Summary of "Zero-dimensional schemes: curves, singularities and tenors" .

In the wide panorama of Algebraic Geometry, a key role is played by Scheme Theory and a very noteworthy class of schemes is the one of zero-dimensional schemes. Indeed, apart from their intrinsic interest, zero-dimensional schemes deserve a particular and careful consideration in light of their several interactions with other fields of Algebraic Geometry. For instance:

i) many problems concerning secant varieties of projective varieties can be translated, via Apolarity Theory, in problems concerning zero-dimensional schemes; see [24] and [30] for more details about this topic;

ii) some intensely studied topics regarding plane algebraic curves, such as freeness and computation of Tjurina and Milnor numbers, are strictly related to the analysis of Jacobian schemes, which are zero-dimensional schemes encoding all the information about the singularities of the curve; see [38] and [43] for more details on free curves and [6] for more details on computations of Tjurina and Milnor number;

iii) zero-dimensional schemes are the constitutive elements of Hilbert schemes of points, a widely studied branch of Algebraic Geometry; see [35] and [36] for more details on Hilbert schemes.

Zero-dimensional schemes allow to establish deeper connections between these three topics; see [19], [20], [17],[18] and [36] for connections between i) and iii), and see [22] and [28] for connections between i) and ii). Also see [26] for a collection of topics about zero-dimensional schemes.

Beyond Algebraic Geometry, some other research fields where zero-dimensional schemes find applications are:

- Commutative Algebra, where they can be used, for instance, for a geometrical approach to Artin algebras and Gorenstein rings; see [37] for a recent state of the art;

- Code Theory for error-correcting codes associated to 0-dimensional schemes; see [10] for some application.

A particularly interesting class of zero-dimensional schemes is represented by fat points, which have long been, and still are, at the core of many Algebraic Geometry problems. Indeed, they represent a powerful tool for the study of many problems, such as the computation of the defectivity of some secant varieties and the study of singular points of projective varieties. These two aspects are among those that will be addressed in this thesis. Nowadays, our knowledge of fat points is certainly very rich, but nonetheless, there are still important open problems associated with them, such as the Gimigliano-Harbourne-Hirschowitz-Segre conjecture and, more in general, the complete classification of fat point schemes with bad postulation. See [30] for an exhaustive state of the art on fat points and see [1], [5], [2], [3], [4], [9], [7], [13], [11] for some applications of fat points to the study of secant varieties.

The main purpose of this thesis is to generalise fat points by introducing a new class of zero-dimensional schemes. In the literature, there already are some examples of such generalisations that broaden, for instance, the definition from $\mathbb{P}^n$ to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. An important example can be found in [25], where fat points in multiprojective spaces are used to study a classical problem concerning the dimension of certain secant varieties of Segre varieties. However, to our knowledge, there are no generalisations that, given the ambient space, specifically pertain to the geometry of the scheme itself. This is the type of generalisation that we are seeking in the thesis. More specifically we state the following question:

Question 1 *How can one define a new class of schemes in $\mathbb{P}^n$ that generalises the class of fat points?*

There are several motivations behind this question, some more related to the topics of this thesis.

The first one sprouts from studying the Jacobian scheme of a plane algebraic curve, which, in recent years, has proven to be of great interest. The two aspects most carefully considered in connection with the Jacobian scheme are the study of free divisors and the computation of Tjurina numbers for isolated singularities; see [27] for more details on the first topic and see [6] for more details on the second one. However, despite the extensive research on these topics, we are not aware of any analysis on the geometric structure of the Jacobian scheme. We address this gap and, carrying on this analysis for ordinary singularities, we note that fat points play an important role, but are not sophisticated enough to provide a satisfactory geometric analysis. For this reason, it is necessary to extend the definition of fat point and consider a broader class of schemes. This concept will be clarified below.

Another motivation comes from the application of algebraic geometry to the study of tensors. Indeed, zero-dimensional schemes have proven to be very useful tools for studying many problems in this context. Some important examples in this regard can be found in [9], [13], [11], [12], [19], [20], [18], [25], [23], [30], where zero-dimensional schemes are used to study and generalise secant varieties of some classical projective varieties, such as Segre, Veronese and Segre-Veronese varieties. In particular, new classes of zero-dimensional schemes can give new information on the geometry of tensors. We will show how to get this information via our new schemes.

