapter, we consider a generalisation of the Cramer-Euler problem: characterise the possible complete intersections lying on a Veronese surface $V_{2,d}$, and more generally on a Veronese variety $V_{n,d}$; recall notation and definition in §1.3 and §1.4. Note that for $d = 1$ the Veronese surface $V_{2,1}$ is the plane $\mathbb{P}^2$, so that our problem in this special case is exactly the Cramer-Euler problem. In Theorem 5.3.5 we completely solve the problem showing that for $d > 2$ the only reduced complete intersections of $\mathbb{P}^{N_{n,d}}$ lying on $V_{2,d}$ are finite sets of either one or two points while, for the Veronese surface $V_{2,2} \subseteq \mathbb{P}^5$, one also has plane conics and their intersections with suitable hypersurfaces. Moreover, in Theorem 5.3.3 we show that, except for the case $d = 2$, the only complete intersections lying on rational normal curves $V_{1,d}$ are the trivial ones, that is one single point or the set of two points. The case $V_{1,2}$, that is of a plane conic, is different. In fact, by cutting with any properly chosen curve, one will produce a complete intersection set of points.

Inspired by these evidences, we formulate Conjecture 5.4.2: the only reduced complete intersections of $V_{n,d}$, $d \geq 3$, are finite sets of either one or two points while for $d = 2$ one also has plane conics and their intersections with suitable hypersurfaces. We also checked the validity of the conjecture for $V_{3,2}$, see Proposition 5.4.1.

In order to prove the main result of this chapter, Theorem 5.3.5, we characterise, in Theorem 5.1.11, the possible Hilbert functions of reduced subvarieties of Veronese varieties. In other words, we characterise all possible Hilbert functions of radical ideals in the Veronese ring $\mathbb{C}[z_0, \dots, z_{N_{n,d}}]/\mathcal{I}(V_{n,d})$, where $\mathcal{I}(V_{n,d})$ is the defining ideal of $V_{n,d}$. Beyond their application to the proof of our theorem, Hilbert functions play a central role in Commutative Algebra and in Algebraic Geometry, for example see [24], [90], and [99]. Indeed, in recent times, Hilbert functions have also been used as tools in other fields. For example, in the study of Waring rank, that is the tensor rank for symmetric tensors, in the paper [36]. Another example is the study of Strassen's Conjecture, a crucial conjecture in complexity theory, now proved to be false in general [98], but still open in the relevant case of symmetric tensors, see [39] and [40]. As a last example, we also mention the study of the identifiability of tensors, which plays a crucial role in Algebraic Statistic, see [9],[10] and [11].

In this chapter, we first characterise the Hilbert functions of reduced subvarieties of $V_{n,d}$. Thus, we generalise 0-sequences and differentiable 0-sequences introduced in [67], see Definition 1.1.21 and Definition 1.1.28. Successively, we give a more effective characterisation for the case of the rational normal curves $V_{1,d}$ in Theorem 5.1.7, so recovering a classical result, and for the case of the surfaces $V_{2,d}$ in Theorem 5.2.4. In [64], Theorem 4.5, a similar characterisation is given in the case of subschemes of $V_{n,d}$, that is, in the case of any ideal in a Veronese ring. However, we consider this characterisation not to be enough effective for our purposes. In fact, given a candidate Hilbert function, one has to solve a kind of interpolation problem to decide whether the given function is the Hilbert function of a reduced subvariety, or subscheme, of $V_{n,d}$, see Remark 5.2.7.

More precisely, the chapter is structured as follows. In §5.1, we recall some basic notions needed later on in the chapter and we study the relationship between $\mathcal{I}(\nu_{n,d}^{-1}(\mathbb{X}))$ and $\mathcal{I}(\mathbb{X})$, where $\mathbb{X} \subseteq V_{n,d}$ is a reduced subvariety of $V_{n,d}$. Using this, we characterise the Hilbert functions of subvarieties lying on $V_{n,d}$ and introduce d-sequences and differentiable d-sequences. Moreover, we use these results to study the Hilbert functions of divisors of $V_{n,d}$ and the Hilbert functions of finite sets of reduced points lying on a rational normal curve $V_{1,d}$, recovering the classical results. In §5.2, we apply Theorem 5.1.4 to the special case of Veronese surfaces, getting a more explicit characterisation for the Hilbert functions of reduced subvarieties of Veronese surfaces in Theorem 5.2.4. In §5.3, we use the previously introduced tools to prove the main theorem of the chapter, characterising all possible complete intersections lying on Veronese surfaces. In §5.4, we study the reduced complete intersection lying on $V_{3,2}$ and we state Conjecture 5.4.2 about reduced complete intersections lying on Veronese varieties.

5.1 Preliminary results

In this section, we introduce the needed basic notions and some preliminary results, including a complete characterisation of the possible Hilbert functions of reduced subvarieties of a Veronese variety; recall definitions and notation in §1.1. The following lemma will be very useful in the rest of the chapter.

Lemma 5.1.1. Let us consider the graded morphism

$$\begin{array}{rccc} \varphi_d: & S & \to & R \\ & a & \mapsto & a \\ & z_i & \mapsto & \underline{x}^{\alpha_i} \end{array}$$

for all $a \in \mathbb{C}$, and for $i \in \{0, \dots, N_{n,d}\}$. Then the following hold:

1. $\ker \varphi_d = \mathcal{I}(V_{n,d})$.
2. $\operatorname{Im} \varphi_d = \bigoplus_{\ell=0}^{\infty} R_{\ell d}$ and, in particular, $\varphi_d(S_t) = R_{td}$.
3. If $\mathbb{X} \subseteq V_{n,d}$, then $(\mathcal{I}(\nu_{n,d}^{-1}(\mathbb{X})))_{td} = \varphi_d(\mathcal{I}(\mathbb{X})_t)$. In particular
$$\varphi_d(\mathcal{I}(\mathbb{X})) = \bigoplus_{s=0}^{\infty} (\mathcal{I}(\nu_{n,d}^{-1}(\mathbb{X})))_{sd}.$$
4. If $\mathbb{X}$ is a subvariety of $V_{n,d}$ and we set $\mathbb{Y} = \nu_{n,d}^{-1}(\mathbb{X})$, then
$$H_{\mathbb{X}}(t) = H_{\mathbb{Y}}(td) \ \forall\, t \geq 0.$$

Proof The proof of (1) is trivial. To prove (2) it is enough to note that any monomial of degree td can be written as product of monomials of degree d. The proof of (3) is a straightforward check of a double inclusion. The proof of (4) follows from the chain of graded isomorphisms

$$S/\mathcal{I}(\mathbb{X}) \cong \frac{S/\mathcal{I}(V_{n,d})}{\mathcal{I}(\mathbb{X})/\mathcal{I}(V_{n,d})} \cong \frac{\psi_d(S/\mathcal{I}(V_{n,d}))}{\psi_d(\mathcal{I}(\mathbb{X})/\mathcal{I}(V_{n,d}))} = \frac{\bigoplus_{s=0}^{\infty} R_{sd}}{\varphi_d(\mathcal{I}(\mathbb{X}))}$$

where $\psi_d : S/\mathcal{I}(V_{n,d}) \to \bigoplus_{s=0}^{\infty} R_{sd}$ is the canonical isomorphism induced by φ_d. ■

Remark 5.1.2. Since φ_d is a ring homomorphism, $\varphi_d(\mathcal{I}(\mathbb{X}))$ is an ideal of

$$\operatorname{Im}\varphi_d = \bigoplus_{s=0}^{\infty} R_{sd}$$

but it is not an ideal of R. Nevertheless, one has

$$(\varphi_d(\mathcal{I}(\mathbb{X}))R)_{td} = (\varphi_d(\mathcal{I}(\mathbb{X})))_{td}.$$

Remark 5.1.3. In the notations of Lemma 5.1.1, if we choose $\mathbb{X} = V_{n,d}$ then we have $\mathbb{Y} = \mathbb{P}^n$ and thus we get that the Hilbert function of $V_{n,d}$ is $H_{V_{n,d}}(t) = \binom{n+td}{n}$.