This thesis is divided into five chapters. Chapter 1 is totally devoted to present the mathematical entities which are the objects of our research and to introduce the tools we will use to describe them. We start by recalling some general definitions and properties of Hilbert functions and fat points and by giving a quick overview on Apolarity Theory and Inverse Systems. After that, we introduce Segre, Veronese and Segre-Veronese varieties in the setting of Waring-like problems, stressing the interchangeability of the algebraic, geometrical and tensorial interpretations of these varieties. We also present the machinery of secant varieties, showing how zero-dimensional schemes can be used to study the defectivity of secant varieties. In particular, we briefly describe how the postulation of zero-dimensional schemes can be studied via the Horace method and the differential Horace method. Finally, we give some definitions about singularities of plane algebraic curves and we recall the Jacobian and Milnor schemes related to a plane algebraic curve.

In Chapter 2 we devote our attention to a special type of zero-dimensional schemes: the Jacobian scheme of a plane algebraic curve. To this purpose we start by proving an algebraic version for plane curves of the famous Mather-Yau theorem, stated in [40], which allows us to simplify the next results. After that, we focus on the Jacobian schemes at ordinary singularities, and this study suggests us the introduction of a new class of schemes answering Question 1: symmetric schemes. In Chapter 2 we give the definition only for the projective plane. We also provide some examples of ordinary singularities whose Tjruina number is strictly less than the Milnor number, so partially recovering, with more algebraic tools, some results of [16] and [39].

In Chapter 3 we give the definition of symmetric scheme for any $\mathbb{P}^n$ and we point out how symmetric schemes are a generalisation of fat points. We also introduce the definition of *superfat points* and we study the geometry of these new schemes. Since it is quite difficult to manage symmetric schemes in $\mathbb{P}^n$, after some general results, we narrow down to symmetric schemes of $\mathbb{P}^2$. After showing some of their properties, we use them to define some new varieties paremeterising symmetric and partially symmetric tensors. We study the defectivity of these varieties and the shape of the tensor parameterised by them.

In Chapter 4 we prove the good postulation of generic unions of 2-squares in $\mathbb{P}^2$. To do that, we use the Horace method ad we provide two different proofs. The two proofs only differ in proving the good postulation with respect to curves of even degrees, which are the hardest ones: in the first proof we use the differential Horace method, while in the second one we avoid to use the differential Horace method and we solve the problem giving an argument based on a particular property of 2-squares.

Finally, in Chapter 5 we deal with the classification of reduced zero-dimensional schemes lying on Veronese varieties. We show how this problem can be considered as a generalisation of the Cramer-Euler problem and we completely solve it for the case of Veronese surfaces. The main tool we use to give our classification is an accurate study of the possible Hilbert functions of reduced points on Veronese surfaces. We conclude the chapter with a Conjecture on complete intersections lying on Veronese varieties, inspired by the case of Veronese surfaces and by other experimental evidences.

From Jacobian schemes to symmetric schemes

The reasons that led us to pose Question 1 arose from studying a particular type of zero-dimensional schemes: the Jacobian scheme of a plane algebraic curve. However, before bringing up the "more sophisticated" Jacobian schemes, let us explore the origin of the idea of using zero-dimensional schemes, in particular fat points, to study the singularities of plane algebraic curves.

The first well-known remark is that if $\mathcal{C} : F = 0$ is a reduced curve of $\mathbb{P}^2$ passing through a point P, then saying that $\mathcal{C}$ has a singular point of multiplicity m at P means that $\mathcal{C}$ contains the fat point mP but not the fat point $(m+1)P$. This is a very rough information, because it does not allow to distinguish different analytical classes of singularities having the same multiplicity. Nevertheless, there are other 0-dimensional schemes contained in $\mathcal{C}$ which could characterise the singularity more carefully. For example, if P is an A_n singularity, then P is a nodal-type singularity if and only if for any $\ell \geq 1$ there is a curvilinear scheme supported at P of length ℓ contained in $\mathcal{C}$, while P is a cuspidal singularity A_{2r} if and only if for any $\ell \leq 2r+1$ there is a curvilinear scheme supported at P of length ℓ contained in $\mathcal{C}$, and no curvilinear scheme supported at P of length $> 2r+1$ is contained in $\mathcal{C}$ (see [31], Theorem 2.3). So, one possible approach to study a singularity is to understand which kind of "maximal" zero-dimensional schemes supported at P is contained in $\mathcal{C}$ but, since the curve $\mathcal{C}$ is 1-dimensional, it might contain curvilinear schemes supported at P of arbitrary lengths. We can undertake another way by using $\mathbb{X}(\mathcal{C})$, the Jacobian scheme of $\mathcal{C}$, which is defined as the subscheme of $\mathbb{P}^2$ associated to the Jacobian ideal

$$\mathbb{J}(\mathcal{C}) := \left(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) \subseteq \mathbb{C}[x_0, x_1, x_2].$$

Indeed, the Jacobian scheme is the zero-dimensional scheme encoding all the information, up to analytical equivalence, of all the singularities of $\mathcal{C}$.

An analogue of the Jacobian algebra $\mathbb{C}[x_0, x_1, x_2]/\mathbb{J}(\mathcal{C})$ can be defined, mutatis mutandis, also in the set of analytic geometry and it is a highly studied topic; see for instance [33] and [34]. As proved by a famous theorem of J. N. Mather and S. S.-T. Yau in [40], the Jacobian algebras of two analytical germs at O, both having an isolated singularity at O, are isomorphic as $\mathbb{C}$-algebras if and only if the two germs are analytically equivalent. Clearly, this result greatly simplifies the study of singularities up to analytical equivalence. However, it has the disadvantage, in case one wishes to work in an algebraic context, of requiring a transition from the algebraic setting to the analytic one. In order to avoid this transition, we prove, in Chapter 2, an analogue of Mather-Yau Theorem in a purely algebraic context.

Theorem 1 *Let $\mathcal{C} : f = 0$, $\mathcal{D} : g = 0$ be reduced algebraic curves in $\mathbb{A}^2$ with a singular point at O. Then the analytical germs of $\mathcal{C}$ and $\mathcal{D}$ at O are analytically equivalent if and only if their (algebraic) Jacobian schemes at O are isomorphic as schemes over $\mathbb{C}$.*

After proving this theorem, we focus on the geometry of Jacobian schemes at ordinary singularities and on their Tjurina and Milnor numbers. In doing so, we remark that these schemes possess a particular symmetry property: each line passing through their support intersects them with the same length. Let us be more precise.

Definition 1 Let Y be a 0-dimensional scheme supported at one point $P \in \mathbb{P}^2$. We say that Y is *k-symmetric* if, for every line r passing through P, $\ell(Y \cap r) = k$. We say that Y is a *k-symmetric local complete intersection* (k-slci for short) if it is a local complete intersection of two curves $\mathcal{D}, \mathcal{E}$ with no tangent in common at P and such that $m_P(\mathcal{D}) = k, m_P(\mathcal{E}) = k$, this implying $\ell(Y) = k^2$.

Clearly a k-slci is k-symmetric.

Theorem 2 *Let P be a multiple ordinary point of multiplicity m for a plane curve $\mathcal{C}$ in $\mathbb{P}^2$ and let Z_P be its Milnor scheme at P and X_P be its Jacobian scheme at P. Then:*

1. *the tangent cones of the derivative curves $\mathcal{C}_x, \mathcal{C}_y$ have no lines in common, hence $Z_P = (\mathcal{C}_x \cap \mathcal{C}_y)_P$ is a $(m-1)$-slci, so that $\mu = \ell(Z_P) = (m-1)^2$;*
2. *X_P is a $(m-1)$-symmetric scheme and $\tau = \ell(X_P) \leq (m-1)^2$;*
3. *in particular, if $\mathcal{C}$ is a union of m distinct lines through P, then $X_P = Z_P$, so that $\ell(X_P) = (m-1)^2$.*

It is precisely this theorem that has inspired Question 1. In fact, the only case in which the Jacobian scheme of an ordinary singularity is a fat point is the case of nodes, that is, double points with two distinct principal tangents. In all other cases, the obtained schemes are not fat points but share with them the symmetry property stated in Definition 1.