The following theorem is an immediate consequence of Lemma 5.1.1.

Theorem 5.1.4. *Let $h(t) : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of a projective variety in $\mathbb{P}^{N_{n,d}}$. Then there exists $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ such that $H_{\mathbb{X}}(t) = h(t)$ if and only there exists $k(t) : \mathbb{N} \to \mathbb{N}$ a Hilbert function of a projective variety in $\mathbb{P}^n$ such that $h(t) = k(dt)$.*

Remark 5.1.5. We note that, if a variety $\mathbb{X}' \subseteq \mathbb{P}^{N_{n,d}}$ has Hilbert function satisfying the conditions of Theorem 5.1.4, this does not mean that $\mathbb{X}'$ lies on a Veronese variety. The theorem only guarantees that there exists *some* subvariety $\mathbb{X}$ of a Veronese variety having the same Hilbert function of $\mathbb{X}'$. Consider, for example, seven generic points in $\mathbb{P}^3$. By genericity they do not lie on a $V_{1,d}$, that is they do not lie on a rational normal curve, but their Hilbert function satisfies the hypothesis of the theorem.

In the case of divisors of $V_{n,d}$ we can be more explicit.

Proposition 5.1.6. *If $\mathbb{X}$ is a divisor of $V_{n,d}$ with $\deg \mathbb{X} = de$, then*

$$H_{\mathbb{X}}(t) = \begin{cases} \binom{n+dt}{n}, & \text{if } t \leq \left\lfloor \frac{e-1}{d} \right\rfloor \\ \binom{n+dt}{n} - \binom{n+dt-e}{n}, & \text{if } t \geq \left\lfloor \frac{e-1}{d} \right\rfloor + 1 \end{cases}$$

Proof Since $\mathbb{X}$ is a divisor and $\deg \mathbb{X} = de$, there exists a (unique) hypersurface $\mathbb{Y} : F = 0$ of degree e in $\mathbb{P}^n$ such that $\nu_{n,d}(\mathbb{Y}) = \mathbb{X}$. For each $t \in \mathbb{N}$ we have the following short exact sequence

$$0 \longrightarrow R(-e)_t \xrightarrow{F} R_t \xrightarrow{\pi} (R/(F))_t \longrightarrow 0$$

and, as a consequence, we get $H_{\mathbb{Y}}(t) = \dim(R/(F))_t = \dim R_t - \dim R(-e)_t$ and this ends the proof.

Remark 5.1.7. As a special case of divisor, one can consider $\mathbb{X} \subseteq V_{1,d} \subseteq \mathbb{P}^d$ a finite set of s reduced points on the rational normal curve of degree d. Using an argument similar to the one used in Proposition 5.1.6 one get the well known result:

$$H_{\mathbb{X}}(t) = \begin{cases} dt+1, & \text{if } t \leq \left\lfloor \frac{s-2}{d} \right\rfloor \\ s, & \text{if } t \geq \left\lfloor \frac{s-2}{d} \right\rfloor + 1 \end{cases}.$$

As we have noticed in §1.1.2, the characterisation of Theorem 1.1.29 can be rephrased: if $(c_t)_{t\in\mathbb{Z}}$ is a sequence of non-negative integers, then there exists a reduced $\mathbb{C}$-algebra A such that $(c_t)_{t\in\mathbb{Z}}$ is the Hilbert function of A if and only if $(c_t)_{t\in\mathbb{Z}}$ is a differentiable 0-sequence. Theorem 5.1.4 suggests us to extend Definition 1.1.21 and 1.1.28 as follows.

Definition 5.1.8. A 0-sequence $(b_t)_{t\in\mathbb{N}}$ is called *d-sequence* if there exists a 0-sequence $(c_t)_{t\in\mathbb{N}}$ such that $b_t = c_{(d+1)t}$.

Definition 5.1.9. A 0-sequence $(b_t)_{t\in\mathbb{N}}$ is called *differentiable d-sequence* if there exists a differentiable 0-sequence $(c_t)_{t\in\mathbb{N}}$ such that $b_t = c_{(d+1)t}$.

Remark 5.1.10. We note that a differentiable d-sequence is necessarily a differentiable 0-sequence.

We can now rephrase Theorem 5.1.4 as follows:

Theorem 5.1.11. *Let $(h_t)_{t\in\mathbb{N}}$ be a sequence of non-negative integers such that $h_0 = 1$ and $h_1 = N_{n,d} + 1$. There exists a projective variety $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if $(h_t)_{t\in\mathbb{N}}$ is a differentiable $(d-1)$-sequence.*

It is natural to ask for an effective characterisation of d-sequences similar to the one of Theorem 1.1.29. The question does not have an answer in general yet, nevertheless one can give an answer in a special case using our results. In the case of $(d-1)$-sequences with $h_1 = \binom{d+2}{2}$, such a characterisation can be easily produced using Theorem 5.2.4 in the next section.

5.2 Hilbert functions of points on Veronese surfaces

In this section, we focus our attention on the case of Veronese surfaces $V_{2,d}$. In particular, we give an effective characterisation of the Hilbert function of any reduced subvariety of $V_{2,d}$ in Theorem 5.2.4.

Notation 5.2.1. Given $d, t, s \in \mathbb{N}$ such that $s \geq d^2t + \frac{d(d+3)}{2}$ we define the following two functions:

$$\mu_1(d,t,s) := d^2t + \frac{d(d+3)}{2} - s$$

$$\mu_2(d,t,s) := \begin{cases} \left\lfloor \frac{2d(t+1)+3-\sqrt{1+8\mu_1(d,t,s)}}{2} \right\rfloor, & \text{if } 1 \le \mu_1(d,t,s) \le \binom{d+1}{2} \\ dt-n, & \text{if } \underset{0\le n\le dt}{\binom{d+1}{2}+dn<\mu_1(d,t,s)\le\binom{d+1}{2}+d(n+1)} \end{cases}$$

We begin with a technical result.

Lemma 5.2.2. *Let $d,t \in \mathbb{N}$. Consider a function $h : \{1,2,\dots,d\} \to \mathbb{N}$ such that there exists $i_0 \in \{1,2,\dots,d\}$ with the properties*

1. *$h(i) = dt+i+1$ for each $1 \le i \le i_0-1$;*
2. *$h(i) \ge h(i+1)$ for each $i_0 \le i \le d-1$.*

Then $h(d) \le \mu_2(d,t,\sum_{i=1}^{d} h(i))$ and moreover the inequality is sharp.

Proof We distinguish four cases depending on the value of

$$p := d^2t + \frac{d(d+3)}{2} - \sum_{i=1}^{d} h(i).$$

For each of them, we give a function $\tilde{h}(i)$ satisfying *1.* and *2.* and such that

$$\tilde{h}(d) = \begin{cases} \left\lfloor \frac{2d(t+1)+3-\sqrt{1+8p}}{2} \right\rfloor, & \text{if } 1 \le p \le \binom{d+1}{2} \\ dt-n, & \text{if } \binom{d+1}{2}+dn < p \le \binom{d+1}{2}+d(n+1), 0 \le n \le dt \end{cases}$$

Then we show that for any function $h'(i)$ satisfying *1.* and *2.* it holds that $h'(d) \le \tilde{h}(d)$. We do this in detail in case $p = \binom{n}{2}$ and for the remaining cases we produce the function $\tilde{h}$.

1. $p = \binom{n}{2}$, $1 \le n \le d+1$

   In this case we set

$$\tilde{h}(i) = \begin{cases} dt+i+1, & \text{if } 1 \le i \le d-n \\ d(t+1)-n+2, & \text{if } d-n+1 \le i \le d \end{cases}.$$

   We have

$$\sum_{i=1}^{d} \tilde{h}(i) = d^2t + \frac{d(d+3)}{2} - \frac{n(n-1)}{2} = d^2t + \frac{d(d+3)}{2} - p = \sum_{i=1}^{d} h(i)$$

   and

$$\tilde{h}(d) = d(t+1)-n+2 = \left\lfloor \frac{2d(t+1)+3-\sqrt{1+8p}}{2} \right\rfloor$$

hence $\tilde{h}(i)$ is as we want. Now let us suppose that there exists

$$h'(i) : \{1,2,\ldots,d\} \to \mathbb{N}$$

satisfying *1.* and *2.* and such that $h'(d) > \tilde{h}(d)$, that is

$$h'(d) = d(t+1) - n + 2 + a, \quad a \geq 1.$$

Since $n \leq d+1$, we have $h'(d) \geq dt + 1 + a \geq dt + 2$. As a consequence (observe that $dt+2$ is the maximum value of $h(1)$), by *1.* and *2.*, it follows that $h'(i)$ is increasing at least until reaching the value $h'(d)$. In particular, if we set $i' = \min\{1 \leq i \leq d \mid h(i) = d(t+1) - n + 2 + a\}$ we have

$$dt + i' + 1 = d(t+1) - n + 2 + a$$

so that $i' = d - n + 1 + a$ and $i_0 \geq i'$. Hence, using again *1.* and *2.*, we get

$$h'(i) = \tilde{h}(i) = dt + i + 1 \qquad \text{if } i \leq d - n$$

$$h'(i) > \tilde{h}(i) \qquad \text{if } i \geq d - n + 1$$

Hence:

$$\sum_{i=1}^{d} h'(i) = \sum_{i=1}^{d-n} \tilde{h}(i) + \sum_{i=d-n+1}^{d} \underbrace{h'(i)}_{>\tilde{h}(i)} > \sum_{i=1}^{d} \tilde{h}(i) = \sum_{i=1}^{d} h(i)$$

and this is a contradiction.

2. $\binom{n}{2} < p < \binom{n+1}{2}$, $1 \leq n \leq d$

   Let $b \in \mathbb{Z}$ be such that $p = \binom{n}{2} + b$. In this case we set

$$\tilde{h}(i) = \begin{cases} dt + i + 1, & \text{if } 1 \leq i \leq d - n + 1 \\ d(t+1) - n + 2, & \text{if } d - n + 2 \leq i \leq d - b \\ d(t+1) - n + 1, & \text{if } d - b + 1 \leq i \leq d \end{cases}.$$

3. $p = \binom{d+1}{2} + d(n+1)$, $0 \leq n \leq dt$

   In this case we set

$$\tilde{h}(i) = dt - n,\ 1 \leq i \leq n.$$

4. $\binom{d+1}{2} + nd < p < \binom{d+1}{2} + (n+1)d$, $1 \leq n \leq dt$

   Let $b \in \mathbb{Z}$ be such that $p = \binom{d+1}{2} + nd + b$. In this case we set

$$\tilde{h}(i) = \begin{cases} dt + 1 - n, & \text{if } 1 \leq i \leq d - b \\ dt - n, & \text{if } d - b + 1 \leq i \leq d \end{cases}$$

■

In the following proposition we characterise Hilbert functions of reduced points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ which arise from Hilbert functions of reduced points in $\mathbb{P}^2$ by sampling with steps of length d.

Proposition 5.2.3. *Let us consider a finite set of reduced points $\mathbb{X} \subseteq \mathbb{P}^{\frac{d(d+3)}{2}}$ and set*

$$t_1 = \max\left\{t \mid H_{\mathbb{X}}(t) = H_{V_{2,d}}(t)\right\}, \quad t_2 = \min\left\{t \mid H_{\mathbb{X}}(t) = |\mathbb{X}|\right\}.$$

Then $H_{\mathbb{X}}(t)$ is a $(d-1)$-sequence if and only if the following conditions hold

1.
$$\mu_2(d, t_1, \Delta H_{\mathbb{X}}(t_1+1)) \geq \left\lceil \frac{\Delta H_{\mathbb{X}}(t_1+2)}{d} \right\rceil;$$

2. *For all $t_1 + 2 \leq t \leq t_2 - 1$*
$$\left\lfloor \frac{\Delta H_{\mathbb{X}}(t)}{d} \right\rfloor \geq \left\lceil \frac{\Delta H_{\mathbb{X}}(t+1)}{d} \right\rceil.$$

Proof First, we assume that there exists $\mathbb{Y} \subseteq \mathbb{P}^2$ such that $H_{\mathbb{Y}}(dt) = H_{\mathbb{X}}(t)$. Let us set

$$H_{\mathbb{X}}(t) = h_t, \quad \Delta H_{\mathbb{X}}(t) = \Delta h_t,$$

and

$$H_{\mathbb{Y}}(t) = k_t, \quad \Delta H_{\mathbb{Y}}(t) = \Delta k_t.$$

Since $(k_t)_{t\in\mathbb{N}}$ is the Hilbert function of a finite set of reduced points of $\mathbb{P}^2$, then by Dubreil theorem (see Theorem 1.1.35) there exists $t' \in \mathbb{N}$ such that the following conditions hold:

- $k_t = \binom{2+t}{t}$ for all $t \leq t'$ and $k_t < \binom{2+t}{t}$ for all $t > t'$, and thus $\Delta k_t = t+1$ for all $t \leq t'$;
- $\Delta k_t \geq \Delta k_{t+1}$ for all $t > t'$, and Δk_t is eventually equal to 0.