We will shortly discuss how Theorem 2 not only inspired Question 1 but also a possible answer to it. Before that, however, we want to emphasise that Theorem 2 also suggests another question:

Question 2 *Do there exist ordinary singularities whose Tjurina number is strictly less than the Milnor number?*

Questions of this kind date back to Zariski and appear quite often in Algebraic and Analytic Geometry; see for instance [6] and [44]. In fact, Question 2 already has a complete answer which can be recovered using some results of [16] and [39]. In Chapter 2, we state the result in the form of following theorem.

Theorem 3 *Let $\mathcal{C}$ be a plane algebraic curve and assume that $P \in \operatorname{Sing}\mathcal{C}$ is a multiple ordinary point of multiplicity $m \geq 2$. Then*

$$\left\lfloor \frac{3m^2 - 2m - 4}{4} \right\rfloor \leq \tau_P(\mathcal{C}) \leq (m-1)^2.$$

Moreover, the bounds are sharp and all the values of $\tau_P(\mathcal{C})$ occur.

Despite Question 2 being fully answered, in Chapter 2 we provide some explicit examples of ordinary singularities whose Tjruina number is strictly less than the Milnor number. Our examples are special cases of a more general class of curves given in [16], but there is a main difference: the approach used in [16] is analytical, while ours is entirely algebraic. We consider the family of curves

$$\mathcal{C}_{b,c} : x^m + y^m + x^b y^c = 0$$

with $b + c > m$, having an ordinary singularity at O and we compute the Tjruina number $\tau_O(\mathcal{C}_{b,c})$ using Gröbner basis. In particular, we prove that for $m \geq 5$ the curves $\mathcal{C}_{b,c}$ attain the lower bound in Theorem 3.

Symmetric schemes and tensors

As anticipated, the symmetric schemes inspired by Theorem 2 give a satisfying answer to Question 1. In Chapter 3 we start by generalising the definition of m-symmetric scheme and m-symmetric local complete intersection as follows.

Definition 2 A 0-dimensional scheme X supported at one point $P \in \mathbb{P}^n$ is said to be

- *m-symmetric* if $\ell(X \cap L) = m$, for every line L passing through P;
- an *m-symmetric local complete intersection* (m-slci for short) if it is a local complete intersection of n hypersurfaces having multiplicity at P equal to m and whose tangent cones at P have no line in common.

The reason why m-symmetric schemes are good candidates to generalise fat points, is that m-fat points are the prime example of m-symmetric schemes and, moreover, any m-symmetric scheme supported at $P \in \mathbb{P}^n$ contains the fat point mP. In other words, fat points are the m-symmetric schemes which are minimal with respect to the schematic inclusion. In light of that, we found quite natural to ask the following questions:

Question 3 *Among all the m-symmetric schemes supported at the same point P, which are the maximal ones with respect to schematic inclusion?*

Question 4 *What is the maximum length of an m-symmetric scheme?*

Since these points are, in some sense, "fatter" than fat points, we call the maximal m-symmetric schemes *m-superfat points* and we answer to both questions thanks to the following theorem.

Theorem 4 *A scheme $X \subseteq \mathbb{P}^n$ is an m-superfat point supported at $P \in \mathbb{P}^n$ if and only if it is an m-slci. Thus, any m-superfat point in $\mathbb{P}^n$ has length m^n and it is a Gorenstein scheme.*

We also stress the existence of a special class of m-superfat points of $\mathbb{P}^n$, that of m-hypercubes, i.e. m-superfat points defined by an ideal of the form $(\ell_1^m, \ell_2^m, \ldots, \ell_n^m)$ for $\ell_1, \ldots, \ell_n \in \mathbb{C}[x_0, \ldots, x_n]_1$ linearly independent linear forms. However, even though up to this moment we mentioned just m-fat points, m-superfat points and m-hypercubes, there are other schemes possessing m-symmetry.

This last remark shows how bad the situation can be in $\mathbb{P}^n$. For this reason, after we have given some general results on symmetric schemes, we narrow down to the case of $\mathbb{P}^2$, where the situation is easier to manage. The first noteworthy result in this direction is the coincidence of 2-superfat schemes of $\mathbb{P}^2$ with 2-squares, i.e. with the 2-hypercubes of $\mathbb{P}^2$.