Note that

$$\Delta h_t = h_t - h_{t-1} = k_{dt} - k_{d(t-1)} = k_{dt} + \sum_{i=1}^{d-1}(k_{dt-i} - k_{dt-i}) - k_{dt-d} =$$

$$= \sum_{i=0}^{d-1}(k_{dt-i} - k_{dt-(i+1)}) = \sum_{i=0}^{d-1} \Delta k_{dt-i}$$

and thus

$$\Delta h_{t+1} = \sum_{i=0}^{d-1} \Delta k_{d(t+1)-i} = \sum_{i=1}^{d} \Delta k_{dt+i}\ . \tag{5.1}$$

Thus, Remark 5.1.3 yields that

$$\sum_{i=1}^{d} \Delta k_{dt+i} = \Delta h_{t+1} = \binom{2+d(t+1)}{2} - \binom{2+dt}{2} = d^2 t + \frac{d(d+3)}{2} \text{ for all } t \le t_1 - 1 \,.$$

Since $\Delta k_t \le t+1$ for all t and

$$\sum_{i=1}^{d} (dt+i+1) = d^2 t + \frac{d(d+3)}{2}$$

it follows that

$$\sum_{i=1}^{d} \Delta k_{dt+i} = d^2 t + \frac{d(d+3)}{2} \text{ for all } t \le t_1 - 1$$

if and only if

$$\Delta k_t = t+1 \text{ for all } t \le dt_1$$

and thus $t' \ge dt_1$. Moreover, since $h_t < \binom{2+dt}{2}$ for $t > t_1$, the same argument shows that

$$\sum_{i=1}^{d} \Delta k_{dt+i} < d^2 t + \frac{d(d+3)}{2} \text{ for all } t \ge t_1$$

and for $t = t_1$ we have

$$\sum_{i=1}^{d} \Delta k_{dt_1+i} < d^2 t_1 + \frac{d(d+3)}{2} \,.$$

As a consequence, there exists a minimum $i_0 \in \{1,2,\dots,d\}$ such that $\Delta k_{dt_1+i_0} < dt_1 + i_0 + 1$ and therefore $t' \le dt_1 + d - 1 = d(t_1+1) - 1$. It follows that

$$\Delta k_{dt_1+i} = dt_1 + i + 1 \tag{5.2}$$

for $1 \le i \le i_0 - 1$ and

$$\Delta k_{dt_1+i_0} \ge \Delta k_{dt_1+i_0+1} \ge \cdots \ge \Delta k_{dt_1+i_0+a} = 0$$

for some $a \in \mathbb{N}$. Moreover

$$\Delta k_{dt_1+i_0} \ge \Delta k_{dt_1+i_0+1} \ge \cdots \ge \Delta k_{dt_1+d} \tag{5.3}$$

and

$$\Delta k_{dt+1} \ge \Delta k_{dt+2} \ge \cdots \ge \underbrace{\Delta k_{dt+d+1}}_{=\Delta k_{d(t+1)+1}} \tag{5.4}$$

for each $t \ge t_1 + 1$. Now, for a fixed $t \ge t_1 + 1$, by (5.4) we have

$$(d-1)\Delta k_{dt+1} \ge \sum_{i=2}^{d} \Delta k_{dt+i},$$

and
$$(d-1)\Delta k_{dt+d} \le \sum_{i=1}^{d-1} \Delta k_{dt+i}.$$
Using formula (5.1), we obtain
$$\Delta k_{dt+1} = \Delta h_{t+1} - \sum_{i=2}^{d} \Delta k_{dt+i} \ge \Delta h_{t+1} - (d-1)\Delta k_{dt+1}$$
and hence
$$\Delta k_{dt+1} \ge \frac{\Delta h_{t+1}}{d}$$
and similarly
$$\Delta k_{dt+d} = \Delta h_{t+1} - \sum_{i=1}^{d-1} \Delta k_{dt+i} \le \Delta h_{t+1} - (d-1)\Delta k_{dt+d}$$
and hence
$$\Delta k_{dt+d} \le \frac{\Delta h_{t+1}}{d}.$$
Moreover, since $(h_t)_{t\in\mathbb{N}}$ and $(k_t)_{t\in\mathbb{N}}$ are integer valued, we have that
$$\Delta k_{dt+1} \ge \left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil, \qquad \Delta k_{dt+d} \le \left\lfloor \frac{\Delta h_{t+1}}{d} \right\rfloor$$
for each $t \ge t_1 + 1$. By (5.2) and (5.3) it follows that the function
$$\begin{array}{rccc} \Delta k_{dt_1+i}: & \{1,2,\ldots,d\} & \to & \mathbb{N} \\ & i & \mapsto & \Delta k_{dt_1+i} \end{array}$$
satisfies the hypothesis of Lemma 5.2.2. Hence, we get that
$$\mu_2(d, t_1\Delta h_{t_1+1}) \ge \Delta k_{dt_1+d} \ge \Delta k_{d(t_1+1)+1} \ge \left\lceil \frac{\Delta h_{t_1+2}}{d} \right\rceil$$
thus proving condition (1). Finally, using (5.4) we get that
$$\Delta k_{dt+d} \ge \Delta k_{dt+d+1} = \Delta k_{d(t+1)+1}$$
for each $t \ge t_1 + 1$, and hence
$$\left\lfloor \frac{\Delta h_{t+1}}{d} \right\rfloor \ge \left\lceil \frac{\Delta h_{t+2}}{d} \right\rceil$$
for each $t \ge t_1 + 1$. We note that this inequality is always verified for $t \ge t_2$ since
$$\left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil = 0$$

for each $t \geq t_2$. Hence, also condition (2) is proved.

Now we assume that conditions (1) and (2) hold and we prove that there exists $\mathbb{Y} \subseteq \mathbb{P}^2$ such that $H_{\mathbb{Y}}(dt) = h_t$. Since

$$k_t = \sum_{i=0}^{t} \Delta k_t,$$

we can construct the Hilbert function $(k_t)_{t\in\mathbb{N}}$ by its first difference Δk_t. For each t let e_t be the unique integer such that

$$\Delta h_t \equiv e_t \pmod d, \quad 0 \leq e_t \leq d-1.$$

We define Δk_t as follows:

- if $0 \leq t \leq dt_1$ we set $\Delta k_t = t+1$;
- if $dt_1 + 1 \leq t \leq d(t_1+1)$ we construct Δk_t according to Lemma 5.2.2;
- if $t \geq d(t_1+1)+1$ we set

$$\Delta k_{dt+i} = \left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil, \quad t \geq t_1+1,\ 1 \leq i \leq e_{t+1}$$
$$\Delta k_{dt+i} = \left\lfloor \frac{\Delta h_{t+1}}{d} \right\rfloor, \quad t \geq t_1+1,\ e_{t+1}+1 \leq i \leq d.$$