Proposition 1 *Every* 2*-superfat scheme* $X \subseteq \mathbb{P}^2$ *is a 2-square, i.e.* $\mathcal{I}_X$ *can be written, up to some projectivity, as* $\mathcal{I}_X = (x_1^2, x_2^2)$.

As we show in some examples, this identification has nothing similar neither in higher dimension nor in higher degree. The other main result about superfat schemes in $\mathbb{P}^2$ is the following theorem, which allows us to relate fat points and superfat points.

Theorem 5 *For every* $P \in \mathbb{P}^2$ *and for any* $m \geq 1$*, the schematic union of all* m*-squares supported at* P *is the fat point* $(2m-1)P$.

A very classical issue when dealing with zero-dimensional schemes is the study of their postulation. For this reason, after analysing the aforementioned properties of 2-squares, we deemed appropriate to investigate the postulation of a generic union of 2-squares.

Question 5 *What is the postulation of a generic union of 2-squares in* $\mathbb{P}^2$*?*

In Chapter 4 we answer this question by showing that a generic union of 2-squares always has good postulation. The proof strategy we use is the "Horace method", introduced by J. Alexander and A. Hirschowitz in several papers, which we briefly recall in Chapter 1. We provide two different proofs, which agree for odd degrees but differ for the even ones. Indeed, the odd degree case can be solved using some simple specialisations, while the even one is more challenging.

In the first proof, we solve the problem by introducing a new specialisation: we collapse two 2-squares together, thus finding a new scheme that, with the help of differential Horace method, allows to bypass the arithmetic obstruction.

The idea of the second proof for even degrees is the following: we start by substituting one of the 2-squares with a double point contained in it, so obtaining a subscheme of the initial scheme and proving by induction that the number of conditions imposed on the degree d curves by this new scheme is one less than the expected number of conditions imposed by the initial scheme. After proving that, we conclude coming back to the original scheme and proving that when we pass from the double point to the 2-square we actually impose one more condition.

As we have already mentioned, zero-dimensional schemes have proven to be very useful in the study of varieties parameterising tensors. For this reason, once enough tools to handle the 2-squares are obtained, it is quite natural to pose the following question:

Question 6 *Is it possible to obtain new information about tensors using our new class of symmetric schemes? If so, what kind of information?*

We partially answer this question for the special case of 2-squares but, as we will recall in the list of open problems at the end of this introduction, we reckon that a general insight of symmetric schemes can provide a considerable information about symmetric and partially symmetric tensors. In our analysis, we consider some embeddings of 2-squares on Veronese and Segre-Veronese varieties, constructing a "bridge" between 2-squares and (partially) symmetric tensor. By doing so, we define new varieties, that we briefly describe here.

- $Q(V_{2,d})$

  We define

$$Q^0(V_{2,d}) := \bigcup_{Q \subseteq \mathbb{P}^2} L(\nu_{2,d}(Q)), \qquad Q(V_{2,d}) = \overline{Q^0(V_{2,d})}$$

where the union is taken on all the 2-squares Q of $\mathbb{P}^2$. Even though we show that $Q(V_{2,d}) = \tau_2(V_{2,d})$, and thus $Q(V_{2,d})$ is an already known variety, this new way of defining it gives a more refined description of the forms in $\tau_2(V_{2,d})$. As a consequence, we can show that $\tau_2(V_{2,d})$ is always contained in $\sigma_4(V_{2,d})$.

- $QQ(V_{2,d})$

  The description given by $Q(V_{2,d})$ highlights that the variety $\tau_2(V_{2,d})$ contains a 1-codimensional subvariety parameterising more particular forms, namely the ones that can be written (up to a projectivity in $\mathbb{P}^2$) as $y_0^{d-2}y_1y_2$. Let $d \geq 3$ and consider the morphism

$$\begin{array}{rccc} \Phi: & \mathbb{P}(T_1) \times \mathbb{P}(T_1) \times \mathbb{P}(T_1) & \to & \tau_2(V_{2,d}) \subseteq \mathbb{P}(T_d) \\ & ([\ell_0], [\ell_1], [\ell_2]) & \mapsto & [\ell_0^{d-2}\ell_1\ell_2] \end{array}.$$

  The *cuckoo variety* $QQ(V_{2,d})$ of $V_{2,d}$ is defined to be the scheme theoretic image of Φ, that is

$$QQ(V_{2,d}) := \operatorname{Im} \Phi.$$

  We show some geometrical properties of $QQ(V_{2,d})$ regarding, for instance, its tangent spaces and their intersections with $QQ(V_{2,d})$.