Under our assumptions, this choice guarantees that Δk_t is the first difference function of a set of reduced points in $\mathbb{P}^2$ (see [83] Proposition 1.1). Moreover, for $t \leq t_1 - 1$, we have

$$\sum_{i=1}^{d} \Delta k_{dt+i} = \sum_{i=1}^{d} (dt+i+1) = d^2 t + \frac{d(d+3)}{2} = \Delta h_{t+1},$$

and, for $t = t_1$ we have by construction

$$\sum_{i=1}^{d} \Delta k_{dt_1+1} = \Delta h_{t_1+1}$$

while for $t \geq t_1+1$ we have

$$\sum_{i=1}^{d} \Delta k_{dt+i} = \sum_{i=1}^{e_{t+1}} \Delta k_{dt+i} + \sum_{i=e_{t+1}+1}^{d} \Delta k_{dt+i} = e_{t+1}\left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil + (d-e_{t+1})\left\lfloor \frac{\Delta h_{t+1}}{d} \right\rfloor =$$
$$= e_{t+1}\left(\frac{\Delta h_{t+1} - e_{t+1}}{d} + 1\right) + (d-e_{t+1})\left(\frac{\Delta h_{t+1} - e_{t+1}}{d}\right) = \Delta h_{t+1}.$$

Hence we have

$$k_{dt} = \sum_{i=0}^{dt} \Delta k_t = \Delta k_0 + \sum_{i=0}^{t-1} \left(\sum_{j=1}^{d} \Delta k_{di+j} \right) = \Delta h_0 + \sum_{i=0}^{t-1} \Delta h_{i+1} = \sum_{i=0}^{t} \Delta h_t = h_t$$

for all t. This concludes the proof. ∎

We now give our effective characterisation of the Hilbert functions of reduced sets of points on Veronese surfaces.

Theorem 5.2.4. *Let $(h_t)_{t\in\mathbb{N}}$ be the Hilbert function of a finite set of m reduced points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ and set*

$$t_1 = \max \left\{ t \mid h(t) = H_{V_{2,d}}(t) \right\} \qquad t_2 = \min \left\{ t \mid h(t) = m \right\}.$$

Then there exists $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^{\frac{d(d+3)}{2}}$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if the following conditions hold

1. $$\mu_2(d, t_1, \Delta h_{t_1+1}) \geq \left\lceil \frac{\Delta h_{t_1+2}}{d} \right\rceil;$$

2. *For all $t_1 + 2 \leq t \leq t_2 - 1$*
$$\left\lfloor \frac{\Delta h_t}{d} \right\rfloor \geq \left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil.$$

Proof It follows from Theorem 5.1.4 and Proposition 5.2.3. ∎

Let us see now two explicit instances of use of Theorem 5.2.4 in determining whether a given Hilbert function can be realised as the Hilbert function of a subvariety of a Veronese surface.

Example 5.2.5. Let us consider the sequence $(h_t)_{t\in\mathbb{N}}$ defined as follows

t	0	1	2	3	4	5	6	7	8	9	10	11
h_t	1	36	120	253	435	666	946	1256	1531	1744	1956	2022

and $h_t = 2022$ for $t \geq 12$. It is easy to check, using Theorem 1.1.29, that this is the Hilbert function of a set of 2022 reduced points in $\mathbb{P}^{35}$. We ask whether there exists $\mathbb{X} \subseteq V_{2,7} \subseteq \mathbb{P}^{35}$ such that $H_{\mathbb{X}}(t) = h_t$ for all $t \geq 0$. To answer, we use Theorem 5.2.4. First we determine t_1 and t_2. Since the Hilbert function of $V_{2,7}$ is $H_{V_{2,7}}(t) = \binom{2+7t}{2}$, we have that

t	0	1	2	3	4	5	6	7	8	9	10	11	12
$H_{V_{2,7}}$	1	36	120	253	435	666	946	1275	1653	2080	2556	3081	3655

so that $t_1 = 6$ and $t_2 = 11$. To determine $\mu_1(7{,}6,\Delta h_{t_1+1})$ we compute Δh_{t_1+1}. We have that

t	0	1	2	3	4	5	6	7	8	9	10	11	12
Δh_t	1	35	84	133	182	231	280	310	275	213	212	66	0

and thus $\mu_1(7{,}6{,}310) = 7^2 \cdot 6 + \frac{7(7+3)}{2} - 310 = 19$. Finally, since $19 \leq \binom{7+1}{2} = 28$, we get

$$\mu_2(7{,}6{,}310) = \left\lfloor \frac{2 \cdot 7(6+1) + 3 - \sqrt{1 + 8 \cdot 19}}{2} \right\rfloor = 44.$$

To check conditions (1) and (2) of Theorem 5.2.4, we compute $\left\lfloor \frac{\Delta h_t}{7} \right\rfloor$ and $\left\lceil \frac{\Delta h_t}{7} \right\rceil$ obtaining the following table

t	0	1	2	3	4	5	6	7	8	9	10	11	12
$\left\lceil \frac{\Delta h_t}{7} \right\rceil$	1	5	12	19	26	33	40	45	40	31	31	10	0
$\left\lfloor \frac{\Delta h_t}{7} \right\rfloor$	0	5	12	19	26	33	40	44	39	30	30	9	0

Since $\mu_2(7{,}6{,}310) = 44$ and $\left\lceil \frac{\Delta h_8}{7} \right\rceil = 40$ condition (1) is satisfied. However, condition (2) is not satisfied for $t = 9$ and hence such an $\mathbb{X}$ does not exist.

Example 5.2.6. Now, we consider the sequence $(h_t)_{t\in\mathbb{N}}$ defined as follows

t	0	1	2	3	4	5	6	7	8	9	**10**	11
h_t	1	36	120	253	435	666	946	1256	1531	1744	**1915**	2022

and $h_t = 2022$ for $t \geq 12$; note that this function coincides with the one of the previous example, but for $t = 10$. We ask whether there exists $\mathbb{X} \subseteq V_{2,7} \subseteq \mathbb{P}^{35}$ such that $H_{\mathbb{X}}(t) = h_t$ for all $t \geq 0$. As in the previous example, we have $t_1 = 6$ and $t_2 = 11$. Moreover, we get

t	0	1	2	3	4	5	6	7	8	9	10	11	12
Δh_t	1	35	84	133	182	231	280	310	275	213	201	107	0
$\left\lceil \frac{\Delta h_t}{7} \right\rceil$	1	5	12	19	26	33	40	45	40	31	29	16	0
$\left\lfloor \frac{\Delta h_t}{7} \right\rfloor$	0	5	12	19	26	33	40	44	39	30	28	15	0

and thus $\mu_1(7{,}6{,}310) = 19$ and $\mu_2(7{,}6{,}310) = 44$. Thus, condition (1) is satisfied and condition (2) is satisfied for $t = 8{,}9{,}10$. Hence, such an $\mathbb{X}$ exists.

Remark 5.2.7. One could deal with both the previous examples without using Theorem 5.2.4, but just using Macaulay's inequalities for Hilbert functions. However, this requires a trial and error approach. In fact, since $d = 7$ one should try to fill, step by step, six gaps between h_i and h_{i+1} satisfying Macaulay's inequalities. Thus, for each possible choice at each step, one should compute the appropriate binomial expansion. Moreover, one should check also that the first difference function of the constructed Hilbert function is still a Hilbert function: doing this will require a very large number of computations.