- $q_2(SV_{d,d})$

  After considering Veronese varieties, we move on to Segre-Veronese varieties and, more precisely, we consider the (d,d)-embeddings

$$\mathbb{P}^1 \times \mathbb{P}^1 \to SV_{d,d} \subseteq \mathbb{P}^{d^2-1}.$$

  Clearly, to do that we need to specify what we mean by a 2-square in $\mathbb{P}^1 \times \mathbb{P}^1$: given a point $P = [a_0, a_1; b_0, b_1]$ we call 2-square of $\mathbb{P}^1 \times \mathbb{P}^1$ supported at P the 0-dimensional subscheme $Q_P \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defined by the bihomogeneous ideal $(\ell_s^2, \ell_t^2) \subseteq \mathcal{R}$, where

$$\ell_s = a_1 s_0 - a_0 s_1, \quad \ell_t = b_1 t_0 - b_0 t_1.$$

  The reason why we choose these subschemes is because they allow, in some sense, to recover the usual 2-squares in $\mathbb{P}^2$. At this point we can define, for any $d \geq 2$, the following varieties:

$$q_2(SV_{d,d}) := \overline{\bigcup_{P \in \mathbb{P}^1 \times \mathbb{P}^1} L(sv_{d,d}(Q_P))}$$

  whose points correspond to partially symmetric tensors of the form

$$(a_0 s_0 + a_1 s_1)^d (b_0 t_0 + b_1 t_1)^d.$$

  We show that $q_2(SV_{d,d})$ and its secant variety $\sigma_2(q_2(SV_{d,d}))$ have the expected dimension for any $d \geq 2$. In particular, for $d = 2$ the secant variety $\sigma_2(q_2(SV_{d,d}))$ fills the whole $\mathbb{P}^8$, thus any partially symmetric tensor in $\mathbb{P}^8$ can be written as the sum of two partially symmetric tensors which depend only on four parameters each.

- $qq_2(SV_{(d,d)})$

  Analogously to the cuckoo varieties $QQ(V_{2,d})$, we define the *cuckoo varieties* $qq_2(SV_{d,d})$ as the image of the morphism

$$\begin{array}{ccc} \mathbb{P}(\mathcal{R}_1^{(1)}) \times \mathbb{P}(\mathcal{R}_1^{(1)}) \times \mathbb{P}(\mathcal{R}_1^{(2)}) \times \mathbb{P}(\mathcal{R}_1^{(2)}) & \to & q_2(SV_{d,d}) \subseteq \mathbb{P}(\mathcal{R}_{d,d}) \\ ([m_s], [n_s], [m_t], [n_t]) & \mapsto & [m_s^{d-1} n_s m_t^{d-1} n_t] \end{array}.$$

  For $d = 2$, $qq_2(SV_{2,2})$ is the Segre Variety $S_{2,2}$, which is well-known to be 2-defective, i.e. $\dim \sigma_2(S_{2,2}) = 7$. This does not happen for $d \geq 3$, as we show in the thesis.

Zero-dimensional schemes on Veronese varieties

In Chapter 5, we change a bit our perspective and we consider the following problem related to the geometry of Veronese varieties:

Question 7 *What are the possible complete intersections lying on a Veronese variety $V_{n,d}$?*

There are several reasons that make this question interesting. Indeed, complete intersections and their algebraic counterpart, regular sequences, play a central role in Commutative Algebra and in Algebraic geometry. We have examples ranging from the more classical and still open Hartshorne conjecture to modern applications in the field of geometry of tensor. In fact, complete intersections have recently been shown to have unexpected applications. For example, in [8] and [15], the strength and the slice rank of polynomials are studied using complete intersections. For a more exhaustive overview on complete intersections we advise to see [32].

Note that, for $d = 1$ and $n = 2$, the Veronese surface $V_{2,1}$ is the plane $\mathbb{P}^2$, so that our problem in this special case is exactly the Cramer-Euler problem, which consists in characterising the sets of points in $\mathbb{P}^2$ that are complete intersections. We answer Question 7 in the case of Veronese surfaces, showing that for $d > 2$ the only reduced complete intersections of $\mathbb{P}^{N_{n,d}}$ lying on $V_{2,d}$ are finite sets of either one or two points while, for the Veronese surface $V_{2,2} \subseteq \mathbb{P}^5$, one also has plane conics and their intersections with suitable hypersurfaces. More precisely, we prove the following theorem.

Theorem 6 *If $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^{N_{2,d}}$ is a reduced complete intersection of type $(a_1, \dots, a_r)$, with $a_1 \leq \cdots \leq a_r$ then one of the following holds:*

1. $(d, r, (a_1, a_2, \dots, a_r)) = (2, 4, (1, 1, 1, 2))$*, that is, $\mathbb{X}$ is a conic lying on $V_{2,2}$;*
2. $(d, r, (a_1, a_2, \dots, a_r)) = (2, 5, (1, 1, 1, 2, a_5))$*, any $a_5 \in \mathbb{N}$, that is, $\mathbb{X}$ is a set of $2a_5$ complete intersection points of a conic lying on $V_{2,2}$ and a hypersurface of degree a_5;*
3. $(d, r, (a_1, a_2, \dots, a_r)) = (d, N_{2,d}, (1, 1, \dots, 1))$ *for any $d \geq 2$, that is, $\mathbb{X}$ is a reduced point;*
4. $(d, r, (a_1, a_2, \dots, a_r)) = (d, N_{2,d}, (1, 1, \dots, 1, 2))$ *for any $d \geq 2$, that is, $\mathbb{X}$ is a set of two reduced points.*

In order to prove this theorem, we characterise the possible Hilbert functions of reduced subvarieties of Veronese varieties. Beyond their application to the proof of our theorem, Hilbert functions play a central role in Commutative Algebra and in Algebraic Geometry, for example see [42], [41], and [14]. In recent times Hilbert functions have also been used as tools in other fields, such as the study of Waring rank, that is the tensor rank for symmetric tensors, see [21], and the study of the identifiability of tensors.

In characterising these Hilbert functions, we generalise the notion of 0-sequences and of differentiable 0-sequences introduced in [29]. We give a more effective characterisation for the case of the rational normal curves $V_{1,d}$, thus recovering a classical result, and for the case of the surfaces $V_{2,d}$.

Moreover, we show that, except for the case $d = 2$, the only complete intersections lying on rational normal curves $V_{1,d}$ are the trivial ones, that is one single point or the set of two points. The case $V_{1,2}$, that is of a plane conic, is different. In fact, by cutting with any properly chosen curve, one will produce a complete intersection set of points. Inspired by this evidence we formulate a conjecture: the only reduced complete intersections of $V_{n,d}$, $d \geq 3$, are finite sets of either one or two points, while for $d = 2$ one also has plane conics and their intersections with suitable hypersurfaces. We also checked the validity of the conjecture for $V_{3,2}$.

Open problems

We list here some open problems related to the topics of this thesis.

1. The Jacobian scheme of a plane curve whose singularities are just double and triple ordinary points is a zero-dimensional scheme whose components are reduced points and 2-squares. What can be said about the freeness of the curve?

2. For $m = n = 2$, all the m-superfat points of $\mathbb{P}^n$ have maximal Hilbert function. This is not true for any other value of $m > 2$ and $n > 2$, but there is some evidence that the *generic* m-superfat point of $\mathbb{P}^n$ has maximal Hilbert function. Is this true?

3. Would it be possible to generalise the varieties $Q(V_{2,d}), QQ(V_{2,d}), q_2(SV_{d,d})$, $qq_2(SV_{22})$ by considering m-symmetric schemes more general than 2-squares? Clearly, this would require a deeper study of m-symmetric schemes.

4. Is it true that any generic union of m-hypercubes in $\mathbb{P}^n$ has good postulation? We just know that for $m = n = 2$.

5. Is it possible to find an "effective" characterisation of the Hilbert functions of subvarieties of $V_{n,d}$ for $n > 3$ similar to the one we found for $n = 2$?

6. Is it true that the only reduced complete intersections lying on a Veronese variety are the ones we listed in our conjecture?

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