5.3 Complete intersections

In this section we focus on the study of complete intersection varieties of $\mathbb{P}^{N_{n,d}}$ which lie on some Veronese variety $V_{n,d}$ for $n = 1$ and $n = 2$; the case $n = 3$ and $d = 2$ is treated in Proposition 5.4.1. For generalities on complete intersections, we refer to [57]. To compare with similar existence and non-existence results for complete intersection on hypersurfaces and their applications, we refer to [38] and [94].

Definition 5.3.1. Let $\mathbb{X}$ be a projective variety, let $\mathcal{I}(\mathbb{X}) = (f_1, f_2, \ldots, f_r)$ be its defining ideal with $f_1, \ldots, f_r$ a minimal set of generators, and let $a_i = \deg f_i$. We say that $\mathbb{X}$ is a *complete intersection of type* $(a_1, \ldots, a_r)$ if $\operatorname{codim} \mathbb{X} = r$.

Proposition 5.3.2. *If $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ is a reduced complete intersection of type $(a_1, \ldots, a_r)$, with $a_1 \leq \ldots \leq a_r$, then $a_1 = 1$. Moreover, either $r = N_{n,d}$ and $a_1 = \ldots = a_{N_{n,d}} = 1$ or $a_i = 2$ for some i.*

Proof Since $\mathbb{X} \subseteq V_{n,d}$, we have that $\mathcal{I}(\mathbb{X}) \supseteq \mathcal{I}(V_{n,d})$. By Theorem 1.4.26, $\mathcal{I}(V_{n,d})$ is generated by the 2×2 minors of a catalecticant matrix, thus, either $a_1 = 1$ or $a_1 = 2$. Assume by contradiction that $a_1 = 2$, that is $a_i > 1$ for all i, and let $\mathcal{I}(\mathbb{X}) = (f_1, \ldots, f_r)$. In this case, it is possible to assume that $\mathcal{I}(V_{n,d}) = (f_1, \ldots, f_q)$ for some $q \leq r$ and, as a consequence, any syzygy of the generators of $\mathcal{I}(V_{n,d})$

gives a syzygy of the generators of $\mathcal{I}(\mathbb{X})$. The determinantal representation of $\mathcal{I}(V_{n,d})$ yields that the generators of $\mathcal{I}(\mathbb{X})$ have a syzygy of linear forms and this is a contradiction since $f_1, \dots, f_r$ are a regular sequence and their syzygies, given by the Koszul complex, only contain elements of degree at least $a_1 = 2$. In conclusion, $a_1 = 1$. To complete the proof, we let p be the largest index such that $a_p = 1$. Assume by contradiction that $p < r$ and $a_{p+1} > 2$. Thus, $\mathcal{I}(V_{n,d}) \subseteq (f_1, \dots, f_p)$ and the latter is the ideal of a linear space Λ of codimension p such that $\Lambda \subseteq V_{n,d}$. Hence a contradiction, since no Veronese variety contains a positive dimensional linear space. In conclusion, either $p = r = N_{n,d}$ or $p < r$ and $a_{p+1} = 2$. ■

As a straightforward application of this proposition, we get the following result.

Theorem 5.3.3. *Let $\mathbb{X} \subseteq V_{1,d} \subseteq \mathbb{P}^d$ be a reduced complete intersection with $d \geq 3$. Then either $\mathbb{X}$ is a point or $\mathbb{X}$ is a set of two points.*

Proof Since $V_{1,d}$ is not a complete intersection and $\dim V_{1,d} = 1$, it must be $\dim \mathbb{X} = 0$. Since $\mathbb{X}$ is a complete intersection, it follows from Proposition 5.3.2 that $\mathbb{X}$ is contained in the intersection of a hyperplane and $V_{1,d}$. In particular, we have $|\mathbb{X}| \leq d$ and using Remark 5.1.7 we get the Hilbert function of $\mathbb{X}$

t	0	1	2	...
$H_{\mathbb{X}}(t)$	1	$\lvert\mathbb{X}\rvert$	$\lvert\mathbb{X}\rvert$	...
$\Delta H_{\mathbb{X}}(t)$	1	$\lvert\mathbb{X}\rvert - 1$	0	...

Finally, since $\mathbb{X}$ is a 0-dimensional complete intersection, $\Delta H_{\mathbb{X}}(t)$ is the Hilbert function of an artinian Gorenstein ideal. Thus, it is symmetric and hence either $\Delta H_{\mathbb{X}}(1) = 0$ or $\Delta H_{\mathbb{X}}(1) = 1$, that is $|\mathbb{X}| = 1$ or $|\mathbb{X}| = 2$. ■

Remark 5.3.4. We note that the proof of Theorem 5.3.3 also shows that any Gorenstein reduced zero-dimensional subscheme of $V_{1,d}$ must be degenerate.

We can now finally describe reduced complete intersection subvarieties of Veronese surfaces.

Theorem 5.3.5. *If $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^{N_{2,d}}$ is a reduced complete intersection of type $(a_1, \dots, a_r)$, with $a_1 \leq \dots \leq a_r$ then one of the following holds:*

1. $(d, r, (a_1, a_2, \dots, a_r)) = (2,4,(1,1,1,2))$, *that is $\mathbb{X}$ is a conic lying on $V_{2,2}$;*
2. $(d, r, (a_1, a_2, \dots, a_r)) = (2,5,(1,1,1,2,a_5))$, *any $a_5 \in \mathbb{N}$, that is $\mathbb{X}$ is a set of $2a_5$ complete intersection points of a conic lying on $V_{2,2}$ and a hypersurface of degree a_5;*
3. $(d, r, (a_1, a_2, \dots, a_r)) = (d, N_{2,d}, (1,1,\dots,1))$ *for any $d \geq 2$, that is $\mathbb{X}$ is a reduced point;*

4. $(d, r, (a_1, a_2, \dots, a_r)) = (d, N_{2,d}, (1,1,\dots,1,2))$ *for any* $d \geq 2$*, that is* $\mathbb{X}$ *is a set of two reduced points.*

Proof Let $\mathcal{I}(\mathbb{X}) = (f_1, \dots, f_r)$ be the ideal of $\mathbb{X}$ and $N := N_{2,d}$. We can suppose without loss of generality that $a_1 \leq a_2 \leq \dots \leq a_r$. Moreover, we let p be the number of a_i equal to 1 and q be the number of a_i equal to 2, that is $a_1 = a_2 = \dots = a_p = 1$, $a_{p+1} = a_{p+2} = \dots = a_{p+q} = 2$. By Proposition 5.3.2 we have that $p \geq 1$. If $p = N$ we are trivially in case (3) thus from now on we suppose $p < N$ and thus, using again Proposition 5.3.2, we also have $q \geq 1$. Since $\mathbb{X} \subseteq V_{2,d}$, $\Delta H_{\mathbb{X}}(t)$ must satisfy conditions (1) and (2) of Theorem 5.2.4. We now use condition (1) to rule out some cases. We start by computing $\Delta H_{\mathbb{X}}(1)$ and $\Delta H_{\mathbb{X}}(2)$. We have that

$$H_{\mathbb{X}}(1) = \dim \mathbb{C}[z_0, \dots, z_N] - \dim(f_1, \dots, f_r)_1 = N - p + 1$$

$$\Delta H_{\mathbb{X}}(1) = H_{\mathbb{X}}(1) - H_{\mathbb{X}}(0) = N - p.$$

Now, let us consider the map

$$\begin{array}{ccc} \mathbb{C}[z_0, \dots, z_N] & \to & \mathbb{C}[z_0, \dots, z_N]/(f_1, \dots, f_p) \cong \mathbb{C}[z_0, \dots, z_{N-p}] \\ f & \mapsto & \tilde{f} \end{array}.$$

Since $(f_1, f_2, \dots, f_r)$ is a regular sequence we have that

$$H_{\mathbb{X}}(2) = \dim \Big(\mathbb{C}[z_0, \dots, z_N]/(f_1, \dots, f_r)\Big)_2 = \dim \Big(\mathbb{C}[z_0, \dots, z_{N-p}]/(\tilde{f}_{p+1}, \dots, \tilde{f}_r)\Big)_2$$

$$= \dim \mathbb{C}[z_0, \dots, z_{N-p}]_2 - q = \binom{N-p+2}{2} - q$$

and hence

$$\Delta H_{\mathbb{X}}(2) = \binom{N-p+2}{2} - q - (N - p + 1) = \frac{(N-p+1)(N-p)}{2} - q.$$

Note that $H_{\mathbb{X}}(1) = N - p + 1 < N + 1 = H_{V_{2,d}}(1)$, thus $t_1 = 0$. Since $t_1 = 0$ we have that

$$\mu_2(d,0,N-p) = \left\lfloor \frac{2d + 3 - \sqrt{1 + 8p}}{2} \right\rfloor,$$

so that by (1) we get

$$\left\lfloor \frac{2d + 3 - \sqrt{1 + 8p}}{2} \right\rfloor \geq \left\lceil \frac{(N-p+1)(N-p)}{2d} - \frac{q}{d} \right\rceil. \tag{5.5}$$

Since $q \leq N - p$, if (p, d, q) is a solution of (5.5), then (p, d) is a solution of

$$\frac{2d + 3 - \sqrt{1 + 8p}}{2} \geq \frac{(N-p)(N-p-1)}{2d}$$

and setting $\alpha = N - p$ the previous inequality yields

$$\underbrace{2d + 3 - \sqrt{(2d+3)^2 - 8(\alpha+1)}}_{f(\alpha)} \geq \underbrace{\frac{\alpha(\alpha-1)}{d}}_{g(\alpha)}. \tag{5.6}$$

A standard calculus argument on $f(\alpha)$ and $g(\alpha)$ shows that if (α, d) is a solution of (5.6) then $\alpha \leq 3$. As a consequence, if (p, d, q) is a solution of (5.5) then $p \geq N-3$, thus one has to look for solutions (p, d, q) only for $p = N-3, N-2, N-1$. Since $r \leq N$, one can easily check that the solutions of (5.5) are the following:

- $(p, d, q) = (N-1, d, 1)$ for each $d \geq 2$.

  In this case $\mathbb{X}$ is a complete intersection of type $(a_1, \ldots, a_N)$ with

  $$a_1 = \cdots = a_{N-1} = 1$$

  and $a_N = 2$, that is $\mathbb{X}$ is a set of two reduced points and we are in case *4*.

- $(p, d, q) = (3,2,1)$.

  In this case $N = 5$ and we know that $a_1 = a_2 = a_3 = 1$ and $a_4 = 2$, thus we distinguish two subcases:

  – If $r = 4$ then $\mathbb{X}$ is a complete intersection of type $(1,1,1,2)$, that is $\mathbb{X}$ is a conic lying on $V_{2,2}$ and we are in case *1*.

  – If $r = 5$ then $\mathbb{X}$ is a complete intersection of type $(1,1,1,2,a_5)$, that is $\mathbb{X}$ is the intersection of a conic lying on $V_{2,2}$ with a hypersurface of degree a_5 and we are in case *2*.

- $(p, d, q) = (3,2,2)$.

  In this case $\mathbb{X}$ is a complete intersection of type $(1,1,1,2,2)$, thus this is a special case of *2*.

- $(p, d, q) = (6,3,3)$.

  In this case $\mathbb{X}$ is a complete intersection of type $(1,1,1,1,1,1,2,2,2)$. We want to show that such $\mathbb{X}$ cannot lie on $V_{2,3}$. Let us suppose by contradiction that $\mathbb{X} \subseteq V_{2,3}$ and set $\mathbb{Y} = \nu_{2,3}^{-1}(\mathbb{X})$. From Lemma 5.1.1 it follows that

  $$H_{\mathbb{Y}}(3) = H_{\mathbb{X}}(1) = N - p + 1 = 9 - 6 + 1 = 4.$$

  Thus, the only way to complete the gaps of $H_{\mathbb{Y}}$ is

t	0	1	2	3	4	5	6	7	8	…
$H_{\mathbb{Y}}(t)$	1	2	3	4	5	6	7	8	8	…

and this shows that $\mathbb{Y}$ lies on a line. As a consequence, $\mathbb{X}$ lies on a rational normal curve. Hence, since $\mathbb{X}$ is a complete intersection and $|\mathbb{X}| = 8$, this is a contradiction by Theorem 5.3.3.

The result is now proved. ■

5.4 More results and open problems

In this section, we show some more results about complete intersections on Veronese varieties. A complete characterisation of the Hilbert functions of subvarieties of Veronese varieties of dimension larger than two seems, presently, to be out of reach. Nevertheless, we can deal with the case of the threefold $V_{3,2}$. This case leads us to formulate Conjecture 5.4.2.

Proposition 5.4.1. *Let $\mathbb{X} \subseteq V_{3,2} \subseteq \mathbb{P}^9$ be a reduced subvariety. Then $\mathbb{X}$ is a complete intersection of type $(a_1, \dots, a_r)$, with $a_1 \leq \cdots \leq a_r$ if and only if $\mathbb{X}$ is one of the following*

- $r = 9, a_1 = \ldots = a_9 = 1$, *that is $\mathbb{X}$ is a reduced point;*
- $r = 9, a_1 = \ldots = a_8 = 1, a_9 = 2$, *that is $\mathbb{X}$ is a set of two reduced points;*
- $r = 9, a_1 = \ldots = a_7 = 1, a_8 = 2, a_9 = b$, *any $b \geq 2$, that is $\mathbb{X} = \mathcal{C} \cap H_b$ for $\mathcal{C} \subseteq V_{3,2}$ a conic and H_b a degree b hypersurface;*
- $r = 8, a_1 = \ldots = a_7 = 1, a_8 = 2$, *that is $\mathbb{X}$ is a conic.*

Proof First we consider the case $\dim \mathbb{X} = 0$. Let $\mathbb{X} \subseteq V_{3,2}$ be a reduced complete intersection of type $(a_1, \dots, a_9)$ with $a_1 \leq a_2 \leq \cdots \leq a_9$. Also let $p, q \in \mathbb{N}$ such that $a_1 = \cdots = a_p = 1$ and $a_{p+1} = \cdots = a_{p+q+1} = 2$. By Proposition 5.3.2 either $p = 9$ or $p \geq 1$ and $q \geq 1$. If $p = 9$ then $\mathbb{X}$ is just a reduced point, thus from now on we suppose $p, q \geq 1$. By the same argument used in the proof of Theorem 5.3.5 we get

$$H_{\mathbb{X}}(1) = 10 - p, \quad H_X(2) = \binom{11-p}{2} - q = \frac{p^2 - 21p + 110}{2} - q.$$

Since $\mathbb{X} \subseteq V_{3,2}$, by Lemma 5.1.1 there exists $\mathbb{Y} = \nu_{3,2}^{-1}(\mathbb{X}) \subseteq \mathbb{P}^3$ such that $H_{\mathbb{X}}(t) = H_{\mathbb{Y}}(td)$ for all $t \geq 0$. In particular, we have that

$$H_{\mathbb{Y}}(2) = 10 - p, \quad H_{\mathbb{Y}}(4) = \frac{p^2 - 21p + 110}{2} - q.$$

Now fix $1 \leq p \leq 8$. By Macaulay's theorem for Hilbert functions (see Theorem 1.1.23) it follows that if the 2-binomial expansion of $H_{\mathbb{Y}}(2)$ is

$$H_{\mathbb{Y}}(2) = \binom{m_2}{2} + \binom{m_1}{1}$$

where $m_2 > m_1$, then

$$H_{\mathbb{Y}}(4) \leq \binom{m_2+2}{4} + \binom{m_1+2}{3} = M(p)$$

On the other hand, since $1 \leq q \leq 9-p$, we have that

$$H_{\mathbb{Y}}(4) \geq \frac{p^2-21p+110}{2} - (9-p) = m(p).$$

Thus, if $M(p) - m(p) < 0$, then $\mathbb{X}$ does not exist. Computing we get the following table

p	$H_{\mathbb{Y}}(2)$	$M(p)$	$m(p)$	$M(p)-m(p)$
1	9	25	37	-12
2	8	19	29	-10
3	7	16	22	-6
4	6	15	16	-1
5	5	9	11	-2
6	4	6	7	-1
7	3	5	4	1
8	2	2	2	0

Hence, $\mathbb{X}$ is either of type (1,1,1,1,1,1,1,2, a_9) or of type (1,1,1,1,1,1,1,1,2). In the first case $\mathbb{X}$ is a set of $2a_9$ reduced points lying on a conic $\mathcal{C} \subseteq V_{3,2}$ and in the second case $\mathbb{X}$ is a set of 2 reduced points. The discussion for reduced 0-dimensional complete intersection is now completed. Now let $\mathbb{X} \subseteq V_{3,2}$ be a positive dimensional reduced complete intersection of type $(a_1, \ldots, a_r)$. For any choice of integers $a_{r+1}, \ldots, a_9$, we can choose suitable hypersurfaces H_{a_i} of degree a_i in such a way that

$$\mathbb{X}' = \mathbb{X} \cap H_{a_{r+1}} \cap \ldots \cap H_{a_9}$$

is a complete intersection of type $(a_1, \ldots, a_r, a_{r+1}, \ldots, a_9)$. Moreover, we can choose the degrees a_i in a such a way that

$$a_i \leq a_{i+1}$$

for all i and $3 \leq a_{r+1}$. Thus $\mathbb{X}' \subseteq V_{3,2}$ is a zero dimensional complete intersection of type $(a_1, \ldots, a_9)$. As a consequence, since we can freely choose the degrees a_i for $i \geq r+1$ we have that $r = 8$ and

$$(a_1, \ldots, a_8, a_9) = (1{,}1{,}1{,}1{,}1{,}1{,}1{,}2, a_9).$$

Hence, $\mathbb{X}$ is of type (1,1,1,1,1,1,2), that is $\mathbb{X}$ is a conic. ■

We can now state the following conjecture, which we already proved for $n \leq 2$ any d and for $n = 3$ and $d = 2$, see Proposition 5.4.1 and Theorems 5.3.3 and 5.3.5.

Conjecture 5.4.2. *Let $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ be a reduced subvariety with $d > 1$. Then $\mathbb{X}$ is a complete intersection of type $(a_1, \dots, a_r)$, with $a_1 \leq \dots \leq a_r$ if and only if*

- *$r = N_{n,d}$, $a_1 = \dots = a_{N_{n,d}} = 1$, any n, d, that is $\mathbb{X}$ is a reduced point;*
- *$r = N_{n,d}$, $a_1 = \dots = a_{N_{n,d}-1} = 1$, $a_{N_{n,d}} = 2$, any n, d, that is $\mathbb{X}$ is a set of two reduced points;*
- *$r = N_{n,d}$, $a_1 = \dots = a_{N_{n,d}-2} = 1$, $a_{N_{n,d}-1} = 2$, $a_{N_{n,d}} = b$, any n, $d = 2$, any $a \geq 2$, that is $\mathbb{X} = \mathcal{C} \cap H_b$ for $\mathcal{C} \subseteq V_{n,2}$ a conic and H_b a degree b hypersurface;*
- *$r = N_{n,d} - 1$, $a_1 = \dots = a_{N_{n,d}-2} = 1, a_{N_{n,d}-1} = 2$, $d = 2$, any n, that is $\mathbb{X}$ is a conic.*

In the case of the Veronese threefold $V_{3,2}$, see proof of Proposition 5.4.1, the complete knowledge of the zero dimensional complete case allows us to complete the proof. This is true in general, as shown by the following Lemma.

Lemma 5.4.3. *If Conjecture 5.4.2 holds for all reduced zero dimensional subvariety of $V_{n,d}$, then it holds for all reduced subvarieties of $V_{n,d}$.*

Proof Let $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ be a reduced complete intersection of type $(a_1, \dots, a_r)$ with $r < N_{n,d}$, that is $\mathbb{X}$ is positive dimensional. Then, for any choice of integers $a_{r+1}, \dots, a_{N_{n,d}}$, we can choose suitable hypersurfaces H_{a_i} of degree a_i in such a way that

$$\mathbb{X}' = \mathbb{X} \cap H_{a_{r+1}} \cap \dots \cap H_{a_{N_{n,d}}}$$

is a complete intersection of type $(a_1, \dots, a_r, a_{r+1}, \dots, a_{N_{n,d}})$. Moreover, we can choose the degrees a_i in a such a way that

$$a_i \leq a_{i+1}$$

for all i and $3 \leq a_{r+1}$. Thus, $\mathbb{X}' \subseteq V_{n,d}$ is a zero dimensional complete intersection of type $(a_1, \dots, a_{N_{n,d}})$. Since we are assuming that the conjecture holds for such an $\mathbb{X}'$ and since we can freely choose the degrees a_i for $i \geq r+1$ we have that $r = N_{n,d} - 1$ and

$$(a_1, \dots, a_{N_{n,d}-1}, a_{N_{n,d}}) = (1, \dots, 1, 2, a_{N_{n,d}})$$

and thus $d = 2$ and $\mathbb{X}$ is a conic. Hence the conjecture holds for $\mathbb{X}$. ∎

